CORE COURSE VI

CLASSICAL ALGEBRA AND THEORY OF NUMBERS

Objectives

1. To lay a good foundation for the study of Theory of Equations.
2. To train the students in operative algebra.

Unit I

Relation between roots & coefficients of Polynomial Equations – Symmetric functions – Sum of the r^{th} Powers of the Roots

Unit II

Newtion's theorem on the sum of the power of the roots-Transformations of Equations – Diminshing, Increasing & Multiplying the roots by a constant - Reciprocal equations - To increase or decrease the roots of the equation by a given quantity.

Unit III

Form of the quotient and remainder – Removal of terms – To form of an equation whose roots are any power – Transformation in general – Descart's rule of sign

Unit IV

Inequalities – elementary principles – Geometric & Arithmetic means – Weirstrass inequalities – Cauchy inequality – Applications to Maxima & Minima.

Unit V

Theory of Numbers – Prime & Composite numbers – divisors of a given number N – Euler's Function (N) and its value – The highest Power of a prime P contained in N! – Congruences – Fermat's, Wilson's & Lagrange's Theorems.

Text Book(s)

1. T.K.Manickavasagam Pillai & others Algebra Volume I.S.V. Publications – 1985 Revised Edition.
2. T.K. Manickavasagam Pillai & others Algebra Volume II, S.V.Publications – 1985 Revised Edition.

Unit I	:	Chapter 6 Section 11 to 13 of (1)
Unit II	:	Chapter 6 Section 14 to 17 of (1)
Unit III	:	Chapter 6 Section 18- 21 & 24 of (1)
Unit IV	:	Chapter 4 of (2)
Unit V	:	Chapter 5 of (2)

References :

1. H.S.Hall and S.R. Knight, Higher Algebra, Prentice Hall of India, New Delhi.
2. H.S. Hall and S.R.Knight, Higher Algebra, McMillan and Co., London, 1948.

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CLASSICAL ALGEBRA AND THEORY OF NUMBERS

UNIT - I

Chapter-6 - Section - II

Relations between the roots and coefficients of equations :-

Let the equation be

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = 0$$

If this equation has the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, then we have

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$$

$$= x^n - \Sigma \alpha_1 x^{n-1} + \Sigma \alpha_1 \alpha_2 x^{n-2} - \cdots + (-1)^n \alpha_1 \alpha_2 \cdots \alpha_n \quad \rightarrow (1)$$

$$= x^n - S_1 x^{n-1} + S_2 x^{n-2} - \cdots + (-1)^n S_n$$

where $S_1, S_2, \ldots, S_n$ are the sum of the products of the roots.

Equating the coefficients of $x^n, x^{n-1}, \ldots, x$ and constant terms, we have

$-p_1 = S_1 =$ sum of the roots taken one at a time.

$(-1)^2 p_2 = S_2 =$ sum of the products of the roots taken two at a time.

$(-1)^3 p_3 = S_3 =$ sum of the products of the roots taken three at a time.

$\vdots$

$(-1)^n p_n = S_n =$ product of the roots

If the equation is (2)

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0$$

divide each term of the equation by a_0, we get

$$x^n + \frac{a_1}{a_0} x^{n-1} + \frac{a_2}{a_0} x^{n-2} + \cdots + \frac{a_{n-1}}{a_0} x + \frac{a_n}{a_0} = 0 \quad \rightarrow (2)$$

So we have,

$$\Sigma \alpha_1 = -\frac{a_1}{a_0}$$

$$\Sigma \alpha_1 \alpha_2 = \frac{a_2}{a_0}$$

$$\Sigma \alpha_1 \alpha_2 \alpha_3 = -\frac{a_3}{a_0}$$

$$\vdots$$

Finally we get:

$$\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

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Examples:-

1. If α, β are the roots of the equation $2x^2+3x+5=0$. Find $\alpha+\beta$, $\alpha\beta$.

Soln.:- Given that $2x^2+3x+5=0$

If $a_0=2$, $a_1=3$ and $a_2=5$

$$\Sigma \alpha = \alpha+\beta = \frac{-a_1}{a_0} \qquad \Sigma \alpha\beta = \frac{a_2}{a_0}$$

$$\therefore \boxed{\alpha+\beta = \frac{-3}{2}} \qquad \boxed{\alpha\beta = \frac{5}{2}}$$

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2. If α, β, γ are the roots of $2x^3+3x^2+5x+6=0$. (3)
Find $\Sigma\alpha$, $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$.

Soln.:-

Given that $2x^3+3x^2+5x+6=0$

If $a_0=2$, $a_1=3$, $a_2=5$, $a_3=6$

$$\therefore \Sigma\alpha = \frac{-a_1}{a_0} = \frac{-3}{2}$$

$$\Sigma\alpha\beta = \frac{a_2}{a_0} = \frac{5}{2}$$

$$\Sigma\alpha\beta\gamma = \frac{-a_3}{a_0} = \frac{-6}{2} = -3$$

===X===

3. Solve the equation $x^3+6x+20=0$ one root being $1+3i$.

Soln.:- Given that $x^3+6x+20=0$

The one root is $1+3i=\alpha$

Take the other root $\boxed{1-3i=\beta}$

If $a_0=1$, $a_1=0$, $a_2=6$, $a_3=20$

$$\Sigma\alpha \therefore \alpha+\beta+\gamma = \frac{-a_1}{a_0}$$

$$1+\cancel{3i}+1-\cancel{3i}+\gamma = \frac{-0}{1}$$

$$2+\gamma=0$$

$$\therefore \boxed{\gamma=-2}$$

===X===

4. Solve the equation $3x^3-23x^2+72x-70=0$, having given the root is $3+i\sqrt{-5}$ (4)

Soln:-

Given that $3x^3-23x^2+72x-70=0$

Given that the root is $\alpha=3+i\sqrt{-5}$

Take the other root is $\beta=3-i\sqrt{-5}$

If $a_0=3$, $a_1=-23$, $a_2=72$, $a_3=-70$

$$\therefore \alpha+\beta+\gamma=\frac{-a_1}{a_0}$$

$$\therefore 3+\cancel{i\sqrt{-5}}+3-\cancel{i\sqrt{-5}}+\gamma=\frac{-(-23)}{3}$$

$$6+\gamma=\frac{23}{3}$$

$$\gamma=\frac{23}{3}-6$$

$$\therefore \boxed{\gamma=\frac{5}{3}}$$

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5. Solve the equation $x^4+4x^3+6x^2+4x+5=0$. Given the root is $\sqrt{-1}$.

Soln:- Given that $x^4+4x^3+6x^2+4x+5=0 \rightarrow (1)$

One root is $\sqrt{-1}=i$, since complex root occur in parts $-i$ is also a root of the given equation.

Since, $x=i$ and $x=-i$

We have $(x-i)(x+i)$ is a factor (1).

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(ii) x^2+1 is a factor of ①.

⑤

$$
\begin{array}{r|l}
 & x^2+4x+5 \\
\hline
x^2+1 & x^4+4x^3+6x^2+4x+5 \\
 & \underline{\cancel{x^4} \quad\quad + x^2} \\
 & \quad (-) \quad\quad (-) \\
 & \quad 4x^3+5x^2+4x+5 \\
 & \quad \underline{\cancel{4x^3} \quad\quad + \cancel{4x}} \\
 & \quad (-) \quad\quad (-) \\
 & \quad\quad 5x^2 \quad +5 \\
 & \quad\quad \underline{5x^2 \quad +5} \\
 & \quad\quad (-) \quad\quad (-) \\
 & \quad\quad\quad 0
\end{array}
$$

$(x-i)(x+i)$

$x^2+ix-ix$

$\therefore x^2+4x+5=0$

$ax^2+bx+c=0$

$$\Rightarrow x = \frac{-4 \pm \sqrt{16-20}}{2}$$

$a=1$
$b=4$
$c=5$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-4 \pm i2}{2}$$

$$\therefore \boxed{x = -2 \pm i}$$

Hence the four roots are

$x = i, -i, -2+i, -2-i$

=====X=====

6. Solve the equation $x^3-12x^2+39x-28=0$, whose roots are in Arithmetic Progression. (A.P) (6)

Soln.:-

Let the roots are $\alpha-d, \alpha, \alpha+d$.

$\therefore$ The sum of the roots

a, a+d, a+2d
a+3d

$$(\alpha-d)+\alpha+(\alpha+d)=\frac{-a_1}{a_0}$$

$$3\alpha = \frac{-(-12)}{1}$$

$$3\alpha = 12$$

$$\therefore \boxed{\alpha = 4}$$

$a_0 = 1$
$a_1 = -12$
$a_2 = 39$
$a_3 = -28$

4	1	-12	39	-28
	0	4	-32	28
	1	-8	7	$\boxed{0}$

$\therefore x^2-8x+7=0$

$(x-1)(x-7)=0$

$\Rightarrow x=1$ & $x=7$

$\therefore$ The roots are $\boxed{x = 4, 1, 7}$.

——X——

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Examples :-

1. Show that the roots of the equation $x^3+px^2+qx+r=0$ are in Arithmetical Progression if $2p^3-9pq+27r=0$. Show that the above condition is satisfied by the equation $x^3-6x^2+13x-10=0$. Hence or otherwise solve the equation.

Proof :-

Let the root of the equation $x^3+px^2+qx+r=0$ be $\alpha-\delta, \alpha, \alpha+\delta$.

We have $a_0=1, a_1=p, a_2=q, a_3=r$

$$\therefore \Sigma\alpha_1 = \alpha-\cancel{\delta}+\alpha+\alpha+\cancel{\delta} = \frac{-a_1}{a_0} = \frac{-p}{1} = -p$$

$$\therefore 3\alpha = -p \Rightarrow \boxed{\alpha = \frac{-p}{3}}$$

$$\Rightarrow \Sigma\alpha_1\alpha_2 = (\alpha-\delta)\alpha+(\alpha-\delta)(\alpha+\delta)+\alpha(\alpha+\delta) = \frac{a_2}{a_0}$$

$$\Rightarrow (\alpha^2-\cancel{\alpha\delta})+\alpha^2+\cancel{\alpha\delta}-\cancel{\alpha\delta}-\delta^2+\alpha^2+\cancel{\alpha\delta} = \frac{q}{1}$$

$$\Rightarrow 3\alpha^2-\delta^2 = q \Rightarrow \delta^2 = 3\alpha^2-q$$

put $\alpha = \frac{-p}{3} \Rightarrow \delta^2 = 3\left(\frac{-p}{3}\right)^2 - q = \boxed{\frac{+p^2}{3}-q=\delta^2}$

~~Substituting~~ α, δ^2 ~~values/wx~~

$$\Rightarrow (\alpha-\delta)\alpha(\alpha+\delta) = \frac{-a_3}{a_0}$$

$$(\alpha^2-\alpha\delta)(\alpha+\delta) = \frac{-r}{1}$$

$$\alpha^3-\cancel{\alpha^2\delta}+\cancel{\alpha^2\delta}-\alpha\delta^2 = -r$$

$$\alpha^3-\alpha\delta^2 = -r$$

Sub. α, δ^2 values in this eqn., we get

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$$\left(-\frac{p}{3}\right)^3 - \left(-\frac{p}{3}\right)\left(\frac{p^2}{3} - q\right) = -r$$

$$\Rightarrow \frac{-p^3}{27} - \left(-\frac{p^3}{9} + \frac{pq}{3}\right) = -r$$

$$\Rightarrow \frac{-p^3}{27} + \frac{p^3}{9} - \frac{pq}{3} = -r$$

$$\Rightarrow \frac{1}{27}\left[-p^3 + 3p^3 - 9pq\right] = -r$$

$$\Rightarrow -p^3 + 3p^3 - 9pq = -27r$$

$$\Rightarrow 2p^3 - 9pq + 27r = 0$$

Hence the proof. $x^3 + px^2 + qx + r = 0$

If the equation $x^3 - 6x^2 + 13x - 10 = 0$

Then, $p = -6$, $q = 13$, $r = -10$

$$\therefore 2p^3 - 9pq + 27r = 2(-6)^3 - 9(-6)(13) + 27(-10)$$
$$= 2(-216) + 54(13) + 27(-10)$$
$$= -432 + 702 - 270$$
$$= -702 + 702$$

$$\therefore 2p^3 - 9pq + 27r = 0$$

$\therefore$ The condition satified by the equation $x^3 - 6x^2 + 13x - 10 = 0$

and $3\alpha = -p \Rightarrow 3\alpha = 6 \Rightarrow \boxed{\alpha = 2}$

$3\alpha^2 - \delta^2 = q \Rightarrow 3(4) - \delta^2 = 13 \Rightarrow \delta^2 = 12 - 13$

$\delta^2 = -1 \Rightarrow \boxed{\delta = \pm i}$

$\therefore$ The roots are $\alpha - \delta$, α, $\alpha + \delta$

$\Rightarrow \boxed{2 - i, 2, 2 + i}$

2. Find the condition that the roots of the equation $ax^3+3bx^2+3cx+d=0$ may be in geometric progression. Solve the equation $27x^3+42x^2-28x-8=0$, whose roots are in geometric progression.

a, a+d, a+2d
a, a, ad, d

Soln:- Let the roots of the given equation $ax^3+3bx^2+3cx+d=0$ are $\frac{k}{r}$, k and kr. then $a_0=a$, $a_1=3b$, $a_2=3c$, $a_3=d$.

$$\therefore \Sigma\alpha_1 = \frac{-a_1}{a_0} \Rightarrow \frac{k}{r}+k+kr = \frac{-3b}{a} \rightarrow (1)$$

$$\Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0} \Rightarrow \frac{k^2}{r}+k^2+k^2r = \frac{3c}{a} \rightarrow (2)$$

$$\Sigma\alpha_1\alpha_2\alpha_3 = \frac{-a_3}{a_0} \Rightarrow \frac{k}{r}\,k\,kr = \frac{-d}{a}$$

$$\Rightarrow k^3 = \frac{-d}{a} \Rightarrow (3)$$

In the equation $27x^3+42x^2-28x-8=0$.

then $a=27$, $3b=42$, $3c=-28$, $d=-8$

$$\therefore (1) \Rightarrow k\left(\frac{1}{r}+1+r\right) = \frac{-42}{27} \Rightarrow (4)$$

$$(2) \Rightarrow k^2\left(\frac{1}{r}+1+r\right) = -\frac{28}{27} \rightarrow (5)$$

$$(3) \Rightarrow \quad k^3 = \frac{8}{27} = \left(\frac{2}{3}\right)^3$$

$$\therefore \boxed{k = \frac{2}{3}}$$

Sub. the value of k in eqn. (4), we get

$$\frac{2}{3}\left(\frac{1}{r}+1+r\right) = \frac{-42}{27}$$

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$$\Rightarrow \frac{1}{r} + 1 + r = -\frac{42}{27} \times \frac{3}{2}$$

$$\Rightarrow \frac{1}{r}\left[1 + r + r^2\right] = -\frac{7}{3}$$

$$\Rightarrow r^2 + r + 1 = -\frac{7}{3} r$$

$$\Rightarrow 3r^2 + 3r + 3 = -7r$$

$$\Rightarrow 3r^2 + 3r + 7r + 3 = 0$$

$$\Rightarrow 3r^2 + 10r + 3 = 0 \qquad 3 \times 3 = 9 \quad 1 \;\; 9$$

$$\Rightarrow 3r^2 + r + 9r + 3 = 0$$

$$\Rightarrow r(3r+1) + 3(3r+1) = 0$$

$$\Rightarrow (3r+1)(r+3) = 0$$

$$\Rightarrow 3r + 1 = 0 \;\;\&\;\; r + 3 = 0$$

$$\Rightarrow 3r = -1 \;\;\&\;\; r = -3$$

$$\Rightarrow \boxed{r = -\frac{1}{3}} \;\;\&\;\; \boxed{r = -3}$$

$\therefore$ The roots are $\frac{k}{r}$, k, kr

~~$\therefore$ + $\frac{2}{3}$ / 12,~~

$k = \frac{2}{3}$ &

$r = -\frac{1}{3}$ $\Rightarrow \frac{2}{3} \times -\frac{3}{1}, \; \frac{2}{3}, \; \frac{2}{3} \times -\frac{1}{3}$

$$\Rightarrow -2, \; \frac{2}{3}, \; -\frac{2}{9} \;//$$

(or)

$k = \frac{2}{3}$ &

$r = -3$ $\Rightarrow \frac{2}{3}\left(-\frac{1}{3}\right), \; \frac{2}{3}, \; \frac{2}{3}(-3)$

$$\Rightarrow -\frac{2}{9}, \; \frac{2}{3}, \; -2 \;//$$

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1. Solve the equation $81x^3-18x^2-36x+8=0$, whose roots are in harmonic progression.

$a_0=81,\ a_1=-18,\ a_2=-36,\ a_3=8$

Soln.:-

Let the roots be α, β, γ. $\quad \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

Then $\frac{2}{\beta} = \frac{1}{\alpha}+\frac{1}{\gamma}$ $\quad \frac{2}{\beta} = \frac{\gamma+\alpha}{\alpha\gamma}$ $\quad \frac{1}{\beta}-\frac{1}{\alpha} = \frac{1}{\gamma}-\frac{1}{\beta}$

i.e., $2\gamma\alpha = \beta\gamma+\alpha\beta \rightarrow (1)$

From the relation between the coefficients and the roots, we have

$$\alpha+\beta+\gamma = \frac{18}{81} \rightarrow (2) \qquad \frac{1}{\beta}+\frac{1}{\beta} = \frac{1}{\gamma}+\frac{1}{\alpha}$$

$$\alpha\beta+\beta\gamma+\gamma\alpha = -\frac{36}{81} \rightarrow (3) \qquad \frac{2}{\beta} = \frac{1}{\gamma}+\frac{1}{\alpha}$$

$$\alpha\beta\gamma = -\frac{8}{81} \rightarrow (4)$$

From (1) and (3), we get

$$\Rightarrow 2\gamma\alpha+\gamma\alpha = -\frac{36}{81}$$

$$\Rightarrow 3\gamma\alpha = -\frac{36}{81}$$

$$\Rightarrow \gamma\alpha = -\frac{4}{27} \rightarrow (5)$$

Sub. this value of $\gamma\alpha$ in eqn. (4), we get

$$\beta\left(-\frac{4}{27}\right) = -\frac{8}{81}$$

$$\therefore \beta = \frac{8}{81}\times\frac{27}{4}$$

$$\Rightarrow \boxed{\beta = \frac{2}{3}}$$

From (2), we have $\alpha+\gamma = \frac{18}{81}-\frac{2}{3}$

$$\alpha+\gamma = -\frac{4}{9} \rightarrow (6)$$

From 5 and 6, we get

$\alpha = \frac{2}{9}$ and $\gamma = -\frac{2}{3}$

$\therefore$ The roots are $\frac{2}{9}, \frac{2}{3}, -\frac{2}{3}$ //

2. If the sum of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ equals the sum of the other two, prove that $p^3 + 8r = 4pq$.

Proof:-

Let the roots of the equation be α, β, γ and δ.

Then $\alpha + \beta = \gamma + \delta \rightarrow (1)$

From the relation of the coefficients and the roots,

we have $a_0 = 1, a_1 = p, a_2 = q, a_3 = r, a_4 = s$

$\sum \alpha_1 = \frac{-a_1}{a_0}$ $\quad \alpha + \beta + \gamma + \delta = -p \rightarrow (2)$

$\sum \alpha_1\alpha_2 = \frac{+a_2}{a_0}$ $\quad \alpha\beta + \underbrace{\alpha\gamma + \alpha\delta}_{\alpha(\gamma+\delta)} + \underbrace{\beta\gamma + \beta\delta}_{\beta(\gamma+\delta)} + \gamma\delta = q \rightarrow (3)$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r \rightarrow (4)$

and $\alpha\beta\gamma\delta = s \rightarrow (5)$

From (1) and (2), we get

$2(\alpha + \beta) = -p \rightarrow (6)$

and (3) $\Rightarrow$ $\alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta) = q$

i.e., $(\alpha\beta + \gamma\delta) + (\alpha + \beta)^2 = q \rightarrow (7)$

(4) $\Rightarrow$ $\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -r$

i.e., $(\alpha + \beta)(\alpha\beta + \gamma\delta) = -r \longrightarrow (8)$

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From (6) and (7), we get

$$\alpha\beta+\gamma\delta+\frac{p^2}{4}=q$$

$$\therefore \alpha\beta+\gamma\delta=q-\frac{p^2}{4} \longrightarrow (9)$$

From (8), we get

$$-\frac{p}{2}(\alpha\beta+\gamma\delta)=-r$$

i.e., $\alpha\beta+\gamma\delta=\frac{2r}{p} \longrightarrow (10)$

Equating (9) and (10), we get

$$q-\frac{p^2}{4}=\frac{2r}{p}$$

i.e., $4pq-p^3=8r$

i.e., $\boxed{p^3+8r=4pq}$

Hence the proof.

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3. Solve the equation $x^4-2x^3+4x^2+6x-21=0$ give that two of its roots are equal in magnitude and opposite in sign.

Soln:-

Let the roots of the equation be $\alpha,\beta,\gamma,\delta$.

Here, $\gamma=-\delta \Rightarrow \gamma+\delta=0 \longrightarrow (1)$

From the relations of the roots and coefficients

$$\alpha+\beta+\gamma+\delta=2 \longrightarrow (2)$$

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=4 \longrightarrow (3)$$

$$\alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta+\alpha\gamma\delta=-6 \longrightarrow (4)$$

and $\alpha\beta\gamma\delta=-21 \longrightarrow (5)$

From ① and ②, we get $\alpha+\beta=2 \longrightarrow ⑥$

③ $\Rightarrow \alpha\beta+\gamma\delta+(\alpha+\beta)(\gamma+\delta)=4$

$\therefore \alpha\beta+\gamma\delta=4 \longrightarrow ⑦$

④ $\Rightarrow \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=-6$

i.e., $\gamma\delta(\alpha+\beta)=-6 \longrightarrow ⑧$

From ⑥ and ⑧, we get $\gamma\delta=-3 \longrightarrow ⑨$

but $\gamma+\delta=0 \Rightarrow \gamma=\sqrt{3}, \delta=-\sqrt{3}$

From ⑦ and ⑨, we get $\alpha\beta=7$.

$\therefore \alpha$ & β are the roots of $x^2-2x+7=0$

$\therefore \alpha=1+\sqrt{-6}, \beta=1-\sqrt{-6}$

$\therefore$ The roots of the equation are

$\pm\sqrt{3}, 1\pm\sqrt{-6}$.//

4. Find the condition that the general biquadratic equation $ax^4+4bx^3+6cx^2+4dx+e=0$ may have two pairs of equal roots.

Soln:- Let the roots be $\alpha, \alpha, \beta, \beta$.

From the relations of coefficients and roots

$\therefore 2\alpha+2\beta=-\frac{4b}{a} \longrightarrow ①$

$\alpha^2+\beta^2+4\alpha\beta=\frac{6c}{a} \longrightarrow ②$

$2\alpha\beta^2+2\alpha^2\beta=-\frac{4d}{a} \longrightarrow ③$

$\alpha^2\beta^2=\frac{e}{a} \longrightarrow ④$

From ①, we get $\alpha+\beta=-\frac{2b}{a} \longrightarrow ⑤$

(15)

From (3), we get $2\alpha\beta(\alpha+\beta) = -\frac{4d}{a}$

$$\therefore \alpha\beta = \frac{d}{b} \longrightarrow (6)$$

From (5) and (6), we get that α, β are the roots of the equation

$$x^2 + \frac{2b}{a}x + \frac{d}{b} = 0.$$

$$\therefore ax^4 + 4bx^3 + bcx^2 + 4dx + e = a\left(x^2 + \frac{2b}{a}x + \frac{d}{b}\right)^2$$

Comparing coefficients

$$bc = a\left(\frac{4b^2}{a^2} + \frac{2d}{b}\right) \text{ and } e = \frac{ad^2}{b^2}$$

$$\therefore 3abc = a^2d + 2b^3 \text{ and } eb^2 = ad^2.$$

This is the required condition that the general biquadratic equation.

Sec: 12 Symmetric function of the roots

If a function involving all the roots of an equation is unaltered in value if any two of the roots are interchanged, it is called a symmetric function of the roots.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation

$f(x) = x^n, p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0$

We have $S_1 = \Sigma \alpha_1 = -p_1$

$$S_2 = \Sigma \alpha_1 \alpha_2 = p_2$$

$$S_3 = \Sigma \alpha_1 \alpha_2 \alpha_3 = -p_3$$

$$\vdots \qquad \vdots$$

$$S_n = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n = (-1)^n p_n$$

Without knowing the values of the roots separately in terms of the coefficients, by using the above relations between the coefficients and the roots of an equation. We can express any symmetric function of the roots in terms of the coefficients of the equation.

Examples:-

1. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, express the value of $\Sigma \alpha^2 \beta$ in terms of the coefficients.

Soln:- From the relations of the roots and coefficients are $\alpha + \beta + \gamma = -p$

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$\alpha\beta+\beta\gamma+\gamma\alpha = q$

$\alpha\beta\gamma = -r$

$\therefore \sum\alpha^2\beta = \alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta$

$= \alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta+3\alpha\beta\gamma-3\alpha\beta\gamma$

$\alpha\beta\gamma$

$=(\alpha^2\beta+\beta^2\alpha+\alpha\beta\gamma)+(\alpha^2\gamma+\gamma^2\alpha+\alpha\beta\gamma)+(\beta^2\gamma+\gamma^2\beta+\alpha\beta\gamma)$
$-3\alpha\beta\gamma$

$=\alpha\beta(\alpha+\beta+\gamma)+\alpha\gamma(\alpha+\beta+\gamma)+\beta\gamma(\alpha+\beta+\gamma)-3\alpha\beta\gamma$

$=(\alpha\beta+\alpha\gamma+\beta\gamma)(\alpha+\beta+\gamma)-3\alpha\beta\gamma.$

$= q(-p)-3(-r)$

$\therefore \boxed{\sum\alpha^2\beta = -qp+3r}$

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2. If $\alpha,\beta,\gamma,\delta$ be the roots of the biquadratic equation $x^4+px^3+qx^2+rx+s=0$, find (1) $\sum\alpha^2$ (2) $\sum\alpha^2\beta\gamma$.
(3)

Soln:-

The relation between the roots and the coefficients are

$\sum\alpha = \alpha+\beta+\gamma+\delta = -p$

$\sum\alpha\beta = \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = q$

$\sum\alpha\beta\gamma = \alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\beta\alpha\delta = -r$

$\alpha\beta\gamma\delta = s$

$(a+b)^2,\ a^2+b^2+2ab$

$\alpha\ \beta\ \gamma\ \delta$

(i) $\sum\alpha^2 = \alpha^2+\beta^2+\gamma^2+\delta^2$

$= \alpha^2+\beta^2+\gamma^2+\delta^2+(2\alpha\beta+2\alpha\gamma+2\alpha\delta+2\beta\gamma+2\beta\delta$
$+2\gamma\delta)-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$=(\sum\alpha)^2-2\sum\alpha\beta = \boxed{p^2-2q = \sum\alpha^2}$

(2) $\sum \alpha^2\beta\gamma = \alpha^2\beta\gamma + \alpha^2\gamma\delta + \alpha^2\delta\beta + \beta^2\alpha\delta + \beta^2\alpha\gamma + \beta^2\gamma\delta$
$\qquad + \gamma^2\alpha\beta + \gamma^2\beta\delta + \gamma^2\alpha\delta + \delta^2\alpha\beta + \delta^2\beta\gamma + \delta^2\alpha\gamma$

$= (\alpha^2\beta\gamma + \beta^2\alpha\gamma + \gamma^2\alpha\beta + \alpha\beta\gamma\delta) + (\beta^2\alpha\delta + \alpha^2\beta\delta$
$\quad + \delta^2\alpha\beta + \alpha\beta\gamma\delta) + (\alpha^2\gamma\delta + \gamma^2\alpha\delta + \delta^2\alpha\gamma + \alpha\beta\gamma\delta)$
$\quad + (\beta^2\gamma\delta + \gamma^2\beta\delta + \delta^2\beta\gamma + \alpha\beta\gamma\delta) - 4\alpha\beta\gamma\delta$

$= \alpha\beta\gamma(\alpha+\beta+\gamma+\delta) + \beta\alpha\delta(\alpha+\beta+\gamma+\delta)$
$\quad + \alpha\gamma\delta(\alpha+\beta+\gamma+\delta) + \beta\gamma\delta(\alpha+\beta+\gamma+\delta) - 4\alpha\beta\gamma\delta$

$= (\alpha\beta\gamma + \beta\alpha\delta + \alpha\gamma\delta + \beta\gamma\delta)(\alpha+\beta+\gamma+\delta) - 4\alpha\beta\gamma\delta$

$= (\sum \alpha\beta\gamma)(\sum\alpha) - 4\alpha\beta\gamma\delta$

$\therefore \boxed{\sum \alpha^2\beta\gamma = pr - 4s}$

---

3. If α, β, γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$, from the equation whose roots are $\alpha\beta, \beta\gamma$ and $\gamma\alpha$.

Soln:- The relations between the roots and coefficients are

$\alpha + \beta + \gamma = -a$

$\alpha\beta + \beta\gamma + \gamma\alpha = b$

$\alpha\beta\gamma = -c$

The required equation is

$(x - \alpha\beta)(x - \beta\gamma)(x - \gamma\alpha) = 0$ $\quad$ ($\because \alpha\beta, \beta\gamma, \gamma\alpha$ are roots

i.e., $(x^2 - x\beta\gamma - x\alpha\beta + \alpha\beta^2\gamma)(x - \gamma\alpha) = 0$

$x^3 - x^2\beta\gamma - x^2\alpha\beta + x\alpha\beta^2\gamma - x^2\gamma\alpha + x\beta\gamma^2\alpha + x\alpha^2\beta\gamma - \alpha^2\beta^2\gamma^2 = 0$

$$\Rightarrow x^3 - x^2(\alpha\beta+\beta\gamma+\gamma\alpha) + (\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)x - \alpha^2\beta^2\gamma^2 = 0$$

$$\Rightarrow x^3 - x^2(\alpha\beta+\beta\gamma+\gamma\alpha) + x\,\alpha\beta\gamma(\alpha+\beta+\gamma) - (\alpha\beta\gamma)^2 = 0$$

$$\Rightarrow x^3 - x^2(+b) + x(-c)(-a) - (-c)^2 = 0$$

$$\Rightarrow x^3 - bx^2 + acx - c^2 = 0$$

i.e., $\boxed{x^3 - bx^2 + acx + c^2 = 0}$ is the required equation.

4) If $\alpha, \beta, \gamma, \delta$ be the roots of the biquadratic equation $x^4 + px^3 + qx^2 + rx + s = 0$, find (i) $\sum \alpha^2\beta^2$ (ii) $\sum \alpha^3\beta$ (iii) $\sum \alpha^4$

Soln.:-

(i) $\sum \alpha^2\beta^2 = \alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2$

$+2[\alpha^2\beta\gamma + \alpha^2\beta\delta + \alpha\beta^2\gamma + \alpha\beta^2\delta + \alpha\beta\gamma\delta$
$+ \alpha^2\gamma\delta + \alpha\beta\gamma^2 + \alpha\beta\gamma\delta + \alpha\gamma^2\delta + \alpha\beta\gamma\delta + \alpha\beta\delta^2$
$+ \alpha\gamma\delta^2 + \beta^2\gamma\delta + \beta\gamma^2\delta + \beta\gamma\delta^2]$

$-2[\alpha^2\beta\gamma + \alpha^2\beta\delta + \alpha\beta^2\gamma + \alpha\beta^2\delta + \alpha^2\gamma\delta + \alpha\beta\gamma^2 + \alpha\gamma^2\delta$
$+ \alpha\beta\delta^2 + \alpha\gamma\delta^2 + \beta^2\gamma\delta + \beta\gamma^2\delta + \beta\gamma\delta^2]$

$-2[\alpha\beta\gamma\delta + \alpha\beta\gamma\delta + \alpha\beta\gamma\delta]$

$$= (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)^2 - 2\sum\alpha^2\beta\gamma - 2(3\alpha\beta\gamma\delta)$$

$$\therefore \sum\alpha^2\beta^2 = (\sum\alpha\beta)^2 - 2\sum\alpha^2\beta\gamma - 6\alpha\beta\gamma\delta$$

$$\sum\alpha^2\beta^2 = q^2 - 2(pr - 4s) - 6s$$

$$\therefore \sum\alpha^2\beta^2 = q^2 - 2pr + 8s - 6s \Rightarrow \boxed{\sum\alpha^2\beta^2 = q^2 - 2pr + 2s}$$

(ii) $\sum \alpha^3\beta = \alpha^3\beta + \alpha^3\gamma + \alpha^3\delta + \beta^3\alpha + \beta^3\gamma + \beta^3\delta + \gamma^3\alpha + \gamma^3\beta + \gamma^3\delta + \delta^3\alpha + \delta^3\beta + \delta^3\gamma$

$$
\begin{aligned}
&= \alpha^3\beta + \alpha^3\gamma + \alpha^3\delta + \beta^3\alpha + \beta^3\gamma + \beta^3\delta + \gamma^3\alpha + \gamma^3\beta + \gamma^3\delta \\
&\quad + \delta^3\alpha + \delta^3\beta + \delta^3\gamma + [\alpha^2\beta\gamma + \alpha^2\gamma\delta + \alpha^2\delta\beta + \beta^2\alpha\delta + \beta^2\alpha\gamma \\
&\quad + \beta^2\gamma\delta + \gamma^2\alpha\beta + \gamma^2\beta\delta + \gamma^2\alpha\delta + \delta^2\alpha\beta + \delta^2\beta\gamma + \delta^2\alpha\gamma] \\
&\quad - [\alpha^2\beta\gamma + \alpha^2\gamma\delta + \alpha^2\delta\beta + \beta^2\alpha\delta + \beta^2\alpha\gamma + \beta^2\gamma\delta + \gamma^2\alpha\beta + \gamma^2\beta\delta \\
&\quad + \gamma^2\alpha\delta + \delta^2\alpha\beta + \delta^2\beta\gamma + \delta^2\alpha\gamma] \\
&= (\alpha^3\beta + \alpha^3\gamma + \alpha^3\delta + \alpha^2\beta\gamma + \alpha^2\gamma\delta + \alpha^2\delta\beta) + (\beta^3\alpha + \beta^3\gamma + \beta^3\delta + \beta^2\alpha\delta \\
&\quad + \beta^2\alpha\gamma + \beta^2\gamma\delta) + (\gamma^3\alpha + \gamma^3\beta + \gamma^3\delta + \gamma^2\alpha\beta + \gamma^2\beta\delta + \gamma^2\alpha\delta) \\
&\quad + (\delta^3\alpha + \delta^3\beta + \delta^3\gamma + \delta^2\alpha\beta + \delta^2\beta\gamma + \delta^2\alpha\gamma) - [\alpha^2\beta\gamma + \alpha^2\gamma\delta \\
&\quad + \alpha^2\delta\beta + \beta^2\alpha\delta + \beta^2\alpha\gamma + \beta^2\gamma\delta + \gamma^2\alpha\beta + \gamma^2\beta\delta + \gamma^2\alpha\delta + \delta^2\alpha\beta \\
&\quad + \delta^2\beta\gamma + \delta^2\alpha\gamma] \\
&= \alpha^2[\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \gamma\delta + \delta\beta] + \beta^2[\alpha\beta + \beta\gamma + \beta\delta + \alpha\delta + \alpha\gamma + \gamma\delta] \\
&\quad + \gamma^2[\alpha\gamma + \gamma\beta + \gamma\delta + \alpha\beta + \beta\delta + \alpha\delta] + \delta^2[\alpha\delta + \beta\delta + \gamma\delta + \alpha\beta \\
&\quad + \beta\gamma + \alpha\gamma] - \sum \alpha^2\beta\gamma \\
&= (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) - \sum \alpha^2\beta\gamma \\
&= (\sum \alpha^2)(\sum \alpha\beta) - \sum \alpha^2\beta\gamma \\
&= (p^2 - 2q)q - (pr - 4s)
\end{aligned}
$$

$$\boxed{\sum \alpha^3\beta = p^2 q - 2q^2 - pr + 4s}$$

(iii) $\sum \alpha^4 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4$

$$
\begin{aligned}
&= \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + 2\alpha^2\beta^2 + 2\alpha^2\gamma^2 + 2\alpha^2\delta^2 + 2\beta^2\gamma^2 \\
&\quad + 2\beta^2\delta^2 + 2\gamma^2\delta^2 - 2[\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 \\
&\quad + \beta^2\delta^2 + \gamma^2\delta^2] \\
&= (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^2 - 2\sum \alpha^2\beta^2 = (\sum \alpha^2)^2 - 2\sum \alpha^2\beta^2 \\
&= (p^2 - 2q)^2 - 2(q^2 - 2pr + 2s) = p^4 + 4q^2 - 4p^2 q - 2q^2 + 4pr - 4s
\end{aligned}
$$

$$\boxed{\sum \alpha^4 = p^4 - 4p^2 q - 2q^2 + 4pr - 4s}$$

5) If α, β, γ are the roots of $x^3+px^2+qx+r=0$, form the equation whose roots are $\beta+\gamma-2\alpha$, $\gamma+\alpha-2\beta$, $\alpha+\beta-2\gamma$.

(21)

Soln:- The relations between the roots and coefficients are

$$\alpha+\beta+\gamma = -p$$
$$\alpha\beta+\beta\gamma+\alpha\gamma = q$$
$$\alpha\beta\gamma = -r$$

In the required equation is

find S_3 and sub. eqn ①

$$x^3 - S_1x^2 + S_2x - S_3 = 0 \quad \to ①$$

$\Rightarrow S_1 =$ Sum of the roots

$= \beta+\gamma-2\alpha+\gamma+\alpha-2\beta+\alpha+\beta-2\gamma$

$S_1 = 0$

$\Rightarrow S_2 =$ Sum of the product of the roots taken two at a time

$= (\beta+\gamma-2\alpha)(\gamma+\alpha-2\beta)+(\gamma+\alpha-2\beta)(\alpha+\beta-2\gamma)$
$\quad + (\beta+\gamma-2\alpha)(\alpha+\beta-2\gamma)$

$= (\beta+\gamma+\alpha-3\alpha).(\gamma+\alpha+\beta-3\beta)+(\gamma+\alpha+\beta-2\beta)$
$\quad .(\alpha+\beta+\gamma-2\gamma)+(\beta+\gamma+\alpha-2\alpha)(\alpha+\beta+\gamma-2\gamma)$

$= (-p-3\alpha)(-p-3\beta)+(-p-3\beta)(-p-3\gamma)$
$\quad +(-p-3\alpha)(-p-3\gamma)$

$= (p+3\alpha)(p+3\beta)+(p+3\beta)(p+3\gamma)+(p+3\alpha)(p+3\gamma)$

$= (p^2+3p\beta+3\alpha p+9\alpha\beta)+(p^2+3\gamma p+3\beta p+9\beta\gamma)$
$\quad +(p^2+3\gamma p+3\alpha p+9\alpha\gamma)$

$= 3p^2+6p\beta+6\alpha p+9\alpha\beta+6\gamma p+9\beta\gamma+9\alpha\gamma$

$= 3p^2+6p(\alpha+\beta+\gamma)+9(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= 3p^2+6p(-p)+9q$

$\therefore \boxed{S_2 = 9q-3p^2}$

Home Work

① If α, β, γ are the roots of the equation $x^3+px^2+qx+r=0$, prove that $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) = r-pq$.

② If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4-4x^2-x+2=0$, find the values of $\Sigma\alpha^2\beta$ and $\Sigma\frac{1}{\alpha^2}$.

Ans:- $\Sigma\alpha^2\beta = -3$ & $\Sigma\frac{1}{\alpha^2} = \frac{17}{4}$

③ If α, β, γ are the roots of the equation $x^3+px^2+qx+r=0$, find the equation whose roots are

(CIA-1)

(i) $\alpha+\beta$, $\beta+\gamma$, $\gamma+\alpha$ (ii) $\alpha(\beta+\gamma)$, $\beta(\gamma+\alpha)$, $\gamma(\alpha+\beta)$

Ans:- (i) $x^3+2px^2+(p^2+q)x+pq-r=0$

(ii) $x^3-2qx^2+(q^2+pr)x+r^2-pqr=0$

Sec: 13 Sum of the powers of the roots of an equation

Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ be the roots of the equation $f(x)=0$. The sum of the r^{th} powers of the roots

i.e., $\alpha_1^r+\alpha_2^r+\cdots+\alpha_n^r$ $S_1 = \Sigma\alpha_1 = \alpha_1+\alpha_2+\alpha_3 \ldots \alpha_n$

$S_r = \Sigma\alpha^r$ $\alpha_1^r+\alpha_2^r+\ldots+\alpha_n^r$

is usually denoted by S_r. We can easily see that S_r constitutes a symmetric function of the roots and hence we can calculate the value of S_r by the methods described in the previous atricle. When r is greater than 4, the calculation of S_r by the previous method becomes tedious and in case, the following method can be used profitably:

$\boxed{\log f(x) = \log[(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)]}$

$= \log(x-\alpha_1)+\log(x-\alpha_2)+\cdots+\log(x-\alpha_n)$

We have $f(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$

Taking logarithms on both sides and differentiating, we get

$$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha_1}+\frac{1}{x-\alpha_2}+\cdots+\frac{1}{x-\alpha_n}$$

$$\frac{x f'(x)}{f(x)} = \frac{x}{x-\alpha_1} + \frac{x}{x-\alpha_2} + \cdots + \frac{x}{x-\alpha_n}$$

$$= \frac{\cancel{x}}{\cancel{x}\left(1-\frac{\alpha_1}{x}\right)} + \frac{1}{1-\frac{\alpha_2}{x}} + \cdots + \frac{1}{1-\frac{\alpha_n}{x}}$$

$$= \left(1-\frac{\alpha_1}{x}\right)^{-1} + \left(1-\frac{\alpha_2}{x}\right)^{-1} + \cdots + \left(1-\frac{\alpha_n}{x}\right)^{-1}$$

$$= 1 + \frac{\alpha_1}{x} + \frac{\alpha_1^2}{x^2} + \cdots + \frac{\alpha_1^n}{x^n} + \cdots$$

$$+ 1 + \frac{\alpha_2}{x} + \frac{\alpha_2^2}{x^2} + \cdots + \frac{\alpha_2^n}{x^n} + \cdots$$

$$+ \cdots$$

$$+ 1 + \frac{\alpha_n}{x} + \frac{\alpha_n^2}{x^2} + \cdots + \frac{\alpha_n^n}{x^n} + \cdots$$

$$= n + (\Sigma \alpha_1)\frac{1}{x} + (\Sigma \alpha_1^2)\frac{1}{x^2} + \cdots + (\Sigma \alpha_1^r)\frac{1}{x^r} + \cdots$$

$$= n + S_1 \frac{1}{x} + S_2 \frac{1}{x^2} + \cdots + S_r \frac{1}{x^r} + \cdots$$

$\therefore S_r =$ Coefficient of $\frac{1}{x^r}$ in the expansion of $\frac{x f'(x)}{f(x)}$.

Example:

1. Find the sum of the cubes of the roots of the equation $x^5 = x^2 + x + 1$.

Soln:- The equation can be written in the form

$$f(x) = x^5 - x^2 - x - 1 \qquad f'(x) = 5x^4 - 2x - 1$$

$\therefore S_3 =$ Coefficient of $\frac{1}{x^3}$ in the expansion of $\frac{x(5x^4 - 2x - 1)}{x^5 - x^2 - x - 1}$

$$= \text{Coefficient of } \frac{1}{x^3} \text{ in } \frac{x^5\left[5 - \frac{2}{x^3} - \frac{1}{x^4}\right]}{x^5\left[1 - \frac{1}{x^3} - \frac{1}{x^4} - \frac{1}{x^5}\right]}$$

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$\therefore S_3 =$ Coefficient of $\frac{1}{x^3}$ in $\dfrac{5-\frac{2}{x^3}-\frac{1}{x^4}}{1-\frac{1}{x^3}-\frac{1}{x^4}-\frac{1}{x^5}}$

$= \quad " \quad " \left(5-\frac{2}{x^3}-\frac{1}{x^4}\right)\left[1-\left(\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}\right)\right]^{-1}$

$= \quad " \quad " \left(5-\frac{2}{x^3}-\frac{1}{x^4}\right)\left\{1+\left(\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}\right)+\left(\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}\right)^2+\dots\right\}$

$(1-x)^{-1} = 1+x+x^2+x^3+\dots$

$= \quad " \quad " \left(5-\frac{2}{x^3}-\frac{1}{x^4}\right)\left[1+\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}+\frac{1}{x^6}+\frac{1}{x^8}+\dots\right]$

$= \quad " \quad " \left[5-\frac{2}{x^3}-\frac{1}{x^4}+\frac{5}{x^3}-\frac{2}{x^6}-\frac{1}{x^7}+\dots\right]$

$\therefore S_3 =$ Coefficient of $\frac{1}{x^3}$ in $\left[5-\frac{2}{x^3}-\frac{1}{x^4}+\frac{5}{x^3}-\dots\right]$

$\therefore S_3 = -2+5$

$\boxed{S_3 = 3}$

2. Show that the sum of the fourth powers of the roots of the equation $x^5+px^3+qx^2+s=0$ is $2p^2$.

Proof:- The equation can be written in the form

$f(x) = x^5+px^3+qx^2+s = 0$

$\therefore f'(x) = 5x^4+3px^2+2qx = 0$

$\therefore S_4 =$ Coefficient of $\frac{1}{x^4}$ in the expansion $\dfrac{x(5x^4+3px^2+2qx)}{x^5+px^3+qx^2+s}$

$= \quad " \quad " \quad \dfrac{x^5\left(5+\frac{3p}{x^2}+\frac{2q}{x^3}\right)}{x^5\left(1+\frac{p}{x^2}+\frac{q}{x^3}+\frac{s}{x^5}\right)}$

$$S_4 = \text{Coefficient of } \frac{1}{x^4} \text{ in the expansion } \left(5+\frac{3p}{x^2}+\frac{2q}{x^3}\right)\cdot\left[1+\left(\frac{p}{x^2}+\frac{q}{x^3}+\frac{s}{x^5}\right)\right]^{-1}$$

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$$= \quad " \quad " \quad \left(5+\frac{3p}{x^2}+\frac{2q}{x^3}\right)\left\{1-\left(\frac{p}{x^2}+\frac{q}{x^3}+\frac{s}{x^5}\right)+\left(\frac{p}{x^2}+\frac{q}{x^3}+\frac{s}{x^5}\right)^2-\cdots\right\}$$

$(1+x)^{-1} = 1-x+x^2+x^3+$

$$= \quad " \quad " \quad \left(5+\frac{3p}{x^2}+\frac{2q}{x^3}\right)\left(1-\frac{p}{x^2}-\frac{q}{x^3}+\frac{s}{x^5}+\frac{p^2}{x^4}+\frac{q^2}{x^6}+\cdots\right)$$

$$= \quad " \quad " \quad \left(5+\frac{3p}{x^2}+\frac{2q}{x^3}-\frac{5p}{x^2}-\frac{3p^2}{x^4}-\frac{2pq}{x^5}-\frac{5q}{x^3}-\frac{3pq}{x^5}-\frac{2q^2}{x^6}+\cdots+\frac{5p^2}{x^4}+\cdots\right)$$

$$\therefore S_4 = \text{Coefficient of } \frac{1}{x^4} \text{ in } \left(-\frac{3p^2}{x^4}+\frac{5p^2}{x^4}\right)$$

$$\therefore S_4 = -3p^2+5p^2$$

i.e., $\boxed{S_4 = 2p^2}$

Hence the proof.

Home Work

Calculate the sum of the "cubes" of the roots of the equation

(i) $x^4+2x+3=0$ (ii) $x^3-6x^2+11x-6=0$

Ans:- (i) $S_3=-6$ (ii) $S_3=36$

Unit-I is over

Classical Algebra And Theory Of Numbers

Unit - II

Chapter: 6 - Section:- 14

Newton's Theorem on the sum of the powers of the roots

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation

$$f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0 \longrightarrow (1)$$

and let be $S_r = \alpha_1^r + \alpha_2^r + \cdots + \alpha_n^r$

So that $S_0 = n$

$$\Rightarrow f(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) \longrightarrow (2) \quad (\because \alpha_1, \alpha_2, \ldots, \alpha_n \text{ are the roots})$$

Taking 'log' we get

$$\Rightarrow \log f(x) = \log\left[(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)\right]$$

$$\Rightarrow \log f(x) = \log(x-\alpha_1) + \log(x-\alpha_2) + \cdots + \log(x-\alpha_n)$$

differentiate on both sides, we get

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \cdots + \frac{1}{x-\alpha_n}$$

$$\Rightarrow f'(x) = \frac{f(x)}{x-\alpha_1} + \frac{f(x)}{x-\alpha_2} + \cdots + \frac{f(x)}{x-\alpha_n} \longrightarrow (3)$$

By actual division, we get

$$\frac{f(x)}{x-\alpha_1} = x^{n-1} + (\alpha_1 + p_1)x^{n-2} + (\alpha_1^2 + p_1\alpha_1 + p_2)x^{n-3} + \cdots$$
$$\cdots + (\alpha_1^{n-1} + p_1\alpha_1^{n-2} + p_2\alpha_1^{n-3} + \cdots + p_{n-1})$$

$$\frac{f(x)}{x-\alpha_2} = x^{n-1} + (\alpha_2 + p_1)x^{n-2} + (\alpha_2^2 + p_1\alpha_2 + p_2)x^{n-3} + \cdots$$
$$\cdots + (\alpha_2^{n-1} + p_1\alpha_2^{n-2} + p_2\alpha_2^{n-3} + \cdots + p_{n-1})$$

$$\vdots \qquad \vdots$$

$$\frac{f(x)}{x-\alpha_n} = x^{n-1} + (\alpha_n + p_1)x^{n-2} + (\alpha_n^2 + p_1\alpha_n + p_2)x^{n-3} + \cdots$$
$$\cdots + (\alpha_n^{n-1} + p_1\alpha_n^{n-2} + p_2\alpha_n^{n-3} + \cdots + p_{n-1})$$

Using these values in eqn. 3, we get

$$f'(x) = nx^{n-1} + (s_1 + np_1)x^{n-2} + (s_2 + p_1 s_1 + np_2)x^{n-3} + \cdots$$
$$\cdots + (s_{n-1} + p_1 s_{n-2} + \cdots + np_{n-1}) \quad \to (4)$$

But, from eqn. (1),

$$\Rightarrow f'(x) = nx^{n-1} + (n-1)p_1 x^{n-2} + (n-2)p_2 x^{n-3} + \cdots + p_{n-2}\, x + p_{n-1} = 0 \quad \to (5)$$

$\therefore (4) = (5) \Rightarrow$

$$nx^{n-1} + (s_1 + np_1)x^{n-2} + (s_2 + p_1 s_1 + np_2)x^{n-3} + \cdots + (s_{n-1} + p_1 s_{n-2} + \cdots + np_{n-1})$$
$$= nx^{n-1} + (n-1)p_1 x^{n-2} + (n-2)p_2 x^{n-3} + \cdots + p_{n-2}\, x + p_{n-1}$$

Equating the coefficients of $x^{n-1}, x^{n-2}, \ldots$ and constant, we get

$$s_1 + p_1 = 0$$
$$s_2 + p_1 s_1 + 2p_2 = 0$$
$$s_3 + p_1 s_2 + p_2 s_1 + 3p_3 = 0$$
$$s_4 + p_1 s_3 + p_2 s_2 + p_3 s_1 + 4p_4 = 0$$
$$\cdots$$
$$\cancel{s_r + p_1 s_{r-1} + p_2 s_{r-2} + \cdots + r p_r = 0}$$
$$\cdots$$
$$\cancel{s_n + p_1 s_{n-1} + p_2 s_{n-2} + \cdots + n p_n = 0}$$
$$s_{n-1} + p_1 s_{n-2} + p_2 s_{n-3} + \cdots + p_{n-2} s_1 + (n-1)p_{n-1} = 0$$

$s_1 + np_1 = (n-1)p_1$
$s_1 + np_1 - (n-1)p_1 = 0$
$s_1 + np_1 - np_1 + p_1 = 0$
$np_{n-1} = p_{n-1}$
$np_{n-1} - p_{n-1} = 0$
$p_{n-1}(n-1) = 0$

Multiply x^{r-n} in eqn. (1), we get,

$$x^{r-n} f(x) = x^r + p_1 x^{r-1} + p_2 x^{r-2} + \cdots + p_n x^{r-n}$$

$x^n\, x^{r-n}$
$x^{n-1}\, x^{r-n}$

Replacing in this identity, x by the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, in succession and adding, we have

$$s_r + p_1 s_{r-1} + p_2 s_{r-2} + \cdots + p_n s_{r-n} = 0$$

put $r = n \Rightarrow s_n + p_1 s_{n-1} + p_2 s_{n-2} + \cdots + p_n s_0 = 0$ $\quad (\because s_0 = n)$

$\Rightarrow s_n + p_1 s_{n-1} + p_2 s_{n-2} + \cdots + np_n = 0$

put $r = n+1 \Rightarrow s_{n+1} + p_1 s_n + p_2 s_{n-1} + \cdots + p_n s_1 = 0$

put $r = n+2 \Rightarrow s_{n+2} + p_1 s_{n+1} + p_2 s_n + \cdots + p_n s_2 = 0$

and so on.

Thus, we get (3)

$S_r + P_1 S_{r-1} + P_2 S_{r-2} + \cdots + r P_r = 0$ if $r < n$

&

$S_r + P_1 S_{r-1} + P_2 S_{r-2} + \cdots + P_n S_{r-n} = 0$ if $r \geq n$

Hence the theorem

Corollary:-

To find the sum of the negative integral powers of the roots of $f(x)=0$, put $x=\frac{1}{y}$ and find the sums of the corresponding positive powers of the roots of the transformed equation.

Examples:-

in this model twentieth power $x^4+ax+b=0$ is $570a^4b^2-4b^5$

1. Show that the sum of the eleventh powers of the roots of $x^7+5x^4+1=0$ is zero.

$n=7$ $r=11$ $r>n$

proof:- Given that $x^7+5x^4+1=0 \rightarrow (1)$

Assume that the equation,

$$x^7+P_1x^6+P_2x^5+P_3x^4+P_4x^3+P_5x^2+P_6x+P_7=0$$

From (1) $\Rightarrow P_1=P_2=P_4=P_5=P_6=0$, $P_3=5$, $P_7=1$. $\rightarrow (2)$

$\therefore S_{11}+P_1S_{10}+P_2S_9+P_3S_8+P_4S_7+P_5S_6+P_6S_5+P_7S_4=0$

Sub. (2) in this equation, From the 2nd eqn. of the Newton's Theorem $(r>n) \Rightarrow (11>7)$

We get, $S_{11}+5S_8+S_4=0 \rightarrow (3)$

$\Rightarrow S_8+P_1S_7+P_2S_6+P_3S_5+P_4S_4+P_5S_3+P_6S_2+P_7S_1=0$ $\because 8>7$

Using eqn. (2) in this equation, we get

$$S_8+5S_5+S_1=0 \rightarrow (4)$$

$\Rightarrow S_5+P_1S_4+P_2S_3+P_3S_2+P_4S_1+5P_5=0$ From the 1st eqn. of Newton's Theorem $\because r<n \Rightarrow 5<7$

Using eqn. (2) in this equation, we get

$$S_5+5S_2=0 \rightarrow (5)$$

$\Rightarrow S_4 + P_1 S_3 + P_2 S_2 + P_3 S_1 + 4P_4 = 0$ $(\because 4 < 7)$

Using ② in this eqn., we get

$S_4 + 5S_1 = 0 \rightarrow$ ⑥

$\Rightarrow S_2 + P_1 S_1 + 2P_2 = 0$

i.e., $S_2 = 0 \rightarrow$ ⑦

Also, $S_1 = 0 \rightarrow$ ⑧

Sub. ⑧ in ⑥ $\Rightarrow S_4 = 0 \rightarrow$ ⑨

Sub. ⑦ in ⑤ $\Rightarrow S_5 = 0 \rightarrow$ ⑩

Sub. ⑧ & ⑩ in ④ $\Rightarrow S_8 = 0 \rightarrow$ ⑪

Sub. ⑨ & ⑪ in ③ $\Rightarrow \boxed{S_{11} = 0}$

$\therefore$ The sum of the eleventh powers of the roots of the eqn. is

$$\boxed{S_{11} = 0.}$$

2. Find $\frac{1}{\alpha^5} + \frac{1}{\beta^5} + \frac{1}{\gamma^5}$, where α, β, γ are the roots of the equation $x^3 + 2x^2 - 3x - 1 = 0$.

Soln.:- put $x = \frac{1}{y}$ in the equation, then the equation become

$\Rightarrow \frac{1}{y^3} + \frac{2}{y^2} - \frac{3}{y} - 1 = 0 \Rightarrow \frac{-1}{y^3}\left[-1 - 2y + 3y^2 + y^3\right] = 0$

$\Rightarrow y^3 + 3y^2 - 2y - 1 = 0 \Rightarrow \boxed{P_1 = 3,\ P_2 = -2,\ P_3 = -1}$

$\therefore$ The roots of the equation are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

$\therefore \frac{1}{\alpha^5} + \frac{1}{\beta^5} + \frac{1}{\gamma^5} = S_5$

$\therefore$ Newton's theorem on the sum of the powers of the roots of the equation, we get

$S_5 + 3S_4 - 2S_3 - S_2 = 0 \rightarrow$ ① $(r > n \Rightarrow 5 > 3$

$S_4 + 3S_3 - 2S_2 - S_1 = 0 \rightarrow$ ②

$S_3 + 3S_2 - 2S_1 - S_0 = 0 \rightarrow$ ③

$S_2 + 3S_1 - 4 = 0 \rightarrow$ ④

$S_1 + 3 = 0 \Rightarrow S_1 = -3 \rightarrow$ ⑤

(5)

Sub (5) in eqn. (4), we get

$\Rightarrow S_2 - 9 - 4 = 0 \Rightarrow S_2 = 13 \rightarrow (6)$

Sub. (5) & (6) in (3), we get

$\Rightarrow S_3 + 3(13) - 2(-3) - 3 = 0 \Rightarrow S_3 + 39 + 6 - 3 = 0$

$\Rightarrow S_3 = -42 \rightarrow (7)$

Sub. (5), (6) & (7) in (2), we get

$S_4 + 3(-42) - 2(13) - (-3) = 0$

$\Rightarrow S_4 - 126 - 26 + 3 = 0 \Rightarrow S_4 - 152 + 3 = 0$

$\Rightarrow S_4 - 149 = 0 \Rightarrow S_4 = 149 \rightarrow (8)$

Sub. (6), (7) & (8) in (1), we get

$S_5 + 3(149) - 2(-42) - (+13) = 0$

$\Rightarrow S_5 + 447 + 84 - 13 = 0 \Rightarrow S_5 + 531 - 13 = 0$

$\Rightarrow S_5 + 518 = 0 \Rightarrow \boxed{S_5 = -518}$

$\therefore \boxed{\frac{1}{\alpha^5} + \frac{1}{\beta^5} + \frac{1}{\gamma^5} = -518}$.

3. If $a+b+c+d = 0$, show that

$$\frac{a^5+b^5+c^5+d^5}{5} = \frac{a^2+b^2+c^2+d^2}{2} \cdot \frac{a^3+b^3+c^3+d^3}{3}$$

Proof:-

Given that $a+b+c+d=0$.

$\therefore a, b, c, d$ are the roots of the equation $x^4 + P_1x^3 + P_2x^2 + P_3x + P_4 = 0$

From Newton's theorem on the sums of powers of the roots,

We get

$S_5 + P_1S_4 + P_2S_3 + P_3S_2 + P_4S_1 = 0 \quad \text{if } 5 > 4 \rightarrow (1)$

$S_4 + P_1S_3 + P_2S_2 + P_3S_1 + 4P_4 = 0 \quad \text{if } 4 = 4 \rightarrow (2)$

$S_3 + P_1S_2 + P_2S_1 + 3P_3 = 0 \quad \text{if } 3 < 4 \rightarrow (3)$

$S_2 + P_1S_1 + 2P_2 = 0 \quad \text{if } 2 < 4 \rightarrow (4)$

$S_1 + P_1 = 0 \quad \text{if } 1 < 4 \rightarrow (5)$

From (5) $\Rightarrow$ $S_1 = 0$ $\quad \left(\because \sum \alpha_1 = a+b+c+d = 0 \quad S_1 = -p_1 \right.$ (6)

Sub. $S_1 = 0$ in (4) $\Rightarrow$ $S_2 + 0 + 2p_2 = 0$

$\Rightarrow S_2 = -2p_2$

Sub. S_1 & S_2 values in (3) $\Rightarrow$ $S_3 + 0 + 0 + 3p_3 = 0$

$\Rightarrow S_3 = -3p_3$

Sub., S_1, S_2 & S_3 values in (1) $\Rightarrow$

$$S_5 + 0 + p_2(-3p_3) + p_3(-2p_2) + 0 = 0$$

$$\Rightarrow S_5 - 3p_2p_3 - 2p_2p_3 = 0$$

$$\Rightarrow S_5 - 5p_2p_3 = 0$$

$$\Rightarrow S_5 = 5p_2p_3$$

$\left(S_2 = -2p_2 \Rightarrow p_2 = -\frac{S_2}{2} \right.$

$\left. S_3 = -3p_3 \Rightarrow p_3 = -\frac{S_3}{3} \right.$

$$\Rightarrow \frac{S_5}{5} = p_2 p_3$$

$$\Rightarrow \frac{S_5}{5} = \left(-\frac{S_2}{2}\right)\left(-\frac{S_3}{3}\right)$$

$$\Rightarrow \frac{S_5}{5} = \frac{S_2}{2} \cdot \frac{S_3}{3}$$

ie., $$\frac{a^5+b^5+c^5+d^5}{5} = \frac{a^2+b^2+c^2+d^2}{2} \cdot \frac{a^3+b^3+c^3+d^3}{3}$$

Hence the proof.

4. Show that the sum of the m^{th} powers, where $m \leq n$, of the roots of the equation $x^n - 2x^{n-1} - 2x^{n-2} - \dots - 2x - 2 = 0$ is $3^m - 1$.

proof:- Given that the equation is

$$x^n - 2x^{n-1} - 2x^{n-2} - \dots - 2x - 2 = 0$$

$\Rightarrow p_1 = p_2 = \dots = p_n = -2$ and $p_{m-1} = -2$

by Newton's Theorem, also $p_m = -2$ & p_n in p_m

$$S_m - 2S_{m-1} - 2S_{m-2} - \dots - m.2 = 0, \text{ if } m \leq n \longrightarrow (1)$$

and $$S_{m-1} - 2S_{m-2} - 2S_{m-3} - \dots - (m-1)2 = 0 \longrightarrow (2)$$

① − ② ⇒ (7)

$$(S_m - 2S_{m-1} - 2S_{m-2} - \cdots - 2m) - (S_{m-1} - 2S_{m-2} - 2S_{m-3} - \cdots - (m-1)2) = 0$$

$$\Rightarrow S_m - 2\cancel{S_{m-1}} - 2\cancel{S_{m-2}} - \cdots - 2m - S_{m-1} + 2\cancel{S_{m-2}} + 2\cancel{S_{m-3}} + \cdots + 2(m-1) = 0$$

$$\Rightarrow S_m - 3S_{m-1} - 2m + 2(m-1) = 0$$

$$\Rightarrow S_m - 3S_{m-1} - 2\cancel{m} + 2\cancel{m} - 2 = 0$$

$$\Rightarrow S_m - 3S_{m-1} - 2 = 0$$

$$\Rightarrow S_m = 2 + 3S_{m-1}$$

$$= 2 + 3(2 + 3S_{m-2})$$

$$= 2 + 3\cdot 2 + 3^2 S_{m-2}$$

$$= 2 + 3\cdot 2 + 3^2(2 + 3S_{m-3})$$

$$= 2 + 3\cdot 2 + 3^2\cdot 2 + 3^3 S_{m-3}$$

$$\vdots$$

$$\therefore S_m = 2 + 3\cdot 2 + 3^2\cdot 2 + 3^3\cdot 2 + \cdots + 3^{m-1} S_1$$

$$= 2 + 3\cdot 2 + 3^2\cdot 2 + 3^3\cdot 2 + \cdots + 3^{m-1}\,2 \qquad \left(\because S_1 = -P_1 \Rightarrow S_1 = -(-2) = 2\right)$$

$$= 2(1 + 3 + 3^2 + 3^3 + \cdots + 3^{m-1})$$

$$= 2\left(\frac{3^m - 1}{3 - 1}\right) \qquad \left(\because 1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}\right)$$

$$= 2\left(\frac{3^m - 1}{2}\right)$$

$$\boxed{S_m = 3^m - 1}$$

Hence the proof.

5. Determine the value of $\phi(\alpha_1) + \phi(\alpha_2) + \cdots + \phi(\alpha_n)$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $f(x)$ and $\phi(x)$ is any rational and integral function of x.

Soln:-

W.K.T. $$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \cdots + \frac{1}{x-\alpha_n}$$

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Multiply $\phi(x)$ on both sides, we get

$$\frac{f'(x)\,\phi(x)}{f(x)} = \frac{\phi(x)}{x-\alpha_1} + \frac{\phi(x)}{x-\alpha_2} + \cdots + \frac{\phi(x)}{x-\alpha_n}$$

Performing the division and retaing only the remainders on both sides of the equation, we have

$$\frac{R_0 x^{n-1} + R_1 x^{n-2} + \cdots + R_{n-1}}{f(x)} = \frac{\phi(\alpha_1)}{x-\alpha_1} + \frac{\phi(\alpha_2)}{x-\alpha_2} + \cdots + \frac{\phi(\alpha_n)}{x-\alpha_n}$$

$$\Rightarrow \frac{R_0 x^{n-1} + R_1 x^{n-2} + \cdots + R_{n-1}}{f(x)} = \frac{\phi(\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n) + \phi(\alpha_2)(x-\alpha_1)(x-\alpha_3)\cdots(x-\alpha_n) + \cdots + \phi(\alpha_n)(x-\alpha_1)\cdots(x-\alpha_{n-1})}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)}$$

$\because f(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$

$$\Rightarrow R_0 x^{n-1} + R_1 x^{n-2} + \cdots + R_{n-1} = \sum \phi(\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)$$

Equating the coefficient of x^{n-1} on both sides, we get

$$\Rightarrow R_0 = \sum \phi(\alpha_1)$$

$$\Rightarrow \sum \phi(\alpha_1) = R_0$$

i.e., $\boxed{\phi(\alpha_1) + \phi(\alpha_2) + \phi(\alpha_3) + \cdots + \phi(\alpha_n) = R_0}$

Home Work

1. If α, β, γ are the roots of $x^3 + qx + r = 0$, prove that
   (i) $3 S_2 S_5 = 5 S_3 S_4$ (ii) $\frac{\alpha^5+\beta^5+\gamma^5}{5} = \frac{\alpha^3+\beta^3+\gamma^3}{3} \cdot \frac{\alpha^2+\beta^2+\gamma^2}{2}$

2. If α, β, γ be the roots of the equation $x^3 - 7x + 7 = 0$,
   Find $\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4}$. Ans:- $3/4$

3. Show that the sum of ninth powers of the roots of the equation $x^3 + 3x + 9 = 0$ is zero

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Section: 15 Transformation of Equations

15.1 Roots with signs changed:-

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = 0$$

Then, $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$

changing $x = -x$, we get

$$(-x)^n + p_1(-x)^{n-1} + p_2(-x)^{n-2} + \dots + p_n = (-x-\alpha_1)(-x-\alpha_2)\dots(-x-\alpha_n)$$

$\therefore$ The roots of the equation.

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \dots \pm p_n = 0$$

are $-\alpha_1, -\alpha_2, \dots, -\alpha_n$.

This is the required transformation.

Problems:-

1. If the roots of $x^3+12x^2+23x+36=0$ are $-1, 4, 9$. Find the equation whose roots are $1, -4, -9$.

Soln:- Give that the equation

$$x^3 + 12x^2 + 23x + 36 = 0. \rightarrow (1)$$

Its root are $-1, 4, 9$.

put $x = -x$ in eqn. (1), we get

$$(-x)^3 + 12(-x)^2 + 23(-x) + 36 = 0$$

$$\Rightarrow -x^3 + 12x^2 - 23x + 36 = 0$$

$$\Rightarrow -(x^3 - 12x^2 + 23x - 36) = 0$$

$$\Rightarrow \boxed{x^3 - 12x^2 + 23x - 36 = 0}$$

This is the required equation of the given roots.

2. Find the equation whose roots are equal in magnitude, but opposite sign to the roots of the equation $x^{10}-12x^8+40x^4-15x+20=0$

Soln.:- Given that the eqn. is (10)

$$x^{10} - 12x^8 + 40x^4 - 15x + 20 = 0$$

put $x = -x$ in this equation, we get

$$\Rightarrow (-x)^{10} - 12(-x)^8 + 40(-x)^4 - 15(-x) + 20 = 0$$

$$\Rightarrow \boxed{x^{10} - 12x^8 + 40x^4 + 15x + 20 = 0}$$

This is the required equation.

3. Multiply the roots of the equation $x^4 + 2x^3 + 4x^2 + 6x + 8 = 0$ by $\frac{1}{2}$.

Soln.:-

Given that $x^4 + 2x^3 + 4x^2 + 6x + 8 = 0 \rightarrow (1)$

Multiply the roots of the eqn. (1) by $\frac{1}{2}$, we get

$$x^4 + \left(\tfrac{1}{2}\right) 2x^3 + \left(\tfrac{1}{2}\right)^2 4x^2 + \left(\tfrac{1}{2}\right)^3 6x + \left(\tfrac{1}{2}\right)^4 \cdot 8 = 0$$

$$\Rightarrow x^4 + x^3 + \frac{1}{4} 4x^2 + \frac{1}{8} 6x + \frac{1}{16} 8 = 0$$

$$\Rightarrow x^4 + x^3 + x^2 + \frac{3}{4} x + \frac{1}{2} = 0$$

$$\Rightarrow \frac{1}{4}\left[4x^4 + 4x^3 + 4x^2 + 3x + 2\right] = 0$$

$$\Rightarrow \boxed{4x^4 + 4x^3 + 4x^2 + 3x + 2 = 0}$$

This is the required equation.

4. Multiply the roots of the equation $x^3 - 3x + 1 = 0$ by 10

Soln.:- Given that $x^3 - 3x + 1 = 0$

Multiply by 10, we get

$x^3 + 0x^2 - 3x + 1 = 0$

$x^3 + (10)(0)x^2 - 3(10)^2 x + (10)^3 = 0$

$$\Rightarrow x^3 - (10)^2 3x + (10)^3 = 0$$

$$\Rightarrow x^3 - (100)3x + 1000 = 0$$

$$\Rightarrow \boxed{x^3 - 300x + 1000 = 0}$$

This is the required equation.

5) Remove the fractional coefficients from the equation
$x^3-\frac{1}{4}x^2+\frac{1}{3}x-1=0$

Soln:- Given that $x^3-\frac{1}{4}x^2+\frac{1}{3}x-1=0 \rightarrow (1)$

Multiply by m in eqn. (1), we get

$$x^3-\frac{m}{4}x^2+\frac{m^2}{3}x\cancel{+\frac{1}{3}m-N=0}-m^3=0$$

put m=12 in this equation, we get L.c.m = 12

$$\Rightarrow x^3-\frac{12}{4}x^2+\frac{(12)^2}{3}x-(12)^3=0$$

$$\Rightarrow x^3-3x^2+\frac{144}{3}x-1728=0$$

$$\Rightarrow \boxed{x^3-3x^2+48x-1728=0}$$

This the required equation.

6) Remove the fractional co-efficient from $x^3-\frac{3}{2}x^2-\frac{1}{16}x+\frac{1}{32}=0$, such that coefficient of the leading down remains unity.

Soln:- Given that the equation is

$$x^3-\frac{3}{2}x^2-\frac{1}{16}x+\frac{1}{32}=0$$

$$\Rightarrow x^3-\frac{3}{2}(m)x^2-\frac{1}{16}(m^2)x+\frac{1}{32}m^3=0$$

~~Multiply~~ put m=32 in this equation, we get

$$\Rightarrow x^3-\frac{3}{2}(32)x^2-\frac{1}{16}(32)^2x+\frac{1}{32}(32)^{\cancel{3}\,2}=0$$

$$\Rightarrow x^3-3(16)x^2-\frac{1}{16}(1024)x+1024=0$$

$$\Rightarrow \boxed{x^3-48x^2-64x+1024=0}$$

This is the required equation.

Home Work

(1) Multiply the roots of the equation is $x^3-6x^2+12x-8=0$ by 10.

Ans:- $x^3-60x^2+1200x-8000$

② Change the sign of the root of the equation.

a) $x^7+4x^5+x^3-2x^2+7x+3=0$ Ans:- $x^7+4x^5+x^3+2x^2+7x-3=0$

b) $x^5+6x^4+6x^3-7x^2+2x-1=0$ Ans:- $x^5-6x^4+6x^3+7x^2+2x+1=0$

Sec: 15.3 Reciprocal roots

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_n=0$$

$$\Rightarrow x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_n=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$$

put $x=\frac{1}{y}$, we get

$$\left(\tfrac{1}{y}\right)^n+p_1\left(\tfrac{1}{y}\right)^{n-1}+p_2\left(\tfrac{1}{y}\right)^{n-2}+\cdots+p_n=\left(\tfrac{1}{y}-\alpha_1\right)\left(\tfrac{1}{y}-\alpha_2\right)\cdots\left(\tfrac{1}{y}-\alpha_n\right)$$

$$\frac{1}{y^n}\left[1+p_1y+p_2y^2+\cdots+p_ny^n\right]=\left(\frac{1-\alpha_1y}{y}\right)\left(\frac{1-\alpha_2y}{y}\right)\cdots\left(\frac{1-\alpha_ny}{y}\right)$$

$$\frac{1}{y^n}\left[p_ny^n+p_{n-1}y^{n-1}+\cdots+p_1y+1\right]=\frac{1}{y^n}(1-\alpha_1y)(1-\alpha_2y)\cdots(1-\alpha_ny)$$

$$p_ny^n+p_{n-1}y^{n-1}+\cdots+p_1y+1=(\alpha_1\cdot\alpha_2\cdots\alpha_n)\left(\tfrac{1}{\alpha_1}-y\right)\left(\tfrac{1}{\alpha_2}-y\right)\cdots\left(\tfrac{1}{\alpha_n}-y\right)$$

$y-\frac{1}{\alpha_1}=0 \quad y-\frac{1}{\alpha_2}$

Hence the roots of the equation $p_ny^n+p_{n-1}y^{n-1}+\cdots+p_1y+1=0$ are $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$.

Sec: 16 Reciprocal equation

If an equation remains unaltered when x is changed into its reciprocal, it is called a reciprocal equation.

Let $x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n=0$ be a reciprocal eqn. → ①

put $x=\frac{1}{x}$ in this equation, we get

$$\Rightarrow p_nx^n+p_{n-1}x^{n-1}+p_{n-2}x^{n-2}+\cdots+p_1x+1=0$$

$$\Rightarrow p_n\left[x^n+\frac{p_{n-1}}{p_n}x^{n-1}+\frac{p_{n-2}}{p_n}x^{n-2}+\cdots+\frac{p_1}{p_n}x+\frac{1}{p_n}\right]=0$$

$$\Rightarrow x^n+\frac{p_{n-1}}{p_n}x^{n-1}+\frac{p_{n-2}}{p_n}x^{n-2}+\cdots+\frac{p_1}{p_n}x+\frac{1}{p_n}=0 \rightarrow ②$$

Comparing eqns. ① & ②, we get

$$p_1=\frac{p_{n-1}}{p_n},\quad p_2=\frac{p_{n-2}}{p_n},\quad \ldots,\quad p_{n-1}=\frac{p_1}{p_n} \text{ and } p_n=\frac{1}{p_n}$$

$\Rightarrow \frac{1}{p_n} = p_n \Rightarrow p_n^2 = 1 \Rightarrow p_n = \pm 1$

Case:(i) $p_n = 1$

Then $p_{n-1} = p_1$, $p_{n-2} = p_2$, ... $p_{n-(n-1)} = p_1 = p_{n-1}$

In this case the coefficients of the terms equidistant from the begining and the end are equal. in magnitude and have the same sign.

Case:(ii) $p_n = -1$

Then, $p_{n-1} = -p_1$, $p_{n-2} = -p_2$, ... , $p_1 = -p_{n-1}$

In this case the terms equidistant from the begining and the end are equal in magnitude but, different sign.

Example:

5 marks (IA-I)

1. Find the roots of the equation $x^5+4x^4+3x^3+3x^2+4x+1=0$.

Soln:-

This is a reciprocal equation of odd degree.

$\therefore (x+1)$ is a factor of $x^5+4x^4+3x^3+3x^2+4x+1=0$

$\Rightarrow x^5+x^4+3x^4+3x^3+3x^2+3x+x+1=0$

$\Rightarrow x^4(x+1)+3x^3(x+1)+3x(x+1)+(x+1)=0$

$\Rightarrow (x+1)[x^4+3x^3+3x+1]=0$

$(x+\frac{1}{x})^2 = z^2$

$x^2+2x\frac{1}{x}+\frac{1}{x^2} = z^2$

$x^2+\frac{1}{x^2} = z^2-2$

$\therefore x+1=0$ and $x^4+3x^3+3x+1=0$

$\boxed{x=-1} \to ①$ $\qquad \to ②$

dividing by x^2 in eqn. ②, we get

$\Rightarrow \frac{x^4+3x^3+3x+1}{x^2} = 0 \Rightarrow x^2+3x+\frac{3}{x}+\frac{1}{x^2}=0$

$\Rightarrow (x^2+\frac{1}{x^2})+3(x+\frac{1}{x})=0$

put $x+\frac{1}{x}=z$ and $x^2+\frac{1}{x^2}=z^2-2$, we get

$\Rightarrow z^2-2+3z=0 \Rightarrow z^2+3z-2=0$

$\Rightarrow z = \frac{-3\pm\sqrt{9-4(1)(-2)}}{2(1)} = \frac{-3\pm\sqrt{9+8}}{2} = \frac{-3\pm\sqrt{17}}{2}$

Hence $x+\frac{1}{x} = \frac{-3\pm\sqrt{17}}{2} \Rightarrow \frac{x^2+1}{x} = \frac{-3\pm\sqrt{17}}{2} \Rightarrow (x^2+1)2 = x(-3\pm\sqrt{17})$

$-x(3\pm\sqrt{17})$

$\Rightarrow 2x^2+2+(+3\pm\sqrt{17})x=0$

ie., $2x^2+(3+\sqrt{17})x+2=0$ From these equations x can be found

and $2x^2+(3-\sqrt{17})x+2=0$

② $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$ Solve the equation. (14)

Soln:-

This is a reciprocal equation of odd degree with unlike signs. Hence $x-1$ is a factor of the left hand side

$\therefore$ ~~[illegible]~~

Give that $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$

$\Rightarrow 6x^5 - 6x^4 + 5x^4 - 5x^3 - 38x^3 + 38x^2 + 5x^2 - 5x + 6x - 6 = 0$

$\Rightarrow 6x^4(x-1) + 5x^3(x-1) - 38x^2(x-1) + 5x(x-1) + 6(x-1) = 0$

$\Rightarrow (x-1)(6x^4 + 5x^3 - 38x^2 + 5x + 6) = 0$

$\Rightarrow x - 1 = 0$ (or) $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0 \to ②$

$\Rightarrow \boxed{x = 1} \to ①$

Dividing eqn ② by x^2, we get

$$\Rightarrow \frac{6x^4 + 5x^3 - 38x^2 + 5x + 6}{x^2} = 0$$

$$\Rightarrow 6x^2 + 5x - 38 + \frac{5}{x} + \frac{6}{x^2} = 0$$

$$\Rightarrow 6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0$$

put $x + \frac{1}{x} = z$ and $x^2 + \frac{1}{x^2} = z^2 - 2$, we get

$\Rightarrow 6(z^2 - 2) + 5z - 38 = 0$

$\Rightarrow 6z^2 - 12 + 5z - 38 = 0$

$\Rightarrow 6z^2 + 5z - 50 = 0$

$6\times -50 = -300$; 20, -15; $20 - 15 = 5$

$\Rightarrow 6z^2 + 20z - 15z - 50 = 0$

$\Rightarrow 2z(3z + 10) - 5(3z + 10) = 0$

$\Rightarrow (2z - 5)(3z + 10) = 0$

$\Rightarrow 2z - 5 = 0$ or $3z + 10 = 0$

$z = 5/2$ or $z = -10/3$

Then $x + 1/x = 5/2$ and $x + 1/x = -10/3$

$$\Rightarrow \frac{x^2+1}{x} = \frac{5}{2}$$

$\Rightarrow 2(x^2+1) = 5x$

$\Rightarrow 2x^2+2-5x=0$

$\Rightarrow 2x^2-5x+2=0$

$\Rightarrow 2x^2-x-4x+2=0$

$\Rightarrow x(2x-1)-2(2x-1)=0$

$\Rightarrow (2x-1)(x-2)=0$

$\Rightarrow 2x-1=0$ (or) $x-2=0$

$\Rightarrow \boxed{x=\frac{1}{2}}$ or $\boxed{x=2}$

$$\frac{x^2+1}{x} = -\frac{10}{3}$$

$3(x^2+1) = -10x$

$3x^2+3+10x=0$

$3x^2+10x+3=0$

$3x^2+9x+x+3=0$

$3x(x+3)+(x+3)=0$

$(3x+1)(x+3)=0$

$3x+1=0$ (or) $x+3=0$

$\boxed{x=-\frac{1}{3}}$ (or) $\boxed{x=-3}$

(15)

$\therefore$ The roots of the equation are $1, \frac{1}{2}, 2, -\frac{1}{3}$ and -3.

3. Solve the equation $x^5-5x^4+9x^3-9x^2+5x-1=0$.

Soln. :-

This is a reciprocal equation of odd degree, with unlike signs. Hence $(x-1)$ is a factor of the left side.

Given that $x^5-5x^4+9x^3-9x^2+5x-1=0$

$\Rightarrow x^5-x^4-4x^4+4x^3+5x^3-5x^2-4x^2+4x+x-1=0$

$\Rightarrow x^4(x-1)-4x^3(x-1)+5x^2(x-1)-4x(x-1)+(x-1)=0$

$\Rightarrow (x-1)[x^4-4x^3+5x^2-4x+1]=0$

$\Rightarrow (x-1)=0$ (or) $x^4-4x^3+5x^2-4x+1=0 \rightarrow (2)$

$\Rightarrow \boxed{x=1} \rightarrow (1)$

Dividing eqn. (2) by x^2, we get

(16)

$$\Rightarrow \frac{x^4-4x^3+5x^2-4x+1}{x^2}=0$$

$$\Rightarrow x^2-4x+5-\frac{4}{x}+\frac{1}{x^2}=0$$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)-4\left(x+\frac{1}{x}\right)+5=0$$

put $x+\frac{1}{x}=z$ and $x^2+\frac{1}{x^2}=z^2-2$, we get

$$\Rightarrow z^2-2-4z+5=0$$

$$\Rightarrow z^2-4z+3=0$$

$$\Rightarrow z^2-z-3z+3=0$$

3: −1, −3

$$\Rightarrow z(z-1)-3(z-1)=0$$

$$\Rightarrow (z-3)(z-1)=0$$

ie., $z=3$, $z=1$

$$\Rightarrow x+\frac{1}{x}=3$$

$$\Rightarrow x^2+1=3x$$

$$\Rightarrow x^2-3x+1=0$$

$$\Rightarrow x=\frac{+3\pm\sqrt{9-4}}{2}$$

$$\Rightarrow \boxed{x=\frac{3\pm\sqrt{5}}{2}}$$

$$x+\frac{1}{x}=1$$

$$x^2+1=x$$

$$x^2-x+1=0$$

$$x=\frac{1\pm\sqrt{1-4}}{2}$$

$$\boxed{x=\frac{1\pm i\sqrt{3}}{2}}$$

∴ The roots of the equation is

$$1,\ \frac{3+\sqrt{5}}{2},\ \frac{3-\sqrt{5}}{2},\ \frac{1+i\sqrt{3}}{2},\ \frac{1-i\sqrt{3}}{2}$$

Home Work

① Solve the equation $x^5+x^4+x^3+x^2+x+1=0$.

Ans:- $-1,\ \frac{1+i\sqrt{3}}{2},\ \frac{1-i\sqrt{3}}{2},\ \frac{-1+i\sqrt{3}}{2},\ \frac{-1-i\sqrt{3}}{2}$

4 Solve the equation $6x^6-35x^5+56x^4-56x^2+35x-6=0$

S Soln.:-

There is no mid term and this is a reciprocal equation of even degree with unlike signs.

We can easily see that (x^2-1) is a factor of the expression on left hand side of the equation.

Given that the equation

$$6x^6-35x^5+56x^4-56x^2+35x-6=0$$

$$\Rightarrow 6(x^6-1)-35x(x^4-1)+56x^2(x^2-1)=0$$

$$\Rightarrow 6(x^6-1+x^4-x^4+x^2-x^2)-35x((x^2)^2-1^2)+56x^2(x^2-1)=0$$

$$\Rightarrow 6(x^6+x^4+x^2-x^4-x^2-1)-35x(x^2-1)(x^2+1)+56x^2(x^2-1)=0$$

$$\Rightarrow 6[x^2(x^4+x^2+1)-(x^4+x^2+1)]-35x(x^2-1)(x^2+1)+56x^2(x^2-1)=0$$

$$\Rightarrow 6[(x^4+x^2+1)(x^2-1)-35x(x^2-1)(x^2+1)+56x^2(x^2-1)=0$$

$$\Rightarrow (x^2-1)[6(x^4+x^2+1)-35x(x^2+1)+56x^2]=0$$

$$\Rightarrow (x^2-1)[6x^4+6x^2+6-35x^3-35x+56x^2]=0$$

$$\Rightarrow (x^2-1)[6x^4-35x^3+62x^2-35x+6]=0$$

ie., $x^2-1=0$ (or) $6x^4-35x^3+62x^2-35x+6=0$

$x^2=1$

$\boxed{x=\pm 1}$

Dividing this eqn. by x^2, we get

$$6x^2-35x+62-\frac{35}{x}+\frac{6}{x^2}=0$$

ie., $6\left(x^2+\frac{1}{x^2}\right)-35\left(x+\frac{1}{x}\right)+62=0$

put $x+\frac{1}{x}=z$ and $x^2+\frac{1}{x^2}=z^2-2$

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$6(z^2-2)-35z+62=0$

$\Rightarrow 6z^2-12-35z+62=0$

$\Rightarrow 6z^2-35z+50=0$

$\Rightarrow 6z^2-15z-20z+50=0$

$6\times 50=300$

$-15 \quad -20$

$\Rightarrow 3z(2z-5)-10(2z-5)=0$

$\Rightarrow (3z-10)(2z-5)=0$

i.e., $3z-10=0$ (or) $2z-5=0$

$z=\frac{10}{3}$

$x+\frac{1}{x}=\frac{10}{3}$

$3(x^2+1)=10x$

$3x^2-10x+3=0$

9

$-1 \quad -9$

a+3/, b5

$3x^2-9x-x+3=0$

$3x(x-3)-(x-3)=0$

$(3x-1)(x-3)=0$

i.e., $3x-1=0$ & $x-3=0$

$\boxed{x=\frac{1}{3}}$ & $\boxed{x=3}$

$z=\frac{5}{2}$

$x+\frac{1}{x}=\frac{5}{2}$

$2(x^2+1)=5x$

$2x^2-5x+2=0$

4

$-1 \quad -4$

$2x^2-x-4x+2=0$

$x(2x-1)-2(2x-1)=0$

$(x-2)(2x-1)=0$

i.e., $x-2=0$ & $2x-1=0$

$\boxed{x=2}$ & $\boxed{x=\frac{1}{2}}$

$\therefore$ The roots of the equation is

$x=1,-1,\frac{1}{3},3,2,\frac{1}{2}$

Home Work

1. Solve the equations

(a) $x^4-10x^3+26x^2-10x+1=0$ $[3\pm\sqrt{8}, 2\pm\sqrt{3}]$

(b) $x^4+3x^3-3x-1=0$ $\left[\pm 1, \frac{-3\pm\sqrt{5}}{2}\right]$

5. Solve the equation $4x^4-20x^3+33x^2-20x+4=0$.

Soln :-

This is a reciprocal equation of even degree with unlike signs.

Given that $4x^4-20x^3+33x^2-20x+4=0$

dividing this equation by x^2, we get

$\Rightarrow 4x^2-20x+33-\frac{20}{x}+\frac{4}{x^2}=0$

$\Rightarrow 4\left(x^2+\frac{1}{x^2}\right)-20\left(x+\frac{1}{x}\right)+33=0$

put $x+\frac{1}{x}=z$ and $x^2+\frac{1}{x^2}=z^2-2$

$\Rightarrow 4z^2-8-20z+33=0$

$\Rightarrow 4z^2-20z+25=0$ $4\times25=100$

$\Rightarrow 4z^2-10z-10z+25=0$ -10 -10

$\Rightarrow 2z(2z-5)-5(2z-5)=0$

$\Rightarrow (2z-5)(2z-5)=0$

i.e., $2z-5=0$ and $2z-5=0$

$z=5/2$ $z=5/2$

$\Rightarrow x+1/x=5/2$

$\Rightarrow 2x^2+2=5x$

$\Rightarrow 2x^2-5x+2=0$

$\Rightarrow 2x^2 - x - 4x + 2 = 0$

$\Rightarrow x(2x-1) - 2(2x-1) = 0$

$\Rightarrow (2x-1)(x-2) = 0$

4

-1 -4

i.e., $2x - 1 = 0$ and $x - 2 = 0$

$\boxed{x = \frac{1}{2}}$ $\boxed{x = 2}$

and also $\boxed{x = \frac{1}{2}}$ and $\boxed{x = 2}$

$\therefore$ The roots of the equation are

$2, \frac{1}{2}, 2, \frac{1}{2}$.

Sec: 17 To increase or decrease the roots of a given equation by a given quantity.

Problems :-

1. Increase by 7 the roots of the equation $3x^4 + 7x^3 - 15x^2 + x - 2 = 0$, Find the transformed equation

Soln. :-

-7	3	7	-15	1	-2
	0	-21	98	-581	4060
-7	3	-14	83	-580	4058
	0	-21	245	-2296	
-7	3	-35	328	-2876	
	0	-21	392		
-7	3	-56	720		
	0	-21			
-7	3	-77			
	0				
	3				

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$\therefore$ The transformed equation is

$$\boxed{3x^4 - 77x^3 + 720x^2 - 2876x + 4058 = 0}$$

(2) Increase by 2, the roots of the equation

$x^4 - x^3 - 10x^2 + 4x + 24 = 0.$

Soln.:-

-2	1	-1	-10	4	24
	0	-2	6	8	-24
-2	1	-3	-4	12	0
	0	-2	10	-12	
-2	1	-5	6	0	
	0	-2	14		
-2	1	-7	20		
	0	-2			
-2	1	-9			
	0				
	1				

$\therefore$ The transformed equation is

$$\boxed{x^4 - 9x^3 + 20x^2 = 0}$$

Home Work

Find the equation whose roots are the equation $4x^5 - 2x^3 + 7x - 3 = 0$, each increased by 2.

Ans:- $4x^5 - 40x^4 + 158x^3 - 308x^2 + 303x - 129 = 0$

Unit - II is over

CLASSICAL ALGEBRA AND THEORY OF NUMBERS

UNIT - III

Chapter-6- Section : 18

Form of the quotient and remainder when a polynomial is divided by a binomial.

Problems:-

① Find the quotient and remainder when $3x^3+8x^2+8x+2$ is divided by $x-4$.

Soln.:-

Given that $3x^3+8x^2+8x+2$ and $x-4=0$

$\Rightarrow x=4$

$$\begin{array}{c|cccc} 4 & 3 & 8 & 8 & 2 \\ & 0 & 12 & 80 & 352 \\ \hline & 3 & 20 & 88 & \boxed{354} \end{array}$$

$\therefore$ The quotient is $3x^2+20x+88$.

The remainder is 354.

② Find the quotient and remainer when $2x^6+3x^5-15x^2+2x-4$ is divided by $x+5$.

Soln.:- $x+5=0 \Rightarrow x=-5$

$$\begin{array}{c|ccccccc} -5 & 2 & 3 & 0 & 0 & -15 & 2 & -4 \\ & 0 & -10 & +35 & -175 & 875 & -4300 & 21490 \\ \hline & 2 & -7 & 35 & -175 & 860 & -4298 & \boxed{21486} \end{array}$$

$\therefore$ The quotient is $2x^5-7x^4+35x^3-175x^2+860x-4298$.

The remainder is 21486.

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3 Diminish the roots of $x^4-5x^3+7x^2-4x+5=0$ by 2.

Soln:- Given that the equation

$$x^4-5x^3+7x^2-4x+5=0$$

2	1	-5	7	-4	5
	0	2	-6	2	-4
2	1	-3	1	-2	$\boxed{1}$
	0	2	-2	-2	
2	1	-1	-1	$\boxed{-4}$	
	0	2	2		
2	1	1	$\boxed{1}$		
	0	2			
2	1	$\boxed{3}$			
	0				
	1				

$\therefore$ The required equation is

$$\boxed{x^4+3x^3+x^2-4x+1=0}$$

Home Work

Diminish the roots of $2x^5-x^3+10x-8=0$ by 5 and find the transformed equation.

Ans:- $2x^5+50x^4+499x^3+2485x^2+6185x+6167=0$

~~(3) Diminish the roots of~~ $\cancel{x^4-5x^3+7x^2-4x+5=0}$ ~~by 2.~~

~~Soln:-~~

$\cancel{1 \quad -5 \quad 7 \quad -4 \quad 5}$

Ans

(3)

(4) Show that the equation $x^4-3x^3+4x^2-2x+1=0$ can be transformed into a reciprocal equation by diminishing the roots by unity. Hence solve the equation.

Soln:-

The operation of diminishing the roots by 1.

Given that the eqn. $x^4-3x^3+4x^2-2x+1=0$

1	1	-3	4	-2	1
	0	1	-2	2	0
1	1	-2	2	0	$\boxed{1}$
	0	1	-1	1	
1	1	-1	1	$\boxed{1}$	
	0	1	0		
1	1	0	$\boxed{1}$		
	0	1			
1	1	$\boxed{1}$			
	0				
	1				

$\therefore$ The transformed equation is $x^4+x^3+x^2+x+1=0$ which is a reciprocal equation.

Dividing this equation by x^2, we get

$$\Rightarrow x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=0$$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)+1=0$$

put $x+\frac{1}{x}=z$ and $x^2+\frac{1}{x^2}=z^2-2$, we get

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$\Rightarrow z^2 - 2 + z + 1 = 0$ (a=1, b=1, c=-1) ④

$\Rightarrow z^2 + z - 1 = 0$

$\Rightarrow z = \frac{-1 \pm \sqrt{1+4}}{2}$ $\quad \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\Rightarrow z = \frac{-1 \pm \sqrt{5}}{2}$

$\therefore x + \frac{1}{x} = \frac{-1+\sqrt{5}}{2}$ and $x + \frac{1}{x} = \frac{-1-\sqrt{5}}{2}$

$\Rightarrow 2(x^2+1) = x(-1+\sqrt{5})$

$\Rightarrow 2x^2 + 2 = -(1-\sqrt{5})x$

$\Rightarrow 2x^2 + (1-\sqrt{5})x + 2 = 0$

$a = 2,\ b = (1-\sqrt{5}),\ c = 2$

$\therefore x = \frac{-(1-\sqrt{5}) \pm \sqrt{(1-\sqrt{5})^2 - 16}}{4}$

$= \frac{(\sqrt{5}-1) \pm \sqrt{1+5-2\sqrt{5}-16}}{4}$

$$\boxed{x = \frac{(\sqrt{5}-1) \pm \sqrt{10-2\sqrt{5}}}{4}}$$

$2(x^2+1) = (-1-\sqrt{5})x$

$2x^2 + 2 = -(1+\sqrt{5})x$

$2x^2 + (1+\sqrt{5})x + 2 = 0$

$a = 2,\ b = 1+\sqrt{5},\ c = 2$

$\therefore x = \frac{-(1+\sqrt{5}) \pm \sqrt{(1+\sqrt{5})^2 - 16}}{4}$

$= \frac{-(1+\sqrt{5}) \pm \sqrt{1+5+2\sqrt{5}-16}}{4}$

$$\boxed{x = \frac{-(1+\sqrt{5}) \pm \sqrt{10+2\sqrt{5}}}{4}}$$

These are roots of the transformed equation.

Home Work

Diminish by 3 the roots of the equation

$x^5 - 4x^4 + 3x^3 - 4x + 6 = 0.$

Ans:- $x^5 + 11x^4 + 42x^3 + 57x^2 - 13x + 60 = 0$

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Section : 19 Removal of terms

(5)

Let $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0$, where $a_0, a_1, \cdots, a_{n-1}, a_n$ are any constant. → (1)

put $y = x - h \Rightarrow x = y + h$ in eqn. (1), we get

$$f(y+h) = a_0 (y+h)^n + a_1 (y+h)^{n-1} + a_2 (y+h)^{n-2} + \cdots$$
$$+ a_{n-1}(y+h) + a_n = 0$$

$$\Rightarrow a_0 \left[y^n + nc_1 y^{n-1} h + nc_2 y^{n-2} h^2 + \cdots + nc_n h^n \right]$$
$$+ a_1 \left[y^{n-1} + (n-1)c_1 y^{n-2} h + (n-1)c_2 y^{n-3} h^2 + \cdots + (n-1)c_{n-1} h^{n-1} \right]$$
$$+ a_2 \left[y^{n-2} + (n-2)c_1 y^{n-3} h + (n-2)c_2 y^{n-4} h^2 + \cdots + (n-2)c_{n-2} h^{n-2} \right]$$
$$+ \cdots$$
$$\cdots$$
$$\cdots$$
$$\cdots$$
$$+ a_{n-1}(y+h) + a_n = 0$$

$\because (x+a)^n = x^n + nc_1 x^{n-1} a + nc_2 x^{n-2} a^2 + \cdots + a^n$

Equating coefficient of $y^n, y^{n-1}, y^{n-2} \cdots$, we get

$$\Rightarrow \boxed{a_0 = 0}$$

$$\Rightarrow nc_1 a_0 h + a_1 = 0 \Rightarrow \boxed{n a_0 h + a_1 = 0}$$

$$\Rightarrow nc_2 a_0 h^2 + (n-1)c_1 h a_1 + a_2 = 0 \Rightarrow \boxed{\frac{n(n-1)}{2} a_0 h^2 + (n-1) h a_1 + a_2 = 0}$$

etc.

If we want to remove the 2^{nd} term in the given equation, we put

$$n a_0 h + a_1 = 0$$

If we want to remove the 3^{rd} term of the given equation, we put

$$\frac{n(n-1)}{2} h^2 a_0 + (n-1) a_1 h + a_2 = 0$$

Problems:-

1. Remove the 2^{nd} term from the equation $x^3 - 6x^2 + 10x - 3 = 0$

Soln. :-

Given that $x^3 - 6x^2 + 10x - 3 = 0$

$\Rightarrow a_0 = 1\ ,\ a_1 = -6\ ,\ a_2 = 10\ ,\ a_3 = -3$ and $n = 3$

W.k.T. to remove the 2^{nd} term condition is

$$n a_0 h + a_1 = 0$$

$\Rightarrow (3)(1)h + (-6) = 0$

$\Rightarrow 3h - 6 = 0 \Rightarrow 3h = 6 \Rightarrow \boxed{h = 2}$

Now,

2	1	-6	10	-3
	0	2	-8	4
2	1	-4	2	$\boxed{1}$
	0	2	-4	
2	1	-2	$\boxed{-2}$	
	0	2		
	1	$\boxed{0}$		
	0			
	1			

$\therefore$ The new removing the 2^{nd} term of the equation is

$$\boxed{x^3 - 2x + 1 = 0}.$$

given ! Solve the equation by removing the 2^{nd} term from the equation is $x^4+20x^3+143x^2+430x+462=0$.

Soln :-

G.T. $x^4+20x^3+143x^2+430x+462=0$

$\Rightarrow a_0=1,\ a_1=20,\ a_2=143,\ a_3=430,\ a_4=462$ and $n=4$.

W.K.T. The condition of the removing 2^{nd} term is

$$na_0h+a_1=0$$

$\Rightarrow (4)(1)h+20=0$

$\Rightarrow 4h=-20 \Rightarrow h=\frac{-20}{4}$

$\Rightarrow \boxed{h=-5}$

Now,

-5	1	20	143	430	462
	0	-5	-75	-340	-450
-5	1	15	68	90	12
	0	-5	-50	-90	
-5	1	10	18	0	
	0	-5	-25		
-5	1	5	-7		
	0	-5			
-5	1	0			
	0				
	1				

$\therefore$ The new removing the 2^{nd} term of the equation

is $x^4+0\cdot x^3-7x^2+0\cdot x+12=0$

i.e., $\boxed{x^4-7x^2+12=0}$

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$\Rightarrow (x^2)^2 - 7(x^2)' + 12 = 0$

put $x^2 = y$ in this equation, we get

$\Rightarrow y^2 - 7y + 12 = 0$

$\Rightarrow (y-3)(y-4) = 0$

12 → −3, −4

$\Rightarrow y-3=0$ (or) $y-4=0$

$\Rightarrow y=3$ $\quad y=4$

$\therefore x^2=3$ $\quad x^2=4$

$x=\pm\sqrt{3}$ $\quad x=\pm 2$

The roots of new transformed equation is

$\boxed{x = 2, -2, \sqrt{3}, \sqrt{3}}$

The roots of the original equation is

$x = x+h$

$\Rightarrow x = 2+(-5),\ x=-2+(-5),\ x=\sqrt{3}+(-5),\ x=-\sqrt{3}+(-5)$

$x = 2-5,\ x=-2-5,\ x=\sqrt{3}-5,\ x=-\sqrt{3}-5$

ie., $\boxed{x=-3, -7, \sqrt{3}-5, -\sqrt{3}-5}$

Home Work

Solve the eqn. by removing the 2nd term from the equation is $x^3+6x^2+12x-19=0$.

Ans: (i) The roots of new eqn. is $x = 3, 3, 3$.

(ii) The roots of original eqn. is $x = 1, 1, 1$.

(9)

Remove the 2^{nd} term from $x^5+5x^4+3x^3+x^2+x+1=0$

Sol:-

Given that $x^5+5x^4+3x^3+x^2+x+1=0$

$\Rightarrow a_0=1,\ a_1=5,\ a_2=3,\ a_3=1,\ a_4=1,\ a_5=1$ and $n=5$

W.K.T the condition of removing 2^{nd} term is

$$na_0h+a_1=0$$

$$(5)(1)h+5=0$$

$$5h=-5$$

$$\boxed{h=-1}$$

Now,

-1	1	5	3	1	1	1
	0	-1	-4	1	-2	1
-1	1	4	-1	2	-1	2
	0	-1	-3	4	-6	
-1	1	3	-4	6	-7	
	0	-1	-2	6		
-1	1	2	-6	12		
	0	-1	-1			
-1	1	1	-7			
	0	-1				
-1	1	0				
	0					
	1					

$\therefore$ The new removing the 2^{nd} term of the equation is

$$x^5+0x^4-7x^3+12x^2-7x+2=0$$

i.e., $\boxed{x^5-7x^3+12x^2-7x+2=0}$

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(4) Transform the equation $x^4-4x^3-18x^2-3x+2=0$ into one which shall want the third term.

Soln.:- Given that $x^4-4x^3-18x^2-3x+2=0$

$\Rightarrow a_0=1,\ a_1=-4,\ a_2=-18,\ a_3=-3,\ a_4=2$ and $n=4$

W.K.T. The condition of the removing 3^{rd} term is

$$\Rightarrow \frac{n(n-1)}{2}h^2a_0+(n-1)a_1h+a_2=0$$

$$\Rightarrow \frac{4(4-1)}{2}h^2(1)+(4-1)(-4)h-18=0$$

$$\Rightarrow \frac{(4)(3)}{2}h^2+(3)(-4)h-18=0$$

$$\Rightarrow \frac{12}{2}h^2-12h-18=0$$

$$\Rightarrow 6h^2-12h-18=0$$

$$\Rightarrow 6(h^2-2h-3)=0$$

$$\Rightarrow h^2-2h-3=0$$

$$\begin{array}{c} -3 \\ \wedge \\ 1 \quad -3 \end{array}$$

$$\Rightarrow h^2+h-3h-3=0$$

$$\Rightarrow h(h+1)-3(h+1)=0$$

$$\Rightarrow (h+1)(h-3)=0$$

$\therefore h+1=0$ (or) $h-3=0$

$\boxed{h=-1}$ (or) $\boxed{h=3}$

(i) Here $h=3$, We get

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3	1	-4	-18	-3	2
	0	3	-3	-63	-198
3	1	-1	-21	-66	-196
	0	3	6	-45	
3	1	2	-15	-111	
	0	3	15		
3	1	5	0		
	0	3			
	1	8			
	0				
	1				

$\therefore$ The transformed equation is

$$\boxed{x^4+8x^3-111x-196=0}$$

Here, $h=-1$, we get

-1	1	-4	-18	-3	2
	0	-1	5	13	-10
-1	1	-5	-13	10	-8
	0	-1	6	7	
-1	1	-6	-7	17	
	0	-1	7		
-1	1	-7	0		
	0	-1			
	1	-8			
	0				
	1				

$\therefore$ The transformed equation is

$$\boxed{x^4-8x^3+17x-8=0}$$

Home Work

Solve the equation by removing the 2^{nd} term from

(i) $x^4-12x^3+48x^2-72x+35=0$

(ii) $x^4+16x^3+83x^2+152x+84=0$

Ans:- (i) $x^4 - 6x^2 + 8 = 0$

The roots of the new eqn. is

$x = 2, -2, \sqrt{2}, -\sqrt{2}$

The roots of Original eqn. is

$x = 5, 1, 3 \pm \sqrt{2}$.

Ans:- (ii) $x^4 - 13x^2 + 36 = 0$

The roots of the new eqn. is

$x = \pm 3, \pm 2$

The roots of Original eqn. is

$x = -1, -7, -2, -6$.

Section: 20

To form an equation whose roots are any power of the roots of a given equation:

The method of forming such equations is illustrated in the following examples.

Example:

Find the equation whose roots are squares of the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = 0$$

Soln.:-

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of equation

i.e., $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$

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put $x = -x$, we get (13)

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots = (-x-\alpha_1)(-x-\alpha_2)(-x-\alpha_3)\cdots(-x-\alpha_n)$$

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots = (x+\alpha_1)(x+\alpha_2)\cdots(x+\alpha_n)$$

$$(x^n + p_2 x^{n-2} + p_4 x^{n-4} + \ldots)^2 - (p_1 x^{n-1} + p_3 x^{n-3} + \ldots)^2$$

$$= (x^2-\alpha_1^2)(x^2-\alpha_2^2)\cdots(x^2-\alpha_n^2)$$

put $x^2 = y$, we get

$$y^n + (2p_2 - p_1^2) y^{n-1} + \ldots = (y-\alpha_1^2)(y-\alpha_2^2)\cdots(y-\alpha_n^2)$$

i.e., $y^n + (2p_2 - p_1^2) y^{n-1} + \ldots = 0$ will have roots are $\alpha_1^2, \alpha_2^2, \ldots, \alpha_n^2$.

Problems:-

(1) Find the equation whose roots are the square of the roots of the equation $x^4 + x^3 + 2x^2 + x + 1 = 0$.

Soln:-

Given that $x^4 + x^3 + 2x^2 + x + 1 = 0 \rightarrow (1)$

Let the roots are $\alpha, \beta, \gamma, \delta$.

Now,

$$x^4 + x^3 + 2x^2 + x + 1 = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta) \rightarrow (2)$$

put $x = -x$, we get

$$x^4 - x^3 + 2x^2 - x + 1 = (-x-\alpha)(-x-\beta)(-x-\gamma)(-x-\delta)$$

$$x^4 - x^3 + 2x^2 - x + 1 = [-(x+\alpha)][-(x+\beta)][-(x+\gamma)][-(x+\delta)]$$

(14)

$\Rightarrow x^4 - x^3 + 2x^2 - x + 1 = (x+\alpha)(x+\beta)(x+\gamma)(x+\delta)$ → (3)

Now,

$$(x^4+2x^2+1)^2 - (x^3+x)^2 = (x^2-\alpha^2)(x^2-\beta^2)(x^2-\gamma^2)(x^2-\delta^2)$$

$$((x^2)^2+2x^2+1)^2 - (x(x^2+1))^2 = (x^2-\alpha^2)(x^2-\beta^2)(x^2-\gamma^2)(x^2-\delta^2)$$ → (4)

put $x^2 = y$ in this equation, we get

$$(y^2+2y+1)^2 - y(y+1)^2 = (y-\alpha^2)(y-\beta^2)(y-\gamma^2)(y-\delta^2)$$

$$(y^2)^2 + (2y)^2 + 1^2 + 2y^2(2y) + 2(2y)(1) + 2(y^2)(1) - y(y^2+1+2y) = (y-\alpha^2)(y-\beta^2)(y-\gamma^2)(y-\delta^2)$$

$$\left.\begin{array}{l} y^4 + 4y^2 + 1 + 4y^3 + 4y + 2y^2 \\ -y^3 - y - 2y^2 \end{array}\right\} = (y-\alpha^2)(y-\beta^2)(y-\gamma^2)(y-\delta^2)$$

i.e., $y^4 + 3y^3 + 4y^2 + 3y + 1 = 0$

$\boxed{y^4 + 3y^3 + 4y^2 + 3y + 1 = 0}$ is the equation whose roots are $\alpha^2, \beta^2, \gamma^2$ & δ^2.

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If $\alpha, \beta, \gamma, \delta$ be the roots of the biquadratic equation $x^4 - px^3 + qx^2 - rx + s = 0$ form an equation whose roots shall be $\alpha^2, \beta^2, \gamma^2, \delta^2$. Hence find the value of $\Sigma\alpha^2$ and $\Sigma\alpha^2\beta^2\gamma^2$.

Soln:

$$x^4 - px^3 + qx^2 - rx + s = 0 \rightarrow (1)$$

Let the roots of this equation be $\alpha, \beta, \gamma, \delta$

Now, $x^4 - px^3 + qx^2 - rx + s = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta) \rightarrow (2)$

put $x = -x$ in eqn (2), we get

$\Rightarrow (-x)^4 - p(-x)^3 + q(-x)^2 - r(-x) + s = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$\Rightarrow x^4 + px^3 + qx^2 + rx + s = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$\Rightarrow x^4 + px^3 + qx^2 + rx + s = [-(x+\alpha)][-(x+\beta)][-(x+\gamma)][-(x+\delta)]$

$\Rightarrow x^4 + px^3 + qx^2 + rx + s = (x+\alpha)(x+\beta)(x+\gamma)(x+\delta)$

Now,

$$(x^4 + qx^2 + s)^2 + (px^3 + rx)^2 = (x^2-\alpha^2)(x^2-\beta^2)(x^2-\gamma^2)(x^2-\delta^2)$$

$$((x^2)^2 + qx^2 + s)^2 + (x(px^2+r))^2 = (x^2-\alpha^2)(x^2-\beta^2)(x^2-\gamma^2)(x^2-\delta^2)$$

$$((x^2)^2 + qx^2 + s)^2 + x^2(px^2+r)^2 = (x^2-\alpha^2)(x^2-\beta^2)(x^2-\gamma^2)(x^2-\delta^2)$$

put $x^2 = y$ in this equation, we get

$$(y^2 + qy + s)^2 + y(py+r)^2 = (y-\alpha^2)(y-\beta^2)(y-\gamma^2)(y-\delta^2)$$

$$\Rightarrow [y^4 + q^2y^2 + s^2 + 2y^2qy + 2qys + 2y^2s] + y[p^2y^2 + r^2 + 2pyr]$$
$$= (y-\alpha^2)(y-\beta^2)(y-\gamma^2)(y-\delta^2)$$

$$y^4+q^2y^2+s^2+2y^3q+2qys+2y^2s+p^2y^3+yr^2+2pry^2$$
$$=(y-\alpha^2)(y-\beta^2)(y-\gamma^2)(y-\delta^2)$$

$$y^4+(2q+p^2)y^3+(q^2+2s+2pr)y^2+(2qs+r^2)y+s^2$$
$$=(y-\alpha^2)(y-\beta^2)(y-\gamma^2)(y-\delta^2)$$

i.e., $\boxed{y^4+(2q+p^2)y^3+(q^2+2s+2pr)y^2+(2qs+r^2)y+s^2=0}$

is the equation whose roots are $\alpha^2, \beta^2, \gamma^2$ & δ^2.

$\Rightarrow a_0=1$, $a_1=2q+p^2$, $a_2=q^2+2s+2pr$ & $a_3=2qs+r^2$

Here, $s_1=\Sigma\alpha^2=\alpha^2+\beta^2+\gamma^2+\delta^2=\dfrac{-a_1}{a_0}$

$$\Rightarrow \Sigma\alpha^2=\frac{-(2q+p^2)}{1}$$

$$\therefore \boxed{\Sigma\alpha^2=-(2q+p^2)}$$

and $s_3=\Sigma\alpha^2\beta^2\gamma^2=\dfrac{-a_3}{a_0}$

$$\therefore \Sigma\alpha^2\beta^2\gamma^2=\frac{-(2qs+r^2)}{1}$$

$$\therefore \boxed{\Sigma\alpha^2\beta^2\gamma^2=-(2qs+r^2)}$$

3. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4+px^3+qx^2+rx+s=0$, prove that
$(\alpha^2+1)(\beta^2+1)(\gamma^2+1)(\delta^2+1)=(1-q+s)^2+(p-r)^2$.

(17)

Soln.:-

Given that the equation is

$$x^4+px^3+qx^2+rx+s=0$$

If the roots of this equation $\alpha, \beta, \gamma, \delta$.

$\therefore x^4+px^3+qx^2+rx+s=(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

put $x=-x$, we get

$\Rightarrow (-x)^4+p(-x)^3+q(-x)^2+r(-x)+s$

$$=(-x-\alpha)(-x-\beta)(-x-\gamma)(-x-\delta)$$

$\Rightarrow x^4-px^3+qx^2-rx+s=[-(x+\alpha)][-(x+\beta)][-(x+\gamma)][-(x+\delta)]$

$\Rightarrow x^4-px^3+qx^2-rx+s=(x+\alpha)(x+\beta)(x+\gamma)(x+\delta)$

$\Rightarrow$ Now, $(x^4+qx^2+s)^2-(px^3+rx)^2=(x^2-\alpha^2)(x^2-\beta^2)(x^2-\gamma^2)\cdot(x^2-\delta^2)$

$\Rightarrow (x^4+qx^2+s)^2-[x(px^2+r)]^2=(x^2-\alpha^2)(x^2-\beta^2)(x^2-\gamma^2)(x^2-\delta^2)$

$\Rightarrow ((x^2)^2+qx^2+s)^2-[x^2(px^2+r)^2]=(x-\alpha^2)(x^2-\beta^2)(x-\gamma^2)(x^2-\delta^2)$

put $x^2=-y$, we get

$\Rightarrow [(-y)^2+q(-y)+s]^2-[(-y)(py+r)^2]$

$$=(-y-\alpha^2)(-y-\beta^2)(-y-\gamma^2)(-y-\delta^2)$$

$\Rightarrow (y^2-qy+s)^2-[(-y)(-(py-r))^2]$

$$=[-(y+\alpha^2)][-(y+\beta^2)][-(y+\gamma^2)][-(y+\delta^2)]$$

$\Rightarrow (y^2-qy+s)^2+y(py-r)^2=(y+\alpha^2)(y+\beta^2)(y+\gamma^2)(y+\delta^2)$

put $y=1$, we get

$$\Rightarrow \boxed{(1-q+s)^2+(p-r)^2=(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2)}$$

Hence the proof

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Sec: 21 Transformation in General

Problems:-

1. If α, β, γ are the roots of the equation $x^3+px^2+qx+r=0$ form the equation whose roots are $\alpha - \frac{1}{\beta\gamma}$, $\beta - \frac{1}{\gamma\alpha}$ and $\gamma - \frac{1}{\alpha\beta}$.

Soln.:-

Given that the eqn. is

$$x^3+px^2+qx+r=0 \rightarrow (1)$$

Given that the roots of this eqn. α, β, γ.

Here, $a_0=1$, $a_1=p$, $a_2=q$, $a_3=r$

Now, $S_1 = \alpha+\beta+\gamma = \frac{-a_1}{a_0} = \frac{-p}{1} = -p$

$$S_2 = \alpha\beta+\beta\gamma+\gamma\alpha = \frac{a_2}{a_0} = \frac{q}{1} = q$$

$$S_3 = \alpha\beta\gamma = \frac{-a_3}{a_0} = \frac{-r}{1} = -r$$

Consider, $\alpha - \frac{1}{\beta\gamma}$, $\beta - \frac{1}{\gamma\alpha}$, $\gamma - \frac{1}{\alpha\beta}$

Now, $\alpha - \frac{1}{\beta\gamma} = \alpha - \frac{\alpha}{\alpha\beta\gamma}$

$$= \alpha - \frac{\alpha}{-r}$$

$x = \alpha$

$$= \alpha + \frac{\alpha}{r}$$

Let $y = x + \frac{x}{r}$

$$\Rightarrow y = \frac{xr + x}{r}$$

$$\Rightarrow ry = xr + x$$

$$\Rightarrow ry = x(1+r)$$

$$\Rightarrow x = \frac{yr}{1+r} \rightarrow (2)$$

Sub. eqn. (2) in eqn. (1), we get

$$\Rightarrow \left(\frac{yr}{1+r}\right)^3 + p\left(\frac{yr}{1+r}\right)^2 + q\left(\frac{yr}{1+r}\right) + r = 0$$

$$\Rightarrow \frac{(yr)^3}{(1+r)^3} + \frac{p(yr)^2}{(1+r)^2} + q\frac{(yr)}{1+r} + r = 0$$

$$\Rightarrow \frac{1}{(1+r)^3}\left[(yr)^3 + p(yr)^2(1+r) + q(yr)(1+r)^2 + r(1+r)^3\right] = 0$$

$$\cancel{\Rightarrow \beta / \; y^3 r^3 + p y^2 r^2 + p y^2 r^3 + q r y (1 + r^2 + 2r)}$$

$$\Rightarrow y^3 r^3 + (1+r) p r^2 y^2 + q r y (1+r)^2 + r(1+r)^3 = 0$$

dividing by 'r' in this equation, we get

$$\Rightarrow y^3 r^2 + (1+r) p y^2 r + (1+r)^2 q y + (1+r)^3 = 0$$

i.e., $\boxed{y^3 r^2 + (1+r) p y^2 r + (1+r)^2 q y + (1+r)^3 = 0}$ is the equation whose roots are $\alpha - \frac{1}{\beta\gamma}$, $\beta - \frac{1}{\gamma\alpha}$, $\gamma - \frac{1}{\alpha\beta}$.

—— x ——

Q. If a, b, c be the roots of the equation

$x^3+px^2+qx+r=0$,

find the equation whose roots are $bc-a^2$, $ca-b^2$, $ab-c^2$.

Sol:- Given that $x^3+px^2+qx+r=0 \rightarrow (1)$

Given that a, b, c are the roots of the equation.

Here, $a_0=1$, $a_1=p$, $a_2=q$, $a_3=r$

$\therefore S_1 = a+b+c = \frac{-a_1}{a_0} = \frac{-p}{1} = -p$

$S_2 = ab+bc+ca = \frac{a_2}{a_0} = \frac{q}{1} = q$

$S_3 = abc = \frac{-a_3}{a_0} = \frac{-r}{1} = -r$

Consider, $bc-a^2$, $ca-b^2$, $ab-c^2$.

Now, $bc-a^2 = \frac{abc}{a} - a^2$

$= \frac{-r}{a} - a^2$

Let $y = \frac{-r}{x} - x^2$

$\Rightarrow xy = -r-x^3$

$\Rightarrow x^3+xy+r=0 \rightarrow (2)$

(1) - (2) we get

$\Rightarrow (x^3+px^2+qx+r)-(x^3+xy+r)=0$

$\Rightarrow x^3+px^2+qx+r-x^3-xy-r=0$

$\Rightarrow px^2+x(q-y)=0$

$$\Rightarrow x(px+q-y)=0$$

ie., $x=0$ (or) $px+q-y=0$

But $x \neq 0$ $\therefore$ $px = -q+y$

$$\cancel{\Rightarrow x = \frac{-(q-y)}{p}}$$

$$\Rightarrow \boxed{x = \frac{y-q}{p}} \longrightarrow (3)$$

Sub. (3) in eqn. (1), we get

$$\Rightarrow \left(\frac{y-q}{p}\right)^3 + p\left(\frac{y-q}{p}\right)^2 + q\left(\frac{y-q}{p}\right) + r = 0$$

$$\Rightarrow \frac{(y-q)^3}{p^3} + \frac{p(y-q)^2}{p^2} + \frac{q(y-q)}{p} + r = 0$$

$$\Rightarrow \frac{1}{p^3}\left[(y-q)^3 + p(y-q)^2 p + q(y-q)p^2 + p^3 r\right] = 0$$

$$\Rightarrow \boxed{(y-q)^3 + p^2(y-q)^2 + p^2 q(y-q) + p^3 r = 0}$$

This is the required equation whose root are $bc-a^2$, $ca-b^2$ & $ab-c^2$.

3. If α, β, γ be the roots of the equation

pvs $x^3 - 6x + 7 = 0$, from an equation whose roots are $\alpha^2+2\alpha+3$, $\beta^2+2\beta+3$, $\gamma^2+2\gamma+3$.

Soln.:- Given that $x^3-6x+7=0 \rightarrow (1)$

Given that α, β, γ are roots of the given equation.

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(22)

Given that the roots $\alpha^2+2\alpha+3$, $\beta^2+2\beta+3$ & $\gamma^2+2\gamma+3$

Consider, $\alpha^2+2\alpha+3$

Let $y = x^2+2x+3$

$\Rightarrow x^2+2x+(3-y)=0 \rightarrow (2)$

$(2)\times x \Rightarrow x^3+2x^2+x(3-y)=0 \rightarrow (3)$

$(3)-(1) \Rightarrow [x^3+2x^2+x(3-y)]-(x^3-6x+7)=0$

$\Rightarrow x^3+2x^2+x(3-y)-x^3+6x-7=0$

$\Rightarrow 2x^2+3x-xy+6x-7=0$

$\Rightarrow 2x^2+9x-xy-7=0$

$\Rightarrow 2x^2+(9-y)x-7=0 \rightarrow (4)$

Using cross multiplication rules by (2) & (4), we get

$$\begin{array}{cccc} & x^2 & x & 1 \\ 1 & 2 & 3-y & 1 \\ 2 & 9-y & -7 & 2 \end{array}$$

$\therefore$ The roots are

$$\frac{x^2}{2(-7)-(3-y)(9-y)} = \frac{x}{2(3-y)-(1)(-7)} = \frac{1}{(1)(9-y)-(2)(2)}$$

$$\Rightarrow \frac{x^2}{-14-(9-y)(3-y)} = \frac{x}{2(3-y)+7} = \frac{1}{9-y-4}$$

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$$\therefore x^2 = -14-(9-y)(3-y)$$
$$= -14-(27-9y-3y+y^2)$$
$$= -14-27+9y+3y-y^2$$
$$= -y^2+12y-41$$

$$x = 2(3-y)+7$$
$$= 6-2y+7$$
$$= 13-2y$$

$$1 = 9-y-4$$
$$= 5-y$$

$$\therefore \frac{a^2}{x^2} = \frac{-y^2+12y-41}{(13-2y)^2} = \frac{1}{5-y}$$

$$\Rightarrow (5-y)(-y^2+12y-41) = (13-2y)^2$$

$$\Rightarrow -5y^2+60y-205+y^3-12y^2+41y = 169+4y^2-52y$$

$$\Rightarrow -5y^2+60y-205+y^3-12y^2+41y-169-4y^2+52y = 0$$

$$\Rightarrow y^3+(-5-12-4)y^2+(60+41+52)y-205-169 = 0$$

$$\Rightarrow \boxed{y^3-21y^2+153y-374=0}$$

This is the required equation whose roots are

$\alpha^2+2\alpha+3$, $\beta^2+2\beta+3$, $\gamma^2+2\gamma+3$.

AIS

4. If α is a root of (24)

$$x^2(x+1)^2 - k(x-1)(2x^2+x+1) = 0$$

Prove that $\frac{\alpha+1}{\alpha-1}$ is also a root.

Proof:- Given that $x^2(x+1)^2 - k(x-1)(2x^2+x+1) = 0$ $\rightarrow$ (1)

Given that the root is α.

Let the remaining roots be β, γ and δ.

Given that the root be $\frac{\alpha+1}{\alpha-1}$

$\therefore$ Let $y = \frac{x+1}{x-1}$

$$y(x-1) = x+1$$

$$xy - y = x+1$$

$$xy - x = y+1$$

$$x(y-1) = y+1$$

$$\boxed{x = \frac{y+1}{y-1}} \rightarrow (2)$$

Sub. (2) in eqn. (1), we get

$$\left(\frac{y+1}{y-1}\right)^2 \left(\frac{y+1}{y-1}+1\right)^2 - k\left(\frac{y+1}{y-1}-1\right)\left(2\left(\frac{y+1}{y-1}\right)^2 + \frac{y+1}{y-1}+1\right) = 0$$

$$\frac{(y+1)^2}{(y-1)^2}\left(\frac{y+1+y-1}{y-1}\right)^2 - k\left(\frac{y+1-y+1}{y-1}\right)\left(\frac{2(y+1)^2}{(y-1)^2} + \frac{y+1+y-1}{y-1}\right) = 0$$

$$\frac{(y+1)^2}{(y-1)^2}\left(\frac{2y}{y-1}\right)^2 - k\,\frac{2}{(y-1)}\left(\frac{2(y+1)^2}{(y-1)^2} + \frac{2y}{(y-1)}\right) = 0$$

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Section: 2

$$\frac{(y+1)^2}{(y-1)^2}\frac{(2y)^2}{(y-1)^2}-k\frac{2}{(y-1)}\left(\frac{2(y+1)^2+2y(y-1)}{(y-1)^2}\right)=0$$

$$\frac{4y^2(y+1)^2}{(y-1)^4}-\frac{2k}{(y-1)^3}\left(2(y+1)^2+2y(y-1)\right)=0$$

$$\frac{4y^2(y+1)^2}{(y-1)^4}-\frac{2k(y-1)}{(y-1)^4}\left(2(y+1)^2+2y(y-1)\right)=0$$

$$\frac{4y^2(y+1)^2}{(y-1)^4}-\frac{2k(y-1)}{(y-1)^4}\left(2(y^2+2y+1)+2y^2-2y\right)=0$$

$$\frac{1}{(y-1)^4}\left[4y^2(y+1)^2-2k(y-1)\left(2y^2+4y+2+2y^2-2y\right)\right]=0$$

$$\Rightarrow 4y^2(y+1)^2-2k(y-1)\left(4y^2+2y+2\right)=0$$

$$\Rightarrow 4y^2(y+1)^2-4k(y-1)(2y^2+y+1)=0$$

$$\Rightarrow 4\left(y^2(y+1)^2-k(y-1)(2y^2+y+1)\right)=0$$

$$\Rightarrow \boxed{y^2(y+1)^2-k(y-1)(2y^2+y+1)=0}$$

We get the same equation as the original equation.

$\therefore \frac{\alpha+1}{\alpha-1}$ is a root of

$$x^2(x+1)^2-k(x-1)(2x^2+x+1)=0$$

Hence the proof.

(25)

Section: 24 Descartes' Rule of signs (26)

An equation $f(x)=0$ has 'n' roots, then the following ideas should be started by showing the roots are +ve, -ve and imaginary.

(i) $f(x)=0$, then the number of changing sign is called the +ve roots of the equation.

(ii) put $x=-x$ in $f(x)$.

i.e., $f(-x)=0$, then the number of changing the signs is called the number of -ve roots of the equation.

(iii) The remaining roots of $f(x)$ is an imaginary roots.

1. Show that the equation $x^7-3x^4+3x^2-1=0$ has atleast 2 imaginary roots.

Soln:- Given that the equation is

$$f(x)=x^7-3x^4+3x^2-1=0 \longrightarrow (1)$$

Consider $+ \; - \; + \; -$

The equation has 3 +ve roots.

put $x=-x$ in eqn. (1), we get

$\Rightarrow f(-x) = (-x)^7 - 3(-x)^4 + 3(-x)^3 - 1 = 0$

$= -x^7 - 3x^4 + 3x^3 - 1 = 0$

Now,

$- \quad - \quad + \quad -$

The equation has 2 -ve roots

$\therefore$ The remaing 2 roots are imaginary.

2. Find the number of real roots of

$$x^7 - x^5 - x^4 - 6x^2 + 7 = 0.$$

Soln:- Given that the equation is

$f(x) = x^7 - x^5 - x^4 - 6x^2 + 7 = 0 \rightarrow (1)$

Consider the signs, $+ \quad - \quad - \quad - \quad +$

The equation has 2 +ve roots.

put $x = -x$ in eq. (1), we get

$f(-x) = (-x)^7 - (-x)^5 - (-x)^4 - 6(-x)^2 + 7 = 0$

$\Rightarrow -x^7 + x^5 - x^4 - 6x^2 + 7 = 0$

Now, $- \quad + \quad - \quad - \quad +$

$\therefore$ The equation has 3 -ve roots.

i.e., The equation has 5 Real roots and the remaining 2 roots are imaginary.

3. Determind completely the nature of the roots of the equation $x^5-6x^2-4x+5=0$.

Soln:-

Given that the equation is

$$f(x)=x^5-6x^2-4x+5=0 \rightarrow ①$$

Consider the signs, $+ \quad - \quad - \quad +$

∴ The equation has 2 +ve roots.

put $x=-x$ in eqn. ①, we get

$$f(-x)=(-x)^5-6(-x)^2-4(-x)+5=0$$

$$\Rightarrow -x^5-6x^2+4x+5=0$$

Now, $- \quad - \quad + \quad +$

∴ The equation has one -ve root.

i.e., The remaing 2 roots are imaginary.

4. Show that the equation $x^7-3x^4+3x^3-1=0$ atleast 4 imaginary root.

Soln. :-

Given that the equation is

$$f(x)=x^7-3x^4+3x^2-1=0 \rightarrow ①$$

Consider the signs,

$+ \quad - \quad + \quad -$

∴ The equation has three +ve roots.

put $x=-x$ in eqn. ①, we get

$\Rightarrow f(-x) = (-x)^7 - 3(-x)^4 + 3(-x)^3 - 1 = 0$

$\Rightarrow -x^7 - 3x^4 - 3x^3 - 1 = 0$

Now,

$- \quad - \quad - \quad -$

$\therefore$ The equation has no -ve roots

i.e., The remaing 4 roots are imaginary.

5. Show that $x^9 + x^8 + x^4 + x^2 + 1 = 0$ has 1 real root which is -ve and 8 imaginary roots.

Soln:-

Given that the equation is

$$f(x) = x^9 + x^8 + x^4 + x^2 + 1 = 0 \longrightarrow ①$$

$\therefore$ Consider the signs,

$+ \quad + \quad + \quad + \quad +$

$\therefore$ The equation has no +ve roots

put $x=-x$ in eqn. ①, we get

$\Rightarrow f(-x) = (-x)^9 + (-x)^8 + (-x)^4 + (-x)^2 + 1 = 0$

$\Rightarrow -x^9 + x^8 + x^4 + x^2 + 1 = 0$

Now,

$- \quad + \quad + \quad + \quad +$

$\therefore$ The equation has 1 -ve root.

i.e., The equation of remaing 8 roots are imaginary.

—x—

Home Work:

1. Find the nature of the roots of these equations,

(i) $x^4 + 15x^3 + 7x - 1 = 0$

$+ \; + \; - \; -$

(ii) $x^5 + 5x - 7 = 0$.

2. Show that $12x^7 - x^4 + 10x^3 - 28 = 0$ has atleast four imaginary roots.

3. Find the number of real roots of the equation

$x^3 + 18x - 6 = 0$.

———X———

Unit-III is over

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Classical Algebra And Theory Of Numbers

Unit - IV (1)

Chapter : 4 Inequalities

Elementary principles :-

The following elementary principles of inequality can be easily be proved :-

i) If $a>b$, then $a+x > b+x$ and $a-x > b-x$ for any x.

ii) If $a>b$, then $-a < -b$

iii) If $a>b$, then $ma > mb$ and $-ma < -mb$. (m is +ve)

iv) If $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$, $\ldots$, $a_n > b_n$, and then $a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_n$.

$a_1 a_2 \cdots a_n > b_1 b_2 \cdots b_n$.

v) If $a>b$, then $a^m > b^m$ and $a^{-m} < b^{-m}$. (m is +ve)

Book Work:

1. If $a>b$, then $1 < \frac{a+x}{b+x} < \frac{a}{b}$ ~~and if $a<b$~~ and if $a<b$ then $a/b < \frac{a+x}{b+x} < 1$, where a, b, c are +ve numbers.

Proof :-

(i) if $a>b$

Consider, $\frac{a+x}{b+x} - a/b = \frac{b(a+x) - a(b+x)}{b(b+x)}$

$$= \frac{ab+bx-ab-ax}{b(b+x)}$$

$$= \frac{bx-ax}{b(b+x)}$$

$\Rightarrow \frac{a+x}{b+x} - a/b = \frac{x(b-a)}{b(b+x)} < 0$

$\therefore \frac{a+x}{b+x} - a/b < 0$

$\frac{a+x}{b+x} < a/b \longrightarrow 1$

Consider, $1 - \frac{a+x}{b+x} = \frac{b+x-a-x}{b+x}$

$$= \frac{b+a}{b+x} < 0$$

$\Rightarrow 1 - \frac{a+x}{b+x} < 0 \Rightarrow 1 < \frac{a+x}{b+x} \longrightarrow 2$

From 1 & 2, we get

$$\boxed{1 < \frac{a+x}{b+x} < \frac{a}{b}, \text{ if } a > b.}$$

(ii) If $a < b$

Consider, $\frac{a}{b} - \frac{a+x}{b+x} = \frac{a(b+x) - b(a+x)}{b(b+x)}$

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②

③

$$\frac{a}{b} - \frac{a+x}{b+x} = \frac{ab+ax-ab-bx}{b(b+x)}$$

$$\Rightarrow \frac{a}{b} - \frac{a+x}{b+x} = \frac{x(a-b)}{b(b+x)} < 0$$

$$\Rightarrow \frac{a}{b} - \frac{a+x}{b+x} < 0$$

$$\Rightarrow \frac{a}{b} < \frac{a+x}{b+x} \longrightarrow ③$$

Consider, $\frac{a+x}{b+x} - 1 = \frac{a+x-b-x}{b+x}$

$$= \frac{a-b}{b+x} < 0$$

$$\Rightarrow \frac{a+x}{b+x} - 1 < 0$$

$$\Rightarrow \frac{a+x}{b+x} < 1 \longrightarrow ④$$

From ③ & ④, we get

$$\boxed{\frac{a}{b} < \frac{a+x}{b+x} < 1, \text{ if } a<b}$$

Hence the proof.

2. Show that $\frac{a_1+a_2+a_3+\cdots+a_n+a_{n+1}}{n+1} \gtrless \frac{a_1+a_2+\cdots+a_n}{n}$, if $a_{n+1} \gtrless \frac{a_1+a_2+\cdots+a_n}{n}$

and $\frac{a_1+a_2+\cdots+a_n+a_{n+1}}{n+1} < \frac{a_1+a_2+\cdots+a_n}{n}$, if $a_{n+1} < \frac{a_1+a_2+\cdots+a_n}{n}$.

proof:- Now, $a_{n+1} > \frac{a_1+a_2+\cdots+a_n}{n}$

$$\Rightarrow n\,a_{n+1} > a_1+a_2+\cdots+a_n$$

add a_{n+1} on both sides, we get

$$n\,a_{n+1}+a_{n+1} > a_1+a_2+\cdots+a_n+a_{n+1}$$

Divide '$n+1$' on both sides, we get

$$\Rightarrow \frac{n\,a_{n+1}+a_{n+1}}{n+1} > \frac{a_1+a_2+\cdots+a_n+a_{n+1}}{n+1}$$

$$\Rightarrow \frac{a_{n+1}\,\cancel{(n+1)}}{\cancel{n+1}} > \frac{a_1+a_2+\cdots+a_n+a_{n+1}}{n+1}$$

$$\Rightarrow \frac{a_1+a_2+\cdots+a_n+a_{n+1}}{n+1} < a_{n+1} > \frac{a_1+a_2+\cdots+a_n}{n}$$

$$\Rightarrow \boxed{\frac{a_1+a_2+\cdots+a_n+a_{n+1}}{n+1} > \frac{a_1+a_2+\cdots+a_n}{n}}$$

Now, $a_{n+1} < \frac{a_1 + a_2 + \cdots + a_n}{n}$ ⑤

$$n\, a_{n+1} < a_1 + a_2 + \cdots + a_n$$

add a_{n+1} on both sides, we get

$$n\, a_{n+1} + a_{n+1} < a_1 + a_2 + \cdots + a_n + a_{n+1}$$

devide 'n+1' on both sides, we get

$$\Rightarrow \frac{n\, a_{n+1} + a_{n+1}}{n+1} < \frac{a_1 + a_2 + \cdots + a_n + a_{n+1}}{n+1}$$

$$\Rightarrow \frac{a_{n+1}\cancel{(n+1)}}{\cancel{n+1}} < \frac{a_1 + a_2 + \cdots + a_n + a_{n+1}}{n+1}$$

$$\Rightarrow \frac{a_1 + a_2 + \cdots + a_n + a_{n+1}}{n+1} > a_{n+1} < \frac{a_1 + a_2 + \cdots + a_n}{n}$$

$$\Rightarrow \boxed{\frac{a_1 + a_2 + \cdots + a_n + a_{n+1}}{n+1} < \frac{a_1 + a_2 + \cdots + a_n}{n}}$$

Hence the proof.

3. Prove that $\frac{1}{2\sqrt{n+1}} < \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$

Proof:-

WKT., $\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{5}{6} < \cdots < \frac{2n-1}{2n} < \frac{2n}{2n+1}$

Let, $U_n = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} \to (1)$

and $U_n < \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n+1} \to (2)$

(1) $\times$ (2) $\Rightarrow$

$$U_n^2 < \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n+1}$$

$$\Rightarrow U_n^2 < \frac{1}{2n+1}$$

$$\Rightarrow U_n < \sqrt{\frac{1}{2n+1}}$$

$$\Rightarrow U_n < \frac{1}{\sqrt{2n+1}} \to (3)$$

Now, $U_n = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}$

Multiply $(2n+1)$ on both sides, we get

$$\Rightarrow (2n+1)\,U_n = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\,(2n+1)$$

$$\Rightarrow (2n+1)\,U_n = 1\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\frac{2n-1}{2n-2}\;\frac{2n+1}{2n} \to (4)$$

Now,

$$(2n+1)\,U_n > \frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdots\frac{2n}{2n-1}\;\frac{2n+2}{2n+1} \to (5)$$

(4) $\times$ (5) $\Rightarrow$

$$U_n^2\cdot(2n+1)^2 > \frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\frac{2n-1}{2n-2}\;\frac{2n+1}{2n}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdots\frac{2n}{2n-1}\;\frac{2n+2}{2n+1}$$

$$\Rightarrow (2n+1)^2 u_n^2 > \frac{2n+2}{2}$$

$$\Rightarrow (2n+1)^2 u_n^2 > \frac{2(n+1)}{2}$$

$$\Rightarrow (2n+1)^2 u_n^2 > n+1$$

$$\Rightarrow u_n^2 > \frac{n+1}{(2n+1)^2}$$

$$\Rightarrow u_n > \sqrt{\frac{n+1}{(2n+1)^2}}$$

$$u_n > \frac{\sqrt{n+1}}{\sqrt{(2n+1)^2}}$$

$$\therefore u_n > \frac{\sqrt{n+1}}{2n+1}$$

$$\Rightarrow u_n > \frac{\sqrt{n+1}}{2n+2}$$

$$u_n > \frac{\sqrt{n+1}}{2(n+1)}$$

$$u_n > \frac{\sqrt{n+1}}{2(\sqrt{n+1})^2}$$

$$\Rightarrow u_n > \frac{1}{2\sqrt{n+1}}$$

$$\Rightarrow \frac{1}{2\sqrt{n+1}} < u_n \longrightarrow (6)$$

From (3) & (6) $\Rightarrow \dfrac{1}{2\sqrt{n+1}} < u_n < \dfrac{1}{\sqrt{2n+1}}$

i.e., $\boxed{\dfrac{1}{2\sqrt{n+1}} < \dfrac{1}{2} \cdot \dfrac{3}{4} \cdot \dfrac{5}{6} \cdots \dfrac{2n-1}{2n} < \dfrac{1}{\sqrt{2n+1}}}$.

—x—

4 If a, b, c are positive and not all equal, then, $(a+b+c)(bc+ca+ab) > 9abc$.

Soln:- Given that a, b, c are +ve and $a \neq b \neq c$. $\Rightarrow a-b \neq 0$, $b-c \neq 0$ & $c-a \neq 0$

$a \neq b$ $b \neq c$ $c \neq a$
$a-b \neq 0$ $b-c \neq 0$ $c-a \neq 0$

Consider,

$$(a+b+c)(bc+ca+ab) - 9abc$$

$$= abc + b^2c + bc^2 + a^2c + abc + ac^2 + a^2b + ab^2 + abc - 9abc$$

$$= ac^2 + ab^2 + bc^2 + ba^2 + cb^2 + ca^2 - 6abc$$

$$= (ac^2 + ab^2 - 2abc) + (bc^2 + ba^2 - 2abc) + (cb^2 + ca^2 - 2abc)$$

$$= a(c^2 + b^2 - 2bc) + b(c^2 + a^2 - 2ca) + c(b^2 + a^2 - 2ab)$$

$$= a(b-c)^2 + b(c-a)^2 + c(b-a)^2$$

$$> 0.$$

$\therefore (a+b+c)(bc+ca+ab) - 9abc > 0$

$\Rightarrow \boxed{(a+b+c)(bc+ca+ab > 9abc.}$

Hence the proof.

—x—

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5. Show that

$(b+c-a)^2 + (c+a-b)^2 + (a+b-c)^2 \geq bc + ca + ab.$

Proof:-

$$
\begin{aligned}
\text{L.H.S.} &= (b+c-a)^2 + (c+a-b)^2 + (a+b-c)^2 \\
&= [b+c+(-a)]^2 + [c+a+(-b)]^2 + [a+b+(-c)]^2 \\
&= b^2+c^2+a^2+2bc+2c(-a)+2(-a)b \\
&\quad +c^2+a^2+b^2+2ca+2a(-b)+2(-b)c \\
&\quad +a^2+b^2+c^2+2ab+2b(-c)+2a(-c) \\
&= b^2+c^2+a^2+2bc-2ac-2ab \\
&\quad +c^2+a^2+b^2+2ac-2ab-2bc \\
&\quad +a^2+b^2+c^2+2ab-2bc-2ac \\
&= (a^2+b^2-2ab)+(b^2+c^2-2bc)+(a^2+c^2-2ac) \\
&\quad +a^2+b^2+c^2 \\
&= (a-b)^2+(b-c)^2+(a-c)^2+a^2+b^2+c^2 \\
&\geq a^2+b^2+c^2 \\
&\geq a^2+b^2+c^2-ab-bc-ca+ab+bc+ca \\
&\geq \frac{2}{2}[a^2+b^2+c^2-ab-bc-ca]+ab+bc+ca \\
&\geq \frac{1}{2}[2a^2+2b^2+2c^2-2ab-2bc-2ca] \\
&\quad +ab+bc+ca \\
&\geq \frac{1}{2}[a^2+b^2+c^2-2ab-2bc-2ca+a^2+b^2+c^2] \\
&\quad +ab+bc+ca
\end{aligned}
$$

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$$\geq \frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right] + ab+bc+ca$$

$$\geq ab+bc+ca.$$

i.e., $\boxed{(b+c-a)^2 + (c+a-b)^2 + (a+b+c)^2 \geq ab+bc+ca}$

Hence the proof.

—×—

6) If x, y, z be real and not all equal, show that $x^3+y^3+z^3 \gtrless 3xyz$ according to $x+y+z \gtrless 0$.

{Hint.: $x^3+y^3+z^3-3xyz = \frac{1}{2}(x+y+z)\left[(x-y)^2+(y-z)^2+(z-x)^2\right]$}

Proof:- Given that x, y, z are real, and $x \neq y \neq z \Rightarrow x-y \neq 0$, $y-z \neq 0$ and $z-x \neq 0$.

Given that $x+y+z > 0$

W.K.T. $x^3+y^3+z^3-3xyz = \frac{1}{2}(x+y+z)\left[(x-y)^2+(y-z)^2+(z-x)^2\right]$

$$> 0$$

$\Rightarrow x^3+y^3+z^3 > 3xyz$, if $x+y+z > 0$

llly W.K.T. $x^3+y^3+z^3-3xyz = \frac{1}{2}(x+y+z)\left[(x-y)^2+(y-z)^2+(z-x)^2\right]$

$$< 0$$

i.e., $x^3+y^3+z^3 < 3xyz$, if $x+y+z < 0$

Hence the proof.

—×—

Many inequalities depend on special arrangements of terms or factors. The following examples will illustrate the method:-

1. Prove that if $n > 2$, $(n!)^2 > n^n$.

Proof:- Given that $n > 2$.

W.K.T.
$$1 \cdot n = n$$
$$2 \cdot (n-1) > n$$
$$3 \cdot (n-2) > n$$
$$4 \cdot (n-3) > n$$
$$\vdots$$
$$(n-2) \cdot 3 > n$$
$$(n-1) \cdot 2 > n$$
$$n \cdot 1 = n$$

~~Now,~~ $\cancel{(n!)^2 = (n!)(n!)}$

$$\cancel{(n!)^2 = [n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1][n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1]}$$

Now, $1 \cdot n \cdot 2(n-1) \cdot 3 \cdot (n-2) \cdot 4 \cdot (n-3) \cdots (n-1) \cdot 2 \cdot n \cdot 1 > n^n$

ie., $n!\, n! > n^n$

ie., $\boxed{(n!)^2 > n^n}$

Hence the proof.

—x—

2. If $a_1, a_2, \dots, a_n$ be an arithmetical progression, show that $a_1^2\, a_2^2 \dots a_n^2 > a_1^n\, a_n^n$. Deduce that if $n > 2$, $(n!)^2 > n^n$.

Proof:-

Given $a_1, a_2, \dots a_r \dots a_n$ are arithmetical progression.

W.K.T.

$$a_r = a_1 + (r-1)d \rightarrow (1)$$

$$a_{n-r+1} = a_1 + (n-r)d \rightarrow (2)$$

(1) × (2) we get

$$\Rightarrow a_r\, a_{n-r+1} = [a_1 + (r-1)d][a_1 + (n-r)d]$$

$$= a_1^2 + (r-1)a_1 d + (n-r)a_1 d + (r-1)(n-r)d^2$$

$$= a_1^2 + a_1 d[r-1+n-r] + (r-1)(n-r)d^2$$

$$= \underline{a_1^2 + (n-1)a_1 d} + (r-1)(n-r)d^2$$

$$a_r\, a_{n-r+1} > a_1^2 + (n-1)a_1 d$$

$$\Rightarrow a_r\, a_{n-r+1} > a_1(a_1 + (n-1)d)$$

$$\Rightarrow a_r\, a_{n-r+1} > a_1\, a_n$$

put $r=1 \Rightarrow a_1\, a_n = a_1\, a_n$

$r=2 \Rightarrow a_2\, a_{n-1} > a_1\, a_n$

$r=3 \Rightarrow a_3\, a_{n-2} > a_1\, a_n$

$\vdots$

$r=n \Rightarrow a_n\, a_1 = a_1\, a_n$

Multiplying all terms, we get (13)

$(a_1 a_n)(a_2 a_{n-1})(a_3 a_{n-2}) \cdots (a_n a_1)$

$> (a_1 a_n)(a_1 a_n)(a_1 a_n) \cdots (a_1 a_n)$

$\Rightarrow \boxed{a_1^2 a_2^2 a_3^2 \cdots a_{n-2}^2 a_{n-1}^2 a_n^2 > a_1^n a_n^n} \rightarrow (3)$

put $a_1 = 1, a_2 = 2, a_3 = 3, \ldots, a_n = n$ in this eqn., we get

$1^2 . 2^2 . 3^2 \cdots n^2 > 1^n n^n$

$\Rightarrow (1.2.3. \cdots n)^2 > 1.n^n$

$\Rightarrow$ ie., $\boxed{(n!)^2 > n^n}$

Hence the proof.

3\. Prove that $1! . 3! . 5! \ldots (2n-1)! > (n!)^n$.

proof:-

WKT., $n-r > r+1$

$n > 2r+1$

$(n-r)! . r! > (n-(r+1))! \; (r+1)!$

put $r=1 \Rightarrow (n-1)! \; 1! > (n-2)! \; 2!$

$r=2 \Rightarrow (n-2)! \; 2! > (n-3)! \; 3!$

$r=3 \Rightarrow (n-3)! \; 3! > (n-4)! \; 4!$

$\vdots \qquad \vdots \qquad \vdots$

$(n-1)! \; 1! > (n-2)! \; 2! > (n-3)! \; 3! > \cdots$

put $n = 2n$ and $r = 0$ in eqn. (1), we get

$(2n-1)! \; 1! > (2n-2)! \; 2! > (2n-3)! \; 3! > \cdots > (2n-n)! \; n!$

$\Rightarrow (2n-1)!\ 1! > (2n-2)!\ 2! > (2n-3)!\ 3!$

$> (2n-4)!\ 4! > \cdots > n!\ n!$

Now, $(2n-1)!\ 1! > n!\ n!$

$(2n-3)!\ 3! > n!\ n!$

$(2n-5)!\ 5! > n!\ n!$

$(2n-7)!\ 7! > n!\ n!$

$\vdots$

$1!\ (2n-1)! > n!\ n!$

Multiplying the all terms, we get

$(2n-1)!\ 1!\ (2n-3)!\ 3!\ (2n-5)!\ 5!\ (2n-7)!\ 7!$

$\cdots 1!\ (2n-1)! > (n!\ n!)(n!\ n!) \cdots (n!\ n!)$

$(1!)^2\ (3!)^2\ (5!)^2 \cdots ((2n-1)!)^2 > (n!)^2 (n!)^2 \cdots (n!)^2$

$(1!\ 3!\ 5! \cdots (2n-1)!)^2 > [(n!)^2]^n$ $(x^2)^3$

$(1!\ 3!\ 5! \cdots (2n-1)!)^2 > [(n!)^n]^2$ $(x^3)^2$

$\Rightarrow$ $\boxed{1!\ 3!\ 5! \cdots (2n-1)! > (n!)^n}$

Hence the proof.

—x—

4. Show that $(x^m+y^m)^n < (x^n+y^n)^m$ if $m>n$.

Proof:- Given that $m>n$

Let $x > y \Rightarrow y/x < 1$

Let $(x^m+y^m)^n - (x^n+y^n)^m = \left[x^m\left(1+\frac{y^m}{x^m}\right)\right]^n - \left[x^n\left(1+\frac{y^n}{x^n}\right)\right]^m$

(14)

(15)

$$(x^m+y^m)^n - (x^n+y^n)^m$$

$$= x^{mn}\left(1+\frac{y^m}{x^m}\right)^n - x^{mn}\left(1+\frac{y^n}{x^n}\right)^m$$

$$= x^{mn}\left[\left(1+\frac{y^m}{x^m}\right)^n - \left(1+\frac{y^n}{x^n}\right)^m\right]$$

$(x+a)^n = x^n + nC_1 x^{n-1} a + nC_2 x^{n-2} a^2 + \cdots$

$nC_n 1^{n-n}$

$$= x^{mn}\left[1^n + nC_1 1^{n-1}\left(\frac{y^m}{x^m}\right) + nC_2 1^{n-2}\left(\frac{y^m}{x^m}\right)^2 + \cdots + \left(\frac{y^m}{x^m}\right)^n\right]$$
$$- \left[1^m + mC_1 1^{m-1}\left(\frac{y^n}{x^n}\right) + mC_2 1^{m-2}\left(\frac{y^n}{x^n}\right)^2 + \cdots + \left(\frac{y^n}{x^n}\right)^m\right]$$

$$= x^{mn}\left[1 + n\left(\frac{y}{x}\right)^m + \frac{n(n-1)}{2!}\left(\frac{y}{x}\right)^{2m} + \cdots + \left(\frac{y}{x}\right)^{mn}\right]$$
$$- \left[1 + m\left(\frac{y}{x}\right)^n + \frac{m(m-1)}{2!}\left(\frac{y}{x}\right)^{2n} + \cdots + \left(\frac{y}{x}\right)^{mn}\right]$$

$$= x^{mn}\left[1 + n\left(\frac{y}{x}\right)^m + \frac{n(n-1)}{2}\left(\frac{y}{x}\right)^{2m} + \cdots + \left(\frac{y}{x}\right)^{mn}\right.$$
$$\left. - 1 - m\left(\frac{y}{x}\right)^n - \frac{m(m-1)}{2}\left(\frac{y}{x}\right)^{2n} - \cdots - \left(\frac{y}{x}\right)^{mn}\right]$$

$$= x^{mn}\left[n\left(\frac{y}{x}\right)^m - m\left(\frac{y}{x}\right)^n + \frac{n(n-1)}{2}\left(\frac{y}{x}\right)^{2m}\right.$$
$$\left. - \frac{m(m-1)}{2}\left(\frac{y}{x}\right)^{2n} + \cdots\right]$$

< 0

i.e., $\boxed{(x^m+y^m)^n < (x^n+y^n)^m}$

Hence the proof.

—— x ——

Arithmetic Mean : (AM)

Let $a_1, a_2, a_3, \ldots, a_n$ be n positive quantities, then the arithmetic mean is the sum of the n quantities deivided by n.

i.e., $\boxed{\dfrac{a_1 + a_2 + a_3 + \cdots + a_n}{n} = AM}$ (say)

Geometric Mean: (GM)

Let $a_1, a_2, a_3, \ldots, a_n$ be n positive quantities, then the Geometric mean is the n^{th} root of the product.

i.e., $\boxed{G.M = (a_1 \cdot a_2 \cdot a_3 \cdots a_n)^{1/n}}$.

Result:-

The arithmetic mean of n positive quantities which are not all equal to one another, is greater than their geometric mean.

proof:- Let $a_1, a_2, a_3, \ldots, a_n$ be 'n' +ve quantities.

$$\text{Arithmetical Mean (AM)} = \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}$$

$$\text{Geometrical Mean (GM)} = (a_1 a_2 a_3 \cdots a_n)^{1/n}$$

Examples:-

1. show that $n^n > 1\cdot3\cdot5\cdots(2n-1)$

proof:-

Consider $1, 3, 5, \ldots, (2n-1)$ are +ve quantities and not all equal.

W.K.T. $A > G$

i.e., $$\frac{1+3+5+\cdots+(2n-1)}{n} > \left(1\cdot3\cdot5\cdots(2n-1)\right)^{1/n} \rightarrow ①$$

Now, $1+3+5+\cdots+(2n-1) = \frac{n[1+(2n-1)]}{2} = \frac{2n^2}{2} = n^2$

$\therefore ① \Rightarrow \quad \frac{n^2}{n} > \left(1\cdot3\cdot5\cdots(2n-1)\right)^{1/n}$

$\Rightarrow \quad n > \left(1\cdot3\cdot5\cdots(2n-1)\right)^{1/n}$

i.e., $\boxed{n^n > 1\cdot3\cdot5\cdots(2n-1)}$

Hence the proof.

2. If x and y are positive quantities whose sum is 4, show that $\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2 \not< 12\frac{1}{2}$.

proof:-

Given that x & y are +ve and $x \neq y$.

Given that $x+y=4 \rightarrow ①$

Now, $\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2 = x^2+\frac{1}{x^2}+2+y^2+\frac{1}{y^2}+2$

$\Rightarrow \left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2 = x^2+y^2+\frac{1}{x^2}+\frac{1}{y^2}+4. \rightarrow ②$

W.K.T. $A > G$

$\Rightarrow \frac{x+y}{2} > (xy)^{1/2}$

$\Rightarrow \frac{x+y}{2} > \sqrt{xy} \Rightarrow \frac{4}{2} > \sqrt{xy}$ ($\because$ eqn. 1

$\Rightarrow 2 > \sqrt{xy} \Rightarrow 2^2 > xy \Rightarrow \boxed{4 > xy}$

Also, $\frac{\frac{1}{x^2}+\frac{1}{y^2}}{2} > \left(\frac{1}{x^2}\cdot\frac{1}{y^2}\right)^{1/2}$

$1/x^2, 1/y^2$

$\Rightarrow \frac{\frac{1}{x^2}+\frac{1}{y^2}}{2} > \frac{1}{xy}$

$\Rightarrow \frac{1}{x^2}+\frac{1}{y^2} > \frac{2}{xy} \Rightarrow \boxed{\frac{1}{x^2}+\frac{1}{y^2} > \frac{1}{2}}$

Consider, $x^2+y^2 = x^2+(4-x)^2$ $\quad x+y=4$

$= x^2+16+x^2-8x$ $\quad y=4-x$

$= 2x^2-8x+16$

$= 2x^2-8x+8+8$

$= 2(x^2-4x+4)+8$

$x^2+y^2 = 2(x-2)^2+8$

$\Rightarrow \boxed{x^2+y^2 \geq 8}$

$\therefore$ 2 $\Rightarrow \left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2 \geq 8+\frac{1}{2}+4$

$\left(x+1/x\right)^2+\left(y+1/y\right)^2 \geq 25$

ie., $\left(x+\frac{1}{x}\right)^2+\left(y+1/y\right)^2 \geq 12\frac{1}{2}$

$\Rightarrow \boxed{\left(x+\frac{1}{x}\right)^2+\left(y+1/y\right)^2 \not< 12\tfrac{1}{2}}$ Hence the proof

Exercises

1. Show that $1+x+x^2+\cdots+x^{2n} \nless (2n+1)\,x^n$.

Proof:-

Consider, $1, x, x^2, \dots, x^{2n}$ are +ve quantities, and not all equal.

W.K.T. $A > G$

$$\therefore \frac{1+x+x^2+\cdots+x^{2n}}{2n+1} > \left(1\cdot x\cdot x^2\cdot\cdots\cdot x^{2n}\right)^{1/2n+1}$$

$$\Rightarrow \frac{1+x+x^2+\cdots+x^{2n}}{2n+1} > \left(x^{0+1+2+3+\cdots+2n}\right)^{1/2n+1}$$

$x^a x^b = x^{a+b}$

$$\Rightarrow \frac{1+x+x^2+\cdots+x^{2n}}{2n+1} > \left(x^{\frac{(2n+1)\cancel{2}n}{\cancel{2}}}\right)^{1/2n+1}$$

$1+2+\cdots+n = \frac{n(n+1)}{2}$

$$\Rightarrow \frac{1+x+x^2+\cdots+x^{2n}}{2n+1} > \left(x^{(\cancel{2n+1})n}\right)^{1/\cancel{2n+1}}$$

$n(2n+1)\frac{1}{(2n+1)}$

$$\Rightarrow \frac{1+x+x^2+\cdots+x^{2n}}{2n+1} > x^n$$

$$\Rightarrow 1+x+x^2+\cdots+x^{2n} > (2n+1)\,x^n$$

i.e., $\boxed{1+x+x^2+x^3+\cdots+x^{2n} \nless (2n+1)\,x^n}$

Hence the proof.

2. Prove that $(1^r+2^r+\cdots+n^r)^n > n^n (n!)^r$.

This is Home Work Problem

Home Work

show that if x, y, z are +ve quantities, then $(x+y+z)^3 > 27xyz$.

$\times$

4. show that $\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} > n$.

proof:-

Consider $a_1, a_2, \ldots, a_n$ are +ve quantities and not all equal.

W.k.T. $A > G$

Let $\frac{a_1}{a_2}, \frac{a_2}{a_3}, \ldots, \frac{a_{n-1}}{a_n}, \frac{a_n}{a_1}$ are +ve quantities.

i.e.,
$$\frac{\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}}{n} > \left(\frac{\cancel{a_1}}{\cancel{a_2}} \cdot \frac{\cancel{a_2}}{\cancel{a_3}} \cdots \frac{\cancel{a_{n-1}}}{\cancel{a_n}} \cdot \frac{\cancel{a_n}}{\cancel{a_1}}\right)^{1/n}$$

i.e.,
$$\frac{\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}}{n} > (1)^{1/n}$$

$$\Rightarrow \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} > n(1)$$

$$\Rightarrow \boxed{\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} > n}$$

Hence the proof.

$\times$

5. show that $(x+y+z)^3 > 27(y+z-x)(z+x-y)(x+y-z)$ if x, y, z are +ve quantities.

proof:-

Consider x, y, z are +ve quantities and not all equals.

W.K.T. $A > G$

Let $(y+z-x)$, $(z+x-y)$, $(x+y-z)$ are +ve quantities and not all equal.

i.e., $$\frac{(y+z-x)+(z+x-y)+(x+y-z)}{3} > \left[(y+z-x)(z+x-y)(x+y-z)\right]^{\frac{1}{3}}$$

$$\Rightarrow \frac{x+y+z}{3} > \left[(y+z-x)(z+x-y)(x+y-z)\right]^{\frac{1}{3}}$$

$$\Rightarrow \left(\frac{x+y+z}{3}\right)^3 > (y+z-x)(z+x-y)(x+y-z)$$

$$\Rightarrow \frac{(x+y+z)^3}{27} > (y+z-x)(z+x-y)(x+y-z)$$

$$\Rightarrow \boxed{(x+y+z)^3 > 27(y+z-x)(z+x-y)(x+y-z)}$$

Hence the proof.

=x=

5. Show that $\frac{s}{s-a_1}+\frac{s}{s-a_2}+\dots+\frac{s}{s-a_n}>\frac{n^2}{n-1}$,

if $s=a_1+a_2+\dots+a_n$. Unless $a_1=a_2=\dots=a_n$.

Proof:- Consider $a_1, a_2, \dots, a_n$ are +ve quantities.

W.K.T. $A>G$

Now, $\frac{s}{s-a_1}, \frac{s}{s-a_2}, \dots, \frac{s}{s-a_n}$ are +ve quantities.

i.e., $$\frac{\frac{s}{s-a_1}+\frac{s}{s-a_2}+\dots+\frac{s}{s-a_n}}{n}>\left(\frac{s}{s-a_1}\cdot\frac{s}{s-a_2}\cdots\frac{s}{s-a_n}\right)^{1/n} \to (1)$$

unless $\frac{s}{s-a_1}=\frac{s}{s-a_2}=\dots=\frac{s}{s-a_n}$

i.e., $a_1=a_2=\dots=a_n$

Also, $$\frac{\frac{s-a_1}{s}+\frac{s-a_2}{s}+\dots+\frac{s-a_n}{s}}{n}>\left(\frac{s-a_1}{s}\cdot\frac{s-a_2}{s}\cdots\frac{s-a_n}{s}\right)^{1/n} \to (2)$$

Multiplying (1) & (2), we get

$$\frac{1}{n^2}\left(\frac{s}{s-a_1}+\frac{s}{s-a_2}+\dots+\frac{s}{s-a_n}\right)\cdot\left(\frac{s-a_1}{s}+\frac{s-a_2}{s}+\dots+\frac{s-a_n}{s}\right)>1$$

$$\frac{1}{n^2}\left(\frac{s}{s-a_1}+\frac{s}{s-a_2}+\dots+\frac{s}{s-a_n}\right)\cdot\left(\frac{ns-(a_1+a_2+\dots+a_n)}{s}\right)>1$$

$$\frac{1}{n^2}\left(\frac{s}{s-a_1}+\frac{s}{s-a_2}+\dots+\frac{s}{s-a_n}\right)\left(\frac{ns-s}{s}\right)>1$$

$$\frac{1}{n^2}\left(\frac{s}{s-a_1}+\frac{s}{s-a_2}+\dots+\frac{s}{s-a_n}\right)(n-1)>1$$

$$\Rightarrow \boxed{\frac{s}{s-a_1}+\frac{s}{s-a_2}+\dots+\frac{s}{s-a_n}>\frac{n^2}{(n-1)}}$$

Hence the proof.

—X—

7. If $x_1 x_2 \cdots x_n = y^n$, show that $(1+x_1)(1+x_2)\cdots(1+x_n) \nless (1+y)^n$

Proof:- Given that $x_1 . x_2 \cdots x_n = y^n \rightarrow (1)$

Let $(1+x_1)(1+x_2)\cdots(1+x_n) = 1 + (x_1+x_2+\cdots+x_n) + \Sigma x_1 x_2 + \Sigma x_1 x_2 x_3 + \cdots + x_1 . x_2 \cdots x_n$ $\rightarrow (2)$

Consider, $x_1, x_2, \ldots, x_n$ are +ve quantities.

Now, $\frac{x_1+x_2+\cdots+x_n}{n} > (x_1 . x_2 \cdots x_n)^{1/n}$

$\Rightarrow \frac{x_1+x_2+\cdots+x_n}{n} > (y^n)^{1/n}$

$\Rightarrow \frac{x_1+x_2+\cdots+x_n}{n} > y$

$\Rightarrow x_1+x_2+x_3+\cdots+x_n \geq ny \rightarrow (3)$

$\Sigma x_1 x_2$ consists of nc_2 terms out of which $(n-1)$ terms will contains x_1, $n-1$ terms contain x_2 factor.

ie., $\frac{\Sigma x_1 x_2}{nc_2} \geq \left(x_1^{n-1} . x_2^{n-1} \cdots x_n^{n-1}\right)^{1/nc_2}$

$\geq \left[(x_1 . x_2 \cdots x_n)^{n-1}\right]^{\frac{1}{\frac{n(n-1)}{2}}}$

$\geq \left[(x_1 . x_2 \cdots x_n)^{\cancel{n-1}}\right]^{\frac{2}{n(\cancel{n-1})}}$

$\geq (y^n)^{2/n}$

$\Rightarrow \frac{\Sigma x_1 x_2}{nc_2} \geq y^2$

$\Rightarrow \Sigma x_1 x_2 \geq nc_2 y^2 \rightarrow (4)$

llly $\Sigma x_1 x_2 x_3 \geq nc_3 y^3 \rightarrow (5)$

$\therefore (2) \Rightarrow (1+x_1)(1+x_2)\cdots(1+x_n) \geq 1 + ny + nc_2 y^2 + nc_3 y^3 + \cdots + y^n$

ie., $(1+x_1)(1+x_2)\cdots(1+x_n) \geq (1+y)^n$

$\Rightarrow \boxed{(1+x_1)(1+x_2)\cdots(1+x_n) \nless (1+y)^n}$

Hence the proof.

If $a_1, a_2, \dots, a_n$ are positive and $(n-1)s = a_1+a_2+\dots+a_n$, then prove that $a_1 a_2 \dots a_n \geq (n-1)^n (s-a_1)(s-a_2)\dots(s-a_n)$.

Proof:- Given that $a_1, a_2, \dots, a_n$ are +ve quantities.

and $(n-1)s = a_1+a_2+\dots+a_n$

$$\Rightarrow a_1 = (n-1)s - a_2 - a_3 \dots - a_n \rightarrow (1)$$

Now, $(s-a_2), (s-a_3), \dots, (s-a_n)$ are +ve quantities.

W.K.T. $A > G$

$$\Rightarrow \frac{(s-a_2)+(s-a_3)+\dots+(s-a_n)}{n-1} \geq \left((s-a_2)(s-a_3)\dots(s-a_n)\right)^{1/n-1}$$

$$\Rightarrow \frac{(n-1)s - a_2 - a_3 - \dots - a_n}{n-1} \geq \left((s-a_2)(s-a_3)\dots(s-a_n)\right)^{1/n-1}$$

$$\Rightarrow \frac{a_1}{n-1} \geq \left((s-a_2)(s-a_3)\dots(s-a_n)\right)^{1/n-1}$$ by eqn (1)

$||^{ly}$ $$\frac{a_2}{n-1} \geq \left((s-a_1)(s-a_3)\dots(s-a_n)\right)^{1/n-1}$$

$$\frac{a_3}{n-1} \geq \left((s-a_1)(s-a_2)(s-a_4)\dots(s-a_n)\right)^{1/n-1}$$

....

$$\frac{a_n}{n-1} \geq \left((s-a_1)(s-a_2)\dots(s-a_{n-1})\right)^{1/n-1}$$

Multiplying all terms, we get

$$\frac{a_1}{n-1}\cdot\frac{a_2}{n-1}\frac{a_3}{n-1}\dots\frac{a_n}{n-1} \geq \left[(s-a_2)(s-a_3)\dots(s-a_n)\right]^{1/n-1}$$
$$\times\left[(s-a_1)(s-a_3)\dots(s-a_n)\right]^{1/n-1}$$
$$\times\left[(s-a_1)(s-a_2)\dots(s-a_n)\right]^{1/n-1}$$
$$\vdots$$
$$\times\left[(s-a_1)(s-a_2)\dots(s-a_{n-1})\right]^{1/n-1}$$

$$\left(\frac{1}{n-1}\right)^n (a_1 a_2 a_3 \cdots a_n) \geq \left[(s-a_1)^{n-1} (s-a_2)^{n-1} \cdots (s-a_n)^{n-1}\right]$$

$$\Rightarrow \quad \frac{a_1 a_2 a_3 \cdots a_n}{(n-1)^n} \geq \left[\left((s-a_1)(s-a_2)\cdots(s-a_n)\right)^{n-1}\right]^{1/n}$$

$$\Rightarrow \quad \frac{a_1 a_2 a_3 \cdots a_n}{(n-1)^n} \geq (s-a_1)(s-a_2)\cdots(s-a_n)$$

$$\Rightarrow \quad \boxed{a_1 a_2 a_3 \cdots a_n \geq (n-1)^n (s-a_1)(s-a_2)\cdots(s-a_n)}$$

Hence the proof.

X

Weirstrass inequalities

If $a_1, a_2, \ldots, a_n$ are positive numbers whose sum is s, then

(i) $(1+a_1)(1+a_2)\cdots(1+a_n) > 1+s$

(ii) $(1-a_1)(1-a_2)\cdots(1-a_n) > 1-s$

$\frac{1}{1-s} > \frac{1}{(1-a_1)(1-a_2)\cdots(1-a_n)}$

proof:- Given that $a_1, a_2, \ldots, a_n$ are +ve numbers.

Given that $a_1 + a_2 + \cdots + a_n = s$.

Now, $(1+a_1)(1+a_2) = 1 + a_1 + a_2 + a_1 a_2$

$> 1 + a_1 + a_2 \quad \because a_1 \,\&\, a_2 > 0$

$\Rightarrow$ multiply $(1+a_3)$ on both sides, we get

$\therefore (1+a_1)(1+a_2)(1+a_3) > (1+a_1+a_2)(1+a_3)$

$(1+a_1)(1+a_2)(1+a_3) > 1 + a_1 + a_2 + a_3 + a_1 a_3 + a_2 a_3$

$> 1 + a_1 + a_2 + a_3$

do this process over and over again, we get

$(1+a_1)(1+a_2)(1+a_3)\cdots(1+a_n) > 1 + a_1 + a_2 + a_3 + \cdots + a_n$

$\Rightarrow (1+a_1)(1+a_2)\cdots(1+a_n) > 1+a_1+a_2+\cdots+a_n$

$\Rightarrow \boxed{(1+a_1)(1+a_2)\cdots(1+a_n) > 1+s}$

(i) is proved

Again, $(1-a_1)(1-a_2) = 1-a_1-a_2+a_1a_2$

Multiplying $> 1-a_1-a_2$

x $(1-a_3)$ on both sides, we get

$$(1-a_1)(1-a_2)(1-a_3) > (1-a_1-a_2)(1-a_3)$$
$$> 1-a_1-a_2-a_3+a_1a_3+a_2a_3$$
$$> 1-a_1-a_2-a_3$$

IIly

$$(1-a_1)(1-a_2)(1-a_3)(1-a_4) > 1-a_1-a_2-a_3-a_4$$

$\vdots$

$$(1-a_1)(1-a_2)\cdots(1-a_n) > 1-a_1-a_2-a_3-\cdots-a_n$$

$$(1-a_1)(1-a_2)\cdots(1-a_n) > 1-(a_1+a_2+a_3+\cdots+a_n)$$

i.e., $\boxed{(1-a_1)(1-a_2)\cdots(1-a_n) > 1-s}$

Hence (ii) is proved

If $a_1, a_2, \ldots, a_n$ are positive and if $a_1, a_2, \ldots, a_n$ are all less than 1, then

(i) $(1+a_1)(1+a_2)\cdots(1+a_n) < \frac{1}{1-s}$, if $s<1$

(ii) $(1-a_1)(1-a_2)\cdots(1-a_n) < \frac{1}{1+s}$

proof:-

Given that $a_1, a_2, \ldots, a_n$ are +ve numbers

Given that $a_1, a_2, \ldots, a_n$ are all <1.

(i) Consider $(1+a_1)(1-a_1) = 1-a_1^2 < 1$

$\Rightarrow (1+a_1)(1-a_1) < 1$

$$\Rightarrow 1+a_1 < \frac{1}{1-a_1}$$

IIIly $(1+a_2)(1-a_2) < 1 \Rightarrow (1+a_2) < \frac{1}{1-a_2}$

$$(1+a_3)(1-a_3) < 1 \Rightarrow (1+a_3) < \frac{1}{1-a_3}$$

....

....

....

$$(1+a_n)(1-a_n) < 1 \Rightarrow 1+a_n < \frac{1}{1-a_n}$$

Multiply these all terms, we get

$$(1+a_1)(1+a_2)\cdots(1+a_n) < \frac{1}{1-a_1}\cdot\frac{1}{1-a_2}\cdots\frac{1}{1-a_n}$$

$$(1+a_1)(1+a_2)\cdots(1+a_n) < \frac{1}{(1-a_1)(1-a_2)\cdots(1-a_n)}$$

ie., $\boxed{(1+a_1)(1+a_2)\cdots(1+a_n) < \frac{1}{1-s} \quad \text{if } s<1}$

(ii) Here $(1+a_1)(1-a_1) < 1 \Rightarrow (1-a_1) < \frac{1}{1+a_1}$

Similarly $(1+a_2)(1-a_2) < 1 \Rightarrow (1-a_2) < \frac{1}{1+a_2}$

....

....

....

$$(1+a_n)(1-a_n) < 1 \Rightarrow (1-a_n) < \frac{1}{1+a_n}$$

Multiply these all terms, we get

$$(1-a_1)(1-a_2)\cdots(1-a_n) < \frac{1}{1+a_1}\cdot\frac{1}{1+a_2}\cdots\frac{1}{1+a_n}$$

$$(1-a_1)(1-a_2)\cdots(1-a_n) < \frac{1}{(1+a_1)(1+a_2)\cdots(1+a_n)}$$

ie., $\boxed{(1-a_1)(1-a_2)\cdots(1-a_n) < \frac{1}{1+s}}$

Hence the proof.

=X=

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Cauchy's inequality

If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are two sets of real numbers, then

$$\Sigma a_i^2 \, \Sigma b_i^2 > (\Sigma a_i b_i)^2.$$

proof:-

Given that $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are two sets of positive real numbers.

Consider quadratic expression

$$ax^2 + 2bx + c > 0$$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

if $(2b)^2 - 4ac < 0$ and $a > 0$

$4(b^2 - ac) < 0$

$\Rightarrow \; b^2 - ac < 0$ and $a > 0$

Consider, $(a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \cdots + (a_n x + b_n)^2$

$$= a_1^2 x^2 + b_1^2 + 2a_1 b_1 x + a_2^2 x^2 + b_2^2 + 2a_2 b_2 x + \cdots + a_n^2 x^2 + b_n^2 + 2a_n b_n x$$

$$= (a_1^2 + a_2^2 + \cdots + a_n^2)x^2 + 2(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)x + (b_1^2 + b_2^2 + \cdots + b_n^2) > 0$$

Here, $b^2 - ac < 0$, hence

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 - (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) < 0$$

$$\Rightarrow (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 < (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$

$$\Rightarrow (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) > (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

i.e., $\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 > \left(\sum_{i=1}^{n} a_i b_i\right)^2$

$$\Rightarrow \boxed{\Sigma a_i^2 \, \Sigma b_i^2 > (\Sigma a_i b_i)^2}$$

Hence the proof.

If x is any positive number different from unity and p and q are positive rationals, then

$$\frac{x^p - 1}{p} > \frac{x^q - 1}{q}, \text{ when } p > q.$$

Proof: Given that $x \neq 1$ is positive number.

Given that p & q are positive rationals and $p > q$.

Consider m be a +ve integer.

Then, $\frac{x^{m+1} - 1}{m+1} - \frac{x^m - 1}{m}$

$$= \frac{x^{m+1} + x^m + x^{m-1} + \cdots + x^2 + x - x^m - x^{m-1} - x^{m-2} - \cdots - x - 1}{m+1}$$

$$- \frac{x^m + x^{m-1} + x^{m-2} + \cdots + x^2 + x - x^{m-1} - x^{m-2} - \cdots - x - 1}{m}$$

$$= \frac{x(x^m + x^{m-1} + x^{m-2} + \cdots + x + 1) - (x^m + x^{m-1} + \cdots + x + 1)}{m+1}$$

$$- \frac{x(x^{m-1} + x^{m-2} + \cdots + x + 1) - (x^{m-1} + x^{m-2} + \cdots + x + 1)}{m}$$

$$= \frac{(x-1)(x^m + x^{m-1} + \cdots + x + 1)}{m+1} - \frac{(x-1)(x^{m-1} + x^{m-2} + \cdots + x + 1)}{m}$$

$$= (x-1)\left[\frac{x^m + x^{m-1} + \cdots + x + 1}{m+1} - \frac{x^{m-1} + x^{m-2} + \cdots + x + 1}{m}\right]$$

$$= \frac{x-1}{m(m+1)}\left[m(x^m + x^{m-1} + \cdots + x + 1) - (m+1)(x^{m-1} + x^{m-2} + \cdots + x + 1)\right]$$

$$= \frac{x-1}{m(m+1)}\left[mx^m + mx^{m-1} + \cdots + mx + m - (mx^{m-1} + mx^{m-2} + \cdots + mx + m + x^{m-1} + x^{m-2} + \cdots + x + 1)\right]$$

$$= \frac{x-1}{m(m+1)}\left[mx^m + mx^{m-1} + \cdots + mx + m - mx^{m-1} - mx^{m-2} - \cdots - mx - m - x^{m-1} - x^{m-2} - \cdots - x - 1\right]$$

$$\frac{x^{m+1}-1}{m+1} - \frac{x^m-1}{m}$$

$$= \frac{x-1}{m(m+1)}\left[mx^m - x^{m-1} - x^{m-2} - \cdots - x - 1\right]$$

$$= \frac{x-1}{m(m+1)}\left[(x^m - x^{m-1}) + (x^m - x^{m-2}) + \cdots (x^m - x) + (x^m - 1)\right]$$

$$= \frac{x-1}{m(m+1)}\left[x^{m-1}(x-1) + x^{m-2}(x^2-1) + \cdots + (x-1)(x^{m-1}+x^{m-2} + \cdots + x + 1)\right]$$

$$= \frac{x-1}{m(m+1)}\left[x^{m-1}(x-1) + x^{m-2}(x+1)(x-1) + \cdots + (x-1)(x^{m-1}+x^{m-2}+\cdots+x+1)\right]$$

$$= \frac{(x-1)^2}{m(m+1)}\left[x^{m-1} + x^{m-2}(x+1) + \cdots + x^{m-1} + x^{m-2} + \cdots + x + 1\right]$$

> 0

ie., $\dfrac{x^{m+1}-1}{m+1} - \dfrac{x^m-1}{m} > 0$

$$\Rightarrow \quad \frac{x^{m+1}-1}{m+1} > \frac{x^m-1}{m}$$

ie., $$\boxed{\frac{x^p-1}{p} > \frac{x^q-1}{q}}$$

where $p = m+1$ & $q = m$.

Hence the proof.

—X—

If x and y are positive unequal numbers and p is any rational number except 1, then

$$px^{p-1}(x-y) > (x^p-y^p) > py^{p-1}(x-y)$$

Proof:- Given that x & y are +ve and $x \neq y$.

Given that $p \neq 1$ is a +ve rational number.

W.k.T. $\frac{x^p-1}{p} > \frac{x^q-1}{q} \rightarrow ①$ $p>q$ and $n \neq m$

put $x=\frac{x}{y}$, $q=1$ in eqn: ①, we get

$$\Rightarrow \frac{\left(\frac{x}{y}\right)^p-1}{p} > \frac{\left(\frac{x}{y}\right)^1-1}{1}$$

$$\Rightarrow \frac{\frac{x^p}{y^p}-1}{p} > \frac{x}{y}-1$$

$$\Rightarrow \frac{x^p-y^p}{py^p} > \frac{x-y}{y}$$

$$\Rightarrow x^p-y^p > py^p\,\frac{x-y}{y}$$

$$\Rightarrow x^p-y^p > py^{p-1}(x-y) \rightarrow ②$$

put $x=y/x$ and $q=1$ in eqn: ①, we get

$$\Rightarrow \frac{\left(\frac{y}{x}\right)^p-1}{p} > \frac{\left(\frac{y}{x}\right)^1-1}{1}$$

$$\Rightarrow \frac{y^p-x^p}{px^p} > \frac{y}{x}-1$$

$$\Rightarrow y^p-x^p > px^p\left(\frac{y-x}{x}\right)$$

$$\Rightarrow y^p - x^p > p x^{p-1}(y-x)$$

$$\Rightarrow -(x^p - y^p) > -p x^{p-1}(x-y)$$

$$\Rightarrow -(x^p - y^p) > -p x^{p-1}(x-y).$$

$$\Rightarrow p x^{p-1}(x-y) > x^p - y^p \rightarrow (3)$$

From equation (2) & (3), we get

$$\boxed{p x^{p-1}(x-y) > x^p - y^p > p y^{p-1}(x-y)}$$

Hence the proof.

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Some problems:-

1. If $a_1, a_2, \dots, a_n$ are n positive numbers not all equal to one another, then

(i) $$\frac{a_1^{p+q} + a_2^{p+q} + \dots + a_n^{p+q}}{n} > \frac{a_1^p + a_2^p + \dots + a_n^p}{n} \times \frac{a_1^q + a_2^q + \dots + a_n^q}{n},$$

if p & q are same signs.

(ii) $$\frac{a_1^{p+q} + a_2^{p+q} + \dots + a_n^{p+q}}{n} < \frac{a_1^p + a_2^p + \dots + a_n^p}{n} \times \frac{a_1^q + a_2^q + \dots + a_n^q}{n},$$

if p & q are opposite signs.

Proof:-

Given that $a_1, a_2, \dots, a_n$ are +ve numbers and $a_1 \neq a_2 \neq \dots \neq a_n$.

(i) Given that p & q are same signs.

Consider $a_1^p - a_2^p$, $a_1^q - a_2^q$ are same signs. and both +ve, both -ve or zero.

$$\therefore (a_1^p - a_2^p)(a_1^q - a_2^q) > 0$$

$$\Rightarrow a_1^p a_1^q - a_1^q a_2^p - a_1^p a_2^q + a_2^p a_2^q > 0$$

$$\Rightarrow a_1^{p+q} + a_2^{p+q} - a_1^p a_2^q - a_1^q a_2^p > 0$$

$$\Rightarrow a_1^{p+q} + a_2^{p+q} > a_1^p a_2^q + a_1^q a_2^p$$

$$\Rightarrow a_1^{p+q} + a_2^{p+q} > \sum a_i^p a_j^q$$

In general,

$$a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q} > \sum a_i^p a_j^q$$

$$\Rightarrow (n-1)(a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}) > \sum a_i^p a_j^p$$

$$\Rightarrow n(a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}) - (a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}) > \sum a_i^p a_j^q$$

$$\Rightarrow n(a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}) - \sum a_i^{p+q} > \sum a_i^p a_j^q$$

$$\Rightarrow n(a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}) > \sum a_i^{p+q} + \sum a_i^p a_j^q$$

$$\Rightarrow n(a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}) > (a_1^p + a_2^p + \cdots + a_n^p) \times (a_1^q + a_2^q + \cdots + a_n^q)$$

$$\Rightarrow \boxed{\frac{n(a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q})}{n^2} > \left(\frac{a_1^p + a_2^p + \cdots + a_n^p}{n}\right) \times \left(\frac{a_1^q + a_2^q + \cdots + a_n^q}{n}\right)}$$

Hence (i) is proved

(ii) Given that p and q are opposite signs.

Consider, $(a_1^p - a_2^p)$, $(a_1^q - a_2^q)$ are opposite signs.

$$\Rightarrow (a_1^p - a_2^p)(a_1^q - a_2^q) < 0$$

$$\Rightarrow a_1^p a_1^q - a_2^p a_1^q - a_1^p a_2^q + a_2^p a_2^q < 0$$

$$\Rightarrow a_1^{p+q} + a_2^p a_2^q < a_2^p a_1^q + a_1^p a_2^q$$

$$\Rightarrow a_1^{p+q} + a_2^{p+q} < \sum a_i^p a_j^q$$

In general, $a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q} < \sum a_i^p a_j^q$

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$$\Rightarrow (n-1)\left(a_1^{p+q}+a_2^{p+q}+\cdots+a_n^{p+q}\right) < \sum a_i^p a_j^q$$

$$\Rightarrow n\left(a_1^{p+q}+a_2^{p+q}+\cdots+a_n^{p+q}\right)-\left(a_1^{p+q}+a_2^{p+q}+\cdots+a_n^{p+q}\right) < \sum a_i^p a_j^q$$

$$\Rightarrow n\left(a_1^{p+q}+a_2^{p+q}+\cdots+a_n^{p+q}\right)-\sum a_i^{p+q} < \sum a_i^p a_j^q$$

$$\Rightarrow n\left(a_1^{p+q}+a_2^{p+q}+\cdots+a_n^{p+q}\right) < \sum a_i^{p+q}+\sum a_i^p a_j^q$$

$$\Rightarrow n\left(a_1^{p+q}+a_2^{p+q}+\cdots+a_n^{p+q}\right) < \left(a_1^p+a_2^p+\cdots+a_n^p\right) \times \left(a_1^q+a_2^q+\cdots+a_n^q\right)$$

$$\Rightarrow \frac{\cancel{n}\left(a_1^{p+q}+a_2^{p+q}+\cdots+a_n^{p+q}\right)}{n^{\cancel{2}}} < \left(\frac{a_1^p+a_2^p+\cdots+a_n^p}{n}\right) \times \left(\frac{a_1^q+a_2^q+\cdots+a_n^q}{n}\right)$$

$$\Rightarrow \boxed{\frac{a_1^{p+q}+a_2^{p+q}+\cdots+a_n^{p+q}}{n} < \left(\frac{a_1^p+a_2^p+\cdots+a_n^p}{n}\right) \times \left(\frac{a_1^q+a_2^q+\cdots+a_n^q}{n}\right)}$$

Hence (ii) is proved.

=X=

2. Show that if a, b, c are three positive unequal quantities, then

$$\frac{a^8+b^8+c^8}{a^3b^3c^3} > \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Proof:- Given that a, b, c are positive numbers and $a \neq b \neq c$.

Consider $\dfrac{a^8+b^8+c^8}{3} = \dfrac{a^{6+2}+b^{6+2}+c^{6+2}}{3} \rightarrow (1)$

W.K.T. $\frac{a_1^{p+q}+a_2^{p+q}+\cdots+a_n^{p+q}}{n} > \left(\frac{a_1^p+a_2^p+\cdots+a_n^p}{n}\right)$

$\times\left(\frac{a_1^q+a_2^q+\cdots+a_n^q}{n}\right)$

use this condition in eqn. ① , we get

$$\frac{a^8+b^8+c^8}{3} > \left(\frac{a^6+b^6+c^6}{3}\right)\left(\frac{a^2+b^2+c^2}{3}\right) \rightarrow ②$$

Now, consider a^6, b^6, c^6 are +ve quantities.

W.K.T. $A > G$

$$\Rightarrow \frac{a^6+b^6+c^6}{3} > (a^6.b^6.c^6)^{1/3}$$

$$\Rightarrow \frac{a^6+b^6+c^6}{3} > \left((a^2)^3.(b^2)^3.(c^2)^3\right)^{1/3}$$

$$\Rightarrow \frac{a^6+b^6+c^6}{3} > \left((a^2.b^2.c^2)^{\cancel{3}}\right)^{\cancel{1/3}}$$

$$\Rightarrow \frac{a^6+b^6+c^6}{3} > a^2.b^2.c^2 \rightarrow ③$$

W.K.T. $a^2+b^2+c^2 > ab+bc+ca \rightarrow ④$

Sub ③ & ④ in eqn. ②, we get

$$\Rightarrow \frac{a^8+b^8+c^8}{\cancel{3}} > (a^2.b^2.c^2)\left(\frac{ab+bc+ca}{3}\right)$$

$$\Rightarrow a^8+b^8+c^8 > a^2b^2c^2\,\frac{abc}{abc}\,(ab+bc+ca)$$

$$\Rightarrow a^8+b^8+c^8 > (a^3b^3c^3)\left(\frac{ab+bc+ca}{abc}\right)$$

$$\Rightarrow \frac{a^8+b^8+c^8}{a^3b^3c^3} > \frac{ab}{abc} + \frac{bc}{abc} + \frac{ca}{abc}$$

$$\Rightarrow \boxed{\frac{a^8+b^8+c^8}{a^3b^3c^3} > \frac{1}{c} + \frac{1}{a} + \frac{1}{b}}$$

Hence the proof.

=X=

3. Prove that $8xyz < (y+z)(z+x)(x+y) < \frac{8}{3}(x^3+y^3+z^3)$.

proof:-

W.k.T.

$$\frac{a_1^{p+q}+a_2^{p+q}+\dots+a_n^{p+q}}{n} > \left(\frac{a_1^p+a_2^p+\dots+a_n^p}{n}\right)\left(\frac{a_1^q+a_2^q+\dots+a_n^q}{n}\right)$$

if p & q are same signs.

Consider x, y, z are +ve quantities.

W.k.T. $A > G$

Now, $\frac{x+y}{2} > (xy)^{1/2}$

$\frac{y+z}{2} > (yz)^{1/2}$

$\frac{z+x}{2} > (zx)^{1/2}$

Multiplying all terms, we get

$$\Rightarrow \frac{x+y}{2} \cdot \frac{y+z}{2} \cdot \frac{z+x}{2} > (xy)^{1/2} \cdot (yz)^{1/2} \cdot (zx)^{1/2}$$

$$\Rightarrow \frac{(x+y)(y+z)(z+x)}{8} > (x^2y^2z^2)^{1/2}$$

$$\Rightarrow (x+y)(y+z)(z+x) > 8((xyz)^2)^{1/2}$$

$$\Rightarrow (x+y)(y+z)(z+x) > 8xyz$$

$$\Rightarrow 8xyz < (x+y)(y+z)(z+x) \rightarrow (1)$$

Now, $\dfrac{x^2+y^2+z^2}{3} = \dfrac{x^{1+1}+y^{1+1}+z^{1+1}}{3}$

$$\Rightarrow \frac{x^2+y^2+z^2}{3} > \left(\frac{x+y+z}{3}\right)\left(\frac{x+y+z}{3}\right) \rightarrow (2)$$

Now, $\dfrac{x^3+y^3+z^3}{3} = \dfrac{x^{2+1}+y^{2+1}+z^{2+1}}{3}$

$$> \left(\frac{x^2+y^2+z^2}{3}\right)\left(\frac{x+y+z}{3}\right)$$

$$> \left(\frac{x+y+z}{3}\right)\left(\frac{x+y+z}{3}\right)\left(\frac{x+y+z}{3}\right)$$

by eqn. (2)

$$\Rightarrow \frac{x^3+y^3+z^3}{3} > \frac{(x+y+z)^3}{27}$$

$$\Rightarrow \frac{(x+y+z)^3}{27} < \frac{x^3+y^3+z^3}{3}$$

$$\Rightarrow (x+y+z)^3 < \frac{27}{3}(x^3+y^3+z^3)$$

$$\Rightarrow (x+y+z)^3 < 9(x^3+y^3+z^3) \rightarrow (3)$$

Consider $(x+y)$, $(y+z)$, $(z+x)$ are positive quantities.

$$\Rightarrow \frac{x+y+y+z+z+x}{3} > \left((x+y)(y+z)(z+x)\right)^{1/3}$$

$$\Rightarrow \frac{2x+2y+2z}{3} > \left[(x+y)(y+z)(z+x)\right]^{1/3}$$

$$\Rightarrow \frac{2}{3}(x+y+z) > \left[(x+y)(y+z)(z+x)\right]^{1/3}$$

$$\Rightarrow \left[\frac{2}{3}(x+y+z)\right]^3 > (x+y)(y+z)(z+x)$$

$$\Rightarrow \frac{8}{27}(x+y+z)^3 > (x+y)(y+z)(z+x)$$

$$\Rightarrow (x+y)(y+z)(z+x) < \frac{8}{27}(x+y+z)^3$$

$$\Rightarrow (x+y)(y+z)(z+x) < \frac{8}{\cancel{27}_{3}}\left[\cancel{9}(x^3+y^3+z^3)\right]$$

by eqn. 3

$$\Rightarrow (x+y)(y+z)(z+x) < \frac{8}{3}(x^3+y^3+z^3) \rightarrow (4)$$

From 1 & 4, we get

$$8xyz < (x+y)(y+z)(z+x) < \frac{8}{3}(x^3+y^3+z^3).$$

Hence the proof.

4. Establish inequality

$$a^4+b^4+c^4 > abc(a+b+c)$$

$\cancel{a^5+b^5+c^5 > abc(ab+bc+ca)}$

proof:- Consider a^3, b^3, c^3 are positive quantities.

W.K.T. $A > G.$

Now, $$\frac{a^3+b^3+c^3}{3} > (a^3 \cdot b^3 \cdot c^3)^{1/3}$$

$$\Rightarrow \frac{a^3+b^3+c^3}{3} > \left((abc)^{\cancel{3}}\right)^{1/\cancel{3}}$$

$$\Rightarrow a^3+b^3+c^3 > 3abc \rightarrow (1)$$

Now, $$\frac{a^4+b^4+c^4}{3} = \frac{a^{3+1}+b^{3+1}+c^{3+1}}{3}$$

$$\Rightarrow \frac{a^4+b^4+c^4}{3} > \left(\frac{a^3+b^3+c^3}{3}\right)\left(\frac{a+b+c}{3}\right)$$

$$\Rightarrow \frac{a^4+b^4+c^4}{3} > \frac{3abc}{3}\left(\frac{a+b+c}{3}\right)$$

$$\Rightarrow \boxed{a^4+b^4+c^4 > abc\,(b+b+c)}$$

Hence the proof.

=X=

5. If a, b, c, d are all greater than zero, show that

$a, b, c, d > 0$

$$a^5+b^5+c^5+d^5 \geq abcd\,(a+b+c+d).$$

Proof:-

W.k.T.

$$\frac{a_1^{p+q}+a_2^{p+q}+\dots+a_n^{p+q}}{n} > \left(\frac{a_1^p+a_2^p+\dots+a_n^p}{n}\right)\left(\frac{a_1^q+a_2^q+\dots+a_n^q}{n}\right)$$

Consider, a^4, b^4, c^4, d^4 are +ve quantities.

W.k.T $\quad A > G$

Now, $$\frac{a^4+b^4+c^4+d^4}{4} > \left(a^4 b^4 c^4 d^4\right)^{1/4}$$

$$\Rightarrow \frac{a^4+b^4+c^4+d^4}{4} > abcd$$

$$\Rightarrow a^4+b^4+c^4+d^4 > 4abcd \rightarrow (1)$$

Now, $$\frac{a^5+b^5+c^5+d^5}{4} = \frac{a^{4+1}+b^{4+1}+c^{4+1}+d^{4+1}}{4}$$

$$> \left(\frac{a^4+b^4+c^4+d^4}{4}\right)\left(\frac{a+b+c+d}{4}\right)$$

$$\Rightarrow \frac{a^5+b^5+c^5+d^5}{\cancel{4}} > \left(\frac{\cancel{4}abcd}{\cancel{4}}\right)\left(\frac{a+b+c+d}{\cancel{4}}\right)$$

$$\Rightarrow \boxed{a^5+b^5+c^5+d^5 > abcd\,(a+b+c+d)}$$

Hence the proof.

5. Show that $\frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + \frac{a^2+b^2}{a+b} > a+b+c$.

Proof:-

W.K.T.

$$\frac{a_1^{p+q} + a_2^{p+q} + \cdots + a_n^{p+q}}{n} > \left(\frac{a_1^p + a_2^p + \cdots + a_n^p}{n}\right)\left(\frac{a_1^q + a_2^q + \cdots + a_n^q}{n}\right)$$

Now, $\frac{b^2+c^2}{2} = \frac{b^{1+1}+c^{1+1}}{2}$

$$\Rightarrow \frac{b^2+c^2}{2} > \left(\frac{b+c}{2}\right)\left(\frac{b+c}{2}\right)$$

$$\Rightarrow \frac{b^2+c^2}{\cancel{2}} \times \frac{\cancel{2}}{b+c} > \frac{b+c}{2}$$

$$\Rightarrow \frac{b^2+c^2}{b+c} > \frac{b+c}{2} \rightarrow ①$$

IIIly $\frac{c^2+a^2}{c+a} > \frac{c+a}{2} \rightarrow ②$

and $\frac{a^2+b^2}{a+b} > \frac{a+b}{2} \rightarrow ③$

① + ② + ③ $\Rightarrow$

$$\therefore \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + \frac{a^2+b^2}{a+b} > \left(\frac{b+c}{2}\right) + \left(\frac{c+a}{2}\right) + \left(\frac{a+b}{2}\right)$$

$$\Rightarrow \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + \frac{a^2+b^2}{a+b} > \frac{2a+2b+2c}{2}$$

$$\Rightarrow \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + \frac{a^2+b^2}{a+b} > \frac{\cancel{2}(a+b+c)}{\cancel{2}}$$

i.e., $\boxed{\frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + \frac{a^2+b^2}{a+b} > a+b+c.}$

Hence the proof

Home Work

show that $\frac{b^4+c^4}{b+c} + \frac{c^4+a^4}{c+a} + \frac{a^4+b^4}{a+b} > 3abc$

factor inequalities quality equalised positive

Applications of Maxima and Minima:

We have proved that the arithmetical ~~prog~~ mean of n positive numbers (which are not equal to one another) is greater than their geometric mean.

$A > G$

i.e., $\frac{a_1+a_2+\cdots+a_n}{n} > (a_1 a_2 \ldots a_n)^{1/n}$

$$\Rightarrow (a_1 a_2 \ldots a_n)^{1/n} < \frac{a_1+a_2+\cdots+a_n}{n}$$

unless $a_1 = a_2 = \cdots = a_n$.

If $a_1 = a_2 = \cdots = a_n$, then this inequality becomes an equality.

From this we can easily draw the following conclusions:

(1) If $a_1, a_2, \ldots, a_n$ are n positive variable.

Such that $a_1+a_2+\cdots+a_n = k$ (a constant)

Then $(a_1 \cdot a_2 \cdots a_n)^{1/n}$ has the maximum value, when $a_1 = a_2 = \cdots = a_n$.

$\therefore$ The maximum value of $(a_1 a_2 \ldots a_n)^{1/n} < \frac{a_1+a_2+\cdots+a_n}{n}$

$$\Rightarrow \boxed{a_1 a_2 \cdots a_n < \left(\frac{k}{n}\right)^n}$$

2) If $a_1\, a_2 \ldots a_n = k_1$ (a Constant)

Then $a_1+a_2+\cdots+a_n$ is least.

If $a_1=a_2=\cdots=a_n$ and the least value of $a_1+a_2+\cdots+a_n$ is $n(k_1)^{1/n}$.

$$\Rightarrow \boxed{a_1+a_2+\cdots+a_n > n(k_1)^{1/n}}$$

Example:

1. Find the greatest value of $a^m b^n c^p \ldots$, when $a+b+c+\cdots$ is constant $m, n, p, \ldots$ being positive integers.

Soln:- Let $k = a^m b^n c^p \ldots$

Then the maximum value of $a^m b^n c^p \ldots$ is

$$\frac{k}{m^m n^n p^p \ldots} = \left(\frac{a}{m}\right)^m \cdot \left(\frac{b}{n}\right)^n \cdot \left(\frac{c}{p}\right)^p \cdots$$

$$= \frac{a}{m}\cdot\frac{a}{m}\cdots m \text{ factors } \frac{b}{n}\cdot\frac{b}{n}\cdots n \text{ factors}$$

$$\times \frac{c}{p}\cdot\frac{c}{p}\cdots p \text{ factors}$$

$\therefore$ The sum of all these factors $= a+b+c+\cdots$

Since there are m factors each $\frac{a}{m}$, n factors each $\frac{b}{n}$, and so on.

$\therefore\ a+b+c\cdots = \lambda$ say (a constant)

$\therefore$ The sum of all factors is constant.

$\therefore$ The product of the factors $\frac{k}{m^m n^n p^p \ldots}$ is greatest, when all the factors are equal.

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Now,

i.e., when $\frac{a}{m} = \frac{b}{n} = \frac{c}{p} = \cdots$

$\therefore$ each is equal to $\frac{a+b+c+\cdots}{m+n+p+\cdots} = \frac{\lambda}{m+n+p+\cdots}$

$\Rightarrow a = \frac{m\lambda}{m+n+p+\cdots}, \; b = \frac{n\lambda}{m+n+p+\cdots}, \; c = \frac{p\lambda}{m+n+p+\cdots}, \cdots$

$\therefore$ The greatest value of k is

$$a^m b^n c^p \cdots = \left(\frac{m\lambda}{m+n+p+\cdots}\right)^m \cdot \left(\frac{n\lambda}{m+n+p+\cdots}\right)^n \cdot \left(\frac{p\lambda}{m+n+p+\cdots}\right)^p \cdots$$

$$\boxed{a^m b^m c^p \cdots = \left(\frac{\lambda}{m+n+p+\cdots}\right)^{m+n+p+\cdots} \cdot m^m \cdot n^n \cdot p^p \cdots}$$

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$x^m x^n$ x^{m+n+p}

2. If the perimeter of a triangle is given, show that the area is greatest when the triangle is equilateral.

Proof:-

Let a, b, c be the sides of the triangle.

Let $a+b+c = 2s$.

Here Δ is the area, then

$$\Delta^2 = s(s-a)(s-b)(s-c)$$

$\therefore s-a+s-b+s-c = 3s-a-b-c$

$= 3s-(a+b+c)$

$= 3s-2s$

$= s$ (a constant)

Hence, the value of $(s-a)(s-b)(s-c)$ is greatest, when $s-a = s-b = s-c$.

Now, $s-a = s-b$

$a = b$

and $s-b = s-c$

$b = c$

and $s-c = s-a$

$c = a$

ie., $a = b = c$

$\therefore$ The value of Δ is greatest when $a=b=c$.

Hence the proof.

3\. Find the maximum value of $(3-x)^5 (2+x)^4$, when x lies between 3 and -2.

Soln:-

Let $p = (3-x)^5 (2+x)^4$

Then, $\frac{p}{5^5 4^4} = \left(\frac{3-x}{5}\right)^5 \times \left(\frac{2+x}{4}\right)^4$

$= \left(\frac{3-x}{5}\right)\left(\frac{3-x}{5}\right) \cdots 5 \text{ factors} \times \left(\frac{2+x}{4}\right)\left(\frac{2+x}{4}\right) \cdots$ 4 factors

$\therefore$ Sum of the factors $= 5\left(\frac{3-x}{5}\right) + 4\left(\frac{2+x}{4}\right)$

$= 3 - x + 2 + x$

$= 5$

Hence, $\frac{p}{5^5 4^4}$ is greatest, when all the factors are equal

$\Rightarrow \frac{3-x}{5} = \frac{2+x}{4}$

$\Rightarrow 4(3-x) = 5(2+x)$

$\Rightarrow 12-4x = 10+5x$

$\Rightarrow 12-10 = 5x+4x$

$\Rightarrow 9x = 2$

$\Rightarrow \boxed{x = 2/9}$

$\therefore$ p is greatest, when $x = 2/9$.

$\therefore$ The greatest value of $p = (3-2/9)^5 \cdot (2+2/9)^4$

$$= \left(\frac{27-2}{9}\right)^5 \left(\frac{18+2}{9}\right)^4$$

$$= \left(\frac{25}{9}\right)^5 \left(\frac{20}{9}\right)^4$$

$$= \frac{(25)^5 (20)^4}{9^5 \cdot 9^4}$$

$$= \frac{(5^2)^5 (4\times 5)^4}{9^9}$$

$$= \frac{5^{10} \cdot 5^4 4^4}{9^9}$$

$$\boxed{\text{The greatest value of } p = \frac{5^{14} \cdot 4^4}{9^9}.}$$

$=\!\!=\!\!\times\!\!=\!\!=$

4. Show that the greatest value of $xyz(d-ax-by-cz)$ is $\frac{d^4}{4^4 abc}$.

Proof:- Let $p = xyz(d-ax-by-cz)$

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Multiply abc on both sides, we get

$$abc\cdot p = xyz(d-ax-by-cz)\cdot abc$$

$$abc\,p = ax\cdot by\cdot cz\,(d-ax-by-cz)$$

The sum of the factors

$$= ax+by+cz+d-ax-by-cz$$
$$= d$$
$$= \text{constant.}$$

Hence, the product p.abc is greatest, when all factors are equal.

i.e., when $ax = by = cz = d-ax-by-cz$

Let $ax = by = cz = d-ax-by-cz = k$

Then, $k = \dfrac{\text{sum of the factors}}{4}$ (no. of factors)

$a^3 b^2$. aaabb

$$k = \frac{d}{4}$$

$$\therefore\ ax = k \Rightarrow ax = \frac{d}{4} \Rightarrow x = \frac{d}{4a}$$

$$by = k \Rightarrow by = \frac{d}{4} \Rightarrow y = \frac{d}{4b}$$

$$cz = k \Rightarrow cz = \frac{d}{4} \Rightarrow z = \frac{d}{4c}$$

Hence, the greatest value of p is

$$= xyz(d-ax-by-cz)$$

$$= \frac{d}{4a}\cdot\frac{d}{4b}\cdot\frac{d}{4c}\left(d-\frac{d}{4}-\frac{d}{4}-\frac{d}{4}\right)$$

$$= \frac{d^3}{4^3 abc}\left(\frac{4d-d-d-d}{4}\right) = \frac{d^3}{4^4 abc}(4d-3d)$$

$$= \frac{d^3}{4^4 abc}(d) = \frac{d^4}{4^4 abc}$$

Hence the proof.

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5. Find the least value of $4x+3y$ for positive values of x, y subject to the condition $x^3y^2=6$.

Soln:-

Given that $x^3y^2=6 \to (1)$

If λ, μ are any constants.

Take, $\lambda x, \lambda x, \lambda x, \mu y, \mu y$ are positive factors

$\therefore$ $\lambda x \cdot \lambda x \cdot \lambda x \cdot \mu y \cdot \mu y$

$= \lambda^3 x^3 \mu^2 y^2$

$= 6\lambda^3\mu^2 = k$ (by eqn (1))

and $\lambda x+\lambda x+\lambda x+\mu y+\mu y = 3\lambda x+2\mu y$

Hence, the least value of $3\lambda x+2\mu y$ is $5(6\lambda^3\mu^2)^{1/5}$.

$\because$ The least value of $a_1+a_2+\dots+a_n$ is $n(k_1)^{1/n}$.

putting $3\lambda=4$ and $2\mu=3$, we get

$3\lambda x+2\mu y = 4x+3y$ is $= 5\left[6\left(\frac{4}{3}\right)^3\left(\frac{3}{2}\right)^2\right]^{1/5}$

$= 5\left[6\times\frac{4}{3}\times\frac{4}{3}\times\frac{4}{3}\times\frac{3}{2}\times\frac{3}{2}\right]^{1/5}$

$\because 3\lambda=4 \Rightarrow \lambda=4/3$

$2\mu=3 \Rightarrow \mu=3/2$

$= 5(32)^{1/5}$

$= 5(2)$

$= 10$

$\therefore$ The least value of $4x+3y$ is 10.

$=\!=\!=\times=\!=\!=$

Q7. If p and q are positive integers, find the maximum value of $x^p y^q$ subject to $x+y=a$, where a is a given positive number and the numbers x and y are positive.

Soln :- Given that p and q are positive integers.

Let $k = x^p y^q$.

Then $\dfrac{k}{p^p q^q} = \dfrac{x^p}{p^p} \cdot \dfrac{y^q}{q^q} = \left(\dfrac{x}{p}\right)^p \cdot \left(\dfrac{y}{q}\right)^q$

$= \dfrac{x}{p}, \dfrac{x}{p}, \ldots p$ factors $\dfrac{y}{q}, \ldots q$ factors

$\therefore$ The sum of all these factors $= p\left(\dfrac{x}{p}\right) + q\left(\dfrac{y}{q}\right)$

$= x+y$

$= a$ (G.T. $x+y=a$)

Hence, product of the factors $\dfrac{k}{p^p q^q}$ is greatest, when all factors are equal, when $\dfrac{x}{p} = \dfrac{y}{q}$. each is equal to

$$= \frac{\text{Sum of factors}}{p+q} = \frac{a}{p+q}$$

$\Rightarrow \dfrac{x}{p} = \dfrac{a}{p+q} \Rightarrow x = \dfrac{ap}{p+q}$

and $\dfrac{y}{q} = \dfrac{a}{p+q} \Rightarrow y = \dfrac{aq}{p+q}$

$\therefore$ The greatest value of k is $= \left(\dfrac{ap}{p+q}\right)^p \cdot \left(\dfrac{aq}{p+q}\right)^q$

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Classical Algebra And Theory Of Numbers

(1)

Unit-V

Theory of Numbers :-

Prime Number :-

A number which is divisible by unity and itself is called Prime Numbers.

A number p is said to be a prime number if its only divisors one and p.

Example : 2, 3, 5, 7, ...

Composite Number :-

A number is said to be composite if it is not prime.

(or)

A number is said to be composite whose devisors other then one and itself.

Example :- 4, 6, 8, 9, ...

Note :-

Two numbers which have no common devise other then one is said to be prime to one another.

Example :- a) 2, 3 b) 3, 4 c) 4, 9.

=x=

Congruences :-

If a and b are any two integers are called congruences with respect to some k integer, such that $\frac{a-b}{m} = k \Rightarrow a = km + b$. It is denoted by $\boxed{a \equiv b \pmod{m}}$.

Example:

1) $3 \equiv 24 \pmod 7$, because $3-24 = -21$ is divisible by 7.

2) $42 \not\equiv 5 \pmod 8$ because $42-5 = 37$ is not divisible by 8.

$42-5$

$\frac{37}{8}$

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Note:-

Every Composite number can be written as product of prime numbers one and only one way.

i.e., $\boxed{N = p^a q^b r^c s^d \ldots}$ (*)

Here, $p, q, r, s, \ldots$ are Prime numbers and $a, b, c, d, \ldots$ are any integers.

2.

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Note:- (*)

No. of devisors $= (a+1)(b+1)(c+1)(d+1)\ldots$

Sum of devisors $= \left(\frac{p^{a+1}-1}{p-1}\right)\left(\frac{q^{b+1}-1}{q-1}\right)\left(\frac{r^{c+1}-1}{r-1}\right)\left(\frac{s^{d+1}-1}{s-1}\right)$

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Problems:-

1. Find the smallest number with 18 divisors.

Soln.:- Let N = smallest number with 18 divisors.

$\therefore N = p^a q^b r^c s^d \ldots$

Now, $18 = 2 \times 3 \times 3$

$$\begin{array}{r|l} 2 & 18 \\ \hline 3 & 9 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$

W.k.T.

No. of divisors $= (a+1)(b+1)(c+1)\cdots$

Here, $3 \cdot 3 \cdot 2 = (a+1)(b+1)(c+1)$

Now, $a+1 = 3 \Rightarrow a = 3-1 = \boxed{2 = a}$

$b+1 = 3 \Rightarrow b = 3-1 = \boxed{2 = b}$

$c+1 = 2 \Rightarrow c = 2-1 = \boxed{1 = c}$

$\therefore N = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \ldots$

ie., $N = 2^2 \cdot 3^2 \cdot 5^1$

$= 4 \cdot 9 \cdot 5$

$N = 180$

$\therefore$ The smallest number with 18 divisors $= 180$.

Find the smallest number with 10 divisors.

Soln.:-

Let N = Smallest no. with 10 divisors

$\therefore N = p^a q^b r^c s^d \ldots$

$$\begin{array}{r|l} 2 & 10 \\ \hline 5 & 5 \\ \hline & 1 \end{array}$$

Now, $10 = 5 \times 2$

W.K.T. No. of divisors $= (a+1)(b+1)(c+1)\cdots$

Here, $5 \times 2 = (a+1)(b+1)$

$\therefore a+1 = 5 \Rightarrow a = 5-1 \Rightarrow \boxed{a = 4}$

$b+1 = 2 \Rightarrow b = 2-1 \Rightarrow \boxed{b = 1}$

$\therefore N = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \ldots$

$= 2^4 \cdot 3^1 = 16 \cdot 3 = 48$

$\therefore$ The smallest number with 10 divisors $= 48$

3. Find the smallest number with 24 divisors.

Soln:- Let N = Smallest no. with 24 divisors.

$\therefore N = p^a q^b r^c s^d \ldots$

$$\begin{array}{r|l} 2 & 24 \\ 2 & 12 \\ 2 & 6 \\ 3 & 3 \\ & 1 \end{array}$$

Now, $24 = 3 \times 2 \times 2 \times 2$

W.K.T. No. of divisors $= (a+1)(b+1)(c+1)(d+1)\ldots$

Here, $3 \times 2 \times 2 \times 2 = (a+1)(b+1)(c+1)(d+1)$

Now, $a+1=3 \Rightarrow a=3-1 \Rightarrow a=2$

$b+1=2 \Rightarrow b=2-1 \Rightarrow b=1$

$c+1=2 \Rightarrow c=2-1 \Rightarrow c=1$

$d+1=2 \Rightarrow d=2-1 \Rightarrow d=1$

$\therefore N = 2^a \, 3^b \, 5^c \, 7^d \, 11^e \ldots$

$= 2^2 \, 3^1 \, 5^1 \, 7^1$

$= 4 \cdot 3 \cdot 5 \cdot 7$

$N = 420$

$\therefore$ $\boxed{\text{The smallest number with 24 divisors} = 420}$

~~4. Find the No. of divisors and sum of divisors of 840.~~

~~**Soln:-**~~

~~W.K.T. $N = p^a q^b r^c s^d \ldots$~~

$$\begin{array}{r|l} 2 & 840 \\ 2 & 420 \\ 2 & 210 \\ 2 & 105 \\ & 1 \end{array}$$

~~Now, $840 = 2 \times 2 \times 2 \times 3 \times 5 \times 7$~~

~~W.K.T. No. of divisors~~

~~$= (a+1)(b+1)(c+1)(d+1)(e+1)(f+1)\ldots$~~

~~Here $7 \times 5 \times 3 \times 2 \times 2 \times 2 = (a+1)(b+1)(c+1)(d+1)(e+1)(f+1)$~~

~~$\therefore a+1$~~

4. Find the number of divisors and sum of divisors of 840.

Soln: W.k.T. $N = p^a q^b r^c s^d \ldots$

$$\begin{array}{r|l} 2 & 840 \\ 2 & 420 \\ 2 & 210 \\ 3 & 105 \\ 5 & 35 \\ 7 & 7 \\ & 1 \end{array}$$

$\therefore 840 = 2 \times 2 \times 2 \times 3 \times 5 \times 7$

Now, $N = 2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1$

W.k.T. No. of divisors $= (a+1)(b+1)(c+1)(d+1)$

$= (3+1)(1+1)(1+1)(1+1)$

$= 4 \cdot 2 \cdot 2 \cdot 2$

$\therefore \boxed{\text{No. of divisors} = 32}$

Here, $p=2$, $q=3$, $r=5$, $s=7$

and $a=3$, $b=1$, $c=1$, $d=1$

$\therefore$ Sum of divisors $= \left(\frac{p^{a+1}-1}{p-1}\right)\left(\frac{q^{b+1}-1}{q-1}\right)\left(\frac{r^{c+1}-1}{r-1}\right)\left(\frac{s^{d+1}-1}{s-1}\right)$

$= \left(\frac{2^{3+1}-1}{2-1}\right)\left(\frac{3^{1+1}-1}{3-1}\right)\left(\frac{5^{1+1}-1}{5-1}\right)\left(\frac{7^{1+1}-1}{7-1}\right)$

$= \left(\frac{2^4-1}{1}\right)\left(\frac{3^2-1}{2}\right)\left(\frac{5^2-1}{4}\right)\left(\frac{7^2-1}{6}\right)$

$= \left(\frac{16-1}{1}\right)\left(\frac{9-1}{2}\right)\left(\frac{25-1}{4}\right)\left(\frac{49-1}{6}\right)$

$= \left(\frac{15}{1}\right)\left(\frac{8}{2}\right)\left(\frac{24}{4}\right)\left(\frac{48}{6}\right)$

$= 15 \times 4 \times 6 \times 8$

$= 2880$

$\therefore \boxed{\text{Sum of divisors} = 2880}$

$\Longrightarrow\times\Longleftarrow$

5. Find the number of divisors and sum of divisors of 1458. 6.
excluding the number itself.

Soln:-

2	1458
3	729
3	243
3	81
3	27
3	9
3	3
	1

W.k.T. $N = p^a q^b r^c s^d \ldots$

Now, $N = 2 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$

$= 2^1 3^6$

$\therefore$ No. of divisors $= (a+1)(b+1)(c+1)\ldots$

$= (1+1)(6+1)$

$= 2 \times 7$

$= 14$

Here, $\left.\begin{array}{l}\text{No. of divisors} \\ \text{excluding itself}\end{array}\right\} = 14 - 1 = 13.$

Here, $p = 2$, $q = 3$ and $a = 1$, $b = 6$

Sum of divisors $= \left(\frac{p^{a+1}-1}{p-1}\right)\left(\frac{q^{b+1}-1}{q-1}\right)$

$= \left(\frac{2^{1+1}-1}{2-1}\right)\left(\frac{3^{6+1}-1}{3-1}\right) = \left(\frac{2^2-1}{1}\right)\left(\frac{3^7-1}{2}\right)$

$= \left(\frac{4-1}{1}\right)\left(\frac{2187-1}{2}\right) = (3)\left(\frac{2186}{2}\right)$

$= (3)(1093) = 3279$

$= 3279.$

Sum of divisors excluding itself $= 3279 - 1458$

$= 1821$

$\therefore$

No. of divisors $= 13$
Sum of divisors $= 1821$

excluding the number itself.

=====X=====

6. Find the number of divisors of 480 excluding 1 and 480.

Soln:-

W.K.T. $N = p^a q^b r^c s^d \dots$

2	480
2	240
2	120
2	60
2	30
3	15
5	5
	1

Now, $N = 2\times2\times2\times2\times2\times3\times5$

$\therefore N = 2^5 \cdot 3^1 \cdot 5^1$

$\therefore$ No. of divisors $= (a+1)(b+1)(c+1)(d+1)\dots$

$= (5+1)(1+1)(1+1)$

$= 6\times2\times2$

$= 24$

No. of divisors excluding 1 and 480 $\Big\} = 24-2 = 22$

$= 22$

Here, $p=2, q=3, r=5$ and $a=5, b=1, c=1$

$\therefore$ Sum of divisors $= \left(\frac{p^{a+1}-1}{p-1}\right)\left(\frac{q^{b+1}-1}{q-1}\right)\left(\frac{r^{c+1}-1}{r-1}\right)$

$= \left(\frac{2^6-1}{2-1}\right)\left(\frac{3^2-1}{3-1}\right)\left(\frac{5^2-1}{5-1}\right)$

$= \left(\frac{2^6-1}{1}\right)\left(\frac{3^2-1}{2}\right)\left(\frac{5^2-1}{4}\right)$

$= \left(\frac{64-1}{1}\right)\left(\frac{9-1}{2}\right)\left(\frac{25-1}{4}\right) = \left(\frac{63}{1}\right)\left(\frac{8}{2}\right)\left(\frac{24}{4}\right)$

$= 63\times4\times6$

$= 1512$

Sum of divisors excluding 1 and 480 $= 1512-1-480$

$= 1031$

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Home Work

1. Find the no. of divisors and sum of divisors of 360.
(i) 24 (ii) 1170.

2. Find the no. of divisors and sum of divisors of 288 excluding the number itself.
Ans:- No. of divisors = 17   531

=x=

Euler's function: $[\phi(N)]$

The number of positive integer less than N and prime to it, is denoted by $\phi(N)$.

Example:-
~~N = +ve~~ integer
$\phi(11) = 10$
$\therefore$ 11 is prime to 10, they have no common divisor.

=x=

Note:
The number of +ve integers less than N and not prime to each is denoted by $\phi(N)$, N is composi[te]

Note: N = composite
W.K.T. $N = p^a q^b r^c s^d \ldots$

Here, $p, q, r, s, \ldots$ are prime
and $a, b, c, d, \ldots$ are +ve integers

$$\therefore \boxed{\phi(N) = N\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\left(1-\frac{1}{r}\right)\left(1-\frac{1}{s}\right)\cdots}$$

=x=

Problems:-

1. Find $\phi(N)$, when $N=240$.

Soln.:-

Given that $N=240$

W.K.T. $N = p^a q^b r^c s^d \ldots$

$\therefore N = 2^4\, 3^1\, 5^1$

$$\begin{array}{r|l} 2 & 240 \\ 2 & 120 \\ 2 & 60 \\ 2 & 30 \\ 3 & 15 \\ 5 & 5 \\ & 1 \end{array}$$

Here $p=2, q=3, r=5$ and

$a=4, b=1, c=1$

Now, $\phi(N) = N\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\left(1-\frac{1}{r}\right)$

$= 240\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$

$= 240\left(\frac{2-1}{2}\right)\left(\frac{3-1}{3}\right)\left(\frac{5-1}{5}\right)$

$= 240\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)$

$= \cancel{240}\left(\frac{4}{\cancel{3\times5}}\right) = 16\times4$

$= 64$

$\therefore \boxed{\phi(N) = 64}$

2. Find the number of integers less than 729.

Soln.:-

W.K.T. $N = p^a q^b r^c s^d \ldots$

Given that $N = 240$ 729.

i.e., $N = 2^0\, 3^6$

$$\begin{array}{r|l} 3 & 729 \\ 3 & 243 \\ 3 & 81 \\ 3 & 27 \\ 3 & 9 \\ 3 & 3 \\ & 1 \end{array}$$

Here, $p=2, q=3$ and $a=0, b=6$

W.K.T. $\phi(N) = N\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)$

$= 729\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)$

$= 729\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)$

$\therefore \boxed{\phi(N) = 243}$

Home Work

1. Find the number of integers less than 720. Ans:- 192.

2. Find $\phi(N)$, when $N = 180$. Ans:- 48.

Divisors of a given number N.

N can be expressed the product of primes and let N be $p^a q^b r^c \ldots$, where $p, q, r, \ldots$ are prime number

Let n be the no. of divisors.

The divisors of N are the terms in the expansion of

$$(1+p+p^2+\cdots+p^a)(1+q+q^2+\cdots+q^b)(1+r+r^2+\cdots+r^c)\cdots$$

Hence, the number of terms in the product will be the number of divisors and we can easily see that the no. of divisors is $(a+1)(b+1)(c+1)\ldots$

The divisors include 1 and the number N itself.

$\therefore$ The sum of all divisors is the sum of all the terms in the continued product.

$$\therefore \boxed{S = \left(\frac{p^{a+1}-1}{p-1}\right)\left(\frac{q^{b+1}-1}{q-1}\right)\left(\frac{r^{c+1}-1}{r-1}\right)\cdots}$$

Reminder:

Euler's function $\phi(N) = N\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\left(1-\frac{1}{r}\right)\cdots$

Corollary: 1

If $N = ab$, where a & b are prime to one another, then

$$\phi(N) = \phi(a)\cdot\phi(b)$$

Corollary: 2

If $a, b, c, d, \ldots, k$ are prime to one another, then

$$\phi(abcd\ldots k) = \phi(a)\cdot\phi(b)\cdot\phi(c)\ldots\phi(k).$$

Corollary: 3

If p is prime, then $\phi(p^r) = p^r\left(1-\frac{1}{p}\right)$

$\phi(p^{10}) = p^{10}\left(1-\frac{1}{p}\right)$

Result:-

If $d_1, d_2, \ldots, d_r$ (including 1 and itself (N)) are the divisors of N, then show that

$\phi(d_1) + \phi(d_2) + \cdots + \phi(d_r) = N$.

proof:-

Given that $d_1, d_2, \ldots, d_r$ are divisors of N.

W.k.T. $N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots, p_n^{a_n}$, where $p_1, p_2, \ldots, p_n$ are primes and $a_1, a_2, a_3, \ldots, a_n$ are integers.

W.k.T. $\phi(N) = \phi(p_1^{a_1}\cdot p_2^{a_2}\ldots p_n^{a_n}) = \phi(p_1^{a_1})\,\phi(p_2^{a_2})\ldots\phi(p_n^{a_n})$

Hence $\phi(d_1), \phi(d_2), \phi(d_3), \ldots, \phi(d_r)$ are the terms in the expansion of

$$\left[1+\phi(p_1)+\phi(p_1^2)+\cdots+\phi(p_1^{a_1})\right]$$
$$\times\left[1+\phi(p_2)+\phi(p_2^2)+\cdots+\phi(p_2^{a_2})\right]$$
$$\times\cdots\cdots\cdots$$
$$\times\left[1+\phi(p_n)+\phi(p_n^2)+\cdots+\phi(p_n^{a_n})\right]$$

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$\therefore \phi(d_1)+\phi(d_2)+\cdots+\phi(d_n)$

$$= [1+\phi(P_1)+\phi(P_1^2)+\cdots+\phi(P_1^{a_1})]$$
$$\times [1+\phi(P_2)+\phi(P_2^2)+\cdots+\phi(P_2^{a_2})]$$
$$\times \cdots\cdots$$
$$\times [1+\phi(P_n)+\phi(P_n^2)+\cdots+\phi(P_n^{a_n})]$$

Now,
$$1+\phi(P_1)+\phi(P_1^2)+\cdots+\phi(P_1^{a_1})$$
$$= 1+P_1\left(1-\frac{1}{P_1}\right)+P_1^2\left(1-\frac{1}{P_1}\right)+\cdots+P_1^{a_1}\left(1-\frac{1}{P_1}\right)$$
$$= 1+P_1\left(\frac{P_1-1}{P_1}\right)+P_1^2\left(\frac{P_1-1}{P_1}\right)+\cdots+P_1^{a_1}\left(\frac{P_1-1}{P_1}\right).$$
$$= 1+P_1-1+P_1^2-P_1+\cdots+P_1^{a_1-1}(P_1-1)$$
$$= 1+P_1-1+P_1^2-P_1+\cdots+P_1^{a_1}-P_1^{a_1-1}$$
$$= P_1^{a_1}$$

$|||^{ly}$
$$1+\phi(P_2)+\phi(P_2^2)+\cdots+\phi(P_2^{a_2}) = P_2^{a_2}$$
$$1+\phi(P_3)+\phi(P_3^2)+\cdots+\phi(P_3^{a_3}) = P_3^{a_3}$$
$$\cdots$$
$$\cdots$$
$$1+\phi(P_n)+\phi(P_n^2)+\cdots+\phi(P_n^{a_n}) = P_n^{a_n}$$

i.e., $\phi(d_1)+\phi(d_2)+\cdots+\phi(d_n) = P_1^{a_1}\cdot P_2^{a_2}\cdot P_3^{a_3}\cdots P_n^{a_n}$

$$\therefore \boxed{\phi(d_1)+\phi(d_2)+\cdots+\phi(d_n) = N}$$

Hence the proof.

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Sec:10 The highest power of a prime p contained in $n!$

The highes power of a prime number p contained in $n!$ is

$$\boxed{\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\cdots+\left[\frac{n}{p^k}\right], \text{ if } \left[\frac{n}{p^{k+1}}\right]=0.}$$

Problems:-

1. Find the highest power of 3 dividing 1000!.

Soln:-

W.K.T. The highest power of a prime no. p divide $n!$ is

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots+\left[\frac{n}{p^k}\right], \text{ where } \left[\frac{n}{p^{k+1}}\right]=0.$$

Here, $n! = 1000!$ and $p=3$.

$$\therefore \left[\frac{n}{p}\right]=\left[\frac{1000}{3}\right]=333$$

$$\frac{n}{p.p} \quad \frac{n/p}{p}$$

$$\left[\frac{n}{p^2}\right]=\left[\frac{1000}{3^2}\right]=\left[\frac{333}{3}\right]=111$$

$$\frac{n}{p^2 p}=\frac{n/p^2}{p}$$

$$\left[\frac{n}{p^3}\right]=\left[\frac{1000}{3^3}\right]=\left[\frac{111}{3}\right]=37$$

$$\left[\frac{n}{p^4}\right]=\left[\frac{1000}{3^4}\right]=\left[\frac{37}{3}\right]=12$$

$$\left[\frac{n}{p^5}\right]=\left[\frac{1000}{3^5}\right]=\left[\frac{12}{3}\right]=4$$

$$\left[\frac{n}{p^6}\right]=\left[\frac{1000}{3^6}\right]=\left[\frac{4}{3}\right]=1$$

$$3^{498} / 1000!$$

$$=333+111+37+12+4+1=498$$

$\therefore$ $\boxed{3^{498} \text{ is the highest power of 3 dividing } 1000!}$

2. Find the highest power of 2, 5, 7, 11, 13 Contained in 1000!

11^{98} 5^{249} 7^{164}

Soln.:- W.k.T. The highest power of a prime number p contained in $n!$ is

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\cdots+\left[\frac{n}{p^k}\right], \text{ where } \left[\frac{n}{p^{k+1}}\right]=0$$

(i) Here, $n! = 1000!$ and $p = 13$

Now, $\left[\frac{n}{p}\right] = \left[\frac{1000}{13}\right] = 76$ $\quad 5^{249}$

$\left[\frac{n}{p^2}\right] = \left[\frac{1000}{13^2}\right] = \left[\frac{76}{13}\right] = 5$ $\quad 5^{49}$

$\therefore$ The highest power of 13 in 1000! is $\quad 7$

$= 76 + 5$

$= 81$

$\therefore$ $\boxed{13^{81} \text{ is the highest power of 13 dividing 1000!}}$

(ii) Here $n! = 1000!$ and $p = 2$

Now, $\left[\frac{n}{p}\right] = \left[\frac{1000}{2}\right] = 500$

$\left[\frac{n}{p^2}\right] = \left[\frac{1000}{2^2}\right] = \left[\frac{500}{2}\right] = 250$

$\left[\frac{n}{p^3}\right] = \left[\frac{1000}{2^3}\right] = \left[\frac{250}{2}\right] = 125$

$\left[\frac{n}{p^4}\right] = \left[\frac{1000}{2^4}\right] = \left[\frac{125}{2}\right] = 62$

$\left[\frac{n}{p^5}\right] = \left[\frac{1000}{2^5}\right] = \left[\frac{62}{2}\right] = 31$

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$$\left[\frac{n}{p^6}\right] = \left[\frac{1000}{2^6}\right] = \left[\frac{31}{2}\right] = 15$$

$$\left[\frac{n}{p^7}\right] = \left[\frac{1000}{2^7}\right] = \left[\frac{15}{2}\right] = 7$$

$$\left[\frac{n}{p^8}\right] = \left[\frac{1000}{2^8}\right] = \left[\frac{7}{2}\right] = 3$$

$$\left[\frac{n}{p^9}\right] = \left[\frac{1000}{2^9}\right] = \left[\frac{3}{2}\right] = 1$$

$\therefore$ The highest power of 2 in 1000! is

$$= 500 + 250 + 125 + 62 + 31 + 15 + 7 + 3 + 1$$

$$= 994$$

$\therefore$ $\boxed{2^{994} \text{ is the highest powers of 2 dividing } 1000!}$

—X—

Note:-

The number of zeros in end of n! is a power of a prime number 5 is

$$\left[\frac{n}{5}\right] + \left[\frac{n}{5^2}\right] + \left[\frac{n}{5^3}\right] + \cdots + \left[\frac{n}{5^k}\right], \text{ where } \left[\frac{n}{5^{k+1}}\right] = 0.$$

—X—

Problems:-

With how many zeros does 79! end?

Soln:-

W.K.T. The highest number of zeros end of a prime number 5 dividing n! is

$\left[\frac{n}{5}\right]+\left[\frac{n}{5^2}\right]+\cdots+\left[\frac{n}{5^k}\right]$, where $\left[\frac{n}{5^{k+1}}\right]=0$.

Here, $n! = 79!$ and $p=5$

Now, $\left[\frac{n}{5}\right] = \frac{79}{5} = 15$

$$\left[\frac{n}{5^2}\right] = \left[\frac{79}{5^2}\right] = \left[\frac{15}{5}\right] = 3$$

$\therefore$ The no. of zeros of a prime 5 dividing 79! is

$= 15+3$

$= 18$

$\therefore$ $\boxed{79! \text{ will end in 18 zeros.}}$

---

19 zeros

2. With how many zeros does (i) 257! (ii) 61! (iii) 82!

14 zeros

Soln:-

W.K.T. The highest no. of zeros end of a prime number 5 dividing $n!$ is

$\left[\frac{n}{5}\right]+\left[\frac{n}{5^2}\right]+\cdots+\left[\frac{n}{5^k}\right]$, where $\left[\frac{n}{5^{k+1}}\right]=0$

Here, (i) $n! = 257!$ and $p=5$

Now, $\left[\frac{n}{5}\right] = \left[\frac{257}{5}\right] = 51$

$$\left[\frac{n}{5^2}\right] = \left[\frac{257}{5^2}\right] = \left[\frac{51}{5}\right] = 10$$

$$\left[\frac{n}{5^3}\right] = \left[\frac{257}{5^3}\right] = \left[\frac{10}{5}\right] = \frac{2}{15}$$

$\frac{n}{5^4} = \frac{257}{5^4}$

$\therefore$ The no of zeros of 5 dividing 257! is

$= 51 + 10 + 2$

$= 63$

$\therefore$ $\boxed{257! \text{ will ends in } 63 \text{ zeros}}$

Sec: 12 Congruences

Congruences with the same moduli possess many properties of qualities. Some of them are given below:

1. If $a \equiv b \pmod m$ and $a_1 \equiv b_1 \pmod m$ and if q, r are integers, then $qa + ra_1 \equiv (qb + rb_1) \pmod m$.

Proof:-

W.k.T. $\frac{a-b}{m} = k \Rightarrow a-b = km \Rightarrow a = b+km$, $a \equiv b \pmod m$

m^{rly}

$\frac{a_1 - b_1}{m} = k_1 \Rightarrow a_1 - b_1 = k_1 m \Rightarrow a_1 = b_1 + k_1 m$

$\Rightarrow a_1 \equiv b_1 \pmod m$

$\therefore qa + ra_1 = q(b+km) + r(b_1 + k_1 m)$

$= qb + qkm + rb_1 + rk_1 m$

$= qb + rb_1 + m(qk + rk_1)$

$= qb + rb_1 + Mm$, where $M = qk + rk_1$

ie., $\boxed{qa + ra_1 \equiv (qb + rb_1) \pmod m}$

Hence the proof.

Corollary: 1

If $a \equiv b \pmod m$; $a_1 \equiv b_1 \pmod m$, then $a + a_1 \equiv b + b_1 \pmod m$ and $a - a_1 \equiv b - b_1 \pmod m$

Corollary: 2

If $a \equiv b \pmod{m}$, $a_1 \equiv b_1 \pmod{m}$, $a_2 \equiv b_2 \pmod{m}$,
then $a + a_1 + a_2 + \cdots \equiv b + b_1 + b_2 + \cdots \pmod{m}$.

=== x ===

2. If $a \equiv b \pmod{m}$; $a_1 \equiv b_1 \pmod{m}$, then
$aa_1 \equiv bb_1 \pmod{m}$.

Proof:-

$a \equiv b \pmod{m} \Rightarrow a = b + km$

$a_1 \equiv b_1 \pmod{m} \Rightarrow a_1 = b_1 + k_1 m$

$$\therefore aa_1 = (b+km)(b_1+k_1m)$$
$$= bb_1 + bk_1m + b_1km + kmk_1m$$
$$= bb_1 + m(kb_1 + k_1b + kk_1m)$$

ie., $\boxed{aa_1 \equiv bb_1 \pmod{m}}$

Hence the proof.

=== x ===

Corollary: 1

If $a \equiv b \pmod{m}$, $a_1 \equiv b_1 \pmod{m}$, $a_2 \equiv b_2 \pmod{m}$, ...
then $a a_1 a_2 \cdots \equiv b\, b_1\, b_2 \cdots \pmod{m}$.

Corollary: 2

If $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$

Corollary: 3

If $a \equiv b \pmod{m}$, then $f(a) \equiv f(b) \pmod{m}$, if $f(x)$
a polynomial in x.

=== x ===

These results show that congruences may be manipulated as regards addition, subtraction and multiplication with integral numbers, just like equations. As regards division a modifications is necesary.

=== x ===

3. If $ax \equiv bx \pmod{m}$ and if h is H.C.F. of x, m, then $a \equiv b \left(\bmod \left(\frac{m}{h}\right)\right)$.

Proof:-

Let $x = ph$, $m = qh$, where p, q are coprime.

$\therefore ax \equiv bx \pmod{m} \Rightarrow ax - bx = km$.

i.e., $aph - bph = kqh$

$\Rightarrow p\cancel{h}(a-b) = kq\cancel{h}$

$\Rightarrow \quad a - b = k\frac{q}{p}$

Here, q is prime to p.

$\therefore$ $a-b$ has q as a factor.

$\Rightarrow \quad a-b = M(q)$

$\Rightarrow \quad a-b = M\left(\frac{m}{h}\right)$

$\Rightarrow \quad \boxed{a \equiv b\left(\bmod\left(\frac{m}{h}\right)\right)}$

Hence the proof.

=== x ===

Corollary:

If $h = 1$, $a \equiv b \pmod{m}$

Thus the rule of cancellation holds for congruence on the condition that the cancelled factor is relatively prime to the modulus.

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4. If $a \equiv b \pmod{m_1}$, $a \equiv b \pmod{m_2}$, $a \equiv b \pmod{m_3}$ ... $a \equiv b \pmod{m_n}$, then $a \equiv b \pmod{m}$, where m is the least multiple of $m_1, m_2, \dots, m_n$.

Proof:- Given that

$$a \equiv b \pmod{m_1} \Rightarrow a-b = \text{a multiple of } m_1$$

$a \equiv b \pmod{m_2}$ ||ly $\Rightarrow a-b = \text{a multiple of } m_2$

$a \equiv b \pmod{m_3}$ $a-b = \text{a multiple of } m_3$

$\cdots$

$\cdots$

$\cdots$

$a \equiv b \pmod{m_n}$ $a-b = \text{a multiple of } m_n$

Therefore, we get,

$a-b = \text{a multiple of } m$, where m is the least common multiple of $m_1, m_2, \dots, m_n$.

$$\therefore \boxed{a \equiv b \pmod{m}}$$

Hence the proof

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Sec: 16 Fermat's Theorem

If 'p' is a prime and 'a' is any number prime to p, then $a^{p-1}-1$ is divisible by p.

proof:-

Given that p is prime number, and 'a' is any ~~prime~~ number prime to p.

Consider,

$(x+a)^n = x^n + nC_1 x^{n-1} a + nC_2 x^{n-2} a^2 + \dots + nC_n x^{n-n} a^n$

$$(a+1)^p = a^p + pC_1\, a^{p-1}(1) + pC_2\, a^{p-2}(1)^2 + \dots + pC_{p-1}\, a^1 (1)^{p-1} + pC_p\, a^0 (1)^p$$

$$(a+1)^p = a^p + pC_1\, a^{p-1} + pC_2\, a^{p-2} + \dots + pC_{p-1}\, a + 1$$

$$\Rightarrow (a+1)^p - a^p - 1 = pC_1\, a^{p-1} + pC_2\, a^{p-2} + \dots + pC_{p-1}\, a.$$

$$\Rightarrow (a+1)^p - (a^p+1) = \text{a multiple of } p$$

$\because pC_1, pC_2 \dots pC_{p-1}$ are all divisible by p

$$\Rightarrow (a+1)^p - (a^p+1) \equiv 0 \pmod{p}$$

$$\Rightarrow (a+1)^p \equiv (a^p+1) \pmod{p}$$

put $a = a-1 \Rightarrow a^p \equiv ((a-1)^p+1) \pmod{p}$

$a = a-2 \Rightarrow (a-1)^p \equiv ((a-2)^p+1) \pmod{p}$

$a = a-3 \Rightarrow (a-2)^p \equiv ((a-3)^p+1) \pmod{p}$

$\dots \qquad \dots$

$\dots \qquad \dots$

$\dots \qquad \dots$

$a = 2 \Rightarrow 3^p \equiv (2^p+1) \pmod{p}$

$a = 1 \Rightarrow 2^p \equiv (1^p+1) \pmod{p}$

Adding the above equations, we get

$\Rightarrow a^p + (a-1)^p + (a-2)^p + \cdots + 3^p + 2^p$

$$\equiv \left[((a-1)^p + 1) + ((a-2)^p + 1) + ((a-3)^p + 1) + \cdots + (2^p \right.$$

$$\left. + (1^p + 1) \right] \pmod p$$

$\Rightarrow$

$a^p + \cancel{(a-1)^p} + \cancel{(a-2)^p} + \cdots + \cancel{3^p} + \cancel{2^p}$

$$\equiv (\cancel{(a-1)^p} + 1 + \cancel{(a-2)^p} + 1 + \cancel{(a-3)^p} + 1 + \cdots$$

$$+ \cancel{2^p} + 1 + 1^p + 1) \pmod p$$

$\Rightarrow a^p \equiv 1^p + 1 + 1 + \cdots + 1$ $(a-1)$ times $\pmod p$

$\Rightarrow a^p \equiv ((a-1) + 1^p) \pmod p$

$\Rightarrow a^p \equiv (a-1+1) \pmod p$

$\Rightarrow a^p \equiv a \pmod p$

$\Rightarrow a^p - a \equiv 0 \pmod p$

$\therefore a^p - a$ is divisible by p.

$\therefore a(a^{p-1} - 1)$ is divisible by p

ie., $\boxed{a^{p-1} - 1 \text{ is divisible by } p}$

Hence the proof.

Method : II

Let $(x+y)^p = x^p + pC_1\, x^{p-1} y + pC_2\, x^{p-2} y^2 + \cdots$

$$+ pC_{p-1}\, x\, y^{p-1} + y^p$$

Here, $pC_1, pC_2, \ldots, pC_{p-1}$ are divisible by p.

$\therefore (x+y)^p \equiv (x^p + y^p) \pmod p \rightarrow (1)$

Now,
$$(x+y+z)^p = (x+y)^p + {}^pC_1 (x+y)^{p-1} z + {}^pC_2 (x+y)^{p-2} z^2 + \cdots + z^p$$

$$\Rightarrow (x+y+z)^p \equiv \left((x+y)^p + z^p\right) \pmod p$$

$$\Rightarrow (x+y+z)^p \equiv (x^p + y^p + z^p) \pmod p \quad \text{(by ①)} \quad \text{②}$$

$\cdots$

$$\Rightarrow (x+y+z+\cdots+w)^p \equiv (x^p + y^p + z^p + \cdots + w^p) \pmod p$$

where $x, y, z, \ldots, w$ are any integers.

Let there be a integers, put each equal to 1.

$$\Rightarrow (1+1+1+\cdots+1 \text{ } a \text{ terms})^p \equiv (1^p + 1^p + 1^p + \cdots + 1^p) \pmod p$$

$$\Rightarrow a^p \equiv (1+1+1+\cdots+1) \pmod p$$

$$\Rightarrow a^p \equiv a \pmod p$$

$$\Rightarrow a^p - a \equiv 0 \pmod p$$

$\therefore$ $a^p - a$ is divisible by p.

$\therefore$ $a(a^{p-1} - 1)$ is divisible by p.

i.e., $\boxed{a^{p-1} - 1 \text{ is divisible by } p}$

Hence the proof.

Method : III

When $a, 2a, 3a, \ldots, (p-1)a$ are divided by p, the remainders are $1, 2, \ldots (p-1)$ in a certain order. since p is prime to a.

Let $a \equiv r_1 \pmod p$

$2a \equiv r_2 \pmod p$

$3a \equiv r_3 \pmod p$

$\vdots$

$(p-1)a \equiv r_{p-1} \pmod p$

Here, $r_1, r_2, \ldots, r_{p-1}$ are $1, 2, 3, \ldots, (p-1)$ in a certain order.

$\therefore a \cdot 2a \cdot 3a \cdots (p-1)a \equiv r_1 r_2 r_3 \cdots r_{p-1} \pmod p$

$\Rightarrow [1 \cdot 2 \cdot 3 \cdots (p-1)]\, a^{p-1} \equiv 1 \cdot 2 \cdot 3 \cdots (p-1) \pmod p$

$\Rightarrow (p-1)!\, a^{p-1} \equiv (p-1)! \pmod p$

$\Rightarrow (p-1)!\, a^{p-1} - (p-1)! \equiv 0 \pmod p$

$\Rightarrow (p-1)! \left[a^{p-1} - 1 \right] \equiv 0 \pmod p$

$\therefore (p-1)!\, (a^{p-1} - 1)$ is divisible by p.

But $(p-1)!$ is not divisible by p. Because p is prime.

$\therefore$ $\boxed{a^{p-1} - 1 \text{ is divisible by } p}$

Hence the proof.

Corollary: 1

$a^p - a$ is divisible by p, if p is prime and 'a' is prime to p.

Corollary: 2

If p is an odd prime and a is prime to p, then $a^{\frac{1}{2}(p-1)} \pm 1$ is divisible by p.

proof:- Given that p is an odd prime and a is prime to p.

Let $a^{p-1} - 1 = (a^{\frac{p-1}{2}})^2 - 1^2$ $\quad x^{ab} = (x^a)^b$

$= \left[a^{\frac{1}{2}(p-1)} - 1\right]\left[a^{\frac{1}{2}(p-1)} + 1\right]$ $\quad (a^{\frac{p-1}{2}})^2$ $\quad a^{\frac{2}{2}(p-1)}$

Here, $a^{p-1} - 1$ is divisible by p.

$\therefore a^{\frac{1}{2}(p-1)} - 1$ or $a^{\frac{1}{2}(p-1)} + 1$ is divisible by p.

i.e., $\boxed{a^{\frac{1}{2}(p-1)} \pm 1 \text{ is divisible by } p.}$

Hence the proof.

Problems:-

Show that if x and y are both prime to the prime number n, then $x^{n-1} - y^{n-1}$ is divisible by n. Deduce that $x^{12} - y^{12}$ is divisible by 1365.

proof:-

Given that x and y are both prime to n.

Here n is prime number.

Let $x^{n-1} - 1 \equiv 0 \pmod{n}$, since x is prime to n & n is prime. $\rightarrow (1)$

IIIly $y^{n-1} - 1 \equiv 0 \pmod{n} \rightarrow (2)$

Subtracting (1) & (2), we get

$\Rightarrow x^{n-1} - 1 - y^{n-1} + 1 \equiv 0 \pmod n$

$\Rightarrow x^{n-1} - y^{n-1} \equiv 0 \pmod n \rightarrow (1)$

$\therefore$ $\boxed{x^{n-1} - y^{n-1} \text{ is divisible by } n}$

Next, we prove that $x^{12} - y^{12}$ is divisible by 1365.

Now, $x^{12} - y^{12} \equiv 0 \pmod{13}$ $\quad (\because x^{13-1} - y^{13-1} \equiv 0 \pmod{13}$, b

$\therefore x^{12} - y^{12}$ is divisible by 13. $\rightarrow (2)$

$x^{12} - y^{12} = (x^6)^2 - (y^6)^2$

$= (x^6 - y^6)(x^6 + y^6)$

but $x^6 - y^6 = x^{7-1} - y^{7-1} \equiv 0 \pmod 7$ (by (1)

$\therefore x^6 - y^6$ is divisible by 7.

ie., $x^{12} - y^{12}$ is divisible by 7 $\rightarrow (3)$

$x^{12} - y^{12} = x^{12} - y^{12} + x^8y^4 + x^4y^8 - x^8y^4 - x^4y^8$

$= x^{12} + x^8y^4 + x^4y^8 - x^8y^4 - x^4y^8 - y^{12}$

$= x^4(x^8 + x^4y^4 + y^8) - y^4(x^8 + x^4y^4 + y^8)$

$= (x^4 - y^4)(x^8 + x^4y^4 + y^8)$

but $x^4 - y^4 = x^{5-1} - y^{5-1} \equiv 0 \pmod 5$

$\therefore x^4 - y^4$ is divisible by 5.

ie., $x^{12} - y^{12}$ is divisible by 5 $\rightarrow (4)$

$x^{12} - y^{12} = (x^6)^2 - (y^6)^2$

$= (x^6 - y^6)(x^6 + y^6)$

$= (x^6 + x^4y^2 + x^2y^4 - x^4y^2 - x^2y^4 - y^6)(x^6 + y^6)$

$= [x^6 + x^4y^2 + x^2y^4 - (x^4y^2 + x^2y^4 + y^6)](x^6 + y^6)$

$x^{12}-y^{12} = [x^2(x^4+x^2y^2+y^4) - y^2(x^4+x^2y^2+y^4)](x^6+y^6)$

$= (x^2-y^2)(x^4+x^2y^2+y^4)(x^6+y^6)$

but $x^2-y^2 = x^{3-1}-y^{3-1} \equiv 0 \pmod 3$

$\therefore x^2-y^2$ is divisible by 3.

i.e., $x^{12}-y^{12}$ is divisible by $3 \to (5)$

by i.e., $x^{12}-y^{12}$ is divisible by $13\times7\times5\times3$ (by (2), (3), (4) & (5))

$\therefore$ $\boxed{x^{12}-y^{12} \text{ is divisible by } 1365}$

Hence the proof.

2. Show that the 8th power of any number is of the form $17m$ or $17m\pm1$.

Proof:-

Let the number be N.

N may be prime to 17 or may not be prime to 17.

If N is not prime to 17, it must be a multiple of 17. Since 17 is a prime number.

In that case N is a multiple of 17.

$\therefore$ $\boxed{N^8 \text{ is of the form } 17m}$

$a-b = km$

$N^8 = 17m$

If N is prime to 17,

$a^{p-1}-1 \equiv 0 \pmod p$

$a^{p-1}-1$ is divisible by p

$\therefore N^{17-1}-1$ is divisible by 17.

i.e., $N^{16}-1$ is divisible by 17

$\Rightarrow (N^8)^2-1$ is divisible by 17.

i.e., $(N^8-1)(N^8+1)$ is divisible by 17.

$\therefore N^8-1$ or N^8+1 is divisible by 17.

$\therefore N^8-1=17m$ or $N^8+1=17m$

$\Rightarrow N^8=17m+1$ or $N^8=17m-1$

Hence, $N^8=17m\pm1$

$\therefore$ $\boxed{N^8 \text{ is one of the form } 17m \text{ or } 17m\pm1}$

Hence the proof

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3. Show that $n^{13}-n$ is divisible by 2730.

proof:-

If $n^{13}-n = n(n^{13-1}-1)$ $\qquad n^{13-1}-1\equiv 0(\text{mod } 13)$

but $n^{13-1}-1$ is divisible by 13 (by fermat's theorem)

i.e., $n^{13}-n$ is divisible by 13 $\rightarrow$ (1)

Then, $n^{13}-n = n(n^{12}-1)$

$= n((n^6)^2-1)$

$= n(n^6-1)(n^6+1)$

but $n^6-1\equiv 0(\text{mod } 7)$ $\qquad n^{7-1}-1\equiv 0(\text{mod } 7)$

$\therefore n^6-1$ is divisible by 7.

i.e., $n^{13}-n$ is divisible by 7. $\rightarrow$ (2)

Then, $n^{13}-n = n(n^{12}-1)$

$= n(n^{12}-1+n^8+n^4-n^8-n^4)$

$= n(n^{12}+n^8+n^4-n^8-n^4-1)$

$= n[n^4(n^8+n^4+1)-(n^8+n^4+1)]$

$= n(n^4-1)(n^8+n^4+1)$

$n^{5-1}-1\equiv 0(\text{mod } 5)$

but $n^4 - 1 \equiv 0 \pmod 5$

$\therefore n^4 - 1$ is divisible by 5.

ie., $n^{13} - n$ is divisible by 5. $\to$ (3)

Then, $n^{13} - n = n(n^{12} - 1)$

$= n(n^6 - 1)(n^6 + 1)$

$= n[n^6 + n^4 + n^2 - n^4 - n^2 - 1](n^6 + 1)$

$= n[n^2(n^4 + n^2 + 1) - (n^4 + n^2 + 1)](n^6 + 1)$

$= n(n^2 - 1)(n^4 + n^2 + 1)(n^6 + 1)$

but $n^2 - 1 \equiv 0 \pmod 3$ $\quad n^{3-1} - 1 \equiv 0 \pmod 3$

$\therefore n^2 - 1$ is divisible by 3.

ie., $n^{13} - n$ is divisible by 3. $\to$ (4)

Then, $n^{13} - n = n(n^2 - 1)(n^4 + n^2 + 1)(n^6 + 1)$

$= n(n-1)(n+1)(n^4 + n^2 + 1)(n^6 + 1)$

but $n - 1 \equiv 0 \pmod 2$ $\quad n^1 = n^{2-1} - 1$

$\therefore n - 1$ is divisible by 2.

ie., $n^{13} - n$ is divisible by 2. $\to$ (5)

$\therefore n^{13} - n$ is divisible by 13, 7, 5, 3 and 2. by (1), (2), (3), (4) & (5)

ie., $n^{13} - n$ is divisible by $13 \times 7 \times 5 \times 3 \times 2$

ie., $\boxed{n^{13} - n \text{ is divisible by } 2730}$

Hence the proof.

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4. Show that $n^7 - n$ is divisible by 42.

Proof:-

If $n^7 - n = n(n^6 - 1)$

but $n^6 - 1 \equiv 0 \pmod 7$ $n^{7-1}-1 \equiv 0 \pmod 7$

$\therefore n^6 - 1$ is divisible by 7. $n^{p-1}-1 \equiv 0 \pmod p$

i.e., $n^7 - n$ is divisible by 7. $\rightarrow$ (1)

Then $n^7 - n = n(n^6 - 1)$

$$= n(n^6 + n^4 - n^4 + n^2 - n^2 - 1)$$

$$= n[(n^6 + n^4 + n^2) - (n^4 + n^2 + 1)]$$

$$= n[n^2(n^4 + n^2 + 1) - (n^4 + n^2 + 1)]$$

$$= n(n^2 - 1)(n^4 + n^2 + 1)$$

but $n^2 - 1 \equiv 0 \pmod 3$ $n^{3-1} - 1 \equiv 0 \pmod 3$

$\therefore n^2 - 1$ is divisible by 3.

i.e., $n^7 - n$ is divisible by 3. $\rightarrow$ (2)

Then, $n^7 - n = n(n^2 - 1)(n^4 + n^2 + 1)$

$$= n(n+1)(n-1)(n^4 + n^2 + 1)$$

but $n - 1 \equiv 0 \pmod 2$ $n^{2-1} - 1 \equiv 0 \bmod (2)$

$\therefore n - 1$ is divisible by 2.

i.e., $n^7 - n$ is divisible by 2. $\rightarrow$ (3)

$\therefore n^7 - n$ is divisible by 7, 3, 2 (by (1), (2) & (3))

i.e., $n^7 - n$ is divisible by $7 \times 3 \times 2$

$\therefore$ $\boxed{n^7 - n \text{ is divisible by } 42}$

Hence the proof.

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Home Work

1. Show that n^5-n is divisible by 30.

2. Show that the 4th power of any number is of the form $5m$ (or) $5m+1$.

3. Show that the 9th power of any number is one of the forms $19m$ (or) $19m\pm1$.

Sec: 17 Wilson's Theorem

If p is a prime number, then $(p-1)!+1$ is divisible by p.

Proof:-

Given that p is a prime number.

Let 'a' is any number of the series $1, 2, \ldots, (p-1)$.

Let $a, 2a, 3a, \ldots, (p-1)a$ are divided by p, The remainders are $1, 2, 3, \ldots, (p-1)$ in some order.

Consider a number 'a_1' among the numbers $1, 2, 3, \ldots, (p-1)$. Such that aa_1 is divisible by p, whose remainder is 1.

i.e., $aa_1 \equiv 1 \pmod p$. The numbers a, a_1 are called associate residues.

Suppose, $a = a_1$ we get

$$a^2 \equiv 1 \pmod p$$

$$a^2 - 1 \equiv 0 \pmod p$$

$$(a+1)(a-1) \equiv 0 \pmod p$$

$\therefore$ Either $(a+1)$ divisible by p (or) $a=1$.

Since, a is less than p, $a+1=p$ (or) $a=1$

ie., $a=p-1$ (or) $a=1$

Hence, numbers which are identical with their associate residues are 1 and $p-1$.

Excluding these 2 numbers 1 and $p-1$, the remaining numbers $2, 3, 4, \ldots, (p-2)$ can be grouped $\frac{p-3}{2}$ pairs of associate residues. Such that the product of each pair is congruent with 1.

$$\therefore\ 2 \cdot 3 \cdot 4 \cdots (p-4)(p-3)(p-2) \equiv 1 \pmod{p} \rightarrow (1)$$

$$\therefore\ 1 \cdot (p-1) \equiv -1 \pmod{p} \rightarrow (2)$$

Multiply (1) & (2), we get,

$$1 \cdot 2 \cdot 3 \cdot 4 \cdots (p-3)(p-2)(p-1) \equiv -1 \pmod{p}$$

$$\Rightarrow\ (p-1)! \equiv -1 \pmod{p}$$

$$\Rightarrow\ (p-1)!+1 \equiv 0 \pmod{p}$$

ie., $\boxed{(p-1)!+1 \text{ is divisible by } p}$

Hence the theorem

n^8-1 is divis by a

Problems:-

Show that $18!+1$ is divisible by 437.

(or)

Prove that $18!+1 \equiv 0 \mod (437)$.

Proof:- By Wilson's Theorem,

$(p-1)!+1 \equiv 0 \pmod p$, here p is prime.

choose 19 is a prime number.

Now, $(19-1)!+1 \equiv 0 \pmod{19}$

$\Rightarrow 18!+1 \equiv 0 \pmod{19}$

$\therefore 18!+1$ is divisible by 19. $\rightarrow$ ①

Then, choose 23 is a prime number.

Now, $(23-1)!+1 \equiv 0 \pmod{23}$

$\therefore 22!+1 \equiv 0 \pmod{23}$

$\Rightarrow 22!+1$ is divisible by 23.

$\Rightarrow 22.21.20.19.18!+1 = M(23)$ km

$\Rightarrow (23-1)\cdot(23-2)(23-3)(23-4)\cdot 18!+1 = M(23)$

$\Rightarrow [M(23)+1\cdot 2\cdot 3\cdot 4]\,18!+1 = M(23)$

$\Rightarrow [M(23)+24]\,18!+1 = M(23)$

$\Rightarrow [M(23)+23+1]\,18!+1 = M(23)$

$\Rightarrow [M(23)+1]\,18!+1 = M(23)$

$\Rightarrow M(23)\,18! + \underline{18!+1} = M(23)$

$n(n^2-1)(n^2+1)$

$\therefore$ $18!+1$ is divisible by 23. $\longrightarrow$ ②

From ① & ②, we get

$\Rightarrow$ $18!+1$ is divisible by $19, 23$.

$\Rightarrow$ $18!+1$ is divisible by 19×23.

i.e., $\boxed{18!+1 \text{ is divisible by } 437}$

$\therefore$ $\boxed{18!+1 \equiv 0 \pmod{437}}$

Hence the proof.

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2. Prove that $712!+1 \equiv 0 \pmod{719}$.

Proof:- W.K.T. By Wilson's Theorem,

$(p-1)!+1$ is divisible by p

$\therefore (p-1)!+1 \equiv 0 \pmod{p}$, p is prime.

Here, 719 is a prime number.

$\therefore (719-1)!+1 \equiv 0 \pmod{719}$

$\Rightarrow 718!+1 \equiv 0 \pmod{719}$

$\Rightarrow 718\cdot 717\cdot 716\cdot 715\cdot 714\cdot 713\cdot 712!+1 \equiv 0 \pmod{719}$

$\Rightarrow (719-1)(719-2)(719-3)(719-4)(719-5)$
$(719-6)\, 712!+1 \equiv 0 \pmod{719}$

$\Rightarrow [M(719)+1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6]\, 712!+1 \equiv 0 \pmod{719}$

$\Rightarrow [M(719)+720]\, 712!+1 \equiv 0 \pmod{719}$

$\Rightarrow [M(719)+719+1]\, 712!+1 \equiv 0 \pmod{719}$

$\Rightarrow [M(719)+1]\, 712! + 1 \equiv 0 \pmod{719}$

$\Rightarrow M(719)\,712! + 712! + 1 \equiv 0 \pmod{719}$

$\therefore$ $712!+1$ is divisible by 719.

i.e., $\boxed{712!+1 \equiv 0 \pmod{719}}$

Hence the proof.

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If p is a prime number and $p = 4m+1$, where m is a positive integer, prove that $((2m)!)^2+1$ is divisible by p.

proof:- Given that p is prime number.

And $p = 4m+1 \rightarrow ①$

W.K.T. By Wilson's Theorem,

$(p-1)! + 1 \equiv 0 \pmod{p}$

using eqn. ① in this equation, we get

$\Rightarrow (4m+1-1)! + 1 \equiv 0 \pmod{p}$

$p = 4m+1$
$p = 2m+2m+1$
$2m+1 = p-2m$
$2m+2 = p-2m+1$

$\Rightarrow 4m! + 1 \equiv 0 \pmod{p}$

$\Rightarrow 4m\,(4m-1)\,(4m-2)\cdots(2m+2)\,(2m+1)\,2m! + 1 \equiv 0 \pmod{p}$.

$\Rightarrow (p-1)\,(p-2)\,(p-3)\cdots(p-2m+1)\,(p-2m)\,2m! + 1 \equiv 0 \pmod{p}$.

$\Rightarrow (p-1)\,(p-2)\,(p-3)\cdots(p-(2m-1))\,(p-2m)\,2m! + 1 \equiv 0 \pmod{p}$.

$\Rightarrow [M(p) + 1\cdot 2\cdot 3\cdots(2m-1)(2m)]\, 2m! + 1 \equiv 0 \pmod{p}$.

$\Rightarrow [M(p) + 2m!]\, 2m! + 1 \equiv 0 \pmod{p}$

$\Rightarrow \; r(p)\, 2m! + (2m!)^2 + 1 \equiv 0 \pmod{p}$

$\Rightarrow \; \therefore$ $\boxed{(2m!)^2 + 1 \text{ is divisible by } p.}$

i.e., $(2m!)^2 + 1 \equiv 0 \pmod{p}$.

Hence the proof.

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Sec: 18 Lagrange's Theorem

If $(x+1)(x+2)\cdots(x+p-1) = x^{p-1} + A_1 x^{p-2} + \cdots + A_{p-2}x + A_{p-1}$ and p is a prime number, then $A_1, A_2, \ldots, A_{p-2}$ are all divisible by p.

proof:- Given that

$$(x+1)(x+2)\cdots(x+p-1) = x^{p-1} + A_1 x^{p-2} + \cdots + A_{p-2}x + A_{p-1} \longrightarrow (1)$$

and p is a prime number.

put $x = x+1$ in eqn. (1), we get,

$$(x+2)(x+3)\cdots(x+p) = (x+1)^{p-1} + A_1(x+1)^{p-2} + \cdots + A_{p-2}(x+1) + A_{p-1} \longrightarrow (2)$$

Multiply $(x+1)$ on both sides in eqn. (2), we get

$$\Rightarrow (x+1)(x+2)(x+3)\cdots\underbrace{}_{(x+p-1)}(x+p)$$

$$= (x+1)\left[(x+1)^{p-1} + A_1(x+1)^{p-2} + \cdots + A_{p-2}(x+1) + A_{p-1}\right]$$

$$\Rightarrow \left(x^{p-1} + A_1 x^{p-2} + \cdots + A_{p-2}x + A_{p-1}\right)(x+p)$$

$$= (x+1)^p + A_1(x+1)^{p-1} + \cdots + A_{p-2}(x+1)^2 + A_{p-1}(x+1)$$

by eqn. (1)

$$\Rightarrow x^p + A_1 x^{p-1} + \cdots + A_{p-2}x^2 + A_{p-1}x + px^{p-1} + A_1 p x^{p-2} + \cdots + A_{p-2}px + A_{p-1}p$$

$$= (x+1)^p + A_1(x+1)^{p-1} + \cdots + A_{p-2}(x+1)^2 + A_{p-1}(x+1)$$

$$\Rightarrow (x+1)^p + A_1(x+1)^{p-1} + \cdots + A_{p-2}(x+1)^2 + A_{p-1}(x+1)$$
$$- x^p - A_1 x^{p-1} - \cdots - A_{p-2}x^2 - A_{p-1}x$$

$$= px^{p-1} + A_1 p x^{p-2} + \cdots + A_{p-2}px + A_{p-1}p$$

$$\Rightarrow \left[(x+1)^p - x^p\right] + A_1\left[(x+1)^{p-1} - x^{p-1}\right] + A_2\left[(x+1)^{p-2} - x^{p-2}\right]$$
$$+ \cdots + A_{p-2}\left[(x+1)^2 - x^2\right] + A_{p-1}\left[(x+1) - x\right]$$

$$= px^{p-1} + A_1 p x^{p-2} + \cdots + A_{p-2}px + A_{p-1}p.$$

$$\Rightarrow \left[x^p + pc_1\, x^{p-1} + pc_2\, x^{p-2} + \cdots + pc_p\, x^0 - x^p\right]$$

$$+ A_1\left[x^{p-1} + (p-1)c_1\, x^{p-2} + (p-1)c_2\, x^{p-3} + \cdots + (p-1)c_{p-1} x^0 - x^{p-1}\right] + \cdots + A_{p-1}(x+1-x)$$

$(x+1)^p$, $(x+1)^{p-1}$

$$= px^{p-1} + A_1 px^{p-2} + A_2 px^{p-3} + \cdots + A_{p-2}\, px + A_{p-1}\, p$$

$$\Rightarrow \left[pc_1\, x^{p-1} + pc_2\, x^{p-2} + \cdots + 1\right]$$

$A_2[p-1]$

$$+ A_1\left[(p-1)c_1\, x^{p-2} + (p-1)c_2\, x^{p-3} + \cdots + 1\right] + \cdots + A_{p-1}$$

$$= px^{p-1} + A_1 px^{p-2} + A_2 px^{p-3} + \cdots + A_{p-2}\, px + A_{p-1}\, p$$

Equating co-efficients of $x^{p-2}, x^{p-3}, \ldots$, we get

$$pc_2 + (p-1)c_1 A_1 = A_1 p$$

$$pc_3 + (p-1)c_2 A_1 + (p-2)c_1 A_2 = A_2 p$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$1 + A_1 + A_2 + \cdots + A_{p-2} + A_{p-1} = A_{p-1}\, p.$$

Since, $(p-1)c_1$, $(p-1)c_2, \ldots$ are all not divisible by p. Because, p is a prime number.

$\therefore$ $\boxed{A_1, A_2, \ldots, A_{p-2} \text{ are all divisible by } p.}$

Hence the theorem

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$(x+a)^n = x^n + nc_1 x^{n-1} a + nc_2 x^{n-2} a^2 + \cdots + nc_n a^n$

Corollary : 1

If $(x+1)(x+2)\cdots(x+p-1) = x^{p-1} + A_1 x^{p-2} + \cdots + A_{p-1}$, then prove that $(p-1)! + 1$ is divisible by p.

proof:- Given that

$$(x+1)(x+2)\cdots(x+p-1) = x^{p-1} + A_1 x^{p-2} + \cdots + A_{p-1} \rightarrow (1)$$

put $x=0 \Rightarrow 1.2.3.\cdots(p-1) = 0+0+\cdots+A_{p-1}$

$\Rightarrow (p-1)! = A_{p-1} \rightarrow (2)$

put $x=1$ in eqn. (1), we get

$\Rightarrow 2.3.4\cdots p = 1+A_1+A_2+\cdots+A_{p-2}+A_{p-1}$

$\Rightarrow p! = (1+A_{p-1}) + (A_1+A_2+\cdots+A_{p-2})$

$\Rightarrow 1+A_{p-1} = p! - (A_1+A_2+\cdots+A_{p-2})$

$\Rightarrow (p-1)!+1 = p! - (A_1+A_2+\cdots+A_{p-2})$ by eqn. (2)

By Lagrange's theorem $A_1, A_2, \cdots A_{p-2}$ are all divisible by p.

$\therefore$ $\boxed{(p-1)!+1 \text{ is divisible by } p}$

This is called Wilson's Theorem.

Hence the proof.

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Corollary: 2

If $(x+1)(x+2)\cdots(x+p-1) = x^{p-1}+A_1x^{p-2}+A_2x^{p-3}+\cdots+A_{p-1}$, then prove that x^p-x is divisible by p.

Proof:- Given that

$(x+1)(x+2)\cdots(x+p-1) = x^{p-1}+A_1x^{p-2}+\cdots+A_{p-1} \rightarrow (1)$

Multiply 'x' on both sides in eqn. (1), we get

$x(x+1)(x+2)\cdots(x+p-1) = x[x^{p-1}+A_1x^{p-2}+\cdots+A_{p-1}]$

$x(x+1)(x+2)\cdots(x+p-1) = x^p+A_1x^{p-1}+A_2x^{p-2}+\cdots+xA_{p-1}$

$$x(x+1)(x+2)\cdots(x+p-1)$$
$$= x^p - x + A_1 x^{p-1} + A_2 x^{p-2} + \cdots + A_{p-2} x^2 + A_{p-1} x + x$$

$$\therefore x^p - x = [x(x+1)(x+2)\cdots(x+p-1)] - [A_1 x^{p-1} + A_2 x^{p-2} + \cdots + A_{p-2} x^2] - [A_{p-1}+1]x$$

$x(x+1)(x+2)\cdots(x+p-1)$ is a product of p consecuti integers, must be divisible by p.

Also, p is a prime, then $A_1, A_2, \ldots, A_{p-1}, (A_{p-1}+1)$ are divisible by p.

$\therefore$ $\boxed{x^p - x \text{ is divisible by } p}$, if p is prime.

This is called Fermat's Theorem.

Hence the proof.

Problems:-

1. show that $10^n + 3\cdot 4^{n+2} + 5 \equiv 0 \pmod 9$.

proof:-

Let $f(n) = 10^n + 3\cdot 4^{n+2} + 5$

put $n=1 \Rightarrow f(1) = 10^1 + 3\cdot 4^{1+2} + 5$
$$= 10^1 + 3\cdot 4^3 + 5$$
$$= 10 + 3(64) + 5$$
$$= 10 + 192 + 5$$
$\therefore f(1) = 207$ is divisible by p.

put $n=2 \Rightarrow f(2) = 10^2 + 3\cdot 4^{2+2} + 5$
$$= 100 + 3(256) + 5$$
$$= 100 + 768 + 5$$
$\therefore f(2) = 873$ is divisible by p.

Here, $f(n) = 10^n + 3\cdot 4^{n+2} + 5 = 9k$

$$\Rightarrow 10^n = 9k - 3\cdot 4^{n+2} - 5$$

$$= 9k - 3\cdot 4^n \cdot 4^2 - 5$$

$$= 9k - 48\cdot 4^n - 5 \rightarrow ①$$

put $n = n+1 \Rightarrow$

$$f(n+1) = 10^{n+1} + 3\cdot 4^{n+1+2} + 5$$

$$= 10^n\cdot 10 + 3\cdot 4^n\cdot 4^3 + 5$$

$$= 10\cdot 10^n + 3\cdot 4^n\cdot 64 + 5$$

$$= 10\cdot 10^n + 192\cdot 4^n + 5$$

$$= 10\,[9k - 48\cdot 4^n - 5] + 192\cdot 4^n + 5 \quad \text{(by eqn ①)}$$

$$= 90k - 480\cdot 4^n - 50 + 192\cdot 4^n + 5$$

$$= 90k - 288\cdot 4^n - 45$$

$$= 9\,[10k - 32\cdot 4^n - 5]$$

$$= M(9)$$

$\therefore$ $\boxed{10^n + 3\cdot 4^{n+2} + 5 \text{ is divisible by } 9}$.

ie., $\boxed{10^n + 3\cdot 4^{n+2} + 5 \equiv 0 \pmod 9}$

Hence the proof.

2. Show that $3\cdot 5^{2n+1} + 2^{3n+1}$ is divisible by 17.

Proof:-

Let $f(n) = 3\cdot 5^{2n+1} + 2^{3n+1}$

put $n=1 \Rightarrow f(1) = 3\cdot 5^{2+1} + 2^{3+1}$

$$= 3\cdot 5^3 + 2^4$$

$$= 3(125) + 16$$

$= 375 + 16 = 391$ is divisible by p.

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put $n=2 \Rightarrow f(2) = 3.5^{4+1} + 2^{6+1}$

$= 3.5^5 + 2^7$

$= 3(3125) + 128$

$= 9375 + 128$

$= 9503$ is divisible by p.

Here, $f(n) = 3.5^{2n+1} + 2^{3n+1} = 17k$

$\Rightarrow 3.5^{2n}.5 + 2^{3n}.2^1 = 17k$

$\Rightarrow 2^{3n}.2^1 = 17k - 15.5^{2n}$

$\Rightarrow 2^{3n} = \frac{1}{2}[17k - 15.5^{2n}] \rightarrow ①$

put $n=n+1 \Rightarrow f(n+1) = 3.5^{2(n+1)+1} + 2^{3(n+1)+1}$

$= 3.5^{2n+2+1} + 2^{3n+3+1}$

$= 3.5^{2n+3} + 2^{3n+4}$

$= 3.5^{2n}.5^3 + 2^{3n}.2^4$

$= 3.5^{2n}(125) + 2^{3n}(16)$

$= 375.5^{2n} + 16.2^{3n}$

$= 375.5^{2n} + \overset{8}{16}[\frac{1}{2}(17k - 15.5^{2n})]$ (by ①)

$= 375.5^{2n} + 8(17k - 15.5^{2n})$

$= 375.5^{2n} + 136k - 120.5^{2n}$

$= 375.5^{2n} - 120.5^{2n} + 136k$

$\Rightarrow 375.5^{2n} - 120.5^{2n} + 136k = (375-120)5^{2n} + 136k$

$= 255.5^{2n} + 136k$

$= 17(15.5^{2n} + 8k)$

(43)

$\Rightarrow 17(15 \cdot 5^{3n} + 8k) = 17(17)$

$\therefore 17(15 \cdot 5^{3n} + 2^{3}k) = 17(17)$

$\therefore \boxed{3 \cdot 5^{2n+1} + 2^{3n+1} \text{ is divisible by } 17}$.

i.e., $3 \cdot 5^{2n+1} + 2^{3n+1} \equiv 0 \pmod{17}$

Hence the proof.

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Home Work :-

show that $3^{2n-1} + 2^{n+1}$ is divisible by 7.

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Unit-V is over