Operations Research.

Defn of Operations Research (OR):

(i) OR is a scientific method of providing executive departments with a quantative basis for decisions under their control.

— P.M. Morse and G.E. Kimball.

Linear Programming Problem (LPP).

Linear Programming is a technique for determining an optimum schedule of interdependen activities in view of the available resources.

Procedure for mathematical formulation of a linear programming problem (LPP).

Step:1 Study the given situation to find the key decision to be made.

Step:2 Identify the variables involved and designate them by symbols x_j $(j = 1, 2, \ldots)$.

Step:3 State the feasible alternatives which generaly are : $x_j \geq 0$ for all j

Step:4 Identify the constraints in the problem and express them as linear inequalities (or) equations, LHS of which are linear functions of the decision variables.

Step:5 Identify the objective function and express it as a linear function of the decision variables.

General Linear Programming problem.

Let z be a linear function on R^n defined by

Maximize/ minimize $Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$ — (a)

where c_j's are constants $j = 1, 2, \ldots n$

$$\left.\begin{array}{l} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \geq (\text{or}) \leq (\text{or}) = b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \geq (\text{or}) \leq (\text{or}) = b_2 \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \geq (\text{or}) \leq (\text{or}) = b_m \end{array}\right\} \text{— (b)}$$

$$x_j \geq 0, \quad j = 1, 2, 3 \ldots n \quad \text{— (c)}$$

The problem of determining an n-tuple $(x_1, x_2, \ldots x_n)$ which makes z a minimum (or maximum) and which satisfies (b) and (c) is called the general linear programming problem.

The linear function (a) is called objective function, of the LPP.

The equations (b) are called constraints of the LPP.

The set of inequalities (c) are called the set of non-negative restrictions of the LPP.

x

Problem: A company makes two kinds of leather belts. Belt A is a high quality belt and belt B is of lower quality. The respective profits are 4.00 and 3.00 per belt. Each belt of type A requires twice as much time as a belt of type B.

~~the Company could make 1000 pe~~

and if all the belts were of type B, the Company could make 1000 perday. The supply of leather is sufficient for only 800 belts perday (Both A and B combined). Belt A requires a fancy buckle and only 400 per day are available. There are only 700 buckles a day available for belt B. Determine the optimal product mix.

problem: 2

A Company has three operational departments (weaving, processing and packing) with capacity to produce three different types of cloths namely suitings, Shirtings and wollens yeilding a profit of Rs 2, Rs 4 and Rs 3 per meter respectively. one meter of suiting requires 3 minutes in weawing, 2 minutes in processing and 1 minutes in packing. similarly one meter of Shirting requires 4 minutes in weawing, 1 minute in processing and 3 minutes in packing. One meter of woollen requires 3 minutes in each department. In a week, total run time of each department is 60, 40 and 80 hours for weaving, processing and packing respectively.

Formulate the linear programming problem to find the product mix to maximize the profit.

Soln:
From the problem,

	Departments			Profit (Rs per metre)
	weaving (in minutes)	pressing (in minutes)	packing (in minutes)	
suitings	3	2	1	2
shirtings	4	1	3	4
woollens	3	3	3	3
Availability (mins)	60x60=3600	40x60=2400	80x60=4800	

Step 1: Key decision : To determine the weekly rate of production for the three types of clothes.

Step 2: Let us designate the weekly production of suitings, shirtings, and wollens by ~~x_1, x_3~~ x_1 meters, x_2 meters and x_3 meters respectively.

Step 3: Since it is not possible to produce negative quntaties, feasible alternatives are sets of values satisfying $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Step 4: The constraints are the limited availability of three operational departments. One meter of suiting requires 3 minutes of weaving. The quantity being x_1 meters, the requirement for suiting alone will be $3x_1$ units. Similarly x_2 meters of shirting and x_3 meters of woollen will be requires $4x_2$ and $3x_2$ minutes respectively.

Thus the total requirement of weaving will be $3x_1 + 4x_2 + 3x_3$ which should not exceed the available 3600 minutes. So, the labour constraint becomes $3x_1 + 4x_2 + 3x_3 \leq 3600$.

Similarly the constraints for processing department is $2x_1 + x_2 + 3x_3 \leq 2400$ and for packing department $x_1 + 3x_2 + 3x_3 \leq 4800$.

Step 5: The objective is to maximize the total profit from sales. Assuming that whatever is produced is sold in the market, the total profit is given by the linear relation

$$z = 2x_1 + 4x_2 + 3x_3.$$

$\therefore$ The LPP is

maximize $z = 2x_1 + 4x_2 + 3x_3$

subject to the constraints

$$3x_1 + 4x_2 + 3x_3 \leq 3600$$

$$2x_1 + x_2 + 3x_3 \leq 2400$$

$$x_1 + 3x_2 + 3x_3 \leq 4800$$

$x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

Solution to the problem : 1

mathematical formulation.

Step 1: The key decision is to determine manufacturing rate per day for two types of belt.

Step 2: Let us designate per day manufacturing is x_1 numbers (or) belts in type A and x_2 belts in type B.

Step 3: Since it is not possible to manufactures negative number of belts, so $x_1 \geq 0$, $x_2 \geq 0$.

Step 4: To manufacture Type A belt requires twice the manufacturing time of type B belt and the total time available for type B belt can make 1000 hrs.

(ie) $2x_1 + x_2 \leq 1000$ (Time constraint).

The leather is available for 800 belts/day (for A + B).

(ie) $x_1 + x_2 \leq 800$

buckles available for Type A belt is 400/day

(ie) $x_1 \leq 400$.

buckles available for Type B belt is 700/day.

(ie) $x_2 \leq 700$.

Step 5: The objective function is profit function.

So, Maximize $Z = 4x_1 + 3x_2$.

$\therefore$ The mathematical formulation of the problem is

maximize $z = 4x_1 + 3x_2$

Subject to the constraints

$2x_1 + x_2 \leq 1000$

$x_1 + x_2 \leq 800$

$x_1 \leq 400$

$x_2 \leq 700$

$x_1 \geq 0, \quad x_2 \geq 0.$

Solve the above LPP by graphical solution.

Consider the constraints,

$2x_1 + x_2 \leq 1000, \quad \text{—} (1) \qquad x_1 + x_2 \leq 800 \text{ —} (2)$

$x_1 \leq 400, \text{—} (3) \qquad x_2 \leq 700 \text{ —} (4).$

From (1), $2x_1 + x_2 = 1000$

Substitue $x_2 = 0$,

$2x_1 + 0 = 1000 \Rightarrow 2x_1 = 1000$

$x_1 = \frac{1000}{2} = 500.$

one point on the eqn $A(500, 0)$.

Sub $x_1 = 0$ $\quad$ $2(0) + x_2 = 1000$

$0 + x_2 = 1000 \Rightarrow x_2 = 1000.$

another point on the eqn $B(0, 1000)$

From (2) $x_1 + x_2 = 800$

Sub $x_2 = 0$ $\quad$ $x_1 + 0 = 800$

$x_1 = 800$

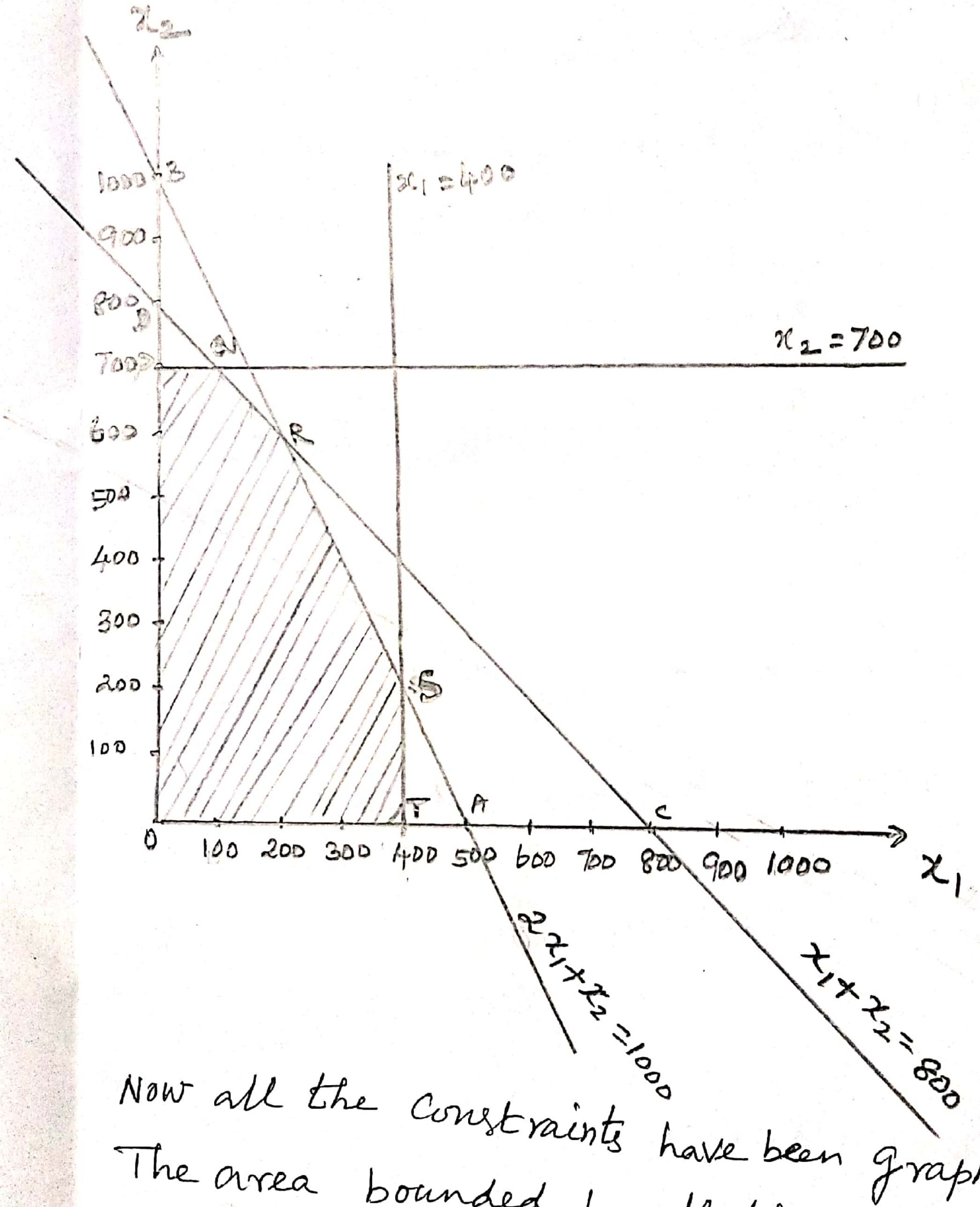
$\Rightarrow C(800, 0)$

~~From~~ Sub $x_1 = 0$, $0 + x_2 = 800$, $x_2 = 800$

$\Longrightarrow D(0, 800)$.

From 3 $x_1 = 400$

and from 4 $x_2 = 700$.



Now all the constraints have been graphed. The area bounded by all the constraints called feasible region (or) solution. ~~by the~~ it is the shaded area OPQRST.

To find the extrem points,

~~To find~~ To find Q, solve equations

$x_1 + x_2 = 800$ and $x_2 = 700$

$x_1 + 700 = 800 \implies x_1 = 800 - 700$
$\implies x_1 = 100.$

$\therefore$ Q is $(100, 700)$.

To find R, solve $2x_1 + x_2 = 1000$ and
$x_1 + x_2 = 800$

$$\begin{array}{rl} & 2x_1 + \cancel{x_2} = 1000 \\ (-) & x_1 + \cancel{x_2} = 800 \\ \hline & x_1 = 200 \\ \hline \end{array}$$

$x_1 + x_2 = 800 \implies 200 + x_2 = 800.$
$x_2 = 800 - 200$
$x_2 = 600.$

$\therefore$ R is $(200, 600)$

To find S, solve $2x_1 + x_2 = 1000$ and
$x_1 = 400$

$2(400) + x_2 = 1000.$
$x_2 = 1000 - 800$
$x_2 = 200$

$\therefore$ S is $(400, 200)$.

Now we compute the z values to the extreme points

$O(0,0)$, $\quad Z = 4x_1 + 3x_2 = 4(0) + 3(0) = 0$.

$\therefore Z = 0$.

$B(0,700)$. $\quad Z = 4x_1 + 3x_2$.

$Z = 4(0) + 3(700) = 0 + 2100$.

$Z = 2100$.

$Q(100,700)$ $\quad Z = 4x_1 + 3x_2$.

$= 4(100) + 3(700)$

$= 400 + 2100 = 2{,}500$

$\therefore Z = 2500$

$R(200,600)$ $\quad Z = 4x_1 + 3x_2$

$= 4(200) + 3(600)$

$= 800 + 1800 = 2{,}600$

$\therefore Z = 2{,}600$

$\cancel{T(400,0)}$

$\cancel{T(0,400)}$ $\quad \cancel{Z = 4x_1 + 3x_2}$

$\cancel{= 4(00)}$

$\cancel{= 4(400) + 3(0)}$.

$\cancel{Z = 1600}$.

$S(400,200)$ $\quad Z = 4x_1 + 3x_2$

$= 4(400) + 3(200)$

$= 1600 + 600$

$Z = 2{,}200$

$T(400,0)$ $\quad Z = 4x_1 + 3x_2$

$Z = 4(400) + 3(0)$.

$Z = 1600$

Z is a maximize function, so we find the maximum value.

Extreme points	Z value ($Z = 4x_1 + 3x_2$).
O (0,0)	0
P (0,700)	2,100
Q (100, 700)	2,500
R (200, 600)	2,600. ← maximum.
S (400, 200)	2,200
T (400, 0)	1,600.

$\therefore$ Maximum $Z = 2,600$.

$x_1 = 200$

$x_2 = 600$.

———— × ————

~~D-S~~ Defn: ~~Surplus~~ Artificial variable:

To obtain initial basic feasible solution, we put the given LPP into its standard form and a non-negative variable is added to the left side of each of equation that lacks the much needed starting basic variables. The so-added variable is called artificial variable.

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Dual problem

Associated with every LPP (max or mini) there always exists another LPP which is based upon the same data and having same solution. The original problem is called Primal problem while the associated one is called its dual problem.

Defn:

4

Basic Solution: Given a System of Simultaneous linear equations in n-unknowns $AX = b$, $(m<n)$; $X^T \in R^n$ where A is an $m \times n$ matrix of rank m. Let B be any $m \times n$ Submatrix, formed by m linearly independent columns of A. Then a solution obtained by setting $n-m$ variables not associated with columns of B, equal to zero and solving the resulting system is called a Basic Solution to the given symtem of equations.

The m variables, which may be all different from zeros, are called Basic ~~vectors~~ variables.

The $m \times m$ ~~non-singlu~~ non-Singular matrix B is called a Basis Matrix and the columns of B as Basis vectors.

Basic feasible solution: A basic solution to the System $AX = b$ is called Basic Feasible if $X_B \geq 0$.

Degenerate Solution: A basic solution to the system $AX = b$ is called Degenerate if one or more of the basic variables vanish.

Optimum Basic feasible solution: A basic feasible solution X_B to the LPP: $\max Z = CX$ Sub to $AX = b$ and $X \geq 0$ is called an optimum basic feasible solution if $Z_0 = C_B X_B \geq Z^*$ where Z^* is the value of the objective function for any feasible solution.

Defn:

Canonical form of LPP.

Maximize $z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

subject to the constraints

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n &\le b_1 \\
a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n &\le b_2 \\
&\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n &\le b_m
\end{aligned}
$$

$x_1, x_2, \dots, x_n \geq 0$.

Defn:

Standard form of LPP.

The general Linear programming problem in the form max (or) mini $z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

subject to the constraints

$$
\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n &= b_2 \\
&\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} &= b_m
\end{aligned}
$$

$x_1, x_2, \dots, x_n \geq 0$.

Slack variable: If the constraint of LPP is '$\le$' type, the inequality constraint can be changed into equation by adding a non-negative variabe, that variable is called slack variable.

Surplus variable: If the constraint of LPP is '$\geq$' type, the inequality constraint can be changed into equation by subtracting a non-negative variable, that variable is called surplus variable.

Defn: **Feasible Solution**: Any solution to a general LPP which also satisfies the non-negative restrictions of the problem is called a feasible solution to the general LPP.

Defn: Optimum Solution: Any feasible solution which optimizes (minimize (or) maximize) the objective functions of a General LPP is called an optimum solution to the general LPP.

Defn: **Slack variable**: Let the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad a_i = 1, 2, \dots k,$$

Then, the non-negative variables x_{n+i} which satisfy

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \quad i = 1, 2, \dots k$$

are called slack variable.

Defn: **Surplus variable**.

Let the constraints of a General LPP be

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad a_i = k+1, k+2, \dots l$$

Then, the non-negative variable x_{n+i} which satisfy

$$\sum_{j=1}^{n} a_{ij} x_j - x_{n+i} = b_i \quad i = k+1, k+2, \dots l$$

are called surplus variable.

What is linear programming?

Linear programming is a mathematical technique for choosing the best alternate from a set of feasible alternatives, in situations where the objective function as well as the restrictions or constraints can be expressed as linear mathematical function.

General Linear programming Problem:

Let z be a linear function on R^n defined by

Maximize (or) Minimize $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ — (1)

where c_j's are constant.

Let (a_{ij}) be an $m \times n$ real matrix and let $\{b_1, b_2, \ldots b_m\}$ be a set of constants

such that

$$(2) \quad \left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq (\text{or}) \leq (\text{or}) = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq (\text{or}) \leq (\text{or}) = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq (\text{or}) \leq (\text{or}) = b_m \end{array}\right.$$

and finally let $x_j \geq 0$ — (3)

The problem of determining an n-tuple $(x_1, x_2, \ldots x_n)$ which makes z a minimum (or) maximum and which ~~satisfies~~ satisfies (2) and (3).

(1) is called Objective function

(2) Constraints

(3) Non-negative restritions.

Defn: Solution: An n-tuple (x_1, x_2, x_3) of real numbers which satisfies the constraints of a general Linear programming is called a solution to the general LPP.

Solve the LPP by graphical method.

maximize $z = 2x_1 + 4x_2$

Subject to the constraints,

$x_1 + 2x_2 \leq 5$

$x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$.

~~Solve the~~

Solution:

Consider the constraint $x_1 + 2x_2 \leq 5$.

we take

$x_1 + 2x_2 = 5$.

put $x_2 = 0$ $\quad$ $x_1 + 2(0) = 5$

$x_1 = 5$

$A(5, 0)$.

put $x_1 = 0$ $\quad$ $0 + 2x_2 = 5$.

$x_2 = 5/2 = 2.5$

$B(0, 2.5)$.

Consider $x_1 + x_2 \leq 4$.

we take $x_1 + x_2 = 4$.

put $x_2 = 0$ $\quad$ $x_1 + 0 = 4$.

$x_1 = 4$.

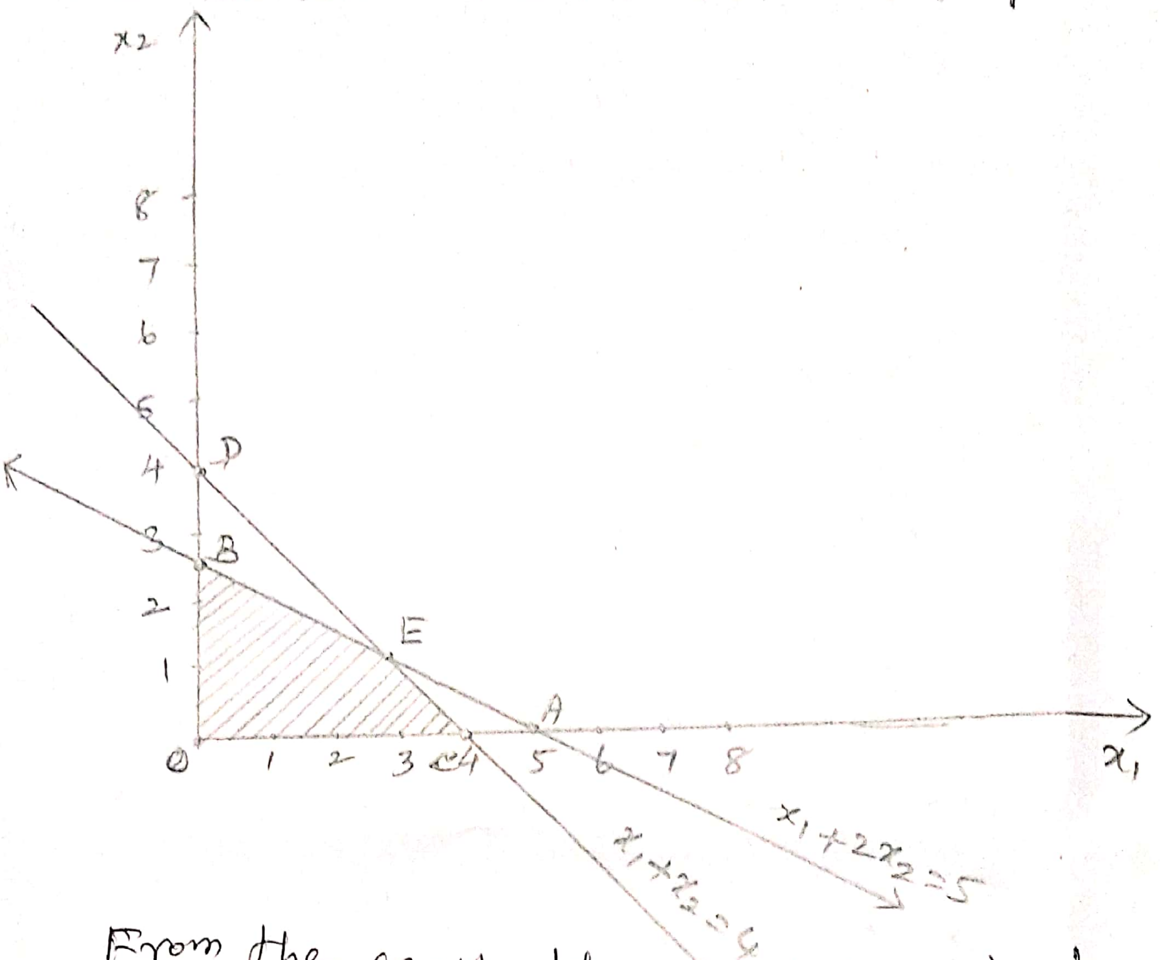
$C(4, 0)$.

put $x_1 = 0$. $\quad$ $0 + x_2 = 4$.

$x_2 = 4$.

$D(0, 4)$.

Now we draw the lines in the graph.



From the graph the feasible region is OBEC.

To find the extreme points

O (0,0), B(0,2.5) and C(4,0).

But we must find E. That point E is intersect point of the equations $x_1+2x_2=5$ and $x_1+x_2=4$.

We solve the equations

$$\begin{array}{r} \cancel{x_1} + 2x_2 = 5 \\ \cancel{x_1} + x_2 = 4 \\ \hline x_2 = 1 \end{array}$$

Sub $x_2 = 1$. in any one eqn

$$x_1 + x_2 = 4.$$
$$x_1 + 1 = 4.$$
$$x_1 = 3.$$

$\therefore$ E is (3,1).

Now we find Z values of extreme points.

extreme points	Z values.
O(0,0)	$Z = 2(0) + 4(0) = 0$.
B(0, 2.5)	$Z = 2(0) + 4(2.5) = 10 \leftarrow$ maximum
E(3,1)	$Z = 2(3) + 4(1) = 10 \leftarrow$ maximum
C(4,0)	$Z = 2(4) + 4(0) = 8$

The given objective function is a profit function.

$\therefore$ The maximum value of $Z = 10$

That arries in two extrem points.

So, the optimum solution exists more that one points.

The solution is $\max Z = 10$
$x_1 = 0, x_2 = 2.5$

The alternate solution is $\max Z = 10$
$x_1 = 3, x_2 = 1$.

Solve the LPP by graphical solution.

(4)

Maximize $z = 6x_1 + x_2$

Subject to the constraints

$2x_1 + x_2 \geq 3$

$x_2 - x_1 \geq 0$

and $x_1, x_2 \geq 0$.

Solution: Consider the constraint

$2x_1 + x_2 \geq 3$.

Now we take $2x_1 + x_2 = 3$.

Sub $x_2 = 0$, $2x_1 + 0 = 3$.

$x_1 = \frac{3}{2} = 1.5$

A(1.5, 0).

Sub $x_1 = 0$

$2(0) + x_2 = 3$.

$x_2 = 3$.

B(0, 3).

Consider $x_2 - x_1 \geq 0$.

We take $x_2 - x_1 = 0 \Rightarrow x_2 = x_1$.

~~Sub $x_1 = 0$~~

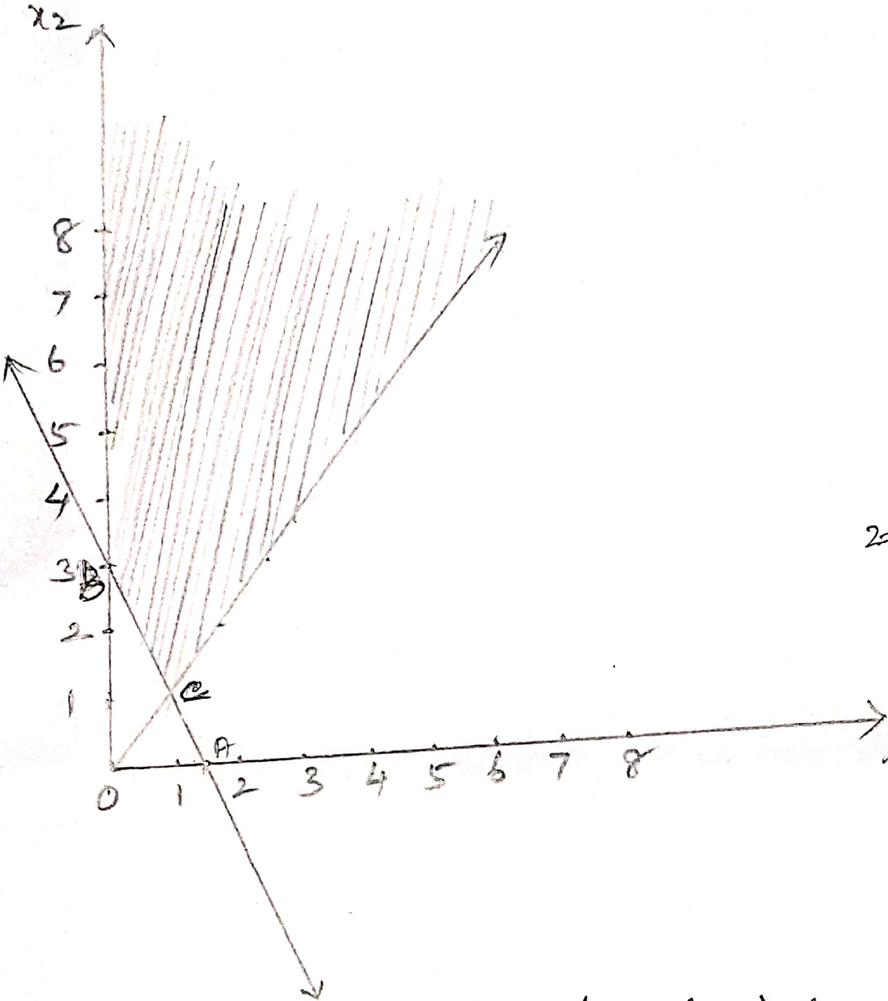
$x_2 = 0 \Rightarrow x_1 = 0$

$x_2 = 1 \Rightarrow x_1 = 1$

$x_2 = 2 \Rightarrow x_1 = 2$.

(ie) The line lies on (0,0), (1,1) (2,2), (3,3) ---

Now we draw the graph.



Solve

$2x_1 + x_2 = 3$.
$-x_1 + x_2 = 0$

$3x_1 = 3$
$x_1 = 1$.

$2x_1 + x_2 = 3$
$2(1) + x_2 = 3$
$\Rightarrow x_2 = 1$.
$\Rightarrow (1,1)$.

(ie) the intersect ~~point~~ point C is $(1,1)$

From this graph, the feasible region unbounded

In this region we have two extreme points B and C.

~~The~~ The value of Z at B & C are

extrem points	Z value.
B $(0,3)$	$Z = 6x_1 + x_2 = 0+3 = 3$.
C $(1,1)$	$Z = 6(1)+1 = 7$.

But there exists number of points in the feasible region for which the value of the objective function is more than 7. ∴ The problem gives unbounded solution. That is it gives infinite number of solutions.

Simplex Method

Simplex Algorithm:

Step 1: Check weather the objective function of the given LPP is to be maximized (or) minimized. If it is to be minimized then we convert it into a problem of maximization it by using the result.

$$\text{minimum } z = -\text{maximum}(-z)$$

Step 2: Check wether all b_i $(i = 1, 2, \dots m)$ are non-negative. If any one of b_i is negative then multiply the corresponding inequation of the constraints by -1, so as to get all b_i $(i = 1, 2, \dots, m)$ non-negative.

Step 3: Convert all the inequations of the constraints in to equations by introducing slack and/or surplus variables in the constraints. put the costs of these variables equal to zero.

Step 4: obtain an initial basic feasible solution to the problem in the form $X_B = B^{-1}b$ and put in the first column of the Simple table.

Steps: Compute the net evaluations $z_j - c_j$ $(j = 1, 2, \cdots, n)$ by using the relation $z_j - c_j = c_B y_j - c_j$

Examine the sign $z_j - c_j$

(i) If all $(z_j - c_j) \geq 0$ then the initial basic feasible solution x_B is an optimum basic feasible solution.

(ii) If ~~atleast~~ one $(z_j - c_j) < 0$, proceed on to the next step.

Step6: If there are more than one negative $z_j - c_j$, then Choose the most negative of them. Let it be $z_r - c_r$ for some $j = r$

(i) If all $y_{ir} \leq 0$ $(i = 1, 2, \cdots, m)$, then there is an unbounded solution to the given problem

(ii) If atleast one $y_{ir} > 0$ $(i = 1, 2, \cdots, m)$ then the corresponding vector y_r enters the basis y_B.

Step 7: Compute the ratios $\left\{ \frac{x_{Bi}}{y_{ir}}, y_{ir} > 0, i=1,2,\cdots,m \right\}$ and choose the minimum of them. Let the minimum of these ratios be x_{BK}/y_{Kr}. Then the vector y_K will level the basis y_B. The common element y_{Kr}, which is in the K^{th} row and r^{th} column is known as the leading ~~(or pivotal element)~~ element (or pivotal element) of the table.

Step 8: Convert the leading element to unity by dividing its row by the leading element itself and all other elements in its column to zeros by making use of the relations:

$$\hat{y}_{ij} = y_{ij} - \frac{y_{Kj}}{y_{Kr}} y_{ir}, \quad i=1,2,\cdots,m+1; \; i \neq K$$

and

$$\hat{y}_{Kj} = \frac{y_{Kj}}{y_{Kr}}, \quad j=0,1,2,\cdots,n.$$

Step 9: Go to step 5 and repeat the computational procedure until either an optimum solution is obtained or there is an indication of an unbounded solution.

Problem :1

Solve the following LPP by Simplex method.

maximize $z = 4x_1 + 10x_2$

Subject to the constraints:

$$2x_1 + x_2 \le 50$$
$$2x_1 + 5x_2 \le 100$$
$$2x_1 + 3x_2 \le 90$$
$$x_1, x_2 \ge 0$$

solution:

Introducing slack variables, the LPP Can be written as

maximize $z = 4x_1 + 10x_2 + 0.x_3 + 0.x_4 + 0.x_5$

subject to the Constraints

$$2x_1 + x_2 + x_3 = 50$$

$$2x_1 + 5x_2 + x_4 = 100$$

$$2x_1 + 3x_2 + x_5 = 90$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

Now we represent the initial simplex table.

		c_j:	4	10	0	0	0
C_B	Y_B		y_1	y_2	y_3	y_4	y_5
0	y_3	50	2	1	1	0	0
0	y_4	100	2	5	0	1	0
0	y_5	90	2	3	0	0	1
$z_j - c_j$		$z = 0$	-4	$-10\uparrow$	0	0	0

In the above table, Two values of $z_j - c_j$ are negative. Now we choose most negative of these two values.

$\therefore$ the most negative of these two values is -10.

The corresponding column vector y_2 enters the basis.

Now, we find the leading element.

Since all the entries of y_2 are positive, we compute $\min\left\{\frac{x_{Bi}}{y_{ir}}, y_{ir} > 0\right\}$;

(ie) $\min\left\{\frac{50}{1}, \frac{100}{5}, \frac{90}{3}\right\} = \frac{100}{5}$. This occurs for the element $y_{22} = 5$. Thus the vector y_4 will leave the Basis y_B and the common element y_{22} becomes the leading element for the first element.

Now, we convert the leading element y_{22} to unity, and all other elements of y_2 to zeroes by making use of the following transformation.

$$\hat{y}_{ij} = y_{ij} - \frac{y_{2j}}{y_{22}} y_{i2} \; ; \quad i = 1,2,3,4, \; i \neq 2$$

and

$$\hat{y}_{2j} = \frac{y_{2j}}{y_{22}} \; ; \quad j = 0,1,2,\cdots 5$$

$\therefore \hat{y}_{21} = \dfrac{y_{21}}{y_{22}} = \dfrac{2}{5}$; $\hat{y}_{20} = \dfrac{y_{20}}{y_{22}} = \dfrac{100}{5} = 20$

$\hat{y}_{10} = y_{10} - \dfrac{y_{20}}{y_{22}} y_{12} = 50 - 20 \times 1 = 30$

$\hat{y}_{30} = \cancel{y_{30} - \dfrac{y_{21}}{y_{22}} y_{32}}\; y_{30} - \dfrac{y_{20}}{y_{22}} y_{32} = 90 - 20 \times 3 = 30$

$\hat{y}_{31} = y_{31} - \dfrac{y_{21}}{y_{22}} y_{32} = 2 - \dfrac{2}{5} \times 3 = \dfrac{4}{5}$

$\hat{y}_{11} = y_{11} - \dfrac{y_{21}}{y_{22}} y_{12} = 2 - \dfrac{2}{5} \times 1 = \dfrac{8}{5}$

$\hat{y}_{14} = y_{14} - \dfrac{y_{24}}{y_{22}} y_{12} = 0 - \dfrac{1}{5} \times 1 = -\dfrac{1}{5}$

and so on.

Now we form the table using these values.

		C_j:	4	10	0	0	0
C_B	y_B	X_B	y_1	y_2	y_3	y_4	y_5
0	y_3	30	$8/5$	0	1	$-1/5$	0
10	y_2	20	$2/5$	1	0	$1/5$	0
0	y_5	30	$4/5$	0	0	$-3/5$	1
$z_j - c_j$		$z = 200$	0	0	0	2	0

In the above table all $z_j - c_j \geq 0$. $\therefore$ The optimum solution exists. $\max z = 200$.

$x_1 = 0,\ x_2 = 20$.

Problem : 2.

Use simplex method to maximize $z = 5x_1 + 4x_2$

Subject to the constraints,

$4x_1 + 5x_2 \le 10,$

$3x_1 + 2x_2 \le 9,$

$8x_1 + 3x_2 \le 12$

$x_1 \geq 0, \quad x_2 \geq 0.$

Solution: ~~The~~ using the slack variables, the LPP can be written as

maximize $z = 5x_1 + 4x_2 + 0.x_3 + 0.x_4 + 0.x_5$

Sub to $4x_1 + 5x_2 + x_3 = 10$

$3x_1 + 2x_2 + x_4 = 9$

$8x_1 + 3x_2 + x_5 = 12$

$x_1, x_2, \ldots, x_5 \geq 0.$

		c_j:	5	4	0	0	0	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	ratio
0	Y_3	10	4	5	1	0	0	$\frac{10}{4} = 5/2$
0	Y_4	9	3	2	0	1	0	$\frac{9}{3} = 3$
0	Y_5	12	8	3	0	0	1	$\frac{12}{8} = \frac{3}{2}$
	$z_j - c_j$	$z = 0$	$-5\uparrow$	-4	0	0	0	

In the above table two $z_j - c_j$ values are negative. So, we go to next iteration.

$z_j - c_j$ values are having $-5, -4$ are negative. So, -5 is the most negative. In the above the first column is pivot column. So y_1 enters the basis. Then find pivot row. Take the ratios to basic feasible solution and pivot column values. In three values $3/2$ is minimum. ∴ third row is pivot row. from this the element 8 is pivot or leading element and y_5 leaves the basis.

Now construct the table,

		c_j:	5	4	0	0	0	
C_B	Y_B	X_B	y_1	y_2	y_3	y_4	y_5	ratio
0	y_3	4	0	$7/2$	1	0	$-1/2$	$4 \times \frac{2}{7} = \frac{8}{7}$
0	y_4	$9/2$	0	$7/8$	0	1	$-3/8$	$\frac{9}{2} \times \frac{8}{7} = \frac{36}{7}$
5	y_1	$3/2$	1	$3/8$	0	0	$1/8$	$\frac{3}{2} \times \frac{8}{3} = 4$
$z_j - c_j$		$z = 15/2$	0	$-17/8$ ↑	0	0	$5/8$	

$z_j - c_j$ having one negative value. so, go to next iteration.

		C_j:	5	4	0	0	0	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	ratio
4	Y_2	$8/7$	0	1	$\frac{2}{7}$	0	$-\frac{1}{7}$	
0	Y_4	$7/2$	0	0	$-1/4$	1	$-1/2$	
5	Y_1	$15/14$	1	0	$-3/28$	0	$5/28$	
$Z_j - C_j$		$Z = 139/14$	0	0	$17/28$	0	$9/28$.	

All $Z_j - C_j \geq 0$. $\therefore$ The above table is optimum.

$$\max Z = 139/14$$
$$x_1 = 5$$
$$x_2 = 4.$$

---

problem : 3

$$\max Z = 107x_1 + x_2 + 2x_3$$

Subject to the Constraints

$$\cancel{14x_1 + x_2 - 6x_4 = 7}$$

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7$$
$$16x_1 + x_2 - 6x_3 \leq 5$$
$$3x_1 - x_2 - x_3 \leq 0$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

Solution:

The given LPP can be written as

$$\max z = 107x_1 + x_2 + 2x_3$$

Sub to

$$\frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 = \frac{7}{3}$$

$$16x_1 + x_2 - 6x_3 \le 5$$

$$3x_1 - x_2 - x_3 \le 0$$

$$x_1, x_2, x_3, x_4 \ge 0.$$

using slack variable

$$\max z = 107x_1 + x_2 + 2x_3 + 0.x_4 + 0.x_5 + 0.x_6$$

Sub to

$$\frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 = 7/3$$

$$16x_1 + x_2 - 6x_3 + x_5 = 5$$

$$3x_1 - x_2 - x_3 + x_6 = 0$$

$$x_1, x_2, \ldots, x_6 \ge 0.$$

		C_j:	107	1	2	0	0	0	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	ratio
0	Y_4	$7/3$	$\frac{14}{3}$	$1/3$	-2	1	0	0	$\frac{7}{3}\times\frac{3}{14}=\frac{1}{2}$
0	Y_5	5	16	1	-6	0	1	0	$\frac{5}{16}$
0	Y_6	0	3	-1	-1	0	0	1	$\frac{0}{3}=0$
$Z_j - C_j$		$Z = 0$	$-107\uparrow$	-1	-2	0	0	0	

Some $Z_j - C_j < 0$ (ie) $Z_j - C_j$ have negative values. we go to next iterative.

From the table, y_6 leaves the Basis y_1 enters the basis.

C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	ratio
C_j:			107	1	2	0	0	0	
0	Y_4	$7/3$	$\cancel{44/3}$ 0	$17/9$	$-4/9$	1	0	$-14/9$	
0	Y_5	5	0	$19/3$	$-2/3$	0	1	$-16/3$	
107	Y_1	0	1	$-1/3$	$-1/3$	0	0	$1/3$	
$z_j - c_j$		$z = 100$	0	$-110/3$	$-113/3 \uparrow$	0	0	$107/3$	

Some $z_j - c_j \le 0$, ~~second Third co~~ $z_3 - c_3$ is the most negative, that is the pivot colum. In the pivot column all the elements are negative. We cannot take ratio. It indicates that there is an unbounded solution to the given LPP.

Problem : 4

Minimize $Z = x_2 - 3x_3 + 2x_5$

Sub to the constraints

$$3x_2 - x_3 + 2x_5 \le 7$$
$$-2x_2 + 4x_3 \le 12$$
$$-4x_2 + 3x_3 + 8x_5 \le 10$$
$$x_2, x_3, x_5 \ge 0.$$

Solution: The LPP can be written as,

Maximize $Z^* = -x_2 + 3x_3 - 2x_5 + 0\cdot x_6 + 0\cdot x_7 + 0\cdot x_8$

Sub to

$$3x_2 - x_3 + 2x_5 + x_6 = 7$$

$$-2x_2 + 4x_3 + 0\cdot x_5 + x_7 = 12$$

$$-4x_2 + 3x_3 + 8\cdot x_5 + x_8 = 10$$

~~$x_2, x_3, x_5, x_6, x_7, x_8 \geq 0$~~

$$x_2, x_3, x_5, x_6, x_7, x_8 \geq 0$$

C_B	Y_B	X_B	C_j: -1 Y_1	3 Y_2	-2 Y_3	0 Y_4	0 Y_5	0 Y_6	ratio
0	Y_4	7	3	1	2	1	0	0	$\frac{7}{1} = 7$
0	Y_5	12	-2	4	0	0	1	0	$\frac{12}{4} = 3$
0	Y_6	10	-4	3	8	0	0	1	$\frac{10}{3} = 3\frac{1}{3}$
$Z_j - C_j$		$Z = 0$	1	$-3\uparrow$	2	0	0	0	

One $Z_j - C_j < 0$. We go to next iteration.

C_B	Y_B	X_B	C_j: -1 Y_1	3 Y_2	-2 Y_3	0 Y_4	0 Y_5	0 Y_6	ratio
0	Y_4	10	$5/2$	0	0	1	$1/4$	0	$10 \times \frac{2}{5} = 4$
3	Y_2	3	$-1/2$	1	2	0	$1/4$	0	—
0	Y_6	1	$-5/2$	0	8	0	$-3/4$	1	—
$Z_j - C_j$		$Z = 9$	$-1/2\uparrow$	0	2	0	$3/4$	0	

one $z_j - C_j < 0$, we go to next iteration.

		C_j:	-1	3	-2	0	0	0	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	ratio
-1	Y_1	4	1	0	4/5	2/5	1/10	0	
3	Y_2	5	0	1	2/5	1/5	3/10	10	
0	Y_6	11	0	0	10	2/5	$-\cancel{1/2}$	1	
$z_j - C_j$		z = 11	0	0	12/5	1/5	8/10	0	

All $z_j - C_j \geq 0$. The table is optimum.

$\therefore$ The solution is

minimize = minimize $z^* = -11$

$x_2 = 4$

$x_3 = 5$

$x_5 = 0$

Solve the LPP max $z = 4x_1 + x_2 + 3x_3 + 5x_4$.

Sub to $4x_1 - 6x_2 - 5x_3 - 4x_3 \geq -20$

$3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$

$8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20$

$x_1, x_2, x_3, x_4 \geq 0$

Soln: The LPP can be written as

max $z = 4x_1 + x_2 + 3x_3 + 5x_4$

sub to

$-4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 20$

$3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$

$8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20$

$x_1, x_2, x_3, x_4 \geq 0$

Define: Transportation Problem (TP)

The transportation Problem deals with the transportation of a single product from several sources (orgins (or) Supply (or) Capacity centers) to several sinks (destinations (or) demand (or) requirement Centres). Ingeneral, let there be m sources $S_1, S_2, \cdots S_m$ with a_i ($i = 1, 2, 3, \cdots, m$) available supplies or capacity at each source i, to be ~~allowed~~ allocated among n destinations $D_1, D_2, \cdots D_n$ with b_j ($j = 1, 2, \cdots, n$) specified requirements at each destination j. Let c_{ij} be the cost of shipping one unit from source i to destination j for each route. Then if x_{ij} be the units shipped per rute from source i to destination j, the promblem is determine the transportation schedule so as to minimize the total transportation cost satisfying the supply and demand conditions.

Mathematical formulation of TP.

$$\text{minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to the constraints

$$x_{i1} + x_{i2} + \cdots + x_{in} = a_i$$

$$i = 1, 2, \cdots, m$$

$$x_{1j} + x_{2j} + \cdots + x_{mj} = b_j, \quad j = 1, 2, \cdots n$$

and $x_{ij} \geq 0$ for all i and j

For a feasible solution to exists, it is necessary that total supply equals total requirement,

(ie)
$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad (\text{Rim Condition})$$

Defn: Degenerate solution: The basic solution ~~In~~ TP is degenerate if the number of occupied (or) allocated cells is not equal to ~~the~~ no of rows + no of columns − 1. ~~is called gene degenerate solution of the TP.~~

(ie) The transportation problem have m rows and n columns then m+n−1 ≠ no of occupied ~~allocated~~ cells.

Defn: unbalanced Transportation problem.

In a TP total no of units supply is not equal to ~~total no of~~ total no of units demand

(ie) Total demand ≠ Total supply.

then the TP called unbalanced TP.

Defn: **Assignmen problem**: The assignment problem is a special case of the transportion problem in which the objective is to assign a number of origins to the equal number of destinations at a minimum Cost (or maximum profit).

Mathematical formulation of Assignmen problem

minimize $z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ and } \sum_{j=1}^{n} x_{ij} = 1;$$

$$x_{ij} = 0 \text{ (or) } 1.$$

for all $i = 1, 2, 3 \ldots n$ and $j = 1, 2, \ldots, n$.

unbalenced assignment problem

In an Assignment problem the number of rows is not equal to number of columns that problem is called unbalenced assignment problem.

(ie) number of rows $\neq$ number of columns.

Transportation table:

		Destination			
	1	2	…	n	Supply
1	x_{11} c_{11}	x_{12} c_{12}	…	c_{1n}	a_1
2	c_{21}	c_{22}	…	c_{2n}	a_2
Origion ⋮	…	…	…	…	⋮
m	c_{m1}	c_{m2}	…	c_{mn}	a_m
Demand	b_1	b_2	…	b_n	

Solve the Transportation problem by North - west corner rule.

	D	E	F	G	Available.
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
~~Available~~ Requirement	200	225	275	250	

	D	E	F	G	available
A	$\boxed{200}$ 11	13	17	14	~~250~~ 50
B	16	18	14	10	300
C	21	24	13	10	400
Requirement	~~200~~ 0	225	275	250	950

	E	F	G	Available
A	$\boxed{50}$ 13	17	14	~~50~~ 0
B	18	14	10	300
C	24	13	10	400
Requirement	~~225~~ 175	275	250	

	E	F	G	
B	$\boxed{175}$ 18	14	10	~~300~~ 125
C	24	13	10	400
	~~175~~ 0	275	250	

	F	G	
B	$\boxed{125}$ 14	10	125 0
C	13	10	400
	~~275~~ 150	250	

	F	G	
C	$\boxed{150}$ 13	10	~~400~~ 250
	~~150~~ 0	250	

	G	
C	$\boxed{250}$	~~250~~ 0
	~~250~~ 0	

	D	E	F	G	available
A	$\boxed{200}$ 11	$\boxed{50}$ 13	17	14	250
B	16	$\boxed{175}$ 18	$\boxed{125}$ 14	10	300
C	21	24	$\boxed{150}$ 13	$\boxed{250}$ 10	400
Requirement	200	225	275	250	

Transportation

$$\text{Cost } Z = 200 \times 11 + 50 \times 13 + 175 \times 18 + 125 \times 14 + 150 \times 13 + 250 \times 10$$

$$Z = 2200 + 650 + 3150 + 1750 + 1950 + 2500$$

$$\therefore Z = 12{,}200.$$

Solve the Transportation problem using
column minima method.

	D_1	D_2	D_3	D_4	Capacity
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	

Solution.

	D_1	D_2	D_3	D_4	
O_1	$\boxed{4}$ 1	2	3	4	~~6~~ 2
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	~~4~~ 0	6	8	6	24

	D_2	D_3	D_4	
O_1	$\boxed{2}$ 2	3	4	~~2~~ 0
O_2	3	2	0	8
O_3	2	2	1	10
	~~6~~ 4	8	6.	

	D2	D3	D4	
O2	3	2	0	8
O3	$\boxed{4}$ 2	2	1	~~10~~ 6
	~~4~~ 0	8	6	

	D3	D4	
O2	$\boxed{8}$ 2	0	~~8~~ 0
O3	2	1	6
	~~8~~ 0	6	

	D4	
O3	$\boxed{6}$ 1	~~6~~ 0
	~~6~~ 0	

	D1	D2	D3	D4	
O1	$\boxed{4}$ 1	$\boxed{2}$ 2	3	4	6
O2	4	3	$\boxed{8}$ 2	0	8
O3	0	$\boxed{4}$ 2	2	$\boxed{6}$ 1	10
	4	6	8	6	

TP Cost $Z = 4\times1 + 2\times2 + 8\times2 + 4\times2 + 6\times1$

$= 4 + 4 + 16 + 8 + 6$

$Z = 38$

Solve the following TP by Least-cost method (or) Matrix Minima method. (3)

	D_1	D_2	D_3	D_4	capacity
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	

Solution

	D_1	D_2	D_3	D_4	
O_1	1	2	3	4	6
O_2	4	3	2	$\boxed{6}$ 0	~~8~~ 2
O_3	0	2	2	1	10
	4	6	8	~~6~~ 0	24

	D_1	D_2	D_3	
O_1	1	2	3	6
O_2	4	3	2	2
O_3	$\boxed{4}$ 0	2	2	~~10~~ 6
	~~4~~ 0	6	8	

	D_2	D_3	
O_1	$\boxed{6}$ 2	3	~~6~~ 0
O_2	3	2	2
O_3	2	2	6
	~~6~~ 0	8	

	D_3	
O_2	[2] 2	~~2~~ 0
O_3	2	6
	~~8~~ 6	

	D_3	
O_3	[6] 2	~~6~~ 0
	~~6~~ 0	

	D_1	D_2	D_3	D_4	capacity
O_1	1	[6] 2	3	4	6
O_2	4	3	[2] 2	[6] 0	8
O_3	[4] 0	2	[6] 2	1	10
Demand	4	6	8	6	

TP Cost $Z = 6\times 2 + 2\times 2 + 6\times 0 + 4\times 0 + 6\times 2$

$= 12 + 4 + 0 + 0 + 12$

$Z = 28$.

Solve the transportation problem by Vogel's Approximation method. (5)

	D_1	D_2	D_3	D_4	supply
O_1	5	3	6	2	19
O_2	4	7	9	1	37
O_3	3	4	7	5	34
Demand	16	18	31	25	

Solution

	D_1	D_2	D_3	D_4	supply	
O_1	5	3	6	2	19	(1)
O_2	4	7	9	[25] 1	~~37~~ 12	(3)
O_3	3	4	7	5	34	(1)
Demand	16	18	31	~~25~~ 0	90	
	(1)	(1)	(1)	(1)		

	D_1	D_2	D_3		
O_1	5	3	6	19	(2)
O_2	[12] 4	7	9	~~12~~ 0	(3)
O_3	3	4	7	34	(1)
	~~16~~ 4	18	31		
	(1)	(1)	(1)		

	P1	P2	P3		
O1	5	$\boxed{18}$ 3	6	~~19~~ 1	(2)
O3	3	4	7	34	(1)
	4	~~18~~ 0	31		
	(2)	(1)	(1)		

	D1	D3		
O1	5	6	1	(1)
O3	$\boxed{4}$ 3	7	~~34~~ 30	(4)
	4	31		
	(1)	(1)		

	D3	
O1	$\boxed{1}$ 6	~~1~~ 0
O3	7	30
	~~31~~ 30	

	D3	
O3	$\boxed{30}$ 7	~~30~~ 0
	~~30~~ 0	

	D1	D2	D3	D4	Supply
O1	5	$\boxed{18}$ 3	$\boxed{1}$ 6	2	19
O2	$\boxed{12}$ 4	7	9	$\boxed{25}$ 1	37
O3	$\boxed{4}$ 3	4	$\boxed{30}$ 7	5	34
Demand	16	18	31	25	

TP cost $Z = 18\times3 + 1\times6 + 12\times4$

$+ 25\times1 + 4\times3 + 30\times7$

$= 54 + 6 + 48 + 25 + 12$

$+ 210.$

$Z = 355$

Assignment Problem

	I	II	III	IV
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

Soln:

7	15	6	0
0	15	1	13
23	4	3	0
9	16	14	0

7	11	5	0
0	11	0	13
23	0	2	0
9	12	13	0

	I	II	III	IV
A	2	6	$\boxed{0}$	$\cancel{0}$
B	$\boxed{0}$	11	$\cancel{0}$	18
C	23	$\boxed{0}$	2	5
D	4	7	8	$\boxed{0}$

The optimum assignment is

$A \rightarrow \text{III}, \; B \rightarrow \text{I}, \; C \rightarrow \text{II}, \; D \rightarrow \text{IV}$

$\therefore$ The assignment cost

$= 17 + 13 + 19 \; 10$

$= 59.$

---

2. Solve the ~~transportation costs~~ Assignment problem,

	A	B	C	D
1	10	25	15	20
2	15	30	5	15
3	35	20	12	24
4	17	25	24	20

Soln:

0	15	5	10
10	25	0	10
23	8	0	12
0	8	7	3

$$\begin{bmatrix} 0 & 7 & 5 & 7 \\ 10 & 17 & 0 & 7 \\ 23 & 0 & 0 & 9 \\ 0 & 0 & 7 & 0 \end{bmatrix}$$

$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \boxed{0} & 7 & 5 & 7 \\ 10 & 17 & \boxed{0} & 7 \\ 23 & \boxed{0} & \cancel{0} & 9 \\ \cancel{0} & \cancel{0} & 7 & \boxed{0} \end{bmatrix} \end{array}$$

The assignments are

$1 \longrightarrow A,\quad 2 \longrightarrow C,\quad 3 \longrightarrow B,\quad 4 \longrightarrow D.$

The assignment cost $= 10 + 5 + 20 + 20$

$= 55$

~~Problem 3~~: Solve the assignment problem.

	A	B	C	D
I				
II				
III				
IV				

problem: 3

Solve the assignment problem

	1	2	3	4	5
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

soln:

7	3	1	5	0
0	9	5	5	4
1	6	7	0	4
4	3	1	0	3
4	0	3	4	0

	1	2	3	4	5
A	7	3	1	5	0
B	0	9	5	5	4
C	1	6	7	0	4
D	4	3	1	0	3
E	4	0	3	4	0

$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
A & 7 & 3 & 1 & 6 & 0 \\
B & 0 & 9 & 5 & 6 & 4 \\
C & 0 & 5 & 6 & 0 & 3 \\
D & 3 & 2 & 0 & 0 & 2 \\
E & 4 & 0 & 3 & 5 & 0 \\
\hline
\end{array}
$$

$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
A & 7 & 3 & 1 & 6 & \boxed{0} \\
B & \boxed{0} & 9 & 5 & 6 & 4 \\
C & \cancel{0} & 5 & 6 & \boxed{0} & 3 \\
D & 3 & 2 & \boxed{0} & \cancel{0} & 2 \\
E & 4 & \boxed{0} & 3 & 5 & \cancel{0} \\
\hline
\end{array}
$$

The Assignments are

$A \longrightarrow 5$, $B \longrightarrow 1$, $C \longrightarrow 4$, $D \longrightarrow 3$,

$E \longrightarrow 2$.

The assignment cost $= 1+0+2+1+5$

$= 9.$

Queueing Theory

Queue (waiting line) : A flow of customers from infinite / finite population towards the service facility forms a queue on account of lack of capability to serve them all at a time.

Elements of a queueing system.

1. **Input process.** Input process described by three factors.

(a) **Size of the queue** : If the total number of potential customers requiring service are only few, then size of the input source is said to be **finite.** on the other hand, if potential customers requiring service are sufficiently large in number, then the input source is considered to be **infinite.**

(b) **Arrival distributions** : The arrival pattern is measured by either mean arrival rate (or) inter-arrival time. These are characterized by the probability distribution associated with this random process. The most common stochastic queueing models assume that arrival rate follow a poisson distribution and/or the inter-arrival times follow an exponential distribution.

(C) **Customers behaviour**: customers reaction upon entering in the system.

Balked: If a customer decides not to enter the queue because of its huge length, he is said to have balked.

Reneged: A customer may enter the queue, but after some time loses patience and decides to leave. In this case he is said have reneged. (2)

Jockey for position: When there are two or more queues, customer may move from one to another for his personal economic gains, that is Jockey for position.

2. **Queue Discipline:** It is a rule according to which customers are selected for service when a queue has been formed. The most common queue discipline are.

(a) FCFS – First Come First Serve.
(or)
FIFO – First Come First Out.

(b) LCFS – Last Come First serve
(or)
LIFS – Last In First Out.

(c) SIRO – Selection for service In Random Order.

3. **Service ~~Mechanism~~ Mechnism:** The service mechanism is concerned with service time and service facilities.

The service facilities can be of the following types.

(a) single queue – one server.

(b) single queue – several servers.

(c) Several queues - one server.

(d) Several queues - several servers.

4. <u>Capacity of the System</u>: The source from which customers are generated may be finite (or) infinite. A finite source limits the customers arriving for service, (ie) there is a finite limit to the maximum queue size. An infinite source is forever "abundant" as in the case of telephone calls arriving at a telephone exchange.

Operating Characteristics of Queueing System.

1. <u>Expected number of customers in the System</u> denoted by $E(n)$ (or) L is the average number of customers in the system, both waiting and in service. Here, n stands for the number of customers in the system.

2. <u>Expected number of customers in the queue</u> denoted by $E(m)$ or L_q is the average number of customers waiting in the queue. Here $m = n-1$, (ie) excluding the customer being served.

3. <u>Expected waiting time in the System</u> denoted by $E(v)$ or W is the average total time spent by a customer in the system. It is generally taken to be the waiting time plus servicing time.

4. Expected waiting time in queue (4) denoted by $E(W)$ (or) W_q is the average time spent by a customer in the queue before the commencement of his service.

5. The serive utilization factor or (busy period) denoted by $\rho = \lambda/\mu$ is the proportion of ~~the~~ time that a server actually spends with the customers.

Where λ - the average number of customers arriving per unit of time.

μ - the average number of customers completing service per unit of time.

Classification of Queuing ~~System~~ models.

Generally queueing models may be specified in the following symbolic form.

$(a/b/c):(d/e)$.

a - inter-arrival time distributions.

b - inter-service time distributions.

c - number of servers.

d - capacity of the system.

e - the queue discipline.

we specify the following letters: (5)

$M \equiv$ poisson arrival (or) departure distribution.

$E_K \equiv$ Erlangian (or) Gamma inter-arrival for service time distribution.

$GI \equiv$ General Input distribution.

G = General service time distribution.

Example: $(M/E_K/1):(\infty/FIFO)$.

Here arrivals follow poisson distribution
service times are Erlangian,
single server
infinite capacity
First in first out queue discipline.

Transient state: A queueing system is said to be a transient state when its operating characteristic are dependent upon time.

Steady-State: If the characteristics of the queueing system becomes independent of time, then an at steady-state condition is said to prevail.

In $P_n(t)$ denotes the probability that there are n customers in the system at time t, then in the steady-state case, we have

$$\lim_{t \to \infty} P_n(t) = P_n \text{ (independent of } t)$$

Model-I

$(M/M/1):(\infty/FCFS)$ ①

Basic characteristics of model (valid only when $\frac{\lambda}{\mu} < 1$)

(i) Probability of no customer in the system is $P_0 = 1 - \rho$ where $\rho = \lambda/\mu$.

(ii) Probability of n customers in the system is $P_n = (1-\rho)\rho^n$

where $\rho = \frac{\lambda}{\mu}$ and $n \geq 0$.

(iii) probability that there are more than n customers in the system is

$$P(>n) = \cancel{\rho^{n+1}} \; \rho^{n+1}$$

(iv) probability that there are more than n customers in the queue is

$$P(>n+1) = \rho^{n+2}$$

(v) Average number of customers in the system is $E(n) = \dfrac{\lambda}{\mu - \lambda} = \dfrac{\rho}{1-\rho}$

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(vi) Average queue length is

$$E(m) = E(n) - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}$$

(vii) Average length of non-empty queue is

$$E(m/m>0) = \frac{E(m)}{P(m>0)} = \frac{\mu}{\mu-\lambda},$$

$$P(m>0) = P(n>1) = \left(\frac{\lambda}{\mu}\right)^2$$

(viii) Average waiting time of customers in the queue is

$$W \text{ or } E(W) = \frac{1}{\lambda} E(m) = \frac{\lambda}{\mu(\mu-\lambda)}$$

(ix) Average waiting time of an arrival who has to wait is

$$E(W/W>0) = \frac{E(W)}{P(W>0)} = \frac{1}{\mu-\lambda}$$

where $P(W>0) = 1 - P(W=0) = 1-(1-\rho) = \rho$

(x) Average waiting time that a customer spends in the system is

$$E(V) = \frac{1}{\lambda} E(n) = \frac{1}{\mu-\lambda}$$

problem: (3)

A TV repairman finds that the time spent on his jobs has an exponential distribution with mean 30 miniutes. If he repairs sets in the order in which they came in, and if the arrival of sets is approximately possion with an average rate of 10 per 8 hour day. what is repairman's expected idle time each day? HOW many jobs are ahead of the average set just brought in?

solution:

Give that

$\lambda = 10$ sets / day

$\mu = 16$ sets / day

$\therefore \rho = \frac{\lambda}{\mu} = \frac{10}{16} = 0.625$

The probability for the repairman to be idle is $P_0 = 1 - \rho = 1 - 0.625 = 0.375$

$\therefore$ Expected idle time perday

$= 8 \times 0.375 = 3$ hours.

$E(n) = \frac{\rho}{1-\rho} = \frac{0.625}{1-0.625} = \frac{5}{3}$ jobs.

Problem: 2 In a super market the average ~~arriv~~ arrival rate of customers is 5 every 30 minutes. The average time it takes to list and calculate the customer's purchases at the cash desk is 4.5 minutes, and this time is exponentially distributed.

(a) How long will the customer expect to wait for service at the cash desk?

(b) What is the chance that the queue ~~legnth~~ length will exceed 5?

(c) What is the probability that the cashier is working?

Solution: Given that

$\lambda = 5$ every 30 minutes

(or) $\frac{1}{6}$ /minute

$\mu = \frac{2}{9}$ /minute

$$\therefore \rho = \frac{\lambda}{\mu} = \frac{1}{6} \times \frac{9}{2} = \frac{3}{4} \text{ (or) } 0.75$$

(a) $E(w) = \dfrac{\lambda}{\mu(\mu - \lambda)} = 13.5$ minutes.

(b) $P(>n+1) = \rho^{5+2} = (0.75)^7$ (or) 0.133

(since $n=5$)

(c) Probability that cashier is working

= Probability of one or more customers in the system

= 1 - probability of no customers in the system

$= 1 - P_0 = \rho = 0.75$

Model-I

$(M/M/1):(\infty/FCFS)$

Basic characteristics of model (valid only when $\frac{\lambda}{\mu}<1$)

(i) Probability of no customer in the system is $P_0 = 1-\rho$ where $\rho = \lambda/\mu$.

(ii) probability of n customers in the system is $P_n = (1-\rho)\,\rho^n$

where $\rho = \frac{\lambda}{\mu}$ and $n \geq 0$.

(iii) probability that there are more than n customers in the system is

$$P(>n) = \cancel{\rho^{n+1}}\ \rho^{n+1}$$

(iv) probability that there are more than n customers in the queue is

$$P(>n+1) = \rho^{n+2}$$

(v) Average number of customers in the system is $E(n) = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}$

(vi) Average queue length is

$$E(m) = E(n) - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}$$

(vii) Average length of non-empty queue is

$$E(m/m>0) = \frac{E(m)}{P(m>0)} = \frac{\mu}{\mu-\lambda},$$

$$P(m>0) = P(n>1) = \left(\frac{\lambda}{\mu}\right)^2$$

(viii) Average waiting time of customers in the queue is

$$W \text{ (or) } E(W) = \frac{1}{\lambda} E(m) = \frac{\lambda}{\mu(\mu-\lambda)}$$

(ix) Average waiting time of an arrival who has to wait is

$$E(w/w>0) = \frac{E(w)}{P(w>0)} = \frac{1}{\mu-\lambda}$$

where $P(w>0) = 1 - P(w=0) = 1-(1-\rho) = \rho$

(x) Average waiting time that a customer spends in the system is

$$E(V) = \frac{1}{\lambda} E(n) = \frac{1}{\mu-\lambda}$$

---

problem:

A TV repairman finds that the time spent on his jobs has an exponential distribution with mean 30 miniutes. If he repairs sets in the order in which they came in, and if the arrival of sets is approximately possion with an average rate of 10 per 8 hour day. what is repairman's expected idle time each day? HOW many jobs are ahead of the average set just brought in?

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$= 8 \times 0.375 = 3$ hours.

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(c) What is the probability that the cashier is working?

Solution: Given that

$\lambda = 5$ every 30 minutes

(or) $\frac{1}{6}$ /minute

$\mu = \frac{2}{9}$ /minute

$$\therefore \rho = \frac{\lambda}{\mu} = \frac{1}{6} \times \frac{9}{2} = \frac{3}{4} \text{ (or) } 0.75$$

(a) $E(w) = \dfrac{\lambda}{\mu(\mu-\lambda)} = 13.5$ minutes.

(b) $P(>n+1) = \rho^{5+2} = (0.75)^7$ (or) 0.133

(since $n=5$)

(c) Probability that cashier is working

= Probability of one or more customers in the system

= 1 − probability of no customers in the system

$= 1 - P_0 = \rho = 0.75$

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Problem: 3 On an average 96 patients ~~per~~ per 24-hour day require the service of an emergency clinic. Also on an average, a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it ~~costs~~ costs the clinic Rs 100 per patient treated to obtain an average servicing time of 10 minutes, and that each minute of decrease in this average time would cost ~~Rs~~ Rs 10 per patient treated. How much would have to be budgeted by the ~~clinic~~ clinic to decrease the average size of the queue from one and one-third patients to half a patient.

Solution: $\lambda = \dfrac{96}{24 \times 60} = \dfrac{1}{15}$

$\mu = \dfrac{1}{10}$ patients per minute.

$\therefore \rho = \dfrac{\lambda}{\mu} = \dfrac{2}{3}$

Average number of patients in the queue are given by

$$E(m) = \frac{\rho^2}{1-\rho} = \frac{(2/3)^2}{1-2/3} = 4/3$$

Fraction of the time which there are no patients is given by

$$P_0 = 1 - \rho = 1 - 2/3 = 1/3$$

Now, when the average size is decreased from 4/3 patients to ½ patients, we are to determine the value of μ. So, we have

$$E(m) = \frac{\lambda^2}{\mu(\mu-\lambda)} \Rightarrow \frac{1}{2} = \frac{(1/15)^2}{\mu(\mu - 1/15)^2}$$

(ie) $\mu = 2/15$ patients/minute.

$\therefore$ Average rate of treatement requirement $= \frac{1}{\mu} = \frac{15}{2} = 7.5$ minutes.

(ie) a decrease in the average rate of treatment is $(10 - 7.5)$ minutes or 2.5 minutes

Budget per patient = Rs $(100 + 2.5 \times 10)$

= Rs 125.

Hence, in order to get the required size of the queue, the budget should be increased from Rs 100 per patient to Rs 125 per patient.

Problem: 4 In a railway marshalling yard, goods trains at a rate of 30 trains/day. Assuming that the inter-arrival time follows an exponential distribution and the service time distribution is also exponential with an average 36 minutes. Calculate the following: (i) the mean queue size (line length) (ii) the probability that the queue size exceeds 10. If the input of trains increases to an average 33/day, what will be the change in (i) and (ii)?

Solution:

$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48}$$

and $\mu = \frac{1}{36}$ trains/minutes

$$\therefore \quad \rho = \frac{\lambda}{\mu} = \frac{36}{48} = 0.75$$

(i) ~~M(m)~~ $E(m) = \frac{\rho}{1-\rho} = \frac{0.75}{1-0.75} = 3$ trains

(ii) $p(\geq 10) = \rho^{10} = (0.75)^{10} = 0.06$

when the input increases to 33 trains/day, we have

$\lambda = \frac{33}{60 \times 24} = \frac{11}{480}$, $\mu = \frac{1}{36}$ trains/minutes, $\therefore \rho = \frac{\lambda}{\mu} = \frac{11}{480} \times 36$

$\rho = 0.83$.

(i) $E(n) = \frac{\rho}{1-\rho} = \frac{0.83}{1-0.83} = 4.8$ (or) 5 trains

(ii) $p(\geq 10) = \rho^{10} = (0.83)^{10} = 0.2$ (Approximately).

Problem (5) At a one-man barber shop, customers arrive according to poisson distribution with a mean arrivale rate of 5 per hour and his hair cutting time was exponentially distributed with an average hair cut taking 10 minutes, It is assumed that because of his excellent reputation, customers were always willing to wait. calculate the following,

(i) Average number of customers in the shop and the average number of customers waiting for a haircut.

(ii) The percent of time an arrival can walk right in without having to wait.

(iii) The percentage of customers who have to wait prior to getting into the barber's chair.

Solution: $\lambda = \frac{5}{60} = \frac{1}{12}$ and $\mu = \frac{1}{10}$ /minute

$\therefore \rho = \frac{\lambda}{\mu} = \frac{10}{12} = 0.83$

(i) $E(n) = \frac{\rho}{1-\rho} = \frac{0.83}{1-0.83} = 4.8$ (or) 5

$E(m) = \frac{\rho^2}{1-\rho} = \frac{(0.83)^2}{1-0.83} = 4$ (App)

(ii) The probability of Queue size being greater than or equal to one, $P(\geq 1) = \rho = 0.833$

$\therefore$ Percentage of customers who have to wait $= 83.3\%$.

(iii) using (ii) the percentage of time an arrival can walk without walking $= 100 - 83.3 = 16.7\%$

Game Theory.

Problem:

Solve the game whose payoff matrix is given by

$$\begin{array}{c|c} & \begin{matrix} B_1 & B_2 & B_3 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} & \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}$$

Solution:

Maximin = Minimax.

$$\begin{array}{cccc} & \begin{matrix} B_1 & B_2 & B_3 \end{matrix} & \text{Row minimum} & \text{maximin} \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} & \begin{bmatrix} 1^{*\uparrow} & 3 & 1^{*\uparrow} \\ 0 & -4^{*} & -3 \\ 1^{\uparrow} & 5^{\uparrow} & -1^{**} \end{bmatrix} & \begin{matrix} 1 \\ -4 \\ -1 \end{matrix} & 1 \\ \text{Column maximum} & \begin{matrix} 1 & 5 & 1 \end{matrix} & & \\ \text{minimum} & 1 & & \end{array}$$

The optimum strategy for player A is A_1

The optimum strategy for player B is B_1 or B_3

The value of the game for player A is 1.

and player B is -1.

problem : 2 Solve the game

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 0 & -4 & -2 \\ A_2 & 3 & -5 & 1 \\ A_3 & -2 & -1 & 6 \\ A_4 & 1 & 0 & 4 \end{array}$$

Soln :

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \text{Row minimum} \\ \hline A_1 & 0^* & -4^* & -2 & -4 \\ A_2 & 3^{\uparrow} & -5^* & 1 & -5 \\ A_3 & -2 & -1 & 6^{\uparrow} & -2^* \\ A_4 & 1 & 0^{\uparrow *} & 4 & 0 \\ \hline \text{Column maxima} & 3 & 0 & 6 & \end{array}$$

max 0

min 0

The optimum strategy for player A is A_4

optimum strategy for player B is B_2

The value of the game for both players A & B is 0.

① Solve the game

$$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 2 & 4 & 5 \\ 10 & 7 & 9 \\ 4 & 6 & 8 \end{bmatrix} \end{array}$$

② Solve the game whose pay off matrix is given by

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \ \cancel{B_4} \\ \text{player A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{bmatrix} \end{array}$$

③ ~~Ha~~ Solve the following game

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{player A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 1 & 7 & 3 & 4 \\ 5 & 6 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{bmatrix} \end{array}$$

Game Theory:

Games without saddle points – mixed strategies

For any 2×2 two person zero-sum game without any saddle point have payoff matrix for player A

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$

The optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$$

and $$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

$$P_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

$$P_2 = 1 - P_1 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

$$q_2 = 1 - q_1 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

and $$\gamma = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

---

problem:

For the game with the following payoff matrix, determine the optimum strategies and the value of the game:

~~P_2 (player II)~~

~~(player I)~~

$$\begin{array}{c|cc} & B_1 & p \\ \hline A_1 & \cancel{5} & \cancel{1} \\ A_2 & \cancel{3} & \cancel{4} \end{array}$$

player B.

player A.

$$\begin{array}{ccc} & B_1 & B_2 \\ A_1 & \left[\begin{matrix} 5 & 1 \\ 3 & 4 \end{matrix}\right. & \\ A_2 & & \end{array}$$

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 5 & 1 \\ A_2 & 3 & 4 \end{array}$$

solution :

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \quad ; \quad p_1 + p_2 = 1.$$

$$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix} \quad ; \quad q_1 + q_2 = 1.$$

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

$$\begin{bmatrix} \overset{a_{11}}{5} & \overset{a_{12}}{1} \\ \overset{a_{21}}{3} & \overset{a_{22}}{4} \end{bmatrix}$$

$$= \frac{4-3}{(5+4)-(1+3)} = \frac{1}{9-4} = \frac{1}{5}.$$

$$p_1 = \frac{1}{5} \qquad p_2 = 1 - p_1 = 1 - 1/5 = \frac{4}{5}.$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{4-1}{(5+4)-(1+3)}.$$

$$= \frac{3}{5}$$

$$q_2 = 1 - q_1 = 1 - \frac{3}{5} = \frac{2}{5}.$$

$$\gamma = \frac{a_{11} a_{22} - a_{12} a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{(5)(4) - (1)(3)}{(5+4) - (1+3)}.$$

$$= \frac{20 - 3}{5} = 17/5.$$

∴ The value of the game is $\frac{17}{5}$.

The optimum mixed strategies is

$$S_A = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} P_1 & P_2 \\ \end{array}$$
$$S_A = \begin{bmatrix} 1/5 & 4/5 \end{bmatrix}$$

$$S_B = \begin{array}{cc} q_1 & q_2 \end{array}$$
$$S_B = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

Solve the ~~gam~~ 2×2 game.

Non-matching player

$$\begin{array}{ll} & \begin{array}{cc} H & T \end{array} \\ \text{matching player} \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 8 & -3 \\ -3 & 1 \end{bmatrix} \end{array}$$

Soln.

$$p_1 = \frac{4}{15} \qquad p_2 = 1 - \frac{4}{15} = \frac{11}{15}$$

$$q_1 = \frac{4}{15} \qquad q_2 = 1 - \frac{4}{15} = \frac{11}{15}$$

value of the game. $\gamma = \frac{1}{15}$.

optimum mixed strategies.

$$S_{match} = \begin{array}{cc} H & T \end{array}$$
$$S_{match} = \begin{bmatrix} 4/15 & 11/15 \end{bmatrix} \qquad S_{non\text{-}match} = \begin{bmatrix} 4/15 & 11/15 \end{bmatrix}$$

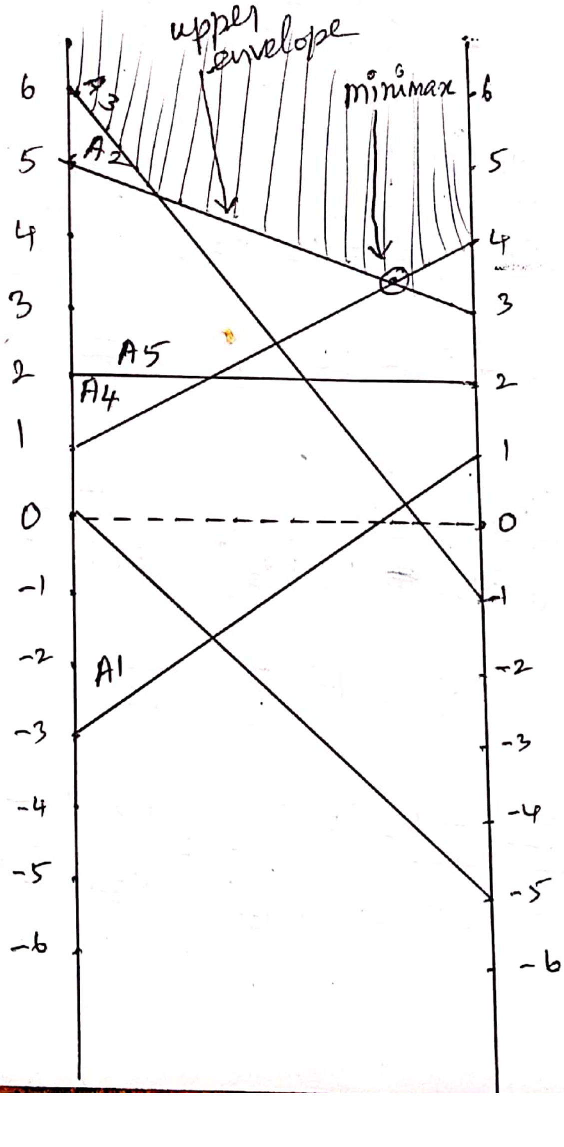
problem :

obtain the optimum strategies for both persons and the value of the game for zero-sum two person game whose payoff matrix is as follows.

player B

$$\text{player } A \quad \begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 1 & -3 \\ A_2 & 3 & 5 \\ A_3 & -1 & 6 \\ A_4 & 4 & 1 \\ A_5 & 2 & 2 \\ A_6 & -5 & 0 \end{array}$$

Solution:



The original 6×2 game reduced to 2×2 game whose pay off matrix is

$$\begin{array}{c} \\ A_2 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

If we now let

$$S_A = \begin{bmatrix} A_2 & A_4 \\ p_1 & p_2 \end{bmatrix} \qquad p_1 + p_2 = 1.$$

$$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix} \qquad q_1 + q_2 = 1.$$

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{1-4}{(3+1)-(5+4)} = \frac{-3}{-5} = \frac{3}{5}$$

$$p_2 = 1 - p_1 = 1 - 3/5 = \frac{2}{5}.$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{1-5}{(3+1)-(5+4)} = \frac{-4}{-5} = \frac{4}{5}$$

$$q_2 = 1 - q_1 = 1 - \frac{4}{5} = \frac{1}{5}.$$

$$r = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{(3)(1)-(5)(4)}{(3+1)-(5+4)} = \frac{-17}{-5} = \frac{17}{5}$$

The optimum ~~Strage~~ strategy for A is

$$S_A = \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \end{array}$$
$$\begin{bmatrix} 0 & 3/5 & 0 & 2/5 & 0 & 0 \end{bmatrix}$$

optimum strategy for B is

$$S_B = \begin{array}{cc} B_1 & B_2 \end{array}$$
$$\begin{bmatrix} \frac{4}{5} & \frac{1}{5} \end{bmatrix}$$

Value of the game

$$\vartheta = \frac{17}{5}$$

Dominance property.

General rules for dominance are

(a) If all the elements of a row, say k^{th}, are less than or equal to the corresponding elements of any other row, say r^{th}, the k^{th} row is dominated by the r^{th} row.

(b) If all the elements of a column, say k^{th} are greater than or equal to the corresponding elements of any column, say r^{th} then k^{th} column is dominated by the r^{th} column.

Problem:

Solve the following game after reducing it to 2×2 game

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{bmatrix} \end{array}$$

Solution:

$$\begin{array}{cccc} & B_1 & B_2 & B_3 \\ A_1 & 1 & 7 & 2 \\ A_2 & 6 & 2 & 7 \\ A_3 & 5 & 1 & 6 \end{array}$$

3^{rd} row dominated by 2^{nd} row. So 3^{rd} row is deleted. The remaining pay-off matrix is

$$\begin{array}{cccc} & B_1 & B_2 & B_3 \\ A_1 & 1 & 7 & 2 \\ A_2 & 6 & 2 & 7 \end{array}$$

3^{rd} column is dominated by 1^{st} column. So 3^{rd} column is deleted. The remaining pay-off matrix is

$$\begin{array}{ccc} & B_1 & B_2 \\ A_1 & 1 & 7 \\ A_2 & 6 & 2 \end{array}$$

In Further, we can not apply dominance property.

~~So we apl~~

Now we have 2×2 game.

(ie)

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 1 & 7 \\ A_2 & 6 & 2 \end{array}$$

$$P_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{2-6}{(1+2)-(7+6)} = \frac{-4}{-10} = \frac{2}{5}$$

$$P_2 = 1 - P_1 = 1 - \frac{2}{5} = 3/5$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{2-7}{(1+2)-(7+6)} = \frac{-5}{-10} = \frac{1}{5}$$

$$q_2 = 1 - q_1 = 1 - 1/5 = \frac{4}{5}$$

$$\gamma = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{(1)(2) - (7)(6)}{(1+2)-(7+6)} = \frac{-40}{-10} = 4$$

The optimum strategy for A is

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 2/5 & 3/5 & 0 \end{bmatrix}$$

for B is

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 1/5 & 4/5 & 0 \end{bmatrix}$$

Value of the game is $\gamma = 4$.

problem: dominance property to solve the following game:

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{player A} \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 6 & 8 & 6 \\ 4 & 12 & 2 \end{bmatrix} \end{array}$$

Solution:

$$\begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 6 & 8 & 6 \\ 4 & 12 & 2 \end{bmatrix} \end{array}$$

1^{st} Column is dominated by 3^{rd} column. delete 1^{st} column.

The remaining pay-off matrix

$$\begin{array}{cc} & \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 8 & 6 \\ 12 & 2 \end{bmatrix} \end{array}$$

2^{nd} Column is dominated by 3^{rd} column. delete 2^{nd} column.

The remaining matrix is

$$\begin{array}{cc} & B_3 \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 6 \\ 2 \end{bmatrix} \end{array}$$

2^{nd} row dominated by 1^{st} row. delete 2^{nd} row.

The remaining matrix is $\begin{array}{cc} & B_3 \\ A_1 & \begin{bmatrix} 6 \end{bmatrix} \end{array}$.

The optimum strategy for A is A_1

The optimum strategy for B is B_3.

Value of the game is 6.

Problem :

Solve the game, whose payoff matrix is

Player B

$$\text{player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \\ \end{array}$$

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 4 & -2 & 3 & -1 \\ A_2 & -1 & 2 & 0 & 1 \\ A_3 & -2 & 1 & -2 & 0 \end{array}$$

Solution :

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 4 & -2 & 3 & -1 \\ -1 & 2 & 0 & 1 \\ -2 & 1 & -2 & 0 \end{bmatrix} \end{array}$$

3^{rd} row is dominated by 2^{nd} row. Delete 3^{rd} row.
The remaining pay-off matrix is

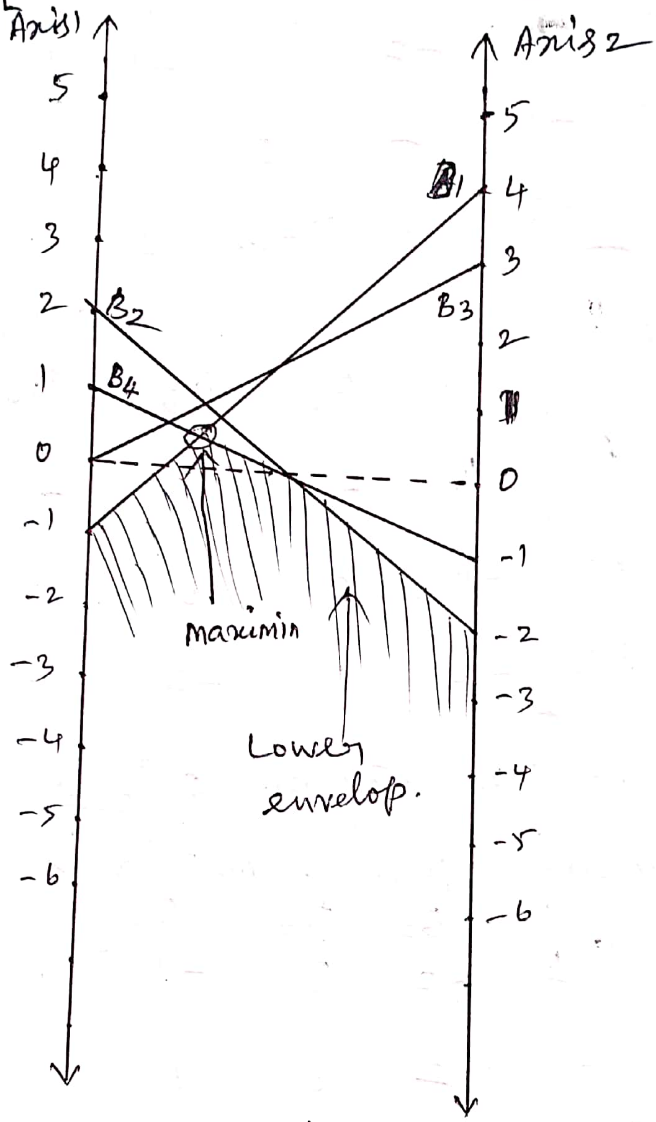
$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 4 & -2 & 3 & -1 \\ -1 & 2 & 0 & 1 \end{bmatrix} \end{array}$$

NOW, we cannot apply Dominance property.

~~Now~~ So, we get the 2×4 game.

No we can apply graphical method.

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 4 & -2 & 3 & -1 \\ -1 & 2 & 0 & 1 \end{bmatrix} \end{array}$$



The pay-off matrix is reduced to 2×2 game

(ie)

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{array}{cc} B_1 & B_4 \end{array} \\ \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$P_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{1-(-1)}{(4+1)-(-1-1)} = \frac{2}{7}$$

$$P_2 = 1 - P_1 = 1 - \frac{2}{7} = \frac{5}{7}$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{1-(-1)}{(4+1)-(-1\cancel{+1})} = \frac{2}{7}$$

$$q_2 = 1 - \frac{2}{7} = \frac{5}{7}.$$

$$\gamma = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{(4)(1) - (-1)(-1)}{(4+1)-(-1-1)} = \frac{3}{7}.$$

$\therefore$ The optimum ~~strategies for~~ strategies for A is.

$$S_A = \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{bmatrix} 2/7 & 5/7 & 0 \end{bmatrix} \end{array}$$

for B is

$$S_B = \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 2/7 & 0 & 0 & 5/7 \end{bmatrix} \end{array}$$

The value of the game is

$$\gamma = \frac{3}{7}$$

Two-person game: When there are two competitors playing a game, it is called a two-person game.

Competitive game: The competetive situations ~~with~~ with two or more competetitors, having conflicting interests and where the action of one depends upon the action taken by the other, are known as competitive games.

Two-person zero-sum game: A game with two players, where a gain of one player equals a loss to the other, is known as a two-person zero-sum game.

Player: The competitors in the game are known as players. A player may be individual or group of individuals or an organisation.

Strategy: A strategy for a player is defined as a set of rules or alernative courses of action available to him in advance, by which player decides the course of action that he should be adopt.

<u>pure Strategy:</u> If the player select the same strategy each time, then it is refferred to as pure-strategy. In this case each player knows exactly what the other player is going to do, the objective of the players is to maximize gains or to minimize losses.

<u>Mixed Strategy.</u> When the players use a combination of strategies and each player always kept guessing as to which course of action is to be selected by the other player at a particular occasion then this is known as mixed strategy. Thus there is a probabilistic a particular situation and objective of the player is to maximize expected gains or to minimize losses.

<u>Optimum Strategy:</u> A course of action or play which puts the player in the most preferred position irrespective of competitors, is called an optimum strategy.

Value of the game: It is the expected payoff of play when all the players of the game follow their optimum strategies. The game is called fair if the value of the game is zero and unfair if it is non-zero.

payoff matrix: when the players select their particular strategies, the payoffs (gains or losses) can be represented in the form of a matrix called payoff matrix.

Rules for determining a saddle point.

Step 1: Select the minimum element of each row of the payoff matrix and mark them [*]

Step 2: select the greatest element of each row of the payoff matrix and mark them [†]

Step 3: If there appears an element in the payoff matrix marks [*] and [†] both, the position of that element is a saddle point of the payoff matrix.

saddle point: maximin value = minimax value.

Use penalty method to maximize $z = 2x_1 + 3x_2$

subject to the constraints

$$x_1 + 2x_2 \leq 4$$

$$x_1 + x_2 = 3 \quad , \quad x_1 \geq 0 \text{ and } x_2 \geq 0.$$

Solution:

The LPP can be written as

Introduce slack variab x_3 and artificial variable x_4

we get,

maximize $z = 2x_1 + 3x_2 + 0.x_3 - Mx_4$.

Subject to the constraints

$$x_1 + 2x_2 + x_3 = 4$$

$$x_1 + x_2 + x_4 = 3.$$

C_j: 2 3 0 $-M$

C_B	Y_B	X_B	y_1	y_2	y_3	y_4	ratio
0	y_3	4	1	2	1	0	$\frac{4}{2} = 2$
$-M$	y_4	3	1	1	0	1	$\frac{3}{1} = 3$
$z_j - c_j$		$z = -3M$	$-M-2$	$-M-3 \uparrow$	0	0	

C_j: 2 3 0 $-M$

C_B	Y_B	X_B	y_1	y_2	y_3	y_4	ratio
3	y_2	2	$1/2$	1	$1/2$	0	$2 \times \frac{2}{1} = 4$
$-M$	y_4	1	$1/2$	0	$-1/2$	1	$1 \times \frac{2}{1} = 2$
$z_j - c_j$		$z = 6 - M$	$-\frac{1}{2} - \frac{M}{2} \uparrow$	0	$\frac{3}{2} + \frac{M}{2}$	0	

C_j : 2 3 0 −M

C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	ratio
3	Y_2	1	0	1	1	−1	
2	Y_1	2	1	0	−1	2	
$Z_j - C_j$		$Z = 7$	0	0	1	1+M	

All $Z_j - C_j \geq 0$, $\therefore$ The final table is optimum.

$$\max z = 7$$

$$x_1 = 2, \quad x_2 = 1.$$

---

problem Solve maximize $z = 3x_1 + 2x_2$

Subject to the constraints $2x_1 + x_2 \leq 2$

$$3x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0.$$

soln: The LPP can be written as,

maximize $z = 3x_1 + 2x_2 + 0 \cdot x_3 - 0 \cdot x_4 - Mx_5$

Subject to the constraints $2x_1 + x_2 + x_3 = 2$

$$3x_1 + 4x_2 - x_4 + x_5 = 12.$$

(where x_3 is a slack variable, x_4 is surplus variable and x_5 is artificial variable).

$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

			C_j: 3	2	0	0	$-M$	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	ratio
0	Y_3	2	2	1	1	0	0	$\frac{2}{1}=2$
$-M$	Y_5	12	3	4	0	-1	1	$\frac{12}{4}=3$
Z_j-C_j		$Z=-12M$	$-3M-3$	$-4M-2$ ↑	0	M	0	

			C_j 3	2	0	0	$-M$	
C_B	Y_B		Y_1	Y_2	Y_3	Y_4	Y_5	
2	Y_2	2	2	1	1	0	0	
$-M$	Y_5	4	-5	0	-4	-1	1	
Z_j-C_j		$Z=-4M+4$	$5M+1$	0	$4M+2$	M	0	

The coefficient of M in each Z_j-C_j is non-negative and an artificial vector appear in the basis, not at the zero level.

$\therefore$ The given LPP does not posses any feasible solution.

Solve the LPP,

minimize $z = 12x_1 + 20x_2$

subject to the constraints

$6x_1 + 8x_2 \geq 100$

$7x_1 + 12x_2 \geq 120$

$x_1, x_2 \geq 0.$

Solution:

This LPP can be written as

maximize $z^* = -12x_1 - 20x_2 + 0 \cdot x_3 - Mx_4 - 0x_5 - Mx_6$.

subject to the constraints

$6x_1 + 8x_2 - x_3 + x_4 = 100$ $\quad -\frac{35}{4}$

$7x_1 + 12x_2 - x_5 + x_6 = 120$ $\quad -\frac{35}{4}$

$x_1, x_2 \ldots x_6 \geq 0.$ $\quad \frac{1}{4}$

where x_3, x_5 are surplus variables

and x_4 and x_6 are artificial variables.

C_j: $-12 \quad -20 \quad 0 \quad -M \quad 0 \quad -M$

C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	ratio
$-M$	Y_4	100	6	8	-1	1	0	0	$\frac{100}{8} = 12\frac{1}{2}$
$-M$	Y_6	120	7	12	0	0	-1	1	$\frac{120}{12} = 10$
$Z_j - C_j$		$Z^* = -220M$	$-13M + 12$	$-20M + 20$	M	0	M	0	

②

		c_j	-12	-20	0	$-M$	0	$-M$	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	ratio
$-M$	Y_4	20	$4/3$	0	-1	1	$2/3$	$-2/3$	$20\times 3/4 = 15$
-20	Y_2	10	$7/12$	1	0	0	$-1/12$	$1/12$	$10\times 12/7 = 17\frac{1}{7}$
$Z_j - C_j$		$Z^* = -20M - 200$	$\frac{-4M}{3}+1/3$	0	M	0	$\frac{-2M}{3}+5/3$	$\frac{5M}{3}-5/3$	

		C_j:	-12	-20	0	$-M$	0	$-M$	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	
-12	Y_1	15	1	0	$-3/4$	$3/4$	$1/2$	$-1/2$	
-20	Y_2	$5/4$	0	1	$\frac{7}{16}$	$\frac{-7}{16}$	$\frac{-9}{24}$	$\frac{9}{24}$	
$Z_j - C_j$		$Z^* = -105$	0	0	$1/4$	$M - 1/4$	$3/2$	$M - 3/2$	

All $Z_j - C_j \geq 0$.

$\therefore$ Minimum $Z = -(-205) = 205$

Use two-phase simplex method to minimize $z = x_1 + x_2$

subject to the constraints $\quad 2x_1 + x_2 \geq 4$

$$x_1 + 7x_2 \geq 7$$

$$x_1, x_2 \geq 0.$$

Solution:

The LPP can be reformulated as

maximise $z_1 = -x_1 - x_2 + 0.x_3 - x_4 + 0.x_5 + x_6$

sub. to $\quad 2x_1 + x_2 - x_3 + x_4 = 4$

$$x_1 + 7x_2 - x_5 + x_6 = 7$$

$$x_1, x_2 \cdots x_6 \geq 0.$$

where

x_3 & x_5 are surplus variables and x_4 & x_6 are artificial variables.

Phase - I

$$\max z_1^* = 0 \cdot x_1 + 0 x_2 + 0 x_3 - x_4 + 0 \cdot x_5 - x_6.$$

		C_j	0	0	0	-1	0	-1	
C_B	Y_B	X_B	y_1	y_2	x_3	y_4	y_5	y_6	ratio
-1	x_4	4	2	1	-3	1	0	0	$\frac{4}{1} = 4$
-1	y_6	7	1	7	0	0	-1	1	$\frac{7}{7} = 1$
$Z_j - C_j$		$Z = -11$	-3	-8	1	0	1	0	

		C_j:	0	0	0	-1	0	-1	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	ratio
-1	Y_4	3	13/7	0	-1	1	1/7	-1/7	$3\times\frac{7}{13}=\frac{21}{13}$
0	Y_2	1	1/7	1	0	0	-1/7	1/7	$1\times\frac{7}{1}=7$
$Z_j - C_j$		$Z_1^* = -3$	-13/7	0	1	0	-1/7	8/7	

		C_j	0	0	0	-1	0	-1	
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	
0	Y_1	2/3	1	0	-7/13	7/13	1/13	-1/13	
0	Y_2	10/3	0	1	1/13	-1/13	-2/13	2/13	
$Z_j - C_j$		$Z_1^* = 0$	0	0	0	1	0	1	

All $Z_j - C_j \geq 0$. Now we goto phase-II.

Phase - II

$$\max Z_1 = -x_1 - x_2 + 0.x_3 + 0.x_5$$

		C_j:					
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	
-1	X_1	2/3	1	0	-7/13	1/13	
-1	Y_2	10/3	0	1	1/13	-2/13	
$Z_j - C_j$		$Z_1 = -31/3$	0	0	6/13	1/13	

All $Z_j - C_j \geq 0$. $\therefore$ $\min z = -(-31/3) = 31/3$.

$x_1 = 2/3$ $x_2 = 10/3$.

Duality in Linear programming.

Every LPP is associated with another LPP called the dual problem of the given LPP. The original (given) LPP is then called the primal problem.

Primal - Dual pairs.

Primal problem	Dual problem.
1. maximize $z = cx$ subject to the constraints $Ax \le b$ and $x \geq 0$	Minimize $z^* = b^T w$ subject to the constraints $A^T w \geq c^T$ and $w \geq 0$
2. maximize $z = cx$ Sub to the constraints $Ax = b$ and $x \geq 0$	minimize $z^* = b^T w$ Sub to the constraints $A^T w \geq c^T$ and w is unrestricted.
3. minimize $z = cx$ subject to the constraints $Ax = b$ and $x \geq 0$	maximize $z^* = b^T w$ sub to the constraints $A^T w \le c^T$ and w is unrestricted
4. maximize (or minimize) $z = cx$ subject to the constraints $Ax = b$ and x is unrestricted	minimize (or maximize) $z^* = b^T w$ subject to the constraints $A^T w = c^T$ and w unrestricted.

Problem: Formulate the dual of the following LPP.

Primal

maximize $z = 5x_1 + 3x_2$

$3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10$

$x_1, x_2 \ge 0.$

Solution

where $c = (5, 3)$.

$b = \begin{pmatrix} 15 \\ 10 \end{pmatrix}$

Dual

~~Soln~~

minimize $z^{\#} = 15w_1 + 10w_2$

Subject to the constraints

$3w_1 + 5w_2 \ge 5$

$5w_1 + 2w_2 \ge 3$

$w_1, w_2 \ge 0.$

Problem Write the Dual of the LPP

primal

minimize $z = 4x_1 + 6x_2 + 18x_3$

$x_1 + 3x_2 \ge 3$

$x_2 + 2x_3 \ge 5$ and $x_1, x_2, x_3 \ge 0.$

Soln

Dual

Maximize $z^{\#} = 3w_1 + 5w_2$

$w_1 \le 4$

$3w_1 + w_2 \le 6$

$2w_2 \le 18$

$w_1, w_2 \ge 0$

Problem: use dual simplex method to solve the following LPP. (1)

maximize $z = -3x_1 - x_2$

Subject to the constraints

$x_1 + x_2 \geq 1,$

$2x_2 + 3x_2 \geq 2$

$x_1, x_2 \geq 0.$

Solution: Using slack variables x_3 and x_4, the given LPP is written as

maximize $z = -3x_1 - x_2$

Subject to the constraints $-x_1 - x_2 \leq -1$

$-2x_2 - 3x_2 \leq -2$

~~x_1, x_2, x_3, x_4~~ $x_1, x_2 \geq 0.$

Standard form

(ie) max $z = -3x_1 - x_2$

Sub to the constraints $-x_1 - x_2 + x_3 = -1$

$-2x_2 - 3x_2 + x_4 = -2.$

$x_1, x_2, x_3, x_4 \geq 0.$

		c_j :	-3	-1	0	0
C_B	Y_B	X_B	Y_1	Y_2	X_3	Y_4
0	Y_3	-1	-1	-1	1	0
0	Y_4	-2	-2	-3	0	1
$z_j - c_j$		$z = 0$	3	1	0	0

②

Here $\min_j \{x_{B_i},\ x_{B_i} < 0\} = \min\{-1, -2\} = -2.$
(x_{B_2})

and $\max_j \left\{ \frac{z_j - c_j}{y_{2j}},\ y_{2j} < 0 \right\}$

$= \max\left\{ \frac{3}{-2}, \frac{1}{-3} \right\} = \frac{-1}{3}$

$= \frac{z_2 - c_2}{y_{22}}$

		c_j:	-3	-1	0	0
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4
0	y_3	$-1/3$	$-1/3$	0	1	$-1/3$
-1	Y_2	$2/3$	$2/3$	1	0	$-1/3$
$z_j - c_j$		$z = -2/3$	$7/3$	0	0	$1/3$

$\min_i \{x_{B_i},\ x_{B_i} < 0\} = -1/3 = x_{B_1}$

and ~~$\max_j \{ \frac{z_j - c_j}{y_{1j}} \}$~~ $\max_j \left\{ \frac{z_j - c_j}{y_{1j}},\ y_{1j} < 0 \right\} =$

$\max\left\{ \frac{7/3}{-1/3}, \frac{1/3}{-1/3} \right\} = -1 = \left(\frac{z_4 - c_4}{y_{14}} \right)$

C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4
0	Y_4	1	1	0	-3	1
-1	Y_2	1	1	1	-1	0
$z_j - c_j$		$z = -1$	2	0	1	0

Since all $z_j - c_j \geq 0$ and all $x_{B_i} \geq 0$, an optimum basic feasible solution has been reached.

$\therefore$ max $z = -1$ and $x_1 = 0$, $x_2 = 1$.

<u>Problem</u> use dual simplex method to solve the following LPP.

$$\text{maximize } Z = -3x_1 - 2x_2$$

Sub to
$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \geq 10$$
$$x_2 \geq 3; \quad x_1, x_2 \geq 0.$$

<u>Solution:</u> Convert all the inequations into '$\leq$' type and then introduce slack variables $x_3 \geq 0$, $x_4 \geq 0$, $x_5 \geq 0$ and $x_6 \geq 0$.

$$\max z = -3x_1 - 2x_2,$$

Sub to $-x_1 - x_2 \leq -1$; $x_1 + x_2 \leq 7$; $-x_1 - 2x_2 \leq -10$;
$x_2 \leq 3$; $x_1, x_2 \ldots x_6 \geq 0$.

~~An initial~~ The LPP can be written as

$$\max z = 3x_1 - 2x_2$$

Sub to $-x_1 - x_2 + x_3 = -1$; $x_1 + x_3 + x_4 = 7$;
$-x_1 - 2x_2 + x_5 = -10$; $x_2 + x_6 = 3$;
$x_1, x_2 \ldots x_6 \geq 0$.

The iterative dual simplex table are

	C_j:		3	-2	0	0	0	0
C_B	Y_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
0	Y_3	-1	-1	-1	1	0	0	0
0	Y_4	7	1	1	0	1	0	0
0	Y_5	-10	-1	-2	0	0	1	0
0	Y_6	3	0	1	0	0	0	1
$Z_j - C_j$		$z = 0$	-3	2	0	0	0	0

$$\min_i \{x_{B_i}, x_{B_i} < 0\} = \min\{-1, -10\} = -10 \ (x_{B_3})$$

and $\max \left\{ \frac{z_j - c_j}{y_{3j}},\ y_{3j} < 0 \right\} = \max \left\{ \frac{+3}{-1}, \frac{+2}{-2} \right\} = -1 \left(\frac{z_1 - c_1}{y_{31}} \right)$.

	c_j		-3	-2	0	0	0	0
C_B	Y_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_3	4	$-1/2$	0	1	0	$-1/2$	0
0	y_4	2	$1/2$	0	0	1	$1/2$	0
-2	y_2	5	$1/2$	1	0	0	$-1/2$	0
0	y_6	-2	$-1/2$	0	0	0	$1/2$	1
$z_j - c_j$		$z = -10$	2	0	0	0	1	0

$\min_i \left\{ x_{B_i},\ x_{B_i} < 0 \right\} = \min \left\{ -2 \right\} = -2 \ (x_{B_4})$

$\max_j \left\{ \frac{z_j - c_j}{y_{4j}},\ y_{4j} < 0 \right\} = \max \left\{ \frac{2}{-1/2} \right\} = -4 \left(\frac{z_1 - c_1}{y_{41}} \right)$

	c_j							
C_B	Y_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_3	6	0	0	1	0	-1	-1
0	y_4	0	0	0	0	1	1	1
-2	y_2	3	0	1	0	0	0	1
-3	y_1	4	1	0	0	0	-1	-2
$z_j - c_j$		$z = -18$	0	0	0	0	3	4

Since all $z_j - c_j \geq 0$ and all $x_{B_i} \geq 0$, an optimum basic feasible solution has been attained.

Hence, $\max z = -18$

$x_1 = 4, \quad x_2 = 3$.