CORE COURSE X

REAL ANALYSIS

Objectives: To enable the students to

1. Understand the real number system and countable concepts in real number system
2. Provide a Comprehensive idea about the real number system.
3. Understand the concepts of Continuity, Differentiation and Riemann Integrals
4. Learn Rolle's Theorem and apply the Rolle's theorem concepts.

UNIT I

Real Number system – Field axioms –Order relation in R. Absolute value of a real number & its properties –Supremum & Infimum of a set – Order completeness property – Countable & uncountable sets.

UNIT II

Continuous functions –Limit of a Function – Algebra of Limits – Continuity of a function –Types of discontinuities – Elementary properties of continuous functions – Uniform continuity of a function.

UNIT III

Differentiability of a function –Derivability & Continuity –Algebra of derivatives – Inverse Function Theorem – Daurboux''s Theorem on derivatives.

UNIT IV

Rolle's Theorem –Mean Value Theorems on derivatives- Taylor's Theorem with remainder- Power series expansion .

UNIT V

Riemann integration –definition – Daurboux's theorem –conditions for integrability – Integrability of continuous & monotonic functions - Integral functions –Properties of Integrable functions - Continuity & derivability of integral functions - The Fundamental Theorem of Calculus and the First Mean Value Theorem.

TEXT BOOK(S)

1. M.K,Singhal & Asha Rani Singhal , A First Course in Real Analysis, R.Chand & Co., June 1997 Edition
2. Shanthi Narayan, A Course of Mathematical Analysis, S. Chand & Co., 1995

UNIT – I - Chapter 1 of [1]
UNIT – II - Chapter 5 of [1]
UNIT – III - Chapter 6 – Sec 1 to 5 of [1]
UNIT – IV - Chapter 8 – Sec 1 to 6 of [1]
UNIT – V - Chapter 6 – Sec 6.2, 6.3, 6.5, 6.7, 6.9 of [2]

REFERENCE(S)

1. Goldberge, Richard R, Methods of Real Analysis, Oxford & IBHP Publishing Co., New Delhi, 1970.

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UNIT-I

FIELD AXIAMS

The set $(R, +, \cdot)$ is a field under $+$ and $\cdot$ if the following axiams are satisfied.

A_1 - Closure under $+$, if $a, b \in R \Rightarrow a+b \in R$

A_2 - Associative under $+$, if $a, b, c \in R$ then

$a+(b+c) = (a+b)+c$.

A_3 - Identity under $+$, if $\exists$ a real no $0 \in R$

$\ni a+0 = 0+a = a, \forall a \in R$.

A_4 - Inverse under $+$

For each $a \in R$, $\exists$ a real no $b \in R$

$\ni a+b = b+a = 0$ Then 'b' is called the inverse of 'a' under $+$.

A_5 - Commutative under $+$ if $a, b \in R \Rightarrow a+b = b+a$.

A_6 - The set $R-\{0\}$ closure under $\cdot$

If $a, b \in R \Rightarrow a.b \in R$.

A_7 - The set $R-\{0\}$ is Associative under $\cdot$

If $a, b, c \in R \Rightarrow a(bc) = (ab)c$

A_8 - The set $R-\{0\}$ is identity under $\cdot$

If $\exists$ a real no $1 \in R \ni a.1 = 1.a = a \ \forall a \in R$.

A_9 - The set $R-\{0\}$ is inverse under $\cdot$

For each $a \in R$, $\exists$ a real no $b \in R$

$\ni a.b = b.a = 1$.

Then 'b' is called the inverse of under $\cdot$

A_{10} - The set $R-\{0\}$ is Commutative under $\cdot$

If $a, b \in R \Rightarrow a.b = b.a$.

A_{11} - Distributive under + and ·

$a(b+c) = ab + ac \quad \forall\, a, b, c \in R$

$(a+b)c = ac + bc \quad \forall\, a, b, c \in R$

The set $(R, +, \cdot)$ satisfies all the axioms 1 to 11 is called a field.

Theorem

Prove that $\sqrt{2}$ is irrational number (or) Prove that there is no rational number whose square is 2.

Proof

Suppose $\sqrt{2}$ is rational number

ie, $\sqrt{2} = \frac{p}{q}$ ----- ① where p and q have no common factor.

Squaring on both sides

$2 = \frac{p^2}{q^2}$ ----- ②

$p^2 = 2q^2$

p^2 = even no

p = even no

$p = 2m$ (Say)

Substuting $p = 2m$ in equ ②

$2 = \frac{(2m)^2}{q^2}$

$2q^2 = (2m)^2$

$2q^2 = 4m^2$

$q^2 = 2m^2$

q^2 = even no

q = even no

$q = 2n$ (Say)

Substuting p and q values in equ ①

$$\sqrt{2} = \frac{2m}{2n}$$

Since p and q have common factor
Our assumption is wrong.
Hence $\sqrt{2}$ is irrational number.

ORDER RELATION

1. Law of Trichotomy

   $a, b \in R$ then only one of the following holds.

   $a < b$, $a = b$, $a > b$.

2. Law of Transitivity

   For each $a, b, c \in R$ if $a > b$, $b > c$ then $a > c$

3. Monotone property for addition

   For each $a, b, c \in R$ and if $a > b$, then $a + c > b + c$

4. Monotone property for multiplication

   For each $a, b, c \in R$ and if $a > b$, $c > 0$ then $ac > bc$.

Absolute value

Definition

If x be a real number then its absolute value is defined by

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

Result 1:-

For every real number $x \in R$, then prove that $|x| = \max\{-x, x\}$

Proof

If $x \geq 0$

$|x| = x$ and $x \geq -x$

If $x < 0$

$|x| = -x$ and $-x \geq x$

In both the cases

$|x|$ is greater than $\{-x, x\}$

ie, $|x| = \max\{-x, x\}$.

Hence the result.

Result ②:-

For every real number $x \in R$ then

Prove that $|x|^2 = x^2 = |-x|^2$

Proof:-

WKT, $|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$

$\Rightarrow |x| = -x$ if $x < 0$

$|x|^2 = x^2$ ---- ①

III^{ly} $|-x|^2 = |x|^2 = x^2$ ---- ②

From equations ① and ②

$|x|^2 = x^2 = |-x|^2$. Hence the result.

Result ③:-

For every $x \in R$, then prove that $|x| = |-x|$

Proof :-

WKT, $|x| = \max\{-x, x\}$

$|-x| = \max\{x, -x\}$

$|-x| = |x|$

Hence the result.

Result ④:-

For every $x, y \in R$ then prove that $|xy| = |x||y|$

Proof:-

WKT, $|xy|^2 = (xy)^2$

$= x^2 y^2$

$= |x|^2 |y|^2$

$|xy|^2 = [|x| |y|]^2$

Square root on both sides

$$|xy| = |x|\,|y|$$

Hence the result.

Result 5 :-

Triangle Inequality (*)

Statement :-

For every $x, y \in R$ then prove that $|x+y| \leq |x| + |y|$

Proof :-

Case 1

If $x + y \geq 0$

then $|x+y| = x+y$

Since $x \leq |x|$, $y \leq |y|$

$\Rightarrow |x+y| \leq |x| + |y|$ ----- (1)

Case 2

If $x + y < 0$

$\Rightarrow -(x+y) > 0$

$\Rightarrow (-x) + (-y) > 0$

By case 1

$\leq |-x| + |-y|$

$|-(x+y)| \leq |x| + |y|$

$|x+y| \leq |x| + |y|$ ----- (2)

Hence the theorem.

Result 6 :-

For every $x, y \in R$, then prove that $|x-y| \geq \big||x| - |y|\big|$

Proof :-

Now, $|x| = |x - y + y|$

$= |(x-y) + y|$

$|x| \leq |x-y| + |y|$ [$\because$ Triangular inequality]

$$|x| - |y| \leq |x-y| \quad \text{-----} \quad ①$$

$|||^{ly}$ $$|y| = |y + x - x|$$

$$= |(y-x) + x|$$

$$|y| \leq |y-x| + |x|$$

$$|y| - |x| \leq |y-x|$$

ie, $$-[|x| - |y|] \leq |y-x|$$

$$\leq |x-y| \quad \text{-----} \quad ②$$

From equations ① & ②

$$|x-y| \geq \big|\, |x| - |y| \,\big|$$ Hence the result.

Result ⑦ :-

For every $x, y \in R$ then prove that $|x-y| \leq |x| + |y|$

Proof :-

$$|x-y| \leq |x + (-y)|$$

$$\leq |x| + |-y|$$

$$|x-y| \leq |x| + |y|$$

Hence the proof.

UPPER BOUND

Definition

Let S be a set of real numbers. A number $u \in R$ is called the upper bound of 'S'.

If $x \leq u \ \forall x \in S$, if $\exists$ an upperbound for a set 'S'

SUPREMUM (LUB)

If S is bounded above then the least of all upper bounds of S is called the least upper bound or Supremum of S. And it is denoted by $Sup(S)$.

LOWER BOUND :- Definition :-

Let S be a set of real numbers. A number $v \in R$ is called the lower bound of 'S'.

If $x \geq v$ $\forall x \in S$, if $\exists$ an lower bound for the set 'S',

Then S is said to be bounded below.

INFIMUM :- (GLB)

The greatest of all the lower bound of the set 'S' is said to be greatest lower bound or an infimum of 'S'.

And it is denoted by Inf (S).

Enemerable Set :-

A set 'S' is said to be enemerable if $\exists$ a one to one mapping from the set N on to the set S.

Countable Set :-

A set 'S' is said to be Countable if it is either finite or enemerable.

Uncountable Set :-

A set 'S' is said to be uncountable if it is not countable.

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(x) State and Prove Archemadian Properties of Real numbers.

Statement

If x and y be any +ve real numbers, then $\exists$ a positive integer n such that $ny > x$.

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Proof :-

Suppose the theorem is wrong

Assume that $ny \leq x$

$\Rightarrow x$ is an upper bound.

$S = \{y, 2y, 3y, \ldots, ny\}$

S Contains a supremum. Say s.

$ny \leq s$

$(n+1)y \leq s$

$ny + y \leq s$

$ny \leq s - y$ is an upper bound.

Our assumption is wrong.

Hence $ny > x$.

Theorem

Every subset of a countable set is countable.

Proof :-

Let A be a countable set and $B \subseteq A$.

To prove that B is countable.

(i) Suppose A is finite

then B is finite $\Rightarrow$ Countable $B \subset A$.

(ii) Suppose A is infinite countable.

and B is infinite subset of A.

Let $A = \{a_1, a_2, \ldots\}$

Each element of B is an a_i for some index i.

Let n_1 be the smallest index

Such that $a_{n_1} \in B$.

Let n_2 be the next smallest index

Such that $a_{n_2} \in B$ and so on

ie, $B = \{a_{n_1}, a_{n_2}, \ldots\}$

Then $f(a_{nk}) = a_{nk}$.

Hence B is one to one correspondence from N on to B.

B is Countable.

ie, Hence the theorem.

Note :-

Every super set of an uncountable set is uncountable.

Proof :-

Let A be an uncountable set and $B \supseteq A$.

Then prove that B is uncountable.

Since A is uncountable.

and $A \subseteq B \Rightarrow$ B is uncountable.

Theorem

Every Collection of Countable set is Countable. (or)

Countable union of Countable set is Countable. (or)

If $A_1, A_2, \ldots$ are countable sets. Then $\bigcup_{i=1}^{\infty} A_i$ are Countable.

Proof :-

Given $A_1, A_2, \ldots, A_n$ are Countable set.

Let us take,

$$A_1 = \{a_{11}, a_{12}, \ldots\}$$

$$A_2 = \{a_{21}, a_{22}, \ldots\}$$

$$\cdots\cdots$$

$$A_n = \{a_{n1}, a_{n2}, \ldots\}$$

Here a_{ij} Stands for j^{th} element of i^{th} row.

Let us define the height of a_{ij}

ie, $i+j$

$$\bigcup_{i=1}^{\infty} A_i = \left\{ \begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{nn} & \end{array} \right\}$$

From this definition the height of a_{11} is 2, height of a_{12} and a_{21} is 3 and so on.

Arranging the no. according to their heights.

$\therefore$ We get $\bigcup_{i=1}^{\infty} A_i$

ie, This Collection is Countable.

Hence the all elements are counted according to the heights.

Hence the theorem.

LIMIT OF A FUNCTION

Let f be a function defined on a neighbourhood N of C if for every $\epsilon > 0$ $\exists$ $\delta > 0$

$\ni 0 < |x - c| < \delta \Rightarrow |f(x) - l| < \epsilon$

In symbol $\lim_{x \to c} f(x) = l$.

Left Sided point

A function f defined on (b, c) if given $\epsilon > 0$, we can find a number $\delta > 0 \ni c - \delta < x < c \Rightarrow |f(x) - l| < \epsilon$

In symbol $\lim_{x \to c-0} f(x) = l$ or $\lim_{\substack{x \to c \\ x < c}} f(x) = l$.

Right Sided point

A function f defined on (c, d) if given $\epsilon > 0$, we can find a number $\delta > 0 \ni c < x < c + \delta \Rightarrow |f(x) - l| < \epsilon$

In Symbol $\lim_{x \to c+0} f(x) = l$ or $\lim_{\substack{x \to c \\ x > c}} f(x) = l$.

Theorem

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} f(x) = m$, then prove that $l = m$.

Proof

Given $\lim_{x \to a} f(x) = l$

ie, For every $\epsilon > 0$, we can find a number $\delta_1 > 0$

$\ni |x - a| < \delta_1 \Rightarrow |f(x) - l| < \epsilon$

|||ly $\lim_{x \to a} f(x) = m$.

ie, For every $\epsilon > 0$, we can find a number $\delta_2 > 0$

$\ni |x - a| < \delta_2 \Rightarrow |f(x) - m| < \epsilon$

Suppose $l \neq m$

$|l - m| > 0$

Let us choose $\epsilon = \frac{1}{2}|l-m|$

Now, $|l-m| = |l-f(x)+f(x)-m|$

$\le |l-f(x)| + |f(x)-m|$

$< \epsilon + \epsilon$

$< 2\epsilon$

$|l-m| < 2\frac{1}{2}|l-m|$

ie, $|l-m| < |l-m|$

Our assume is wrong

$\therefore l-m.$

Hence the theorem

ALGEBRA OF LIMITS

Theorems

① If $\lim_{x\to a} f(x)=l$ and $\lim_{x\to a} g(x)=m$ then prove that $\lim_{x\to a} (f+g)(x) = l+m$

Proof

Given $\lim_{x\to a} f(x)=l$

ie, For every $\epsilon>0$, $\exists\ \delta_1>0 \ni |x-a|<\delta_1 \Rightarrow |f(x)-l|<\frac{\epsilon}{2}$ ---- ①

III'y $\lim_{x\to a} g(x) = m$

ie, For every $\epsilon>0$, $\exists\ \delta_2>0 \ni |x-a|<\delta_2 \Rightarrow |g(x)-m|<\frac{\epsilon}{2}$ ---- ②

Let us choose $\delta = \min(\delta_1, \delta_2)$

From eqns ① & ②

For every $\epsilon>0$, $\exists$ a $\delta>0 \ni |x-a|<\delta$ then $|x-a|<\delta_1$ and $|x-a|<\delta_2$ both true

Now, $|(f+g)(x)-(l+m)| = |f(x)+g(x)-l-m|$

$= |f(x)-l+g(x)-m|$

$\le |f(x)-l|+|g(x)-m|$

$< \frac{\epsilon}{2}+\frac{\epsilon}{2}$

$< \frac{2\epsilon}{2}$

$\therefore |(f+g)(x)-(l+m)| < \epsilon$

Hence, $\lim_{x\to a} (f+g)(x) = l+m$

(2) Let f and g be defined on the neighbourhood of c. If $\lim_{x\to c} f(x)=l$ and $\lim_{x\to c} g(x)=m$, then prove that $\lim_{x\to c} (fg)(x) = lm$.

Proof

Now, $|(fg)(x)-lm| = |f(x)g(x)-mf(x)+mf(x)-lm|$

$= |f(x)[g(x)-m] + m[f(x)-l]|$

$\leq |f(x)[g(x)-m]| + |m[f(x)-l]|$

$\therefore |(fg)(x)-lm| \leq |f(x)|\,|g(x)-m| + |m|\,|f(x)-l|$ ---- (1)

Since $\lim_{x\to c} f(x) = l$

For given $\epsilon = 1 > 0$, $\exists\ \delta_1 > 0 \ni |x-c| < \delta_1 \Rightarrow |f(x)-l| < 1$

$\Rightarrow |f(x)| = |f(x)-l+l|$

$\leq |f(x)-l| + |l|$

$|f(x)| < 1+|l|$ ----- (2)

Since $\lim_{x\to c} f(x) = l$.

For given $\epsilon>0$, $\exists\ \delta_2 > 0 \ni |x-c| < \delta_2 \Rightarrow |f(x)-l| < \frac{\epsilon}{2|m|}$ ---- (3)

Again, Since $\lim_{x\to c} g(x) = m$

For given $\epsilon>0$, $\exists\ \delta_3 > 0 \ni |x-c| < \delta_3 \Rightarrow |g(x)-m| < \frac{\epsilon}{2[1+|l|]}$ ---- (4)

Choose $\delta = \min(\delta_1, \delta_2, \delta_3)$

Substituting eqns (2), (3) & (4) in (1)

$$|(fg)(x)-lm| \leq [1+|l|]\frac{\epsilon}{2[1+|l|]} + |m|\frac{\epsilon}{2|m|}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

ie, $|(fg)(x) - lm| < \epsilon$ whenever $|x-c| < \delta$.

Hence $\lim_{x\to c} (fg)(x) = lm$.

(3) If $\lim_{x \to c} g(x) = m$, then prove that $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{m}$.

Proof

Now, $\left| \frac{1}{g(x)} - \frac{1}{m} \right| = \left| \frac{m - g(x)}{m\, g(x)} \right|$

$$\left| \frac{1}{g(x)} - \frac{1}{m} \right| = \frac{|m - g(x)|}{|m|\,|g(x)|} \quad \text{-----} \; (1)$$

Since $\lim_{x \to c} g(x) = m$.

ie, For every $\epsilon > 0$, $\exists\, \delta > 0 \ni |x - c| < \delta \Rightarrow |g(x) - m| < \frac{\epsilon}{2} |m|^2$ ----- (2)

Since $\epsilon = \frac{|m|}{2}$

We can find $\delta_2 > 0 \ni |x - c| < \delta_2 \Rightarrow |g(x) - m| < \frac{|m|}{2}$ ----- (3)

Now, $|m| = |m - g(x) + g(x)|$

$$\leq |m - g(x)| + |g(x)|$$

$$|m| \leq \frac{|m|}{2} + g(x)$$

ie, $|m| - \frac{|m|}{2} \leq g(x)$

$$\frac{|m|}{2} \leq g(x)$$

$$\therefore \frac{1}{g(x)} \leq \frac{|m|}{2} \quad \text{-----} \; (4)$$

Choose $\delta = \min(\delta_1, \delta_2)$

Substituting eqns (2) & (4) in (1)

$$\left| \frac{1}{g(x)} - \frac{1}{m} \right| \leq \frac{\frac{\epsilon}{2} |m|^2}{|m| \frac{|m|}{2}}$$

$$< \frac{\epsilon}{2} |m|^2 \times \frac{2}{|m|^2}$$

$$\left| \frac{1}{g(x)} - \frac{1}{m} \right| < \epsilon \text{ , whenever } |x - c| < \delta.$$

Hence $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{m}$.

(4) If $\lim_{x \to c} f(x) = l$ and $\lim_{x \to c} g(x) = m$, then prove that $\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \frac{l}{m}$. (provided $m \neq 0$)

Proof

Given that $\lim_{x\to c} f(x) = l$ & $\lim_{x\to c} g(x) = m$

Now, $\lim_{x\to c} \left(\frac{f}{g}\right)(x) = \lim_{x\to c} \frac{f(x)}{g(x)}$

$$= \lim_{x\to c} f(x) \cdot \frac{1}{g(x)}$$

$$= \lim_{x\to c} f(x) \cdot \lim_{x\to c} \frac{1}{g(x)}$$

$$= l \cdot \frac{1}{m}.$$

$\therefore \lim_{x\to c} \left(\frac{f}{g}\right)(x) = \frac{l}{m}.$

Hence the proof.

⑤ If $\lim_{x\to c} f(x) = l$, then prove that $\lim_{x\to c} |f(x)| = |l|$

Proof.

Since $|x-y| \geq \big||x| - |y|\big|$

Given $\lim_{x\to c} f(x) = l$.

ie, For every $\epsilon > 0$, $\exists\ \delta > 0 \ni |x-c| < \delta \Rightarrow |f(x) - l| < \epsilon$

Now, $\big||f(x)| - |l|\big| \leq |f(x) - l| < \epsilon$, $\forall\ |x-c| < \delta$

Hence $\lim_{x\to c} |f(x)| = |l|$

⑥ Suppose that there is a $\delta > 0 \ni h(x) = 0$, whenever $0 < |x-c| < \delta$, then prove that $\lim_{x\to c} h(x) = 0$

Proof

For every $\epsilon > 0$ we may choose a number $\delta > 0$

$\ni 0 < |x-c| < \delta \Rightarrow |h(x)| = 0$

$\Rightarrow |h(x) - 0| = 0 \Rightarrow 0 < \epsilon$

$0 < |x-c| < \delta \Rightarrow |h(x) - 0| < \epsilon$

Hence $\lim_{x\to c} h(x) = 0$

(7) Suppose that there is a $\delta > 0 \ni f(x) = g(x)$ whenever $0 < |x-c| < \delta$, then prove that $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$

Proof

For every $\epsilon > 0$, we may choose a number $\delta > 0$

$\ni f(x) = g(x)$ whenever $|x-c| < \delta$

Let us define $h(x) = f(x) - g(x)$

For every $\epsilon > 0$, $\exists\ \delta > 0 \ni h(x) = 0$, whenever $0 < |x-c| < \delta$

$$\Rightarrow |h(x) - 0| = 0 < \epsilon$$

$$\Rightarrow \lim_{x \to c} h(x) = 0$$

$$\Rightarrow \lim_{x \to c} (f(x) - g(x)) = 0$$

$$\Rightarrow \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = 0$$

$$\Rightarrow \lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

Hence the proof.

CONTINUOUS FUNCTION

Definition

Let f be a function defined on I. Then f is continuous at $x_0 \in I$ if given $\epsilon > 0$ we can find a number $\delta > 0$

$\ni |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.

In symbol $\lim_{x \to x_0} f(x) = f(x_0)$

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

$$-\delta < x - x_0 < \delta \quad -\epsilon < f(x) - f(x_0) < \epsilon$$

$$x_0 - \delta < x < x_0 + \delta \Rightarrow f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon.$$

Right Continuous function

Let f be a function defined on I.

Given $\epsilon > 0$, we can find a number $\delta > 0$

$\ni \quad x_0 < x < x_0 + \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$

In symbol, $\lim_{x \to x_0 + 0} f(x) = f(x_0)$ or $\lim_{\substack{x \to x_0 \\ x_0 > 0}} f(x) = f(x_0)$

Left Continuous function

Let f be a function defined on I.

Given $\epsilon > 0$, we can find a number $\delta > 0$

$\ni x_0 - \delta < x < x_0 \Rightarrow |f(x) - f(x_0)| < \epsilon$

In symbol, $\lim\limits_{x \to x_0 - 0} f(x) = f(x_0)$ or $\lim\limits_{\substack{x \to x_0 \\ x < x_0}} f(x) = f(x_0)$

Theorems

1. A function f defined on $I \subseteq R$ is continuous at $p \in I$ iff for every sequence $\langle p_n \rangle$ in $I \to p$. we have $\lim\limits_{n \to \infty} f(p_n) = f(p)$

Proof

Necessary Condition

Let f be a continuous at $p \in I$ and $\langle p_n \rangle \to p$.

To prove : $\lim\limits_{n \to \infty} f(p_n) = f(p)$

Since f is continuous at p.

For every $\epsilon > 0$ $\exists$ $\delta > 0$ $\ni |x - p| < \delta \Rightarrow |f(x) - f(p)| < \epsilon$ ---- (1)

and $\langle p_n \rangle \to p$

$\therefore \lim\limits_{n \to \infty} p_n = p$.

For given $\delta > 0$, we can find a number $m \in N$

$\ni |p_n - p| < \delta \quad \forall n \geq m$.

By taking $x = p_n$ in eqn (1)

$|p_n - p| < \delta \Rightarrow |f(p_n) - f(p)| < \epsilon \quad \forall n \geq m$

ie, $\lim\limits_{n \to \infty} f(p_n) = f(p)$

Sufficient Condition

Suppose that f is not continuous at $p \in I$.

We prove that $\exists$ a sequence $\langle p_n \rangle \to p$.

Where $\lim\limits_{n \to \infty} f(p_n) \neq f(p)$.

Since f is not continuous at p.

For given $\epsilon > 0$, we can find a number $\delta > 0$

$\ni |x-p| < \delta \Rightarrow |f(x) - f(p)| > \epsilon$.

By taking $\delta = \frac{1}{n}$, we find for each +ve integer $n \in N$

$\exists$ some m $\ni |p_n - p| < \frac{1}{n} \Rightarrow |f(x) - f(p)| > \epsilon$

ie, $\langle p_n \rangle \to p \Rightarrow \langle f(p_n) \rangle \not\to f(p)$

$$\Rightarrow \lim_{n \to \infty} f(p_n) \neq f(p)$$

Hence the theorem.

(2) (X) A function f defined on R is continuous R iff for each open set G on R, $f^{-1}(G)$ is open in R.

Proof

Necessary Condition

Let $f : R \to R$ is continuous

Consider any open set G in R.

To prove $f^{-1}(G)$ is open set in R

If $f^{-1}(G) = \phi \Rightarrow$ It is open in R.

Otherwise $f^{-1}(G) \neq \phi$

$\exists$ $x_0 \in f^{-1}(G)$

$\Rightarrow f(x_0) \in G$.

Since G is an open set. If it is neighbourhood of each of its points.

In particular $f(x_0) \in G$.

For some $\epsilon > 0$ $\ni (f(x_0) - \epsilon, f(x_0) + \epsilon) \in G$.

Now f is continuous at $x_0 \in R$

For every $\epsilon > 0$, $\exists$ $\delta > 0$

$\ni |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.

ie, $x_0 - \delta < x < x_0 + \delta \Rightarrow f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon$.

$\therefore x \in (x_0 - \delta, x_0 + \delta) \Rightarrow f(x) \in \left(f(x_0) - \epsilon, f(x_0) + \epsilon\right)$

$\Rightarrow$ Each f image of $x \in (x_0 - \delta, x_0 + \delta)$ Contained in $(f(x_0) - \epsilon, f(x_0) + \epsilon) \in G$.

$\Rightarrow (x_0 - \delta, x_0, \delta) \in f^{-1}(G)$

$\therefore f^{-1}(G)$ is neighbourhood of each of its points.

ie, $f^{-1}(G)$ is open set.

Sufficient Condition

Given $f^{-1}(G)$ is open set in R when G is open set in R

To prove : f is continuous on R.

Since $(f(x_0) - \epsilon, f(x_0) + \epsilon)$ is an open interval Containing $f(x_0)$.

ie, $f(x_0) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$

$\therefore x_0 \in f^{-1}(f(x_0) - \epsilon, f(x_0) + \epsilon)$

Since $f^{-1}(G)$ is open when G is open.

$\Rightarrow x_0$ is a point of $f^{-1}(f(x_0) - \epsilon, f(x_0) + \epsilon)$

By definition of open set $\exists$ a $\delta > 0$

$\ni (x_0 - \delta, x_0 + \delta) < f^{-1}(f(x_0) - \epsilon, f(x_0) + \epsilon)$

Thus we have form a $\delta > 0$

$\ni x \in (x_0 - \delta, x_0 + \delta) \Rightarrow f(x) \in \left(f(x_0) - \epsilon, f(x_0) + \epsilon\right)$

ie, $x_0 - \delta < x < x_0 + \delta \Rightarrow f(x) - \epsilon < f(x) < f(x_0) + \epsilon$

ie, $\lim_{x \to x_0} f(x) = f(x_0)$

Hence f is continuous at $x_0 \in R$.

Since x_0 is arbitrary then f is continuous on R

Hence the theorem.

Theorems

(1) Let f & g be defined on I and are continuous at $p \in I$. Then prove that $(f+g)$ is continuous at p.

Proof

Let f & g be a continuous at $p \in I$.

By definition, If $\langle p_n \rangle \to p$. Then

$$\lim_{n\to\infty} f(p_n) = f(p) \quad \text{and} \quad \lim_{n\to\infty} g(p_n) = g(p)$$

Now,

$$\lim_{n\to\infty} (f+g)(p_n) = \lim_{n\to\infty} \left(f(p_n) + g(p_n) \right)$$

$$= \lim_{n\to\infty} f(p_n) + \lim_{n\to\infty} g(p_n)$$

$$\lim_{n\to\infty} (f+g)(p_n) = f(p) + g(p)$$

Hence $f+g$ is continuous at p.

(2) Let f & g be defined on I. If f & g are continuous at $p \in I$. Then prove that fg is continuous at $p \in I$.

Proof.

Let f & g be continuous at $p \in I$.

By definition, If $\langle p_n \rangle \to p$. Then

$$\lim_{n\to\infty} f(p_n) = f(p) \quad \text{and} \quad \lim_{n\to\infty} g(p_n) = g(p)$$

Now,

$$\lim_{n\to\infty} (fg)(p_n) = \lim_{n\to\infty} \left(f(p_n) . g(p_n) \right)$$

$$= \lim_{n\to\infty} f(p_n) . \lim_{n\to\infty} g(p_n)$$

$$= f(p)\, g(p)$$

$$\Rightarrow \lim_{n\to\infty} (fg)(p_n) = (fg)(p)$$

Hence fg is continuous at p.

3 If f is continuous at $p \in I$ and $c \in \mathbb{R}$. Then prove that cf is continuous at $p \in I$.

Proof

Let f be continuous at $p \in I$.

By definition, If $\langle p_n \rangle \to p$. Then $\lim_{n\to\infty} f(p_n) = f(p)$

Now, $\lim_{n\to\infty} [(cf)(p_n)] = \lim_{n\to\infty} [c.\ f(p_n)]$

$= \lim_{n\to\infty} c \cdot \lim_{n\to\infty} f(p_n)$

$= c f(p)$

$\Rightarrow \lim_{n\to\infty} [(cf)(p_n)] = (cf)(p)$

Hence cf is continuous.

4 Let f & g be a function defined on I and Let $g(p) \neq 0$. If f & g are continuous at $p \in I$. Then prove that $\lim_{n\to\infty} \left(\frac{f}{g}\right)(p_n) = \left(\frac{f}{g}\right)(p)$.

Proof

Given f & g are continuous at $p \in I$.

By definition, If $\langle p_n \rangle \to p$. Then

$\lim_{n\to\infty} f(p_n) = f(p)$ & $\lim_{n\to\infty} g(p_n) = g(p) \neq 0$

Now, $\lim_{n\to\infty} \left[\left(\frac{f}{g}\right)(p_n)\right] = \lim_{n\to\infty} \left[\frac{f(p_n)}{g(p_n)}\right]$

$= \dfrac{\lim_{n\to\infty} f(p_n)}{\lim_{n\to\infty} g(p_n)}$

$= \dfrac{f(p)}{g(p)}, \quad g(p) \neq 0$

$\Rightarrow \lim_{n\to\infty} \left[\left(\frac{f}{g}\right)(p_n)\right] = \left(\frac{f}{g}\right)(p)$

Hence the theorem.

(5) If f is continuous at p. Then show that $|f|$ is continuous at $p \in I$

Proof

Given f is continuous at $p \in I$.

By definition, If $\langle p_n \rangle \to p$. Then $\lim_{n\to\infty} f(p_n) = f(p)$

Now, $\lim_{n\to\infty} \left[|f|(p_n) \right] = \lim_{n\to\infty} |f(p_n)|$

$$= \left| \lim_{n\to\infty} f(p_n) \right|$$

$$= |f(p)|$$

$$\Rightarrow \lim_{n\to\infty} \left[|f|(p_n) \right] = |f|(p)$$

Hence $|f|$ is continuous at $p \in I$.

(6) Let f & g be a function defined on I & J respectively. If f is continuous at $p \in I$ and g is continuous at $f(p) \in J$. Then prove that $(g \circ f)$ is continuous at p.

Proof

Given f is continuous at $p \in I$.

By definition, If $\langle p_n \rangle \to p$. Then $\lim_{n\to\infty} f(p_n) = f(p)$

ie, $\langle f(p_n) \rangle \to f(p) \in J$

ie, $\langle f(p_n) \rangle$ is sequence in J continuous at $f(p)$

Since g is continuous at $f(p)$.

$\therefore \langle g(f(p_n)) \rangle$ is sequence converges to $g(f(p))$

ie, $\langle (g \circ f)(p_n) \rangle \to (g \circ f)(p)$ whenever $\langle p_n \rangle \to p$

Hence $(g \circ f)$ is continuous at p.

Hence the theorem.

Intermediate Theorem

Statement

If f be continuous on $[a,b]$ and c be any real number between $f(a)$ & $f(b)$ then $\exists$ a real number x in (a,b) such that $f(x) = c$.

Proof

First let us prove that lemma

Lemma 1:

If f be continuous on $[a,b]$ and $f(a) < 0 < f(b)$ then $\exists$ a point $x \in (a,b) \ni f(x) = 0$

proof

Let S be a subset of $[a,b]$ defined by

$$S = \{ x / a \leq x \leq b \text{ and } f(x) < 0 \}$$

Since $f(a) < 0$ and $a \in S$

$\therefore S \neq \phi$.

Since S is the subset of a closed and bounded interval $[a,b]$.

It has a supremum (say u)

ie, $\sup S = u$

Clearly $a \leq u \leq b$

To prove : $f(u) = 0$

ie, To prove the following

(i) $u \neq a$ (ii) $u \neq b$ (iii) $f(u) \not> 0$ (iv) $f(u) \not< 0$

Step 1 :

To prove $u \neq a$

Since $f(a) < 0$ & f is continuous at 'a'

$\exists$ a number $\delta_1 > 0$, $\ni f(x) < 0$, whenever $a \leq x \leq a+\delta$

$\Rightarrow [a, a+\delta_1) \subset S$

$\sup \geq a + \delta_1$

ie, $u \geq a + \delta_1$

$\Rightarrow u > a$ for $\delta_1 > 0$

$\Rightarrow u \neq a$.

Step 2 :-

To prove : $u \neq b$.

Since $f(b) > 0$ and f is Continuous at 'b'

$\exists$ a $\delta_2 > 0 \ni f(x) > 0$ whenever $b - \delta_2 < x < b$.

$\Rightarrow \operatorname{Sup} S \leq b - \delta_2$

ie, $u \leq b - \delta_2$

$\Rightarrow u < b$

$\Rightarrow u \neq b$.

Step 3 :- To prove : $f(u) \ngtr 0$

Suppose $f(u) > 0$ & f be Continuous

$\exists$ a $\delta_3 > 0 \ni f(x) > 0$ whenever $u - \delta_3 < x < u$

Since $u = \operatorname{Sup} S$

$\exists\ x_0 \in S \ni u - \delta_3 < x_0 < u \Rightarrow f(x_0) > 0$ for some $x_0 \in S$.

$\Rightarrow \Leftarrow$

Our assumption is wrong.

Hence $f(u) \ngtr 0$.

Step 4 :- To prove : $f(u) \nless 0$

Suppose $f(u) < 0$ & f be Continuous

$\exists$ a $\delta_4 > 0 \ni f(x) < 0$ whenever $u < x < u + \delta_4$

then $f(x_1) < 0$ for some $x_1 \in [u, u + \delta_4)$

ie, $x_1 > u$.

$\Rightarrow \Leftarrow$

Since S must be a supremum

Hence $f(u) \nless 0$

From step ①, ②, ③ and ④
we get $f(u) = 0$
Thus $\exists$ a point $x \in (a, b) \ni f(x) = 0$
Hence the Lemma (1)

Lemma 2:

If f is continuous on $[a, b]$ & if $f(a) > 0 < f(b)$
then $\exists$ a point $x \in (a, b) \ni f(x) = 0$

Proof

Since f is continuous on $[a, b]$
$\therefore$ $-f$ is continuous on $[a, b]$
& if $(-f(a)) < 0 < (-f(b))$
$\therefore$ By Lemma 1
$\exists$ a point $x \in (a, b) \ni -f(x) = 0$
$\therefore f(x) = 0$

Main proof of the theorem

If c lies between $f(a)$ & $f(b)$
ie, $f(a) < c < f(b)$ or $f(a) > c > f(b)$
Let g be a function defined on $[a, b]$
By setting $g(x) = f(x) - c \quad \forall x \in [a, b]$
Then g is continuous on $[a, b]$
& $g(a) = f(a) - c$, $g(b) = f(b) - c$
Since c lies between $f(a)$ & $f(b)$,
$\therefore g(a)$ & $g(b)$ are opposite sign.

Then by Lemma ① & ②
$\exists$ a point $x_0 \in [a, b] \ni g(x_0) = 0$
$\Rightarrow f(x_0) - c = 0$
$\Rightarrow f(x_0) = c$

Since x_0 is arbitrary point
ie, $f(x) = c \quad \forall x \in [a, b]$
Hence the theorem.

INVERSE FUNCTION THEOREM

Statement

Let f be a continuous 1-1 function on $[a,b]$, then f^{-1} is also continuous.

Proof

Since f is continuous 1-1 function on $[a,b]$

$f(a) \neq f(b)$ for $a \neq b$.

Without loss of generality let us take $f(a) < f(b)$

$f(a) = c$ & $f(b) = d$.

Show that F from $f : [a,b] \to [c,d]$.

Then $= [c,d] \to [a,b]$ also 1-1.

Now we want to prove that g is continuous.

For $y_1 \neq y_2$ in $[c,d]$.

we prove that $\exists$ a x_1 and x_2 in $[a,b]$

$\ni f(x_1) = y_1$, $f(x_2) = y_2$.

Now, $y_1 = y_2 \Rightarrow f(x_1) = f(x_2)$

$y_1 \neq y_2 \Rightarrow f(x_1) \neq f(x_2)$

$\Rightarrow x_1 \neq x_2$ [$\because f$ is 1-1

Again f is continuous on $[a,b]$ and

$f(x_1) = y_1$, $f(x_2) = y_2$.

ie, $a < x_1 < x_2 < b \Rightarrow f(a) < f(x_1) < f(x_2) < f(b)$

Let $y_0 \in (c,d)$ then $\exists$ a unique $x_0 \in (a,b)$

$\ni f(x_0) = y_0$ and $(x_0 - \epsilon, x_0 + \epsilon) \subset (a,b)$

$\therefore x_0 - \epsilon < x_0 < x_0 + \epsilon$.

$\Rightarrow f(x_0 - \epsilon) < f(x_0) < f(x_0 + \epsilon)$

Put $f(x_0 - \epsilon) = y_1$, $f(x_0 + \epsilon) = y_2$, $f(x_0) = y_0$.

then $y_1 < y_0 < y_2$

Let $\delta = \min\{y_0 - y_1, y_2 - y_0\} > 0$

ie, $(y_0 - \delta, y_0 + \delta) \subset (y_1, y_2)$

Then $|y - y_0| < \delta \Rightarrow y_0 - \delta < y < y_0 + \delta$

with choice of 'δ' $\quad y_1 < y < y_2$

$$\Rightarrow g(y_1) < g(y) < g(y_2)$$

$$\Rightarrow x_0 - \epsilon < x < x_0 + \epsilon$$

$$\Rightarrow g(y_0) - \epsilon < g(y) < g(y_0) + \epsilon$$

$$\Rightarrow |g(y) - g(y_0)| < \epsilon$$

Hence g is continuous.

UNIFORM CONTINUOUS

The function f defined on I is said to be uniformly continuous on I if given $\epsilon > 0$ $\exists$ a $\delta > 0$ $\ni |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

THEOREMS

① A function f is uniformly continuous on I, then prove that it is continuous.

Proof

Given f is uniformly continuous on I.

By definition, If given $\epsilon > 0$ $\exists$ $\delta > 0$

$\ni |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

Let x_0 be any point on I.

Putting $y = x_0 \in I$

$\therefore |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.

$\Rightarrow f$ is continuous at x_0.

Since x_0 is arbitrary

Hence f is continuous on I.

ie, Hence the proof.

(2) If f is continuous on the closed and bounded interval I, then prove that f is uniformly continuous on I. (15)

Proof

Suppose f is not uniformly continuous on I.

$\therefore$ For every $\epsilon > 0$ $\exists$ $\delta > 0$

$\ni |x-y| < \delta \Rightarrow |f(x) - f(y)| > \epsilon$ $\forall x, y \in I$.

In particular for each +ve integer n,
we can find $x_n, y_n \in I$

$\ni |x_n - y_n| < \frac{1}{n}$ & $|f(x_n) - f(y_n)| > \epsilon$ ----(1)

Since $\langle x_n \rangle$, $\langle y_n \rangle$ are sequences in I.

WKT, Every sequence on a closed interval I has a convergence subsequence.

$\therefore$ $\exists$ a subsequences $\langle x_{n_k} \rangle$, $\langle y_{n_k} \rangle$ x_n, y_n respectively.

Let x_0, y_0 be the points of I

$\ni$ $x_{n_k} \to x_0$; $y_{n_k} \to y_0$ ----- (2)

From equ (1)

$|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$ & $|f(x_{n_k}) - f(y_{n_k})| > \epsilon$ ---- (3)

From the 1^{st} inequality if (3)

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} y_{n_k}.$$

ie, $x_0 \to y_0$.

From the 2^{nd} inequality if (3)

we find that $\langle f(x_{n_k}) \rangle$ and $\langle f(y_{n_k}) \rangle$ converges to different limit

ie, f is not continuous on I.

$\therefore$ Hence the theorem.

TYPES OF DISCONTINUITIES

Let f be a function defined on I and if f be discontinuity at $p \in I$ then

(i) f has a removable discontinuity at $p \in I$ if

$\lim_{x \to p} f(x)$ exists $\neq f(p)$

(ii) f has a discontinuity of the 1^{st} kind from left at p if $\lim_{x \to p-0} f(x)$ exists $\neq f(p)$

(iii) f has a discontinuity of the 1^{st} kind from right at p if $\lim_{x \to p+0} f(x)$ exists $\neq f(p)$

(iv) f has a discontinuity on the 1^{st} kind if

$\lim_{x \to p} f(x)$ exists $\neq f(p)$

(v) f has a discontinuity of the 2^{nd} kind from left at p if $\lim_{x \to p} f(x)$ does not exists $\neq f(p)$

(vi) f has a discontinuity of the 2^{nd} kind from right at p if $\lim_{x \to p+0} f(x)$ does not exists $\neq f(p)$

(vii) f has a discontinuity of the 2^{nd} kind if

$\lim_{x \to p} f(x)$ does not exists $\neq f(p)$

Derivatives

(i) Derivable from left at $x=b$.

Let f be function defined on $[a,b]$

if $\lim\limits_{x\to b-0} \dfrac{f(x)-f(b)}{x-b}$ exists and equal to $f'(b)$.

Then f is derivable from left at $x=b$.

(ii) Derivable from right at $x=a$.

Let f be a function defined on $[a,b]$

if $\lim\limits_{x\to a+0} \dfrac{f(x)-f(a)}{x-a}$ exists and equal to $f'(a)$.

Then f is derivable from right at $x=a$.

(iii) Derivable at $x=x_0$

Let f be a function defined on $[a,b]$

if $\lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0}$ exists and equal to $f'(x_0)$

Then f is derivable at $x=x_0$.

Theorems

① Let f be a function defined on I if f be derivable at $x_0 \in I$, then it is continuous at $x_0 \in I$.

(or) Prove that, Every derivable function is continuous.

Proof

Since f is derivable at $x=x_0$

$\therefore \lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0}$ exists and equal to $f'(x_0)$

Now, $f(x)-f(x_0) = \dfrac{f(x)-f(x_0)}{x-x_0} \times x-x_0$ if $x \neq x_0$

Applying $\lim\limits_{x\to x_0}$ on both sides

$$\lim_{x\to x_0} [f(x)-f(x_0)] = \lim_{x\to x_0} \left[\frac{f(x)-f(x_0)}{x-x_0} \times (x-x_0)\right]$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \times \lim_{x \to x_0} (x - x_0)$$

$$= f'(x_0) \times 0$$

$$\therefore \lim_{x \to x_0} \left[f(x) - f(x_0) \right] = 0$$

$$\text{ie, } \lim_{x \to x_0} f(x) - \lim_{x \to x_0} f(x_0) = 0$$

$$\therefore \lim_{x \to x_0} f(x) = f(x_0)$$

Hence f is continuous at $x_0 \in I$.

Note

The Converse of the above theorem does not hold.

For example,

$$f(x) = 0 \quad \text{if } x \le 0$$
$$= x \quad \text{if } x > 0$$

In this example f is Continuous at all points.

But it is not derivable at $x = 0$.

$$\lim_{\substack{x \to 0 \\ x < 0}} \frac{f(x) - f(0)}{x - 0} = f'(0)$$

$$\lim_{\substack{x \to 0 \\ x > 0}} \frac{f(x) - f(0)}{x - 0} = f'(0)$$

$$\operatorname{limit}_{\substack{x \to 0 \\ x > 0}} \frac{x - 0}{x - 0} = f'(0)$$

$$f'(0-0) \ne f'(0) = f'(0+0)$$

2 If a function f is derivable at x_0, then for each real number c in $[a,b]$, the function cf is derivable at x_0 and $(cf)'(x_0) = cf'(x_0)$

Proof

Given f is derivable at x_0

$$\therefore \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

Now, $$\frac{(cf)(x) - (cf)(x_0)}{x - x_0} = \frac{cf(x) - cf(x_0)}{x - x_0}$$

$$\frac{(cf)(x) - (cf)(x_0)}{x - x_0} = \frac{c\,[f(x) - f(x_0)]}{x - x_0}$$

Applying $\lim_{x \to x_0}$ on both sides

$$\lim_{x \to x_0} \frac{(cf)(x) - (cf)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{c\,[f(x) - f(x_0)]}{x - x_0}$$

$$(cf)'(x_0) = \lim_{x \to x_0} c \times \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow (cf)'(x_0) = c f'(x_0)$$

Hence the theorem.

3) Let f and g be defined on I and if f and g are derivable at $x_0 \in I$, then prove that $f+g$ is derivable at x_0.

Proof

Since f and g are derivable at x_0.

ie, $$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \quad \& \quad \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0)$$

Now, $$\frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} = \frac{f(x) + g(x) - [f(x_0) + g(x_0)]}{x - x_0}$$

$$= \frac{f(x) - f(x_0) + g(x) - g(x_0)}{x - x_0}$$

$$\frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0}$$

Applying $\lim_{x \to x_0}$ on both sides.

$$\therefore \lim_{x \to x_0} \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} = \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \right]$$

$$(f+g)'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

$$(f+g)'(x_0) = f'(x_0) + g'(x_0)$$

Hence $f+g$ is derivable at x_0.

④ Let f and g be defined on I and if f and g are derivable at $x_0 \in I$, then prove that fg is derivable at x_0 and $(fg)'(x_0) = f'(x_0) g(x_0) + f(x_0) g'(x_0)$

Proof

Since f and g are derivable at x_0.

ie, $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$ & $\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0)$

Now, $$\frac{(fg)(x) - (fg)(x_0)}{x - x_0} = \frac{f(x) g(x) - [f(x_0) g(x_0)]}{x - x_0}$$

$$= \frac{f(x) g(x) - f(x_0) g(x_0) + f(x_0) g(x) - f(x_0) g(x)}{x - x_0}$$

$$= \frac{[f(x) - f(x_0)] g(x) + [g(x) - g(x_0)] f(x_0)}{x - x_0}$$

$$= \frac{[f(x) - f(x_0)] g(x)}{x - x_0} + \frac{[g(x) - g(x_0)] f(x_0)}{x - x_0}$$

Applying $\lim_{x \to x_0}$ on both sides.

$$\lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} = \lim_{x \to x_0} \left\{ \frac{[f(x) - f(x_0)]}{x - x_0} g(x) + \frac{[g(x) - g(x_0)] f(x_0)}{x - x_0} \right\}$$

$$\therefore (fg)'(x_0) = \lim_{x \to x_0} \frac{[f(x) - f(x_0)] g(x)}{x - x_0} + \lim_{x \to x_0} \frac{[g(x) - g(x_0)] f(x_0)}{x - x_0}$$

ie, $(fg)'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \times \lim_{x \to x_0} g(x)$

$+ \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} \times \lim_{x \to x_0} f(x_0)$

$(fg)'(x_0) = f'(x_0)\, g(x_0) + g'(x_0)\, f(x_0)$

Hence fg is derivable at $x_0 \in I$.

5 Let f be derivable at x_0 and $f(x_0) \neq 0$, then prove that $\frac{1}{f}$ is derivable at x_0 and $\left(\frac{1}{f}\right)'(x_0) = \frac{-f'(x_0)}{[f(x_0)]^2}$

Proof

Since f is derivable at x_0 and also Continuous at x_0.

ie, $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$

and $\lim_{x \to x_0} f(x) = f(x_0) \neq 0$

Now,

$$\frac{\left(\frac{1}{f}\right)(x) - \left(\frac{1}{f}\right)(x_0)}{x - x_0} = \frac{1}{x - x_0}\left[\frac{1}{f(x)} - \frac{1}{f(x_0)}\right]$$

$$\frac{\left(\frac{1}{f}\right)(x) - \left(\frac{1}{f}\right)(x_0)}{x - x_0} = \frac{1}{x - x_0}\left[\frac{f(x_0) - f(x)}{f(x)\, f(x_0)}\right]$$

Apply $\lim_{x \to x_0}$ on both sides.

$$\lim_{x \to x_0} \frac{\left(\frac{1}{f}\right)(x) - \left(\frac{1}{f}\right)(x_0)}{x - x_0} = \lim_{x \to x_0}\left\{\frac{-[f(x) - f(x_0)]}{x - x_0} \cdot \frac{1}{f(x) f(x_0)}\right\}$$

$$f'(x_0) = -\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} \frac{1}{f(x)} \cdot \lim_{x \to x_0} \frac{1}{f(x_0)}$$

$$= -f'(x_0)\, \frac{1}{f(x_0)}\, \frac{1}{f(x_0)}$$

ie, $f'(x_0) = \frac{-f'(x_0)}{[f(x_0)]^2}$

Hence the theorem.

(6) Let f and g are derivable at $x_0 \in I$ and let $g(x_0) \neq 0$ then prove that $\frac{f}{g}$ is derivable at x_0 and (6)

$$\left(\frac{f}{g}\right)'(x_0) = \frac{g(x_0) f'(x_0) - f(x_0) g'(x_0)}{[g(x_0)]^2}$$

Proof

Since f and g are derivable at x_0

ie, $\lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$ & $\lim\limits_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0)$

Now,

$$\frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(x_0)}{x - x_0} = \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}}{x - x_0}$$

$$= \frac{1}{x - x_0}\left[\frac{f(x) g(x_0) - g(x) f(x_0)}{g(x)}\right]$$

$$= \frac{1}{x - x_0}\left[\frac{f(x) g(x_0) + f(x_0) g(x_0) - f(x_0) g(x_0) - f(x_0) g(x)}{g(x)\; g(x_0)}\right]$$

$$= \frac{1}{x - x_0}\left[\frac{[f(x) - f(x_0)] g(x_0) - [g(x) - g(x_0)] f(x_0)}{g(x) g(x_0)}\right]$$

$$\therefore \frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(x_0)}{x - x_0} = \left\{\frac{[f(x) - f(x_0)]\, g(x_0)}{x - x_0} - \frac{[g(x) - g(x_0)]\, f(x_0)}{x - x_0}\right\} \frac{1}{g(x)\, g(x_0)}$$

Appling $\lim\limits_{x \to x_0}$ on both sides.

$$\lim_{x \to x_0} \frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(x_0)}{x - x_0} = \lim_{x \to x_0} \left[\frac{[f(x) - f(x_0)]\, g(x_0)}{x - x_0} - \frac{[g(x) - g(x_0)]\, f(x_0)}{x - x_0}\right] \frac{1}{g(x)\, g(x_0)}$$

ie, $\left(\frac{f}{g}\right)'(x_0) = \lim_{x \to x_0} \left[\frac{[f(x)-f(x_0)]\, g(x_0)}{x-x_0} - \frac{[g(x)-g(x_0)]\, f(x_0)}{x-x_0} \right] \times \lim_{x \to x_0} \frac{1}{g(x)\, g(x_0)}$ (7)

$$= \left\{ \lim_{x\to x_0} \frac{[f(x)-f(x_0)]\, g(x_0)}{x-x_0} - \lim_{x\to x_0} \frac{[g(x)-g(x_0)]\, f(x_0)}{x-x_0} \right\} \times \lim_{x\to x_0} \frac{1}{g(x)} \cdot \lim_{x\to x_0} \frac{1}{g(x_0)}$$

$$= \left[\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \times \lim_{x\to x_0} g(x_0) - \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} \times \lim_{x\to x_0} f(x_0) \right] \times \frac{1}{g(x_0)} \times \frac{1}{g(x_0)}$$

ie, $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\, g(x_0) - g'(x_0)\, f(x_0)}{[g(x_0)]^2}$

Hence the theorem.

CHAIN RULE

Statement

Let f and g be functions such that the range of f is contained in the domain of g. If f is derivable at x_0 and g is derivable at $f(x_0)$, then prove that $(g \circ f)$ is derivable at x_0 and

$(g\circ f)'(x_0) = g'(f(x_0))\, f'(x_0)$.

Proof

Since the range of f is contained in the domain of g.

ie, $g\circ f$ has the same domain as that of 'f'.

To prove that, $\lim_{x\to x_0} \frac{(g\circ f)(x) - (g\circ f)(x_0)}{x-x_0}$ exists

and equal to $g'(f(x_0)) . f'(x_0)$

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ie, To prove that, $\lim_{h\to 0} \frac{(g\circ f)(x_0+h)-(g\circ f)(x_0)}{h}$

exists and equal to $g'(f(x_0))f'(x_0)$

Define a function F by setting

$$F(h) = \begin{cases} \frac{g(f(x_0+h))-g(f(x_0))}{f(x_0+h)-f(x_0)} & \text{if } f(x_0+h)-f(x_0)\neq 0 \\ g'(f(x_0)) & \text{if } f(x_0+h)-f(x_0)=0 \end{cases}$$

In terms of F, we have,

$$G(h) = \frac{(g\circ f)(x_0+h)-(g\circ f)(x_0)}{h} \times \frac{f(x_0+h)-f(x_0)}{f(x_0+h)-f(x_0)}$$

$$= \frac{(g\circ f)(x_0+h)-(g\circ f)(x_0)}{f(x_0+h)-f(x_0)} \times \frac{f(x_0+h)-f(x_0)}{h}$$

ie, $G(h) = F(h)\left[\frac{f(x_0+h)-f(x_0)}{h}\right]$ whenever $h\neq 0$ ----- (1)

Equation (1) holds whenever $f(x_0+h)-f(x_0)=0$ then for each side of equ (1) is zero.

Since f is derivable at x_0.

ie, $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists and equal to $f'(x_0)$

From equ (1) we prove that

$\lim_{h\to 0} G(h)$ exists and equal to $g'(f(x_0))f'(x_0)$

ie, $\lim_{k\to 0} \frac{g(f(x_0)+k)-g(f(x_0))}{k} = g'(f(x_0))f'(x_0)$ ----- (2)

$\lim_{h\to 0} F(h)$ exists and equal to g

⑨

② $\Rightarrow$ Given $\epsilon > 0$, we can find a number $\delta > 0$

$\ni$ $0 < |k| < \delta$ then $\left| \dfrac{g(f(x_0)+k) - g(f(x_0))}{k} - g'(f(x_0)) \right| < \epsilon$ ---③

Also since f is derivable at x_0 and Continuous at x_0.

$\therefore$ We can find a number $\delta' > 0$

$\ni$ $|h| < \delta'$ then $|f(x_0+h) - f(x_0)| < \epsilon$ -----④

Now let us consider any number h

$\ni$ $|h| < \delta'$ and $f(x_0+h) - f(x_0) = 0$

then $|F(h) - g'(f(x_0))| < \epsilon$ ----⑤

From the definition of f, On the otherhand

$$f(x_0+h) - f(x_0) \neq 0$$

then $f(x_0+h) - f(x_0) = K$ (say)

$$\Rightarrow f(x_0+h) = f(x_0) + K.$$

We have,

$$F(h) = \frac{g(f(x_0)+h) - g(f(x_0))}{f(x_0+h) - f(x_0)}$$

$$= \frac{g(f(x_0)+K) - g(f(x_0))}{f(x_0)+K - f(x_0)}$$

ie, $F(h) = \dfrac{g(f(x_0)+K) - g(f(x_0))}{K}$ -----⑥

$\therefore$ $|F(h) - g'(f(x_0))| < \epsilon$ provided $|K| < \delta$ ----⑦

$|F(h) - g'(f(x_0))| < \epsilon$ provided $|f(x_0+h) - f(x_0)| < \delta$

From equs ⑤ and ⑦

we find that

if $|K| < \delta'$ then $|F(h) - g'(f(x_0))| < \epsilon$

ie, $\lim_{h \to 0} F(h)$ exists and equal to $g'(f(x_0))$

Hence $\lim_{h\to 0} G(h) = \lim_{h \to 0} F(h) \cdot \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$

$$\therefore (g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0).$$

Hence the theorem.

INVERSE FUNCTION THEOREM FOR DERIVATIVES.

Statement

Let f be a continuous 1-1 function defined on I and let f be derivable at x_0 with $f(x_0) \neq 0$ then the inverse of f is derivable at $f(x_0)$ and its derivative is $\frac{1}{f'(x_0)}$

Proof

Let $f : x \to y$ be a mapping

If g be the inverse of f [ie, $g = f^{-1}$]

Then $g : y \to x \ni f(x) = y \Leftrightarrow g(y) = x$.

Now, Let $f(x_0) = y_0$ So that $g(y_0) = x_0$.

Let $y_0 + k$ be any point of y different from y_0

Since f is 1-1, $\exists$ a unique $x_0 + h \; (\neq x_0)$

$\ni f(x_0 + h) = y_0 + k$

By the definition of g

$g(y_0 + k) = x_0 +$

$$\left.\begin{array}{l} \text{Then, } f(x_0) = y_0, \; f(x_0+h) = y_0 + k \\ \text{and } g(y_0) = x_0, \; g(y_0 + k) = y_0 + h \end{array}\right\} \text{-----} (1)$$

$\&$ $k \neq 0 \Rightarrow h \neq 0$ ----- (2)

It can be easily seen that $k \to 0$ and $h \to 0$.

Since f is derivable at x_0 and also continuous at x_0.

By using the theorem "If f be a continuous 1-1 function on the closed interval, then f^{-1} is also continuous".

ie, g is continuous at y_0.

ie, $\lim_{k \to 0} (g(y_0+k) - g(y_0)) = 0$

$\therefore \lim_{k \to 0} g(y_0+k) = g(y_0)$

ie, $\lim_{h \to 0} (x_0 + h - x_0) = 0$

$\lim_{h \to 0} h = 0$ ----- (3)

Now, Let $k \neq 0$, then

$$\frac{g(y_0+k) - g(y_0)}{k} = \frac{x_0+h-x_0}{k}$$

$$= \frac{h}{y_0+k-y_0}$$

$$= \frac{h}{f(x_0+h) - f(x_0)}$$

$$\frac{g(y_0+k) - g(y_0)}{k} = \frac{1}{\frac{f(x_0+h) - f(x_0)}{h}}$$

Applying $\lim_{k \to 0}$ on both sides.

$$\lim_{k \to 0} \frac{g(y_0+k) - g(y_0)}{k} = \lim_{k \to 0} \frac{1}{\frac{f(x_0+h) - f(x_0)}{h}}$$

ie, $g'(y_0) = \frac{1}{f'(x_0)}$ [$\because k \to 0$ then $h \to 0$

Hence the theorem.

DARBOUX'S THEOREM

Statement

Let f be defined and derivable on $[a,b]$. If $f'(a)f'(b)<0$, then $\exists$ a real number c between a and b such that $f'(c)=0$

Proof

Case (i)

Let $f'(b)>0$ and $f'(a)<0$

then $f'(a)f'(b)<0$

This can be derived in 6 steps.

Step ①:

Since $f'(a)<0$ $\exists$ $h_1>0$

$\ni$ $f(x)<f(a)$ $\forall$ $x\in[a,a+h[$

Since f is derivable at 'a'

ie, $\lim_{x\to a+0}\frac{f(x)-f(a)}{x-a}=f'(a)$

Taking $\epsilon=-f'(a)$ [$\because$ $f'(a)<0$, we can find a number $h_1>0$

$\ni a<x<a+h_1 \Rightarrow \left|\frac{f(x)-f(a)}{x-a}-f'(a)\right|<\epsilon$

ie, $-\epsilon<\frac{f(x)-f(a)}{x-a}-f'(a)<\epsilon$ whenever $a<x<a+h$.

$\therefore$ $f'(a)-\epsilon<\frac{f(x)-f(a)}{x-a}<f'(a)+\epsilon$.

From the 2^{nd} part of the inequality

we get $f'(a)+\epsilon=f'(a)-f'(a)$.

$\therefore$ $f'(a)+\epsilon=0$

ie, $\frac{f(x)-f(a)}{x-a}<0$

$\therefore$ $f(x)-f(a)<0$

$\therefore f(x) < f(a)$

Step ②

If $f'(b) > 0$, $\exists\ h_2 > 0$

$\ni f(x) < f(b) \quad \forall x \in]b-h_2, b]$

Since f is derivable at b

ie, $\lim_{x \to b-0} \dfrac{f(x) - f(b)}{x-b} = f'(b)$

Taking $\epsilon = f'(b)$, we can find a number $h_2 > 0$

$\ni b - h_2 < x < b$ then $\left| \dfrac{f(x) - f(b)}{x-b} - f'(b) \right| < \epsilon$

ie, $-\epsilon < \dfrac{f(x) - f(b)}{x-b} - f'(b) < \epsilon$

$$f'(b) - \epsilon < \frac{f(x) - f(b)}{x-b} < f'(b) + \epsilon.$$

From the 1^{st} part of inequality

$f'(b) - \epsilon < \dfrac{f(x) - f(b)}{x-b}$ whenever $b - h_2 < x < b$.

ie, $0 < \dfrac{f(x) - f(b)}{x-b}$ when $x < b$

$\therefore 0 > f(x) - f(b)$ ($\because$ when $x < b$

$\Rightarrow f(b) > f(x)$

$\Rightarrow f(x) < f(b)$

Step ③

Since f is derivable on $[a, b]$ and also continuous on $[a, b]$

And consequently it attains its supremum as well as inf on $[a, b]$

Now, By step ①, $\inf f \neq f(a)$

and by step ②, $\inf f \neq f(b)$

$\Rightarrow f$ does not attain its infimum at the end points.

$\therefore \exists$ a real number c in (a, b)

$\ni \inf f = f(c)$

Step ④

To prove that $f'(c) \not> 0$.

Suppose $f'(c) > 0$ then $Lf'(c) > 0$ and by step ②

we can find a number $h_3 > 0$

$\ni f(x) < f(c) \quad \forall x \in]c-h_3, c[$

Which is a Controduction to our assumption

Because $f(c)$ is the $\inf f$ on $[a, b]$.

Hence $f'(c) \not> 0$.

Step ⑤

To prove : $f'(c) \not< 0$.

Suppose $f'(c) < 0$ then $Rf'(c) < 0$ and by step ①

we can find a number $h_4 > 0$

$\ni f(x) < f(c) \quad \forall x \in]c, c+h_4[$

Which is a Controduction to our assumption.

Because $f(c)$ is the $\inf f$ on $[a, b]$.

Hence $f'(c) \not< 0$.

Step ⑥

By the step ④ and ⑤

$f'(c) \not> 0$ & $f'(c) \not< 0$.

$\Rightarrow f'(c) = 0$

Case (ii)

Let $f'(a) > 0$ and $f'(b) < 0$

If g be the function $-f$.

Then g is derivable on $[a,b]$

and $g'(a) = -f'(a) < 0$

$g'(b) = -f'(b) > 0$

By Case ①, $\exists$ a real number d in $[a,b]$

$\ni \; g'(d) = 0$

ie, $-f'(d) = 0$

$\Rightarrow f'(d) = 0.$

Hence the theorem.

UNIT - IV

ROLLE'S THEOREM

Statement

Let f be a function defined on $[a,b]$

$\ni$ if (i) f is Continuous on $[a,b]$

(ii) f is derivable on $]a,b[$

(iii) $f(a) = f(b)$

Then $\exists$ a real number c between a and b

$\ni \; f'(c) = 0.$

Proof

Since f is Continuous on $[a,b]$.

$\therefore$ It is Closed and bounded on $[a,b]$

ie, Supremum and Infimum exist.

Let $\text{Sup } f = M$ & $\text{Inf } f = m$.

Case (i)

If $M = m$.

Then, f is Constant function

$$f'(x) = 0 \quad \forall x \in [a, b]$$

$$f'(c) = 0 \quad \forall c \in [a, b]$$

Case (ii)

If $M \neq m$.

Since $f(a) = f(b)$

$\therefore$ Atleast one of the numbers M and m are different from $f(a)$ and also from $f(b)$.

Assume that $M \neq f(a)$

ie, $M \neq f(b)$ also.

$\therefore \exists$ a real number $c \in (a, b)$

$\ni f(c) = M$

Since $f(c)$ is supremum of f on $[a, b]$

$\therefore f(x) \leq f(c) \quad \forall x \in [a, b]$ ----- ①

In particular, $f(c-h) \leq f(c)$ $\forall$ any real number $h > 0$.

$$\ni \frac{f(c) - f(c-h)}{h} \geq 0.$$

$$\text{ie, } \frac{f(c-h) - f(c)}{-h} \geq 0 \quad \forall c-h \in [a, b]$$

Taking $\lim\limits_{h \to 0}$ on both sides

$$Lf'(c) \geq 0 \quad \text{------ ②}$$

From equ ① and $f(c)$ is supremum.

We have $f(c+h) \leq f(c)$

For any real number $h > 0$

$$\ni \frac{f(c) - f(c+h)}{h} \geq 0$$

Taking $\lim\limits_{h \to 0}$ on both sides.

POWER SERIES EXPANSION

① Prove that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

Proof

Let $f(x) = \sin x$

$f^n(x) = \sin\left(x + \frac{n\pi}{2}\right), \forall x \in \mathbb{R}$

Thus for each $n \in \mathbb{N}$, f^n is defined in the interval $[-h, h]$.

By lagrange's remainder after n terms we have,

$$R^n(x) = \frac{x^n}{n!} f^n(\theta x), \text{ where } 0 < \theta < 1$$

ie, $$R^n(x) = \frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right)$$

Now for all $x \in \mathbb{R}$.

To prove that $\lim_{n \to \infty} R^n(x) = 0$.

ie, To prove that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

Let $a_n = \frac{x^n}{n!}$ & $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$, $\forall n \in \mathbb{N}$

Then $$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}$$

$$= \frac{x^n x}{(n+1)n!} \times \frac{n!}{x^n}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{x}{n+1}$$

ie, $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x}{n+1}$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and equal to zero

Thus we find that for each h, the function f

has a mechlarin Series expanssion for each $x \in [-h,h]$ (27)

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots$$

$f(x) = \sin x$ $\quad f(0) = 0$

$f'(x) = \cos x$ $\quad f'(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$

$f'''(x) = -\cos x$ $\quad f'''(0) = -1$

$$\therefore f(x) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \dots$$

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Hence the proof.

(2) Prove that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

Proof

Let $f(x) = \cos x$

$f^n(x) = \cos\left(x + \frac{n\pi}{2}\right)$, $\forall x \in R$.

Thus for each $n \in N$, f^n is defined in the interval $[-h,h]$.

By lagrange's remainder after n terms

we have, $R^n(x) = \frac{x^n}{n!} f^n(\theta x)$, where $0 < \theta < 1$

ie, $R^n(x) = \frac{x^n}{n!} \cos\left(\theta x + \frac{n\pi}{2}\right)$

Now for all $x \in R$

To prove that $\lim_{n \to \infty} R^n(x) = 0$

ie, To prove that $\lim_{n \to \infty} \frac{x^n}{n!} = 0$

Let $a_n = \frac{x^n}{n!}$ & $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

Then $\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}$

$= \frac{x^n x}{(n+1)\, n!} \times \frac{n!}{x^n}$

$\therefore \frac{a_{n+1}}{a_n} = \frac{x}{n+1}$

ie, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x}{n+1}$

$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and equal to zero.

Thus we find that for each h, the function f has a mechlarin series expanssion for each $x \in [-h, h]$.

$f(x) = \cos x$ $\quad f(0) = 1$

$f'(x) = -\sin x$ $\quad f'(0) = 0$

$f''(x) = -\cos x$ $\quad f''(0) = -1$

$f'''(x) = \sin x$ $\quad f'''(0) = 0$

$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots$

$\therefore \cos x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \dots$

ie, $\cos x = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + \dots$

$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

Hence the proof.