CORE COURSE V

SEQUENCES AND SERIES

OBJECTIVES :

1. To lay a good foundation for classical analysis
2. To study the behavior of sequences and series.

Unit I

Sequences – Bounded Sequences – Monotonic Sequences – Convergent Sequence – Divergent Sequences – Oscillating sequences

Unit II

Algebra of Limits – Behavior of Monotonic functions

Unit III

Some theorems on limits – subsequences – limit points : Cauchy sequences

Unit IV

Series – infinite series – Cauchy's general principal of convergence – Comparison – test theorem and test of convergence using comparison test (comparison test statement only, no proof)

Unit V

Test of convergence using D Alembert's ratio test – Cauchy's root test – Alternating Series – Absolute Convergence (Statement only for all tests)

Book for Study

Dr. S.Arumugam & Mr.A.Thangapandi Isaac Sequences and Series – New Gamma Publishing House – 2002 Edition.

Unit I	: Chapter 3 : Sec. 3.0 – 3.5 Page No : 39-55
Unit II	: Chapter 3 : Sec. 3.6, 3.7 Page No:56 – 82
Unit III	: Chapter 3 : Sec. 3.8-3.11, Page No:82-102
Unit IV	: Chapter 4 : Sec. (4.1 & 4.2) Page No : 112-128.
Unit V	: Relevant part of Chapter 4 and Chapter 5: Sec. 5.1 & 5.2 Page No:157-167.

Book for Reference

1. Algebra – Prof. S.Surya Narayan Iyer
2. Algebra – Prof. M.I.Francis Raj

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II. B.Sc., Mathematics
SEQUENCES AND SERIES (16SCCMM5)
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05.08.20

Unit – I

SEQUENCES

Definition:-

Let $f : N \to R$ be a function and let $f(n) = a_n$. Then $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called the sequence in R determined by the function f and is denoted by (a_n) (or) $\{a_n\}$. a_n is called the n^{th} term of the sequence.

The range of the function f, which is a subset of R is called the range of the sequence.

Examples:-1: The function $f : N \to R$ given by $f(n) = n$ determines the sequence $1, 2, 3, \ldots n, \ldots$

Here $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ ..... $a_n = n$ ....

Example: 2: The function $f : N \to R$ given by $f(n) = n^2$ determines the sequence $1^2, 2^2, 3^3, \ldots n^2$.

then $1, 4, 9, \ldots n^2$

Here, $a_1 = 1$, $a_2 = 4$, $a_3 = 9$, $a_n = n^2$

Example : 3, The function $f : N \to R$ given by $f(n) = (-1)^n$ determines the sequence.

$$-1, 1, -1, 1, -1, \ldots\ldots (-1)^n \ldots\ldots$$

$$a = -1.$$

Thus the terms of a sequence need not be distinct.

Note :- The range of this sequence is $\{1, -1\}$. Thus we see that the range of a sequence may be finite or infinite.

Example :- 4 The sequence $((-1)^{n+1})$ is given by $f(n) = (-1)^{n+1}$ determines the sequence.

$$(-1)^{1+1}, (-1)^{2+1}, (-1)^{3+1} \ldots\ldots (-1)^{n+1}$$

$$-1^2, -1^3, -1^4 \ldots\ldots (-1)^{n+1} = 1, -1, 1, -1, \ldots (-1)^{n+1}$$

$a_1 = 1$, $a_2 = -1$, $a_3 = 1$ $a_n = (-1)^{n+1}$

Note :-

The sequence $((-1)^n)$ and $((-1)^{n+1})$ are different.

Example : 5 The constant function $f: N \to R$ given by $f(n) = 1$ determine the sequence

$(1)^1$

1, 1, 1, ..., 1 Convergent

Here,

$a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_n = 1$

This sequence is called a Constant sequence.

Constant Sequence :-

If $f(n) = a$ then 'a' is a Constant such that $f: N \to R$ is a function and a sequence is called a Constant sequence.

Example :- 6 find the Range of a $f: N \to R$ given by $f(n) = \begin{cases} \frac{1}{2}(-n) & \text{if } n \text{ is even} \\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd} \end{cases}$

When n is even

$f(n) = \frac{1}{2}(-2), \frac{1}{2}(-4), \frac{1}{2}(-6) \ldots \frac{1}{2}(-n) \ldots$

$= -1, -2, -3 \ldots\ldots \frac{1}{2}(-n) \ldots$

When n is odd.

$f(n) = \frac{1}{2}(-1), \frac{1}{2}(+3-1), \frac{1}{2}(+5-1) \dots \frac{1}{2}(+n-1) \dots$

$= 0, 1, 2, 3, \dots$

$\therefore$ The sequence is

$\langle f(n) \rangle = \dots -n \dots -3, -2, -1, 0, 1, 2, 3 \dots n$

$= Z$

The range of this sequence is Z.

Example :- 7 The function $f : N \to R$ given by $f(n) = \frac{n}{n+1}$ determine the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \frac{n}{n+1}, \dots$

Example : 8 The function $f : N \to R$ given by $f(n) = \frac{1}{n}$ determines the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \dots$

Example :- 9 The function $f : N \to R$ given by $f(n) = 2n + 3$ determines the sequence $5, 7, 9, 11 \dots$

Example :- 10 Let $x \in R$, the function $f : N \to R$ given by $f(n) = x^{n-1}$ determines the geometric sequence $1, x, x^2, \dots x^n, \dots$

Example: 11 The sequence $(-n)$ is given by $-1, -2, -3, \ldots, -n, \ldots$ the range of this sequence is the set of all negative integers.

Example: 12 Let $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$, Then $a_3 = a_2 + a_1 = 2$; $a_4 = a_3 + a_2 = 3$ and so on. We thus obtain the sequence $1, 1, 2, 3, 5, 8, 13, \ldots$. This sequence is called Fibonacci's sequence.

Exercises: - problem

Write the first five terms of each of the following sequences: -

(a). $\left(\frac{(-1)^n}{n}\right)$

Solution: -

Given $\left(\frac{(-1)^n}{n}\right)$

$a = 1, 2, 3, 4, 5$

$= \frac{(-1)^1}{1}, \frac{(-1)^2}{2}, \frac{(-1)^3}{3}, \frac{(-1)^4}{4}, \frac{(-1)^5}{5}$

$= -1, 1/2, -1/3, 1/4, -1/5$

$a_1 = -1$, $a_2 = 1/2$, $a_3 = -1/3$, $a_4 = 1/4$, $a_5 = -1/5$ //

(b). $\left(\frac{2}{3}\left(1-\frac{1}{10^n}\right)\right)$

$= \frac{2}{3}\left(1-\frac{1}{10^1}\right), \frac{2}{3}\left(1-\frac{1}{10^2}\right), \frac{2}{3}\left(1-\frac{1}{10^3}\right), \frac{2}{3}\left(1-\frac{1}{10^n}\right)$

$\frac{2}{3}\left(1-\frac{1}{10^5}\right)$

$= \frac{2}{3}\left(\frac{10-1}{10}\right), \frac{2}{3}\left(1-\frac{1}{100}\right), \frac{2}{3}\left(1-\frac{1}{1000}\right) \frac{2}{3}\left(1-\frac{1}{10,000}\right),$

$\frac{2}{3}\left(1-\frac{1}{10,0000}\right)$

$= \frac{\cancel{2}}{3}\,\frac{\overset{3}{\cancel{9}}}{\underset{5}{\cancel{10}}}, \frac{\cancel{2}}{\cancel{3}}\,\frac{\overset{33}{\cancel{99}}}{\underset{50}{\cancel{100}}}, \frac{\cancel{2}}{\cancel{3}}\,\frac{\overset{333}{\cancel{999}}}{\underset{500}{\cancel{1000}}}, \frac{\cancel{2}}{\cancel{3}}\,\frac{\overset{3333}{\cancel{9999}}}{\underset{5000}{\cancel{10,000}}},$

$\frac{\cancel{2}}{\cancel{3}}\,\frac{\overset{33333}{\cancel{99999}}}{\underset{50000}{\cancel{1,00000}}}$

$= \frac{3}{5}, \frac{33}{50}, \frac{333}{500}, \frac{3333}{5000}, \frac{33333}{50000}$

$= \times =$

Home work :- (05.08.2020)

(c). $\left(\frac{\cos nx}{n^2+x^2}\right)$, (d) $\left(\frac{(-1)^{n+1}}{n!}\right)$

$= \times =$

Today class is online (End). And next class tomorrow

06.08.2020 Sub: Sequences and Series II . B.Sc., Maths (1)

Online Class in notes

3.2 Bounded sequences

Upper bound sequence :-

A sequence (a_n) is said to be bounded above if there exist $K \in R$ such that $a_n \le K, \forall n \in N$.

Then k is called an upper bound of this sequence

Ex:- 1 $(-n)$

$(-n) = -1, -2, -3, \ldots$

$= -1 > -2 > -3$

$\therefore$ Upper bound of $(-n) = -1$

$K \in R \ni : a_n \ge K, \forall n \in N$

Then K is called lower bound of the sequence

Ex:-

$(n) = 1, 2, 3, \ldots$

$1 < 2 < 3, \ldots$

$\therefore$ Lower bound of $(n) = 1$

Bounded sequence :- definition

If a sequence is both bounded above and below is called a bounded sequence.

Ex:- $((-1)^n)$

$((-1)^n) = -1, 1, -1, 1, \ldots$

Upper bound = 1

Lower bound = -1

Note : 1. Any constant sequence is a bounded sequence.

Note : 2, A sequence (a_n) is bounded if and only if there exist a real number $k \geq 0$, such that $|a_n| \leq k \; \forall n$.

Least upper bound (l.u.b)

Let $A \subseteq R$ and $u \in R$, u is called the Least upper bound (or) supremum of A if,

i). u is an upper bound of A.

ii). If $v < u$ then v is not an upper bound of A.

Greatest lower bound (G.L.b)

Let $A \subseteq R$ and $l \in R$ l is called the greatest lower bound (or) infimum of A if,

i). l is a Lower bound of A

ii). If $m > l$ then m is not a lower bound of A.

Ex : 1

Sequence (n) = 1, 3, 5, 6

g.l.b → 1

l.u.b → 6

∴ sequence is a bounded.

Ex: 2 :- $(n) = 1, 2, 3, \ldots$

$g.l.b \to 1$

$l.u.b \to$ does not exists

$\therefore$ (n) is bounded below but not bounded above.

Ex: 3 :-

$(-n) = -1, -2, -3, \ldots$

$g.l.b \to$ does not exists

$(-n)$ is bounded above but not bound below $l.u.b \to -1$.

Ex: 4 :- $((-1)^{n+1}) = 1, -1, 1, -1, \ldots$

$g.l.b = -1$

$l.u.b = 1$

$((-1)^{n+1})$ is bounded sequence.

Ex: 5 :- $(1/n) = 1, 1/2, 1/3, \ldots 1/\infty$

$g.l.b = 0$

$l.u.b = 1$

$(1/n)$ is bounded sequence.

Lower bound :- A sequence (a_n) said to bounded below if there exist $K \in R$ g: $a_n \geq K \; \forall \; n \in N$, then K is called on lower bound of this sequence (a_n).

Home work :-

1). Determine which of the sequence given in example of 3.1 are
(a). bounded above (b) bounded below (c) bounded

2). Give examples of sequences (a_n) such that
(a). (a_n) is bounded above but not bounded below.
(b). (a_n) is bounded below but not bounded above
(c). (a_n) is a bounded sequences.
(d). (a_n) is neither bounded above not bounded below.

3). Determine the l.u.b and g.l.b of the following sequences if they exist.

(a). $2, -2, 1, -1, 1, -1, \dots$

(b). $1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \dots, \frac{1}{\sqrt{n}}, \dots$

(c). $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$

(d). $1, -1, 2, -2, 3, -3, \dots, n, -n, \dots$

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Thanks to all.

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07.08.20 ①

II. B.Sc, Maths (Unit - I, Continue) 07.08.2020

3.3 Monotonic sequences

Definition: A sequence (a_n) is said to be monotonic increasing, if $a_n \leq a_{n+1}$ for all n. (a_n) is said to be monotonic decreasing, if $a_n \geq a_{n+1}$ for all n.

Then (a_n) is said to be strictly monotonic increasing if $a_n < a_{n+1}$ for all n and strictly monotonic decreasing, if $a_n > a_{n+1}$ for all n. (a_n) is said to be monotonic if it is either monotonic increasing (or) monotonic decreasing.

Ex:-1; $1, 2, 2, 2, 3, 3, 4, 4, 4, 4, \ldots$ is a monotonic increasing sequence.

Ex: 2; $1, 2, 3, 4, \ldots, n, \ldots$ is a strictly monotonic increasing sequence.

Ex: 3; $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n}, \ldots$ is a strictly monotonic decreasing sequence.

Ex: 4; The sequence (a_n) given by $1, -1, 1, -1, 1, \ldots$ is neither monotonic increasing not decreasing.

Hence, (a_n) is not a monotonic sequence.

Ex:- 5 ;

$\left(\frac{2n-7}{3n+2}\right)$ is a monotonic increasing sequence.

Proof:-

Given, $a_n = \left(\frac{2n-7}{3n+2}\right)$, $a_{n+1} = \frac{2(n+1)-7}{3(n+1)+2}$

$$a_n - a_{n+1} = \frac{2n-7}{3n+2} - \frac{2(n+1)-7}{3(n+1)+2}$$

$$= \frac{2n-7}{3n+2} - \frac{2n+2-7}{3n+3+2}$$

$$= \frac{2n-7}{3n+2} - \frac{2n-5}{3n+5}$$

$$= \frac{-25}{(3n+2)(3n+5)} < 0 .$$

$\therefore$ $a_n < a_{n+1}$.

The given sequence is Monotonic increasing sequence

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①. Show that (a_n) is a monotonic sequence, then

(✗). $b_n = \left(\frac{a_1+a_2+\dots\dots+a_n}{n}\right)$ is also a monotonic sequence.

Proof:-

Case (i) Let (a_n) be a monotonic increasing sequence $\Rightarrow a_n \leq a_{n+1}$.

(ii), $a_1 \le a_2 \le a_3 \le \dots\dots \le a_n \le \dots\dots \longrightarrow (1)$

Let $b_n = \dfrac{a_1 + a_2 + \dots\dots a_n}{n}$

$b_{n+1} = \dfrac{a_1 + \dots\dots + a_n + a_{n+1}}{n+1}$

Now, $b_{n+1} - b_n = \dfrac{a_1 + a_2 + \dots\dots a_n + a_{n+1}}{n+1} - \dfrac{a_1 + a_2 + \dots + a_n}{n}$

$$= \frac{n(a_1 + a_2 + \dots + a_n + a_{n+1}) - (n+1)(a_1 + a_2 + \dots + a_n)}{n(n+1)}$$

$$= \frac{n(a_1 + a_2 + \dots\dots + a_n + a_{n+1} - n(a_1 + \dots a_n) - (a_1 + \dots + a_n)}{n(n+1)}$$

$$= \frac{n a_{n+1} - (a_1 + \dots\dots a_n)}{n(n+1)}$$

$$= \frac{n a_{n+1} - a_n}{n(n+1)} \quad \because \text{by } (1)$$

$\geq 0 \quad [\because (a_n)$ is increasing sequence$]$

$\therefore b_n \le b_{n+1}$

$\therefore b_n$ is monotonic increasing sequence.

Case (ii)

Let (a_n) be a monotonic decreasing sequence.

$\Rightarrow a_n \geq a_{n+1}$

$a_1 \geq a_2 \geq a_3 \geq \dots\dots \geq a_n \geq \dots\dots \rightarrow (2)$

Let $$b_n = \frac{a_1 + a_2 + \dots + a_n}{n}$$

$$b_{n+1} = \frac{a_1 + a_2 + \dots + a_n + a_{n+1}}{n+1}$$

$$b_{n+1} - b_n = \frac{a_1 + a_2 + \dots + a_n + a_{n+1}}{n+1} - \frac{a_1 + a_2 + \dots + a_n}{n}$$

$$= \frac{n(a_1 + a_2 + \dots + a_n + a_{n+1}) - (n+1)(a_1 + \dots + a_n)}{n(n+1)}$$

$$= \frac{n(a_1 + a_2 + \dots + a_n + a_{n+1}) - n(a_1 + \dots + a_n) - (a_1 + \dots + a_n)}{n(n+1)}$$

$$= \frac{n a_{n+1} - (a_1 + \dots + a_n)}{n(n+1)}$$

$$= \frac{n a_{n+1} - a_n}{n(n+1)} \quad \because \text{By (2)}$$

≤ 0 $\quad [\because (a_n)$ is decreasing sequence$]$

$b_n \geq b_{n+1}$

$\therefore$ b_n is Monotonic decreasing sequence.

Hence from Case (i) and Case (ii) (b_n) is a Monotonic sequence.

(2) If (a_n) and (b_n) are monotonic increasing (decreasing) sequence then show that (a_n+b_n) is also monotonic increasing (decreasing) sequence.

(*) V.V.I

Exercises Sums

Proof :-

Case (i)

Let (a_n) and (b_n) monotonic increasing sequence. → (1)

$(a_n) = a_1, a_2, \ldots\ldots$ And

$(b_n) = b_1, b_2, \ldots\ldots$

∵ Monotonic increasing sequence

$a_1 < a_2 < \ldots\ldots < a_n < \ldots\ldots$

$b_1 < b_2 < \ldots\ldots < b_n < \ldots\ldots$

$a_n + b_n = a_1+b_1,\ a_2+b_2, \ldots\ldots, a_n+b_n \ldots\ldots$

$a_{n+1} + b_{n+1} = a_2+b_2,\ a_3+b_3, + \ldots\ldots + a_n+b_n,\ a_{n+1}+b_{n+1}, \ldots\ldots$

$(a_{n+1}+b_{n+1}) - (a_n+b_n) = a_{n+1} + b_{n+1} - (a_1+b_1)$ → (2)

∴ By (1) $a_{n+1} > a_n$ and $b_{n+1} > b_n$

∴ (2) ⇒ $(a_{n+1} + b_{n+1}) - (a_n + b_n) > 0$

∴ $a_{n+1} + b_{n+1} > a_n + b_n$

Hence (a_n+b_n) is monotonic increasing sequence (↑).

Case (ii) :-

Let (a_n) and (b_n) monotonic decreasing sequence → (3)

$(a_n) = a_1, a_2, \ldots$ and $(b_n) = b_1, b_2, \ldots$

$\therefore$ Monotonic decreasing sequence

$a_1 > a_2 > \ldots > a_n > \ldots$

$b_1 > b_2 > \ldots > b_n > \ldots$

$a_n + b_n = a_1 + b_1, a_2 + b_2, \ldots a_n + b_n \ldots$

$a_{n+1} + b_{n+1} = a_2 + b_2, a_3 + b_3 + \ldots + a_n + b_n, a_{n+1} + b_{n+1}, \ldots$

$(a_{n+1} + b_{n+1}) - (a_n + b_n) = a_{n+1} + b_{n+1} - (a_1 + b_1)$ → (4)

By (3) $a_{n+1} < a_n$ and $b_{n+1} < b_n$

(4) $\Rightarrow (a_{n+1} + b_{n+1}) - (a_n + b_n) < 0$

$\therefore a_{n+1} + b_{n+1} < a_n + b_n$

Hence, $(a_n + b_n)$ is monotonic decreasing sequence($\downarrow$).

Exercises: sums.

(3). If (a_n) is monotonic increasing sequence show that (λa_n) is increasing, if λ is possitive and (λa_n) is decreasing, if λ is Negative.

(*) V.V.I

Proof:-

(i). Let (a_n) monotonic increasing sequence

$\Rightarrow a_n \leq a_{n+1}$

(ie), $a_1 < a_2 < a_3 < \ldots < a_n \ldots$

Now let $\lambda > 0$

$(\lambda a_n) = \lambda a_1, \lambda a_2, \dots \lambda a_n \dots$

$\because$ (a_n) is monotonic increasing sequence

$\lambda a_1 < \lambda a_2 < \lambda a_3 \dots < \lambda a_n < \dots$

Hence, (a_n) is monotonic increasing sequence, If > 0.

(ii). Let $\lambda < 0$

(ii), $\lambda = -\lambda$

$\Rightarrow a_n \geq a_{n+1}$

(ii), $a_1 > a_2 > a_3 > \dots > a_n > \dots$

Now, let $\lambda < 0$

$(-\lambda a_n) = -\lambda a_1 > -\lambda a_2 > -\lambda a_3 \dots > -\lambda a_n \dots$

Hence (a_n) is monotonic decreasing sequence.

Exercises sums :-

③. Determine which of the following sequence are monotonic.

(a). $(\log n)$

Solution:

$= \log 1, \log 2, \log 3, \log 4, \log 5$

$= 0, 0.3010, 0.4771, 0.6021, 0.6990$

$\therefore$ Monotonic increasing sequence, (b). $\left((-1)^{n+1} n\right)$

Solution :-

$$= \left((-1)^2\, 1\right), \left((-1)^3\, 1\right), \left((-1)^4\, 1\right), \left((-1)^5\, 1\right), \left((-1)^6\, 1\right)$$

$$= 1, -1, 1, -1, 1$$

$\therefore \left((-1)^{n+1} n\right)$ is not monotonic sequence.

(c). $\left(2 + \frac{1}{n}\right)$

Solution :-

$$= \left(2 + \frac{1}{1}\right), \left(2 + \frac{1}{2}\right), \left(2 + \frac{1}{3}\right), \left(2 + \frac{1}{4}\right), \left(2 + \frac{1}{5}\right)$$

$$= 2+1, \frac{4+1}{2}, \frac{6+1}{3}, \frac{8+1}{4}, \frac{10+1}{5}$$

$$= 3, \frac{5}{2}, \frac{7}{3}, \frac{9}{4}, \frac{11}{5}$$

$$= 3, 2.5, 2.33, 2.25, 2.2$$

$\therefore \left(2 + \frac{1}{n}\right)$ is decreasing sequence $(\downarrow)$.

(d). $\left(\frac{1}{2n}\right)$

Solution :-

$$= \frac{1}{2(1)}, \frac{1}{2(2)}, \frac{1}{2(3)}, \frac{1}{2(4)}$$

$$= \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$$

$\therefore$ Monotonic increasing sequence $(\uparrow)$.

(e). $\frac{1}{n!}$

Solution :-

$$= \frac{1}{1}, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}$$

$$= 1, \frac{1}{2}, \frac{1}{8}, \frac{1}{24}$$

$\therefore$ Monotonic is decreasing sequence.

Home work :- (07.08.2020)

(e). $\frac{1}{n!}$

(f). $\left(\frac{(-1)^n}{n}\right)^n$

(g). .6, .66, .666, ...

(h). 2, 1.9, 1.8, ...

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Thanks to all

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08.08.20

II. B.Sc., Maths (Sub: Sequences and Series)

3.4. Convergent Sequences

Definition:- A sequence (a_n) is said to converge to a number l if given $\varepsilon > 0$ there exist a positive integer m such that $|a_n - l| < \varepsilon$ for all $n \geq m$.

We say that l is the limit of the sequence and we write $\lim_{n \to \infty} a_n = l$ (or) $a_n \to l$.

Note: 1

$(a_n) \to l$ if and only if (iff) given $\varepsilon > 0$ there exists a natural number m such that $a_n \in (l - \varepsilon, l + \varepsilon)$ for all ($\forall$) $n \geq m$ (ie), All but a finite number of terms of the sequence lie within the interval $(l - \varepsilon, l + \varepsilon)$.

Note: 2

The above definition does not give any method of finding the limit of a sequence. In many cases, by observing the sequence carefully, we guess whether the limit exists (or) not and also the value of the limit.

Theorem:- 3.1

A sequence cannot converge to two different limits.

Proof:- Let (a_n) be a convergent sequence.

If possible, let l_1 and l_2 be two distinct limits of (a_n).

Let $\varepsilon > 0$ be given,

Since $(a_n) \to l_1$, there exist a natural number n_1 such that

$|a_n - l_1| < \frac{1}{2}\varepsilon \ \forall \ n \geq n_1 \longrightarrow (1)$

Since $(a_n) \to l_2$, there exist a natural number n_2 such that

$|a_n - l_2| < \frac{1}{2}\varepsilon \ \forall \ n \geq n_2 \longrightarrow (2)$

Let $m = \max\{n_1, n_2\}$.

Then $|l_1 - l_2| = |l_1 - a_m + a_m - l_2|$

$\leq |a_m - l_1| + |a_m - l_2|$

$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$ (by 1 and 2)

$= \varepsilon$.

$\therefore |l_1 - l_2| < \varepsilon$ and this is true for every $\varepsilon > 0$.

Clearly, this is possible if and only if $l_1 - l_2 = 0$.

Hence, $\boxed{l_1 = l_2}$.

Problems:-

1). $\lim_{n \to \infty} \frac{1}{n} = 0$ (or) $\left(\frac{1}{n}\right) \to 0$.

Proof:- Let $\varepsilon > 0$ be given, Then, $\left|\frac{1}{n}-0\right| = \frac{1}{n} < \epsilon$ if $n > \frac{1}{\epsilon}$.

Hence, if we choose m to be any natural number such that $m > \frac{1}{\epsilon}$,

Then $\left|\frac{1}{n}-0\right| < \epsilon \ \forall \ n \geq m$.

$\therefore \lim\limits_{n\to\infty} \frac{1}{n} = 0$.

Note : 3 If $\epsilon = 1/100$, then m can be chosen to be any natural number greater than 100.

In this example the choice of m depends on the given ϵ and $[1/\epsilon]+1$ is the smallest value of m that satisfies the requirements of the definition.

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2). The constant sequence 1, 1, 1, ... converges to 1.

Proof :- Let $\epsilon > 0$ be given,

Let the given sequence be denoted by (a_n).

Then $a_n = 1 \ \forall \ n$.

$\therefore |a_n - 1| = |1-1| = 0 < \epsilon \ \forall \ n \in N$.

$\therefore |a_n - 1| < \epsilon \ \forall \ n \geq m$,

where m can be chosen to be any natural number.

$\therefore \lim\limits_{n\to\infty} a_n = 1$. //

Note: 4 The choice of m does not depend on the given ε.

3). $\lim_{n \to \infty} \frac{n+1}{n} = 1$.

Proof:- Let $\varepsilon > 0$ be given.

Now, $\left| \frac{n+1}{n} - 1 \right| = \left| 1 + \frac{1}{n} - 1 \right| = \left| \frac{1}{n} \right|$.

$\therefore$ If we choose m to be any natural number greater than $\frac{1}{\varepsilon}$ we have,

$$\left| \frac{n+1}{n} - 1 \right| < \varepsilon \ \forall\, n \geq m.$$

$$\therefore \lim_{n \to \infty} \frac{n+1}{n} = 1.$$

4). $\lim_{n \to \infty} \frac{1}{2^n} = 0$.

Proof:- Let $\varepsilon > 0$ be given.

Then $\left| \frac{1}{2^n} - 0 \right| = \frac{1}{2^n} < \frac{1}{n}$ (Since $2^n > n \ \forall\, n \in N$).

$$\therefore \left| \frac{1}{2^n} - 0 \right| < \varepsilon \ \forall\, n \geq m.$$

Where m is any natural number greater than $1/\varepsilon$.

$$\therefore \lim_{n \to \infty} \frac{1}{2^n} = 0. \ //.$$

5). The sequence $((-1)^n)$ is not convergent.

Proof:-

Suppose ~~that~~ the sequence $((-1)^n)$ converges to l.

Then, given $\varepsilon > 0$, there exist a natural number m such that

$|(-1)^n - l| < \varepsilon \ \forall \ n \geq m$.

$$\therefore |(-1)^m - (-1)^{m+1}| = |(-1)^m - l + l - (-1)^{m+1}|$$

$$\leq |(-1)^m - l| + |(-1)^{m+1} - l|$$

$$< \varepsilon + \varepsilon = 2\varepsilon. \longrightarrow (1)$$

But, $|(-1)^m - (-1)^{m+1}| = 2. \longrightarrow (2)$

From (1) & (2), we get.

$\therefore 2 < 2\varepsilon$.

(ie), $1 < \varepsilon$, which is a contradiction

Since $\varepsilon > 0$ is arbitrary.

$\therefore$ The sequence $((-1)^n)$ is not convergent. //.

Theorem :- 3.2

Any convergent sequence is a bounded sequence.

Proof:-

Let (a_n) be a convergent sequences.

Let $\lim_{n \to \infty} a_n = l$.

Let $\varepsilon > 0$ be given.

Then there exists $m \in N$ such that $|a_n - l| < \epsilon \;\forall\, n \geq m$.

$\therefore |a_n| < |l| + \epsilon \;\forall\, n \geq m$.

Now, let $K = \max\{|a_1|, |a_2|, \ldots |a_{m-1}|, |l| + \epsilon\}$.

Then, $|a_n| \leq K \;\forall\, n$.

$\therefore (a_n)$ is a bounded sequence. //.

Note: The converse of the above theorem is not true. for example, the sequence $((-1)^n)$ is a bounded sequence. However it is not a convergent sequence.

Exercises:- Problems.

(1). Any convergent sequence is a bounded sequence.

Proof:-

let (a_n) be a convergent sequence

let $\lim\limits_{n \to \infty} a_n = l$

And let $\epsilon > 0$ then "0" be given then there exist $m \in N$

(ie), $|a_n - l| < \epsilon \;\forall\, n \geq m$

$\therefore |a_n| - |l| < \epsilon$

$|a_n| < \epsilon + |l|$

$|a_n| < |l| + \epsilon \;\forall\, n \geq m$

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Now, let $K = \max\{|a_1|, |a_2|, \ldots |a_{m-1}|, |l|+\varepsilon\}$

Then, $\boxed{|a_n| \le K \ \forall n}$

$\therefore$ (a_n) is a bounded sequence. //.

---

(2). Pf any bounded sequence is not a Convergent sequence.

Proof:-

We prove is theorem by an examples.

$$((-1)^n) = -1, 1, -1, 1, \ldots\ldots$$

Here, $g.l.b = -1$

$l.u.b = +1$

$\therefore$ $(-1)^n$ is bounded, but it has two different limits $(-1, 1)$.

$\therefore$ $(-1)^n$ is not convergent.

Prove that the sequence $(n2)$ is not Convergent

$$n^2 = 1, 4, 9, 16, \ldots\ldots$$

$g.l.b = 1$

$l.u.b =$ does not exist

(n^2) is bounded but it has two different limits 1 and ∞.

$\therefore (n)$ is not convergent. //.

---

(3). Prove that $\lim_{n \to \infty} \frac{1}{n^2} = 0$.

<u>Proof</u>:

let $\varepsilon > 0$ be given,

by def. $\left|\frac{1}{n^2} - 0\right| < \left|\frac{1}{n^2}\right|$

$= \frac{1}{n^2} < \frac{1}{n} < \varepsilon \; \forall n$, if $n > \frac{1}{\varepsilon}$

Hence,

$\lim_{n \to \infty} \frac{1}{n^2} = 0$. //.

---

(4). Prove that $\lim_{n \to \infty} \left(1 + \frac{1}{n!}\right) = 1$.

<u>Proof</u> :-

let $\varepsilon > 0$ be given,

$\left|1 + \frac{1}{n!} - 1\right| = \left|\frac{1}{n!}\right| \Rightarrow \frac{1}{n!} < \frac{1}{n} < \varepsilon \; \forall \; n \in N$

if $n > 1/\varepsilon$

$\therefore \frac{1}{n} < \varepsilon \Rightarrow \frac{1}{m} < \varepsilon \; \forall \; m \in N$

$\therefore \left|1 + \frac{1}{n!} - 1\right| < \varepsilon \; \forall \; n \geq m$.

Hence,

$\lim_{n \to \infty} \left(1 + \frac{1}{n!}\right) = 1$ //.

---

(5). Prove that $\lim_{n\to\infty} \frac{2n+1}{2n} = 1$

Proof:-

let $\epsilon > 0$ be given, by defn.

$$\left| \frac{2n+1}{2n} - 1 \right| = \left| \frac{1}{2n} \right| = \frac{1}{2n} < \frac{1}{n} < \epsilon \ \forall\ n \in N$$

If $n > 1/\epsilon$.

$$\frac{1}{n} < \epsilon \Rightarrow \frac{1}{m} < \epsilon \ \forall\ m \in N$$

$$\left| \frac{2n+1}{2n} - 1 \right| < \epsilon \ \forall\ m N$$

Hence, $\lim_{n\to\infty} \frac{2n+1}{2n} = 1$ //.

---

(6). Prove that the following sequences are not convergent.

(a). $((-1)^n n)$

Proof:- Suppose that $((-1)^n n)$ convergent to l

then given $\epsilon > 0$; there exist $(\exists) \in N \ni : |(-1)^n n - l| < \epsilon \ \forall\ n \geq m$

$$|(-1)^n n - l| < \epsilon \ \forall n \geq m$$

$$|(-1)^m m - (-1)^{m+1}(m+1)| = |(-1)^m m - l - (-1)^{m+1}(m+1) + l|$$

$$\leq |(-1)^m m - l| + |(-1)^{m+1}(m+1) + l|$$

$\leq \epsilon + \epsilon$

$\leq 2\epsilon \longrightarrow (1)$

But $|(-1)^m m - (-1)^{m+1}(m+1)| = |(-1)^m m - (-1)^m(-1)(m+1)|$

$= |(-1)^m m + (-1)^m (m+1)|$

$= |(-1)^m m + (-1)^m m + (-1)^m|$

$= |(-1)^m (m+m+1)|$

$= |(-1)^m (2m+1)|$

$= |2(-1)^m m + (-1)^m \cdot 1|$ $\quad \because m > 0$

$= 2m+1$

$|(-1)^n| = 1 \qquad > 3 \longrightarrow (2)$

From (1) and (2) we get,

$3 < 2\epsilon$, $3 \geq 2\epsilon$, $3 \neq 2\epsilon$

This is Contradiction.

Hence, $((-1)^n n)$ is not convergent. //.

Home Work (08.08.2020) $\rightarrow$ Sequence to Convergent in all definition at reading only.

= x =

Thanks to All.

= x =

10.08.2020

II. B.Sc, Maths And Sub: (Sequences and Series)

3.5 Divergent And Oscillating Sequences

We now proceed to classify sequences, which are not Convergent as follows:-

(i). Sequences diverging to ∞

(ii). Sequences diverging to $-\infty$

(iii). Finitely oscillating sequences.

(iv). Infinitely oscillating sequences.

Definition:- A sequence (a_n) is said to diverge to ∞ if given any real number $K > 0$, there exists $m \in N$ such that $a_n > K$ for all $n \geq m$. In symbols we write $(a_n) \to \infty$ (or) $\lim_{n \to \infty} a_n = \infty$.

Note:-

$(a_n) \to \infty$ if and only if (iff) given any real number $K > 0$ there exists $m \in N$ such that $a_n \in (K, \infty)$ for all $n \geq m$.

Examples:-

1). $(n) \to \infty$

Proof:- Let $K > 0$ be any given real number.

Choose m to be any natural number, such that $m > K$.
Then $n > K \;\forall\; n \geq m$.

$\therefore$ $\boxed{(n) \rightarrow \infty}$.

2). $(n^2) \rightarrow \infty$

Proof :- Let $K > 0$ be given real number,

Choose $m \in N$ there exist $m > \sqrt{K}$

Then,

$n \geq m$, $n > \sqrt{K} \;\forall\; n \geq m$

(ie). $n^2 > K \;\forall\; n > m$

$\Rightarrow a_n \in (K, \infty) \;\forall\; n \geq m$.

Hence, $\boxed{(n^2) \rightarrow \infty}$.

3). $(2^n) \rightarrow \infty$

Proof :- let $K > 0$ be any given real number.

Then $2^n > K \Leftrightarrow n \log 2 > \log K$.

$\Leftrightarrow n > (\log K)/\log 2$

Hence, if we choose m to be any natural number such that $m > (\log K)/\log 2$,

Then, $2^n > K \;\forall\; n \geq m$.

$\therefore$ $\boxed{(2^n) \rightarrow \infty}$.

Definition:- A sequence (a_n) is said to diverge to $-\infty$, if given any real number $K < 0$ there exists $m \in N$ such that $a_n < K \ \forall \ n \geq m$. In symbols we write $\lim_{n \to \infty} a_n = -\infty$

(or) $(a_n) \to -\infty$.

Note:-

$(a_n) \to -\infty$ if and only if given any real number $K < 0$, there exists $m \in N$ such that $a_n \in (-\infty, K) \ \forall \ n \geq m$.

A sequence (a_n) is said to be divergent if either $(a_n) \to \infty$ (or) $(a_n) \to -\infty$.

Theorem: 1

$$(a_n) \to \infty \text{ iff } (-a_n) \to -\infty.$$

Proof:-

Let $(a_n) \to \infty$.

Let $K < 0$ be any given real number.

Since $(a_n) \to \infty$ there exists ($\exists$) $m \in N$ such that

$a_n > -K \ \forall \ n \geq m$.

$\therefore -a_n < K \ \forall \ n \geq m$.

$\therefore (-a_n) \to -\infty$.

|||ly, If $(-a_n) \to -\infty$,

Then $(a_n) \to \infty$.

Thanks to All

12.28.2020 ①

II. B.Sc, Maths, Sub: (Sequences and Series)

Notes on Continue

Theorem: 2

If $(a_n) \to \infty$ and $a_n \neq 0 \; \forall \; n \in N$ then $\left(\frac{1}{a_n}\right) \to 0$.

Proof:

Let $(a_n) \to \infty$ and $a_n \neq 0 \; \forall \; n \in N$,

Then $\varepsilon > 0$ be given

$\because (a_n) \to \infty$ there exist $m \in N$, $a_n < \frac{1}{\varepsilon} \; \forall \; n \geq m$

(ie), $\frac{1}{a_n} < \varepsilon$, $\forall \; n \geq m$

$\therefore \left|\frac{1}{a_n}\right| < \varepsilon \; \forall \; n \geq m$

$\Rightarrow \left|\frac{1}{a_n} - 0\right| < \varepsilon \; \forall \; n \geq m$

So $\boxed{\left(\frac{1}{a_n}\right) \to 0}$

Examples:-

The Convergent of the above theorem is not true given an example.

Solution:-

let $(a_n) = \left(\frac{(-1)^n}{n}\right)$

$$a_n = \frac{(-1)^n}{n}$$

$\therefore (a_n) \to 0$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{(-1)^n}{n}$$

$\left(\frac{1}{a_n}\right) = \frac{n}{(-1)^n}$, then $\lim_{n \to \infty} a_n = 0$

$$= \{-1, 2, -3, 4, \ldots\}$$

$$= (1/a_n)$$

And neither divergent to $+\infty$ not $-\infty$

$\therefore \left(\frac{1}{a_n}\right) \to +\infty$

Hence, if $(a_n) \to 0$ then $(1/a_n)$ need not convergent. (or) diverge.

Theorem:- 3

If sequence $(a_n) \to 0$ and $a_n > 0$ $\forall n \in N$ then $\left(\frac{1}{a_n}\right) \to \infty$.

Proof:-

let $(a_n) \to 0$ and $a_n > 0 \; \forall \; n \in N$

Then, ~~$\varepsilon > 0$~~, & $K > 0$, be any given real number.

Since $(a_n) \to 0$ there exists $m \in N$ such that $|a_n| < 1/K \; \forall \; n \geq m$.

$\therefore \; a_n < 1/K \; \forall \; n \geq m$ (since $a_n > 0$).

$\therefore \; 1/a_n > K \; \forall \; n \geq m$.

$\therefore \; \boxed{(1/a_n) \to \infty}$.

Theorem : 4

Any sequence (a_n) diverging to ∞ is bounded below but not bounded above.

Proof:-

let $(a_n) \to \infty$.

Then for any $K > 0$, $K \in R$ there exists $m \in N$ such that $a_n > K, \; \forall \; n \geq m \longrightarrow (1)$

$\therefore$ K is not upper bound of (a_n)

$\therefore$ (a_n) is not bounded above.

Now,

$$l = \min \{a_1, a_2, \ldots, a_m \ldots k\}$$

from (1), $a_n \geq l \; \forall \; n$.

$\therefore$ $\boxed{(a_n) \text{ bounded below}}$

Hence the proof.

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Thanks to All

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13.08.2020

II. B.Sc, Maths, 4 Sub: Sequences and Series

Notes on Continue

Definition:- Oscillating sequence:-

A sequence (a_n) which is neither convergent nor divergent to ∞ (or) $-\infty$ is said to be an oscillating sequence. An oscillating sequence which is bounded is said to be finitely oscillating.

An oscillating sequence which is unbounded is said to be infiniately oscillating.

Theorem: 5 :-

Any sequence (a_n) diverging to $-\infty$ is bounded above but not bounded below.

Proof:-

Let $(a_n) \to -\infty$.

Then for any $k>0$, $k \in R$ there exists $m \in N$ such that $a_n < k \;\forall\; n \geq m \longrightarrow (1)$

$\therefore$ k is not a lower bound of the sequence (a_n).

$\therefore$ (a_n) is not bounded below.

Now,

$$u = \max\{a_1, a_2, \ldots\ldots a_m, k\}$$

from (1), $a_n \leq u \; \forall n$

$\therefore (a_n)$ is bounded above.

Hence the proof

Note:-

The Converge of the above theorem not true. for example.

Proof:-

Let $f: N \to R$ be defined as

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{2}n & \text{if } n \text{ is even} \end{cases}$$

determine the sequence $0, 1, 0, 2, 0, 3, \ldots\ldots$ which is bounded below and not bounded above.

Also, for any $K > 0$, $K \in R$, then find $m \in N$ there exists $a_n > K \; \forall n \geq m$.

Hence, This sequence does not diverge to ∞.

III'y, Let $f: N \to R$ be defined as.

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{1}{2}n & \text{if } n \text{ is even} \end{cases}$$

determine the sequence $0, -1, 0, -2, 0, -3, \ldots$ which is bounded above and not bounded below.

Also,

for any $k < 0$, $k \in R$, we cannot find $m \in N$ there exists $a_n < k \ \forall \ n \geq m$.

Hence,

This sequence does not diverge to $-\infty$.

Example for oscillating sequence :-

Example : 1

$$((-1)^n) = -1, 1, -1, 1, \ldots$$

$g.l.b = -1$

$l.u.b = +1$

$\therefore ((-1)^n)$ is bounded.

Also, $((-1)^n)$ is not convergent

Hence,

$((-1)^n)$ is a finitly oscillating sequences.

Ex : 2

Let $f : N \to R$ defined by

$$f(n) = \begin{cases} \frac{1}{2} n & \text{if } n \text{ is even} \\ \frac{1}{2}(1-n) & \text{if } n \text{ is odd.} \end{cases}$$

<u>Proof:-</u>

If n is even $f(n) = 1, 2, 3, \ldots$

If n is odd $f(n) = 0, -1, -2, \ldots$

$f(n)$ determine this sequence $0, +1, -1, 2, -2, 3, -3, \ldots$

Hence,

The sequence is neither bounded below nor bounded above

Hence, It cannot converge (or) diverge to $\pm\infty$.

$\therefore$ The sequence is infinitely oscillating.

<u>Home work:- 13.08.2020</u>

1). Discuss the behaviour of each of the following sequences.

(a). $(n!)$, (b) $1, 1/2, 2, 1/3, 3, \ldots 1/n, n, \ldots$

(c). $((-1)^n 5)$, (d) $((-1)^n + 5)$, (e) $(-n^2)$, (f) $(\sqrt{n})$

(g). $(\cos n\pi)$, (h). $(\sin n\pi/2)$.

(N) (Exercises sums in (2) and (3) is already explained and notes give to all.)

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Unit - I . Completed

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Thanks to All.

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14.08.20

II. B.Sc, Maths, Sub: Sequences and Series

Unit - II

The Algebra of Limits

Theorem: 1

If $(a_n) \to a$ and $(b_n) \to b$ then $(a_n + b_n) \to a + b$.

Proof:-

Let $\varepsilon > 0$ be given.

Now $|a_n + b_n - a - b| = |a_n - a + b_n - b|$

$\leq |a_n - a| + |b_n - b| \longrightarrow$ ①

Since $(a_n) \to a$, there exists a natural number n_1, such that,

$|a_n - a| < \frac{1}{2}\varepsilon \ \forall \ n \geq n_1. \longrightarrow$ ②

Since $(b_n) \to b$, there exists a natural number n_2, such that,

$|b_n - b| < \frac{1}{2}\varepsilon \ \forall \ n \geq n_2. \longrightarrow$ ③

Let $m = \max\{n_1, n_2\}$.

Then, $|a_n + b_n - a - b| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \ \forall \ n \geq m$.

$|a_n + b_n - a - b| < \varepsilon$ [By ①, ② + ③].

Hence; $\boxed{(a_n + b_n) \to a+b}$

2. Prove that $(a_n - b_n) \to a - b$.

Proof:-

Given $(a_n) \to a$ and $(b_n) \to b$ then $(a_n - b_n) \to a - b$

Let $\varepsilon > 0$ be given.

And $(a_n - b_n) \to a - b$

Now
$$|a_n - b_n - a + b| = |a_n - a - b_n + b|$$
$$\leq |a_n - a| + |b_n - b| \to 1.$$

Since, $(a_n) \to a$, there exists a natural number n_1, such that,
$$|a_n - a| < \frac{1}{2}\varepsilon \ \forall \ n \geq n_1$$
(i). $|a_n - a| < \frac{\varepsilon}{2} \ \forall \ n \geq n_1 \longrightarrow 2$

Since $(b_n) \to b$, there exists a natural number n_2, such that
$$|b_n - b| < \frac{1}{2}\varepsilon \ \forall \ n \geq n_2$$
(i), $|b_n - b| < \frac{\varepsilon}{2} \ \forall \ n \geq n_2 \longrightarrow 3.$

Let
$$m = \max\{n_1, n_2\}$$
Then
$$|a_n - b_n - a + b| \leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \ \forall \ n \geq m.$$
$\therefore |a_n - b_n - a + b| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ \forall \ n \geq m$

(i). $|a_n - b_n - a + b| < \varepsilon$ [By 1, 2, 3]

Hence, $\boxed{(a_n - b_n) \to a - b}$

Theorem: 3

If $(a_n) \to a$ and $K \in R$ then $(Ka_n) \to Ka$.

Proof:-

let $K = 0$,

Then (Ka_n) is the constant sequence $0, 0, 0, \ldots$ and hence the result is trivial.

Now, let $K \neq 0$.

Then $|Ka_n - Ka| = |K|\,|a_n - a| \longrightarrow$ 1

let $\varepsilon > 0$ be given.

Since $(a_n) \to a$, there exists $m \in N$ such that

$|a_n - a| < \frac{\varepsilon}{|K|} \ \forall\ n \geq m. \longrightarrow$ 2

$\therefore |Ka_n - Ka| < \varepsilon \ \forall\ n \geq m$

[By 1 & 2].

$\therefore \boxed{(Ka_n) \to Ka}$.

Theorem:-4

If $(a_n) \to a$ and $(b_n) \to b$ then $(a_n b_n) \to ab$.

Proof:- Let $\varepsilon > 0$ be given.

Now, $|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab|$

$$\le |a_n b_n - a_n b| + |a_n b - ab|$$

$$= |a_n|\,|b_n - b| + |b|\,|a_n - a| \quad \rightarrow (1)$$

Also, since $(a_n) \to a$, (a_n) is a bounded sequence.

$\therefore$ There exists a real number $K > 0$ such that

$|a_n| \le K \;\forall n. \longrightarrow (2)$

Using (1) and (2) we get

$$|a_n b_n - ab| \le K|b_n - b| + |b|\,|a_n - a| \quad \rightarrow (3)$$

Now,

since $(a_n) \to a$. there exists a natural number n_1 such that

$$|a_n - a| < \frac{\varepsilon}{2|b|} \;\forall\; n \ge n_1 \quad \rightarrow (4)$$

Since $(b_n) \to b$, there exists a natural number n_2, such that

$$|b_n - b| < \frac{\varepsilon}{2K} \;\forall\; n \ge n_2 \quad \rightarrow (5)$$

Let $m = \max\{n_1, n_2\}$.

Then,

(5)

$$|a_n b_n - ab| < K\left(\frac{\varepsilon}{2K}\right) + |b|\left(\frac{\varepsilon}{2|b|}\right) = \varepsilon \ \forall\, n \geq m$$

[By (3), (4) and (5)]

Hence, $\boxed{(a_n b_n) \to ab.}$ //.

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Thank you

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II. B.Sc, Maths, Sub: Sequences and series

Unit - II (Continue) in notes

Theorem: 5

If $(a_n) \to a$ and $a_n \neq 0 \ \forall \ n$ and $a \neq 0$, then $\left(\frac{1}{a_n}\right) \to \frac{1}{a}$.

Proof:-

Let $\varepsilon > 0$ be given.

We have $\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a_n - a}{a_n a}\right|$

$$\Rightarrow \frac{1}{|a_n||a|} |a_n - a| \longrightarrow (1)$$

Now, $a \neq 0$.

Hence $|a| > 0$.

Since $(a_n) \to a$ there exists $n_1 \in N$ such that

$|a_n - a| < \frac{1}{2}|a| \ \forall \ n \geq n_1$.

Hence $|a_n| > \frac{1}{2}|a| \ \forall \ n \geq n_1 \longrightarrow (2)$

Using (1) and (2), we get

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{2}{|a|^2} |a_n - a| \ \forall \ n \geq n_1 \longrightarrow (3).$$

Now,

Since $(a_n) \to a$ there exists $n_2 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{1}{2} \varepsilon |a|^2 \ \forall \ n \geq n_2 \longrightarrow (4)$$

Let $m = \max\{n_1, n_2\}$

$$\therefore \left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{2|a|^2 \varepsilon}{|a|^2 \, 2} = \varepsilon \ \forall \ n \geq m$$

[By (3) and (4)].

$$\therefore \boxed{\left(\frac{1}{a_n}\right) \to \frac{1}{a}}.$$

Note:-

Even if $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ do not exist, $\lim\limits_{n\to\infty} (a_n + b_n)$ and $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ may exists.

Example:-

Let $a_n = ((-1)^n)$ and $b_n = ((-1)^{n+1})$.

Clearly, $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ do not exist.

Now $(a_n + b_n)$ is the Constant sequence $0, 0, 0, \ldots\ldots$

each of $(a_n\ b_n) \to 0$. $(a_n b_n) \to -1$ and $(a_n / b_n) \to -1$.

Theorem: 6

If $(a_n) \to a$ then $(|a_n|) \to |a|$.

Proof:-

Let $\varepsilon > 0$ be given.

Now, $||a_n| - |a|| \leq |a_n - a| \to (1)$

Since $(a_n) \to a$, there exists $m \in N$ such that

$|a_n - a| < \varepsilon \ \forall \ n \geq m$.

Hence from (1) we get $||a_n| - |a|| < \varepsilon \ \forall \ n \geq m$.

Hence $\boxed{(|a_n|) \to |a|}$.

Theorem: 7

If $(a_n) \to a$ and $a_n \geq 0 \ \forall \ n$ then $a \geq 0$.

Proof:- Suppose $a < 0$. Then $-a > 0$.

Choose ε such that $0 < \varepsilon < -a$,

So that $a + \varepsilon < 0$.

Now,

Since $(a_n) \to a$, there exists $m \in N$ such that

$|a_n - a| < \varepsilon \ \forall \ n \geq m$.

$\therefore a - \varepsilon < a_n < a + \varepsilon \ \forall \ n \geq m$.

Now,

Since $a+\epsilon < 0$,

We have $a_n < 0 \ \forall \ n \geq m$.

Which is a contradiction.

Since $a_n \geq 0$.

Hence $\boxed{a \geq 0}$.

Note:-

In the above theorem if $a_n > 0 \ \forall \ n$, we cannot say that $a > 0$.

Example:-

Consider the sequence $\left(\frac{1}{n}\right)$.

Here $\frac{1}{n} > 0 \ \forall \ n$ and $\left(\frac{1}{n}\right) \to 0$.

Theorem: 8

If $(a_n) \to a$, $(b_n) \to b$ and $a_n \leq b \ \forall \ n$, then $a \leq b$.

Proof:-

Since $a_n \leq b_n$,

We have $b_n - a_n \geq 0 \ \forall \ n$. $\longrightarrow$ (1)

We know that,

$$(a_n \pm b_n) \to a \pm b$$

(1) $\Rightarrow$ $|b_n (-a_n)| \to b-a$

We know that,

If $(a_n) \to a$ & $a_n \geq 0 \ \forall \ n$ then $a \geq 0$.

$\therefore \quad b-a \geq 0$

$\Rightarrow \quad b \geq a \quad \Rightarrow \quad a \leq b$

Hence $\boxed{a \leq b}$

Theorem: 9

(*) imp

If $(a_n) \to l$, $(b_n) \to l$ and $a_n \leq c_n \leq b_n \; \forall n$, then $(c_n) \to l$.

Proof:-

let $(a_n) \to l$ and $(b_n) \to l$.

$a_n \leq c_n \leq b_n \; \forall n. \longrightarrow (1)$.

$\therefore \quad (c_n) \to l$

let $\varepsilon > 0$ be given

$|c_n - l| < \varepsilon$

$\because (a_n) \to l$ there exists $n_1 \in N$ such that

$l - \varepsilon < a_n < l + \varepsilon \; \forall n \geq n_1 \longrightarrow (2)$ And

$l-\varepsilon$ and $l+\varepsilon$ → $(b_n) \to l$ there exists $n_2 \in N$ such that

$l - \varepsilon < a_n < l + \varepsilon \; \forall n \geq n_2 \longrightarrow (3)$

let $m = \max\{n_1, n_2\} \longrightarrow (4)$

$\therefore \; l - \varepsilon < a_n \leq c_n \leq b_n < l + \varepsilon \; \forall \; n \geq m$

[By (2), (3) & (4)]

$\therefore \; l - \varepsilon < c_n < l + \varepsilon \; \forall \; n \geq m$

$\therefore \; |c_n - l| < \varepsilon \; \forall \; n \geq m$.

$\therefore \; \boxed{(c_n) \to l}$.

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II. B.Sc, Maths, Sub: Sequences and Series

Unit - II (Continue notes)

Theorem : 13

Let $(a_n) \to \infty$. Then

(i). If $c > 0$, $(ca_n) \to \infty$.

(ii). If $c < 0$, $(ca_n) \to -\infty$.

Proof:-

(i). Let $c > 0$

Then $K > 0$ be any given real number.

Since $(a_n) \to \infty$, there exists $m \in N$ such that

$$a_n > K/c \;\; \forall \; n \geq m.$$

$$\therefore \; ca_n > K \;\; \forall \; n \geq m.$$

$$\therefore \; \boxed{(ca_n) \to \infty}.$$

(ii). Let $c < 0$.

Then $K < 0$ be any given real number. then $K/c > 0$.

$\therefore$ There exists $m \in N$ such that $a_n > K/c \;\; \forall \; n \geq m$.

$\therefore \; ca_n < K \;\; \forall \; n \geq m$

Since $c < 0$.

$$\therefore \; \boxed{(ca_n) \to -\infty} \;\; //.$$

Theorem: 14

If $(a_n) \to \infty$ and (b_n) is bounded then $(a_n + b_n) \to \infty$.

Proof:-

Since (b_n) is bounded,

There exists a real number $m < 0$ such that

$$b_n > m \;\forall\; n \longrightarrow (1)$$

Now,

let $k > 0$ be any real number.

Since $m < 0$, $k - m > 0$.

Since $(a_n) \to \infty$,

There exists $n_0 \in N$ such that

$$a_n > k - m \;\forall\; n \geq n_0 \longrightarrow (2)$$

$\therefore a_n + b_n > k - m + m = k \;\forall\; n \geq n_0$

[By (1) and (2)].

$\therefore \boxed{(a_n + b_n) \to \infty}$

//

1). Show that $\lim\limits_{n \to \infty} \dfrac{3n^2 + 2n + 5}{6n^2 + 4n + 7} = \dfrac{1}{2}$.

<u>Solution</u>:-

$$a_n = \frac{3n^2 + 2n + 5}{6n^2 + 4n + 7} = \frac{3 + \frac{2}{n} + \frac{5}{n^2}}{6 + \frac{4}{n} + \frac{7}{n^2}}$$

Now,

$$\lim_{n \to \infty} \left(3 + \frac{2}{n} + \frac{5}{n^2}\right) = 3 + 2$$

$$\lim_{n \to \infty} \frac{1}{n} + 5 \; ; \; \lim_{n \to \infty} \frac{1}{n^2} .$$

$$\Rightarrow 3 + 0 + 0 = 3 .$$

$|||^{ly}$,

$$\lim_{n \to \infty} \left(6 + \frac{4}{n} + \frac{7}{n^2}\right) = 6 .$$

$$\therefore \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\left(3 + \frac{2}{n} + \frac{5}{n^2}\right)}{\left(6 + \frac{4}{n} + \frac{7}{n^2}\right)}$$

$$\Rightarrow \frac{\lim\limits_{n \to \infty} \left(3 + \frac{2}{n} + \frac{5}{n^2}\right)}{\lim\limits_{n \to \infty} \left(6 + \frac{4}{n} + \frac{7}{n^2}\right)}$$

$$= \frac{3}{6} = \frac{1}{2}$$

//.

2) Show that $\lim_{n\to\infty}\left(\frac{1^2+2^2+\dots\dots+n^2}{n^3}\right)=\frac{1}{3}$.

<u>Solution:-</u>

We know that $1^2+2^2+\dots\dots+n^2$

$$=\frac{n(n+1)(2n+1)}{6}$$

$$\therefore \lim_{n\to\infty}\frac{1^2+2^2+\dots\dots+n^2}{n^3}$$

$$=\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}$$

$$=\lim_{n\to\infty}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$=\frac{1}{3}.$$

=== * ===

Thank you

=== * ===

21.08.20

II. B.Sc., Maths, & Sub: Sequences and Series

Unit - II (Continue notes)

3). Show that $\lim_{n \to \infty} \frac{n}{\sqrt{(n^2+1)}} = 1$.

(X).

Solution:-

$$\lim_{n \to \infty} \frac{n}{\sqrt{(n^2+1)}} \Rightarrow \lim_{n \to \infty} \frac{1}{\sqrt{\left(1+\frac{1}{n^2}\right)}}$$

$$a_n = \frac{n}{\sqrt{(n^2+1)}}$$

$$\therefore \lim_{n \to \infty} a_n = 1$$

Now,

$$a_n = \frac{n}{\sqrt{(n^2+1)}} \Rightarrow \frac{n}{\sqrt{n^2}\sqrt{1+1/n^2}}$$

$$= \frac{\cancel{n}}{\cancel{n}\sqrt{1+1/n^2}} \Rightarrow \frac{1}{\sqrt{1+1/n^2}}$$

$$= \frac{1}{\sqrt{1+1/\infty^2}} \Rightarrow 1$$

$$\lim_{n \to \infty} a_n = 1$$

$\lim_{n \to \infty} \frac{1}{\sqrt{1+1/n^2}}$ then if $(a_n) \to a$ and $a_n \neq 0 \; \forall n$.

Then $(1/a_n) \to 1/a$,

$$\therefore \frac{1}{\sqrt{1+1/\infty}} = \frac{1}{\sqrt{1}} \Rightarrow \frac{1}{1} = 1. //$$

4) (i) Show that $(a_n) \to 0$ and (b_n) is bounded then $(a_n b_n) \to 0$.

Solution:-

Let $(a_n) \to 0$, then (b_n) is bounded

$\therefore$ (b_n) is bounded there exists $k > 0$ such that

$$|b_n| \leq K \ \forall n$$

$$\therefore \ |a_n b_n| \leq K |a_n|$$

(ii). $(a_n) \to 0$ there exists $m \in N$, such that

$$|a_n| < \varepsilon/k \quad \forall n \geq m .$$

$$\therefore \ |a_n b_n| < k \ \varepsilon/k \ \forall n \geq m .$$

$$|a_n b_n| < \varepsilon \ \forall n \geq m$$

$$\therefore \ (a_n b_n) \to ab$$

(ii). $|a_n b_n| < \varepsilon \ \forall n \geq m$

Hence $\boxed{(a_n b_n) \to 0}$ //.

5). Show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

Solution :-

$$|\sin n| \le 1 \ \forall\ n.$$

$\therefore$ $(\sin n)$ is a bounded sequences.

Also $\left(\frac{\sin n}{n}\right) \to 0$, and $\left(\frac{1}{n}\right) \to 0$.

$\therefore$ $\left(\frac{\sin n}{n}\right) \to 0$.

---

6). Show that $\lim_{n\to\infty} (a^{1/n}) = 1$, where $a > 0$ is any real number.

Solution:-

Given $\lim_{n\to\infty} (a^{1/n}) = 1$, where $a > 0$.

Case (i)

let $a = 1$

Then $a^{1/n} = 1^{1/n} = 1 \ \forall\ n$

Hence $\boxed{(a^{1/n}) = 1}$

Case (ii)

let $a > 1$

Then $a^{1/n} > 1$

let $a^{1/n} = 1 + h_n$, where $h_n > 0$

$\therefore a = (1+h^n)^n$

$= 1 + nh_n + \dots + h_n^n$

$a > 1 + nh_n$

$\therefore h_n < \frac{a-1}{n}$.

$\therefore a < h_n < \frac{a-1}{n}$

Hence $\lim_{n\to\infty} h_n = 0$

$\therefore (a^{1/n}) = (1+h_n) = (1+0) = 1$

(ie), $\boxed{(a^{1/n}) \to 1}$

<u>Case (iii)</u>

let $0 < a < 1$

Then $1/a > 1$

$\therefore (1/a)^{1/n} \to 1$ [By case (i)]

$\therefore \left(\frac{1}{a^{1/n}}\right) \to 1$

$\therefore (a^{1/n}) \to 1$

$$\boxed{\lim_{n\to\infty} (a^{1/n}) = 1}$$

Hence case (i), (ii) & (iii) are $\lim_{n\to\infty} (a^{1/n}) = 1$ //

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Class: II. B.Sc, Maths & Sub: Sequences and series

Unit - II (Continue notes)

(1)

7) Show that $\lim_{n \to \infty} (n^{1/n}) = 1$.

(X).

Solution:-

Clearly, $n^{1/n} \geq 1 \ \forall n$.

Let $n^{1/n} \geq 1 + h_n$,

Where $h_n \geq 0$.

Then $n = (1+h_n)^n$

$= 1 + nh_n + nC_2 h_n^2 + \ldots\ldots + h_n^n$.

$> \frac{1}{2} n(n-1) h_n^2$.

$\therefore h_n^2 < \frac{2}{n-1}$.

$\therefore h_n < \frac{\sqrt{2}}{n-1}$.

Since, $\frac{\sqrt{2}}{n-1} \to 0$ and $h_n \geq 0$,

$(h_n) \to 0$.

$\therefore (n^{1/n}) = (1+h_n) \to 1$

(ie), $(n^{1/n}) = (1+0) \to 1$.

$\therefore \boxed{(n^{1/n}) = 1}$ //.

8) Show that $\lim_{n\to\infty}\left(\frac{1}{\sqrt{(2n^2+1)}}+\frac{1}{\sqrt{(2n^2+2)}}+\cdots\cdots+\frac{1}{\sqrt{(2n^2+n)}}\right)=\frac{1}{\sqrt{2}}$.

Solution:-

let $a_n = \frac{1}{\sqrt{(2n^2+1)}}+\frac{1}{\sqrt{(2n^2+2)}}+\cdots\cdots+\frac{1}{\sqrt{(2n^2+n)}}$

Then, we have the inequality is,

$$\frac{n}{\sqrt{(2n^2+n)}} \leq a_n \leq \frac{n}{\sqrt{(2n^2+1)}}.$$

$$\therefore \frac{1}{\sqrt{(2+\frac{1}{n})}} \leq a_n \leq \frac{1}{\sqrt{(2+\frac{1}{n^2})}}.$$

Now,

$$\lim_{n\to\infty}\frac{1}{\sqrt{(2+\frac{1}{n})}}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{(2+\frac{1}{n^2})}} \Rightarrow \frac{1}{\sqrt{2}}.$$

$$\therefore \boxed{\lim_{n\to\infty} a_n = \frac{1}{\sqrt{2}}}$$

//.

9) Show that if (a_n) is a sequence diverging to ∞ and (b_n) is a sequence diverging to $-\infty$ then $(a_n + b_n)$ need not be a divergent sequence.

Solution:-

let $(a_n) = (n)$ And

$(b_n) = (-n)$.

Clearly,

$(a_n) \to \infty$ And $(b_n) \to -\infty$.

However $(a_n + b_n)$ is the constant sequence $0, 0, 0, \ldots$ which converges to 0.

Home work of date:- 17.08.2020 to 24.08.2020.

1. Evaluate the limits of the following sequences as $n \to \infty$.

(a). $\left(\frac{3n-4}{2n+7}\right)$; Answer:- $\frac{3}{2}$.

(b). $\left(\frac{4-2n+6n^2}{7-6n+9n^2}\right)$; Answer:- $\frac{2}{3}$.

(c). $\left(\frac{(n^2+3)(n^3+9)}{(n+1)(n^4+6)}\right)$; Answer:- 1.

(d). $\left(\frac{n^2+n+1}{n^3+2}\right)$; Answer:- 0 .

(e). $\left(\sqrt{(n^2+n)}-n\right)$; Answer:- $\frac{1}{2}$.

(f). $\left((-1)^n/n\right)$; Answer:- 0 .

(g). $\sqrt{\frac{3n^2-5n+4}{2n-7}}$; Answer:- $\sqrt{3}/2$.

(h). $\frac{n^2}{\sqrt{(n^4+3n^2+1)}}$; Answer:- 1 .

(i). $\left(\frac{1+2+3+\cdots\cdots+n}{n^2}\right)$; Answer:- $1/2$.

(j). $\left(\frac{1^3+2^3+3^3+\cdots\cdots+n^3}{n^4}\right)$; Answer:- $1/4$.

---

(7). Prove the following:-

(a). $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{(n^2+1)}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{(n^2+n)}}\right)=1$.

(b). $\lim\limits_{n\to\infty} \frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots+\frac{1}{(2n)^2}=0$.

(c). $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{(n+1)}}+\cdots+\frac{1}{\sqrt{(2n)}}\right)=\infty$.

3.7. Behaviour of Monotonic Sequences

Notes :- (or) Statement :-

(i). A monotonic increasing sequence which is bounded above converges to its . l.u.b .

(ii). A monotonic increasing sequence which is not bounded above diverges to ∞ .

(iii). A monotonic decreasing sequence which is bounded below converges to its g.l.b .

(iv). A monotonic decreasing sequence which is not bounded below diverges to $-\infty$.

*

Thank you !

*

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①

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Unit - II (Continue notes)

Proof:-

(i) Let (a_n) be a monotonic increasing sequence which is bounded above.

Let K be the l.u.b of the sequence.

Then,

$$a_n \leq K \ \forall n. \longrightarrow (1)$$

Now,

Let $\varepsilon > 0$ be given.

$\therefore$ $K - \varepsilon < K$ and hence $K - \varepsilon$ is not an upper bound of (a_n).

Hence, there exists a_m such that $a_m > K - \varepsilon$.

Now, since (a_n) is monotonic increasing $a_n \geq a_m \ \forall n \geq m$.

Hence $a_n > K - \varepsilon \ \forall n \geq m \longrightarrow (2)$

$\therefore$ $K - \varepsilon < a_n \leq K \ \forall n \geq m$ [by (1) and (2)] we get.

$\therefore$ $|a_n - K| < \varepsilon \ \forall n \geq m$.

$\therefore$ $\boxed{(a_n) \rightarrow K}$

(ii) proof :- (ii)

Let (a_n) be a monotonic increasing sequence which is not bounded above.

Let $K > 0$ be any real number.

Since (a_n) is not bounded,

There exists $m \in N$ such that $a_m > K$.

Also, $a_n \geq a_m \ \forall \ n \geq m$.

$\therefore \ a_n > K \ \forall \ n \geq m$.

$\therefore \ \boxed{(a_n) \rightarrow \infty}$ //.

proof :- (iii)

Proof of (iii) is similar to that of (i).

Let (a_n) be a monotonic decreasing sequence which is bounded below.

Let K be the G.l.b of the sequence

Then $a_n \geq K \ \forall \ n. \longrightarrow (1)$

Now,

Let $\epsilon > 0$ be given.

$\therefore \ K - \epsilon < K$ and hence $K - \epsilon$ is not an G.l.b of (a_n).

Hence,

There exists a_m such that $a_m < K - \varepsilon$.

Now,

Since (a_n) is monotonic decreasing

$$a_n \leq a_m \ \forall \ n \geq m.$$

Hence,

$$a_n < K - \varepsilon \ \forall \ n \geq m \longrightarrow (2)$$

$\therefore \ K - \varepsilon > a_n > K \ \forall \ n \geq m$ [By (1) and (2)] we get.

$\therefore \ |a_n - K| > \varepsilon \ \forall \ n \geq m$

$\therefore \boxed{(a_n) \rightarrow K}$.

//.

Proof :- (iv) ;

Proof of (iv) is similar to that of (ii).

Let (a_n) be a monotonic decreasing sequence which is not bounded below.

Let $K < 0$ be any real number.

$\because$ (a_n) is not bounded there exists $m \in \mathbb{N}$ such that

$$a_m < K.$$

Also,

$$a_n \leq a_m \ \forall \ n \geq m$$

$$a_n < K \ \forall \ n \geq m$$

$\therefore \boxed{(a_n) \rightarrow -\infty}$ //.

Note :- 1

The above theorem shows that a monotonic sequence either converges or diverges. Thus, a monotonic sequence cannot be an oscillating sequence.

(1). Problem :-

Let $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$. Show that $\lim\limits_{n\to\infty} a_n$ exists and lies between 2 and 3.

Solution :-

Clearly (a_n) is a monotonic increasing sequence. $\boxed{a_n \leq a_{n+1}}$

Also,

$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}.$$

$$= 1 + \left(\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \right)$$

$$= 1 + 2\left(1 - \frac{1}{2^n}\right)$$

$$= 3 - \frac{1}{2^{n-1}} < 3.$$

$\therefore a_n < 3$.

$\therefore (a_n)$ is bounded above.

$\therefore \lim_{n\to\infty} a_n$ exists.

Also,

$$2 < a_n < 3 \ \forall\ n$$

$$\therefore\ 2 \leq \lim_{n\to\infty} a_n \leq 3.$$

(ie). $\lim_{n\to\infty} a_n$ exists and lies between 2 and 3. //.

Hence the result.

Note: 2

The limit of the above sequence is denoted by e..

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(1)

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(2). Show that the sequence $\left(1+\frac{1}{n}\right)^n$ Converges.

Solution:-

Let $a_n = \left(1+\frac{1}{n}\right)^n$, By binomial theorem,

$$a_n = 1+1\ \frac{n(n-1)}{2!}\ \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\ \frac{1}{n^3} + \cdots + \frac{1}{n^n}.$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots$$

$$+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$$

Also, $a_n = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}$

$$\leq 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$$

$$= 1+\left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right)$$

$$= 1+2\left(1-\frac{1}{2^n}\right)$$

②

$$= 3 - \frac{1}{2^{n-1}} < 3.$$

$\therefore a_n < 3$

$\therefore (a_n)$ is bounded above.

Also,

$$a_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+\dots$$

$$\dots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\dots\left(1-\frac{n}{n+1}\right).$$

$$> 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\dots$$

$$\dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-1}{n}\right).$$

$\therefore \boxed{a_{n+1} > a_n}$

$\therefore (a_n)$ is monotonic increasing.

$\therefore (a_n)$ is a convergent sequence. //.

---

③. Show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{1!}+\dots+\frac{1}{n!}\right) = e.$

Solution:-

Let $a_n = \left(1+\frac{1}{n}\right)^n$ and $b_n = 1+\frac{1}{1!}+\dots+\frac{1}{n!}$.

Then $a_n < b_n \ \forall n$

$$\therefore \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \longrightarrow (1)$$

Now,

let $m > n$.

$$a_m = \left(1+\frac{1}{m}\right)^m$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{m}\right)+\frac{1}{3!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)+\cdots\cdots$$

$$+\frac{1}{n!}\left(1-\frac{1}{m}\right)\cdots\left(1-\frac{n-1}{m}\right)+\cdots\cdots+\frac{1}{m!}\left(1-\frac{1}{m}\right)$$

$$\cdots\cdots\left(1-\frac{m-1}{m}\right).$$

$$> 1+1+\frac{1}{2!}\left(1-\frac{1}{m}\right)+\cdots\cdots+\frac{1}{n!}\left(1-\frac{1}{m}\right)\cdots\left(1-\frac{n-1}{m}\right).$$

Fixing n and taking limit as $m\to\infty$, we get.

$$\lim_{m\to\infty} a_m \geq 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!} = b_n.$$

Now,

Taking limit as $n\to\infty$, we get

$$\lim_{m\to\infty} a_m \geq \lim_{m\to\infty} b_n \longrightarrow (2)$$

[By (1) and (2)] we get

$$\therefore \boxed{\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = e}$$

(4). Let $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$ Show that (a_n) Converges.

Solution:-

$$a_{n+1} - a_n .$$

$$= \left(\frac{1}{n+2} + \dots + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \dots + \frac{1}{n+n}\right).$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} - \frac{1}{2n+2} > 0 \ \forall n$$

$\therefore a_{n+1} > a_n \ \forall n$.

$\therefore (a_n)$ is a monotonic increasing sequence.

Also, $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$.

$$< \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n} = 1 \ \forall n.$$

$\therefore (a_n)$ is bounded above.

$\therefore (a_n)$ Converges.

*//

Thank you!

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Unit - II (Continue notes).

(5) Let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Show that (a_n) diverges to ∞.

Solution:-

Clearly, (a_n) is a monotonic increasing sequence.

Now, let $m = 2^n - 1$.

$$a_m = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n - 1}$$

$$= 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \cdots + \left(\frac{1}{2^{n-1}} + \cdots + \frac{1}{2^n - 1}\right).$$

$$> 1 + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right).$$

$$= 1 + (n-1)\frac{1}{2} \Rightarrow \frac{1}{2}(n+1).$$

$\therefore a_m > \frac{1}{2}(n+1)$.

$\therefore (a_n)$ is not bounded above.

Hence $\boxed{(a_n) \to \infty}$.

//.

(b). Prove that $\left(\frac{n!}{n^n}\right)$ Converges.

(X)

Solution:-

let $a_n = \frac{n!}{n^n}$.

Then, $\frac{a_n}{a_{n+1}} = \frac{n!}{n^n} \frac{(n+1)^{n+1}}{(n+1)!}$

$= \left(\frac{n+1}{n}\right)^n > 1.$

$\therefore a_n > a_{n+1} \; \forall \; n \in N.$

$\therefore (a_n)$ is a monotonic decreasing sequence.

Also,

$a_n > 0 \; \forall \; n \in N.$

$\therefore \; (a_n)$ is bounded below.

$\therefore$ $\boxed{(a_n) \text{ Converges}}$.

//.

---

(7). Discuss the behaviour of the geometric sequence (r^n).

(X). 10 Marks

Solution:-

Case (i) Let $r = 0$.

Then (r^n) reduces to the Constant sequence 0,0,0,.......

And,

Hence, Converges to 0.

Case (ii),

let $r=1$.

In this case (r^n) reduces to the constant sequence 1, 1, 1, ... and

Hence, Converges to 1.

Case (iii),

let $0 < r < 1$.

In this Case, (r^n) is a monotonic decreasing sequence, and $(r^n) > 0 \ \forall \ n \in N$.

$\therefore (r^n)$ is monotonic decreasing and bounded below and hence (r^n) Converges.

Let $(r^n) \to l$

Since, $r^n > 0 \ \forall \ n, l > 0. \longrightarrow (1)$

We claim that $l = 0$.

Let $\varepsilon > 0$ be given.

Since $(r^n) \to l$,

There exists $m \in N$ such that,

$l < r^n < l + \varepsilon \ \forall \ n \geq m$.

Fix $n > m$.

Then,

$$l < r^{n+1} \longrightarrow (2)$$

Also $r^{n+1} = r \cdot r^n < r(l+\varepsilon) \longrightarrow (3)$

$\therefore l < r(l+\varepsilon)$

$\because$ [By (2) and (3)] we get,

$\therefore l < \left(\frac{r}{1-r}\right)\varepsilon$.

Since this is true for every $\varepsilon > 0$, we get

$$l \leq 0 \longrightarrow (4)$$

[By (1) and (4)] we get.

$\therefore l = 0$.

<u>Case (iv)</u> :-

~~Let~~ ~~ca~~ Let $-1 < r < 0$.

Then $r^n = (-1)^n |r|^n$,

where $0 < |r| < 1$.

By Case (iii) $(|r|^n) \to 0$.

Also,

$((-1)^n)$ is a bounded sequence.

$\therefore$ ~~$((-1)^n)$~~ $((-1)^n |r|^n)$ Converges to 0.

$\therefore (r^n) \to 0$.

Case (v) :-

let $r = -1$.

In this case (r^n) reduces to $-1, 1, -1, 1, -1, \ldots\ldots$ which oscillates finitely.

Case (vi) :-

let $r > 1$.

Then,

$$0 < \frac{1}{r} < 1 \text{ and}$$

Hence

$$\left(\frac{1}{r^n}\right) \to 0, \quad [\text{By Case (iii)}]$$

$\therefore (r^n) \to \infty$.

Case (vii) :-

let $r < -1$.

Then the terms of the sequence (r^n) are alternatively positive and negative.

Also, $|r| > 1$ and

Hence by Case (vi),

$$(|r|^n) \text{ is unbounded.}$$

$\therefore (r^n)$ oscillates infinitely.

Thus,

(i). (r^n) Converges if $-1 < r \leq 1$.

(ii). (r^n) diverges if $r > 1$.

(iii). (r^n) oscillates if $r \leq -1$. //.

Thank you!

28/08/20

2

Class: II. B.Sc, Maths, Sub: Sequences and Series.

Unit - II (Continue notes).

⑧ Show that if $|r| < 1$ then $(nr^n) \to 0$.

Solution:-

The result is trivial if $r = 0$.

Let $0 < |r| < 1$.

Then,

$$|r| = \frac{1}{1+p}, \quad p > 0.$$

$$\therefore \quad |r|^n = \frac{1}{(1+p)^n}$$

$$= \frac{1}{1 + np + \frac{n(n-1)}{2!}p^2 + \dots}$$

$$< \frac{2}{n(n-1)p^2}$$

$$\therefore \quad |nr^n| < \frac{2}{(n-1)p^2}$$

Now,

let $\varepsilon > 0$ be given.

Then,

$$\frac{2}{(n-1)p^2} < \varepsilon$$

provided $n > 1 + \dfrac{2}{p^2 \varepsilon}$

$\therefore |nr^n| < \varepsilon$ if $n > 1 + \frac{2}{p^2 \varepsilon}$.

$\therefore \boxed{\lim_{n\to\infty} nr^n = 0}$.

//.

---

(9). Show that $\lim_{n\to\infty} \frac{\log n}{n^p} = 0$ if $p > 0$.

(x)

Solution:-

We have $e^p > 1$.

Since, $e > 1$.

$\therefore \frac{1}{e^p} < 1$.

$\therefore \left(\frac{n}{(e^p)^n}\right) \to 0$.

$\therefore$ Given $\varepsilon > 0$, be given.

There exists a natural number m such that

$$\frac{n}{e^{p.n}} < \frac{\varepsilon}{e^p} \ \forall\ n \geq m. \longrightarrow (1)$$

Now,

let g be the positive integer such that,

$$g \leq \log n < (g+1). \longrightarrow (2)$$

$\therefore \frac{\log n}{n^p} < \frac{g+1}{n^p}$.

(3)

$$\leq \frac{g+1}{(e^{g})^{p}}$$

Since, $e^{g} \leq n$ [By (2)] we get,

$$= \frac{e^{p}(g+1)}{e^{p}(g+1)}$$

$$< e^{p}\left(\frac{\varepsilon}{e^{p}}\right)$$ Provided $g+1 \geq m$, [using (1)] we get

$\therefore \frac{\log n}{n^{p}} < \varepsilon$ provided $g+1 \geq m$.

Now,

if $n \geq e^{m}$,

Then

$$\log n \geq m.$$

But, $g+1 > \log n$, [By (2)] we get

$$\therefore n \geq e^{m} \Rightarrow g+1 \geq m.$$

$$\therefore \frac{\log n}{n^{p}} < \varepsilon$$

(ie), Provided $n \geq e^{m}$.

$$\therefore \boxed{\lim_{n \to \infty} \frac{\log n}{n^{p}} = 0}$$

//.

(10). (X) 10 Marks

Let (a_n) and (b_n) be two sequences of positive terms such that $a_{n+1} = \frac{1}{2}(a_n + b_n)$ and $b_{n+1} = \sqrt{(a_n b_n)}$. Prove that (a_n) and (b_n) converge to the same limit.

Solution:-

Let (a_n) and (b_n) be sequence of positive terms.

Then,

Term of (a_{n+1}) is in Arthmetic mean and

Term of (b_{n+1}) is in Geometric mean.

$\therefore$ By hypothesis, a_{n+1} and b_{n+1} are respectively the Arthmetic mean and Geometric mean.

Then,

Terms of lies between a_n and b_n.

Also,

We know that $A.M \geq G.M$.

Hence,

$$a_{n+1} \geq b_{n+1} \longrightarrow (1)$$

Moreover, the A.M and G.M of two numbers lie between the two numbers.

$$\therefore a_n \geq a_{n+1} \geq b_n \;\forall\, n \in N. \longrightarrow (2)$$

And

$$a_n \geq b_{n+1} \geq b_n \;\forall\, n \in N. \longrightarrow (3)$$

$\therefore a_n \geq a_{n+1} \geq b_{n+1} \geq b_n \ \forall n \in N$. [By ② and ③] we get

$\therefore (a_n)$ is a monotonic decreasing sequence, and

(b_n) is a monotonic increasing sequence.

Further, $a_n \geq b_n \geq b_1 \ \forall n \in N$.

And,

$$b_n \leq a_n \leq a_1 \ \forall n \in N.$$

$\therefore (a_n)$ is a monotonic decreasing sequence and bounded below by b_1,

~~By~~, And

$\therefore$ ~~by and~~ (b_n) is a monotonic increasing sequence and bounded above, by a_1.

$\therefore (a_n) \to l$ and $(b_n) \to m$.

Now,

$$a_{n+1} = \frac{1}{2}(a_n + b_n).$$

Taking, limit as $n \to \infty$,

We get, $l = \frac{1}{2}(l+m)$.

$\therefore \boxed{l = m}$.

Thank you!!!

29/08/20

Class: II. B.Sc, Maths, & Sub: Sequences and Series. (1)

Unit - II (Continue notes)

(11). Let (a_n) be a sequence of positive terms such that $a_1 < a_2$ and $a_{n+2} = \frac{1}{2}(a_{n+1} + a_n)$. Then show that (a_{2n-1}) is a monotonic increasing sequence and (a_{2n}) is a decreasing sequence and both converge to the common limit $\frac{1}{3}(a_1 + 2a_2)$. Hence deduce that (a_n) converges to the same limit.

(X) 10. Marks

Solution:-

We have $a_{n+2} = \frac{1}{2}(a_{n+1} + a_n)$ and $a_1 < a_2$.

$\therefore a_3 = \frac{1}{2}(a_2 + a_1)$ and $a_1 < a_2$. $\longrightarrow$ (1)

$\therefore a_1 < a_3 < a_2 \longrightarrow$ (2)

Also,

$a_4 = \frac{1}{2}(a_3 + a_2)$ and $a_3 < a_2$

[By (1) and (2)], we get

$\therefore a_3 < a_4 < a_2 \longrightarrow$ (3)

$\therefore a_1 < a_3 < a_4 < a_2$

[By (2) and (3)], we get

Proceeding as above, we get

$a_1 < a_3 < a_5 < a_6 < a_4 < a_2$ and so on.

$\therefore (a_{2n})$ is a monotonic decreasing sequence bounded below by a_1.

And, (a_{2n-1}) is a monotonic increasing sequence bounded above by a_2.

$\therefore (a_{2n}) \to l$ and $(a_{2n-1}) \to m$.

Now,

$$a_{2n+2} = \frac{1}{2}\left(a_{2n+1} + a_{2n}\right)$$ [by 1], we get

Taking limit as $n \to \infty$, we get

$$l = \frac{1}{2}(m + l).$$

$$\therefore l = m.$$

Now,

let $\varepsilon > 0$ be given.

Since $(a_{2n}) \to l$, there exists $n_1 \in N$ such that,

$$|a_{2n} - l| < \varepsilon \quad \forall\ n \geq n_1.$$

|||ly,

There exists $n_2 \in N$ such that,

$$|a_{2n-1} - l| < \varepsilon \quad \forall\ n \geq n_2.$$

Let $m = \max\{n_1, n_2\}$.

Then $|a_n - l| < \varepsilon \; \forall \; n \geq m$.

$\therefore (a_n) \to l$.

Now,

$$a_{n+2} = \frac{1}{2}(a_{n+1} + a_n)$$

$$a_{n+1} = \frac{1}{2}(a_n + a_{n-1}).$$

....

....

....

$$a_4 = \frac{1}{2}(a_3 + a_2).$$

$$a_3 = \frac{1}{2}(a_2 + a_1).$$

Adding, we get

$$a_{n+2} + \frac{1}{2} a_{n+1} = \frac{1}{2}(a_1 + 2a_2).$$

Taking limit as $n \to \infty$, we get.

$$l + \frac{1}{2} l = \frac{1}{2}(a_1 + 2a_2),$$

(ie), $\boxed{l = \frac{1}{3}(a_1 + 2a_2)}$.

//.

(12). Prove that the sequence (a_n) defined by $a_1 = \sqrt{K}$ and $a_{n+1} = \sqrt{(K+a_n)}$ where $K > 0$ Converges to the positive root of $x^2 - x - K = 0$.

Solution:-

First we prove that $a_n < a_{n+1}$ $\forall$ $n \in N$.

$$a_2 = \sqrt{(K+a_1)}$$

$$= \sqrt{(K+\sqrt{K})} > \sqrt{K} = a_1.$$

$\therefore$ $a_2 > a_1$.

Now,

Let us assume that $a_{m+1} > a_m$ for some $m \in N$.

Then $a_{m+2} = \sqrt{(K+a_{m+1})} > \sqrt{(K+a_m)}$

$$= a_{m+1}.$$

$\therefore$ $a_{m+2} > a_{m+1}$.

$\therefore$ By induction we get,

$$a_{n+1} > a_n \ \forall \ n \in N.$$

$\therefore$ (a_n) is a monotonic increasing sequence,

Now,

$a_{n+1} > a_n$.

$\therefore \sqrt{(k+a_n)} > a_n$.

$\therefore a_n^2 - a_n - k < 0$.

$\therefore a_n$ lies between the root of $x^2 - x - k = 0$.

$\therefore a_n < \alpha$,

Where α is the positive root of $x^2 - x - k = 0$.

Thus, (a_n) is bounded above.

$\therefore (a_n)$ Converges to l.

Clearly, $0 < l \le \alpha$.

Now,
$$a_{n+1} = \sqrt{(k+a_n)}.$$

Taking the limit as $n \to \infty$, we get

$$l = \sqrt{(k+l)}.$$

$\therefore l^2 - l - k = 0$.

$\therefore l$ is the positive root of $x^2 - x - k = 0$.

$\therefore \boxed{l = \alpha.}$

//.

(6)

Home Work 2 :- (29/08/2020).

(1). Let (a_n) be a sequence of positive terms such that $a_1 < a_2$ and $a_{n+2} = \sqrt{(a_{n+1} a_n)}$. Then show that (a_{2n-1}) is a monotonic increasing sequence and (a_{2n}) is a monotonic decreasing sequence and both converge to the common limit $(a_1 a_2^2)^{1/3}$. Hence deduce that (a_n) converges to the same limit.

(2). Prove that $\left(\frac{an+d}{bn+c}\right)$ is a monotonic increasing (or) decreasing (or) a constant sequence according as $bd < ac$, $bd > ac$, $bd = ac$.

(3). Show the sequence (a_n) given by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{(2a_n)}\ \forall\ n \geq 1$, converges to 2.

═ * ═

Completed in Unit - II.

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═ * ═

Thank you!!!

01/09/20

2.

Class : II. B.Sc, Maths, & Sub: Sequence and series.

Unit - III.

3.8 Some theorems on limits.

Theorem: 1

(1). Cauchy's first limit theorem.

If $(a_n) \to l$ then $\left(\frac{a_1+a_2+\cdots\cdots+a_n}{n}\right) \to l$.

Proof:-

Case(i) :- Let $l=0$.

Let $b_n = \frac{a_1+a_2+\cdots\cdots+a_n}{n}$.

Let $\varepsilon > 0$ be given.

Since, $(a_n) \to 0$, there exists $m \in N$ such that

$$|a_n| < \frac{1}{2}\varepsilon \ \forall \ n \geq m. \longrightarrow (1)$$

Now,

let $n \geq m$.

Then,

$$|b_n| = \left|\frac{a_1+a_2+\cdots\cdots+a_m+a_{m+1}+\cdots+a_n}{n}\right|$$

$$\leq \frac{|a_1|+|a_2|+\cdots\cdots+|a_m|}{n} \quad \frac{|a_{m+1}|+\cdots+|a_n|}{n}$$

$$= \frac{k}{n} + \frac{|a_{m+1}|+\cdots\cdots+|a_n|}{n}.$$

where $k = |a_1| + \dots\dots + |a_m|$

$$< \frac{k}{n} + \left(\frac{n-m}{n}\right)\frac{\varepsilon}{2} \quad \text{[by (1)], we get}$$

$$< \frac{k}{n} + \frac{\varepsilon}{2},$$

Since, $\frac{n-m}{n} < 1 \longrightarrow$ (2)

Now,

Since $\left(\frac{k}{n}\right) \to 0$, there exists $n_0 \in N$ such that

$$\frac{k}{n} < \frac{1}{2}\varepsilon \ \forall\ n \geq n_0 . \longrightarrow (3)$$

Let $n_1 = \max\{m, n_0\}$.

Then, $|b_n| < \varepsilon \ \forall \ n \geq n_1$ [using (2) and (3)], we get

$\therefore (b_n) \to 0$.

Case (ii) :- Let $l \neq 0$.

Since, $(a_n) \to l$, $(a_n - l) \to 0$.

$$\therefore \left(\frac{(a_1-l)+(a_2-l)+\dots+(a_n-l)}{n}\right)$$

$$\therefore \left(\frac{a_1+a_2+\dots\dots+a_n-nl}{n}\right) \to 0$$

$$\therefore \left(\frac{a_1+a_2+\dots\dots+a_n}{n} - l\right) \to 0$$

$\therefore \left(\frac{a_1+a_2+\cdots+a_n}{n}\right) \to l$.

$\therefore$ Hence $\boxed{(b_n) \to l}$. //.

Note :- 1

The converges of the Cauchy's first theorem is not true. And consider the sequence $(a_n) = ((-1)^n)$.

Example :- 1

let $(a_n) = (-1)^n$.

Solution:-

$$(b_n) = \frac{a_1+a_2+\cdots+a_n}{n}$$

$$b_1 = \frac{a_1}{1} = \frac{-1}{1} = -1.$$

$$b_2 = \frac{a_1+a_2}{2} = \frac{-1+1}{2} = 0.$$

$$b_3 = \frac{a_1+a_2+a_3}{3} = -1/3.$$

$$b_4 = \frac{a_1+a_2+a_3+a_4}{4} = 0.$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\therefore b_n = \begin{cases} 0 \text{ if } n \text{ is even} \\ -1/n \text{ if } n \text{ is odd.} \end{cases}$$

$\therefore (b_n) \to 0$, then (a_n) is not converges. //.

Hence, converse of the Cauchy's first theorem is not true.

(2). Cesaro's theorem:-

(X) If $(a_n) \to a$ and $(b_n) \to b$ then

$$\left(\frac{a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1}{n}\right) \to ab.$$

Proof:-

let $C_n = \dfrac{a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1}{n}$.

Now,

put $a_n = a + r_n$,

So that $(r_n) \to 0$.

Then,

$$C_n = \frac{(a+r_1) b_n + (a+r_2) b_{n-1} + \dots + (a+r_n) b_1}{n}$$

$$= \frac{a(b_1 + \dots + b_n)}{n} + \frac{r_1 b_n + \dots + r_n b_1}{n}$$

Now,

by Cauchy's first limit theorem,

$$\left(\frac{b_1 + b_2 + \dots + b_n}{n}\right) \to b.$$

$$\therefore \left(\frac{a(b_1 + b_2 + \dots + b_n)}{n}\right) \to ab.$$

Hence it is enough if we prove that $\left(\dfrac{r_1 b_n + \dots + r_n b_1}{n}\right) \to 0$.

Now,

Since, $(b_n) \to b$,

(b_n) is a bounded sequence.

$\therefore$ There exists a real number $K > 0$, such that

$$|b_n| \leq K \ \forall \ n.$$

$$\therefore \left| \frac{r_1 b_n + \ldots\ldots + r_n b_1}{n} \right| \leq K \left| \frac{r_1 + \ldots\ldots + r_n}{n} \right| .$$

Since, $(r_n) \to 0$,

$$\left(\frac{r_1 + \ldots\ldots + r_n}{n} \right) \to 0 .$$

$$\therefore \left(\frac{r_1 b_n + \ldots\ldots r_n b_1}{n} \right) \to 0 \ //.$$

Hence the theorem.

=== * ===

Thank you !!!

=== * ===

02/09/20

Class : II. B.Sc., Maths, & Sub: Sequences and Series

Unit – III, (Continue notes)

(3) Cauchy's second limit theorem :-

Let (a_n) be a sequence of positive terms. Then

$\lim\limits_{n\to\infty} a_n^{1/n} = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ provided the limit on the right hand side exists, whether finite (or) infinite.

Proof :-

Case (i)

$$\text{Let } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l, \text{ finite.}$$

Let $\varepsilon > 0$ be any given real number.

Then there exists $m \in N$ such that,

$$l - \frac{1}{2}\varepsilon < \frac{a_{n+1}}{a_n} < l + \frac{1}{2}\varepsilon \ \forall\ n \geq m.$$

Now,

choose $n \geq m$.

Then

$$l - \frac{1}{2}\varepsilon < \frac{a_{m+1}}{a_m} < l + \frac{1}{2}\varepsilon$$

$$l - \frac{1}{2}\varepsilon < \frac{a_{m+2}}{a_{m+1}} < l + \frac{1}{2}\varepsilon$$

$$\cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots$$

$$l - \frac{1}{2}\varepsilon < \frac{a_n}{a_{n-1}} < l + \frac{1}{2}\varepsilon$$

Multiplying these inequalities, we obtain

$$\left(l - \frac{1}{2}\varepsilon\right)^{n-m} < \frac{a_n}{a_m} < \left(l + \frac{1}{2}\varepsilon\right)^{n-m}.$$

$$\therefore a_m \frac{\left(l - \frac{1}{2}\varepsilon\right)^n}{\left(l - \frac{1}{2}\varepsilon\right)^m} < a_n < a_m \frac{\left(l + \frac{1}{2}\varepsilon\right)^n}{\left(l + \frac{1}{2}\varepsilon\right)^m}.$$

$$\therefore k_1\left(l - \frac{1}{2}\varepsilon\right)^n < a_n < k_2\left(l + \frac{1}{2}\varepsilon\right)^n.$$

Where k_1 and k_2 are some constants.

$$\therefore k_1^{1/n}\left(l - \frac{1}{2}\varepsilon\right) < a_n^{1/n} < k_2^{1/n}\left(l + \frac{1}{2}\varepsilon\right) \longrightarrow (1)$$

Now,

$$\left(k_1^{1/n}\left(l - \frac{1}{2}\varepsilon\right)\right) \to l - \frac{1}{2}\varepsilon$$

Since $\left(k_1^{1/n}\right) \to 1$.

$\therefore$ There exists $n_1 \in N$ such that,

$$\left(l - \frac{1}{2}\varepsilon\right) - \frac{1}{2}\varepsilon < k_1^{1/n}\left(l - \frac{1}{2}\varepsilon\right) < \left(l - \frac{1}{2}\varepsilon\right) + \frac{1}{2}\varepsilon \quad \forall\, n \geq n_1 \longrightarrow (2).$$

lll^{ly},

There exists $n_2 \in N$ such that,

$$\left(l+\tfrac{1}{2}\varepsilon\right)-\tfrac{1}{2}\varepsilon < k_2^{1/n}\left(l+\tfrac{1}{2}\varepsilon\right) < \left(l+\tfrac{1}{2}\varepsilon\right)+\tfrac{1}{2}\varepsilon$$

$\forall\ n \geq n_2$

$\hookrightarrow$ 3

let $n_0 = \max\{m, n_1, n_2\}$.

Then, $l-\varepsilon < k_1^{1/n}\left(l-\tfrac{1}{2}\varepsilon\right) < a_n^{1/n}$ ~~&~~

$k_2^{1/n}\left(l+\tfrac{1}{2}\varepsilon\right) < l+\varepsilon \quad \forall\ n \geq n_0$

[By 1, 2 and 3] we get.

$\therefore\ l-\varepsilon < a_n^{1/n} < l+\varepsilon \quad \forall\ n \geq n_0$.

Hence, $\left(a_n^{1/n}\right) \to l$.

<u>Case (ii)</u> :-

Let $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \infty$.

Then $\lim\limits_{n\to\infty} \dfrac{(1/a_n)}{(1/a_{n+1})} = 0$ $\left[\because \text{ if } (a_n)\to\infty \text{ \& } a_n \neq 0 \text{ so } (1/a_n)\to 0\right]$.

$\because$ By Case(i) $(1/a_n)^{1/n} \to 0$

$\therefore\ (a_n^{1/n}) \to \infty$ //.

Hence the proof.

(4). **Theorem :- 4**

Let (a_n) be any sequence and $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = l$.

If $l > 1$, then $(a_n) \to 0$.

Proof :-

Let K be any real number such that

$1 < K < l$.

Since, $$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = l.$$

There exists $m \in N$ such that,

$$l - \varepsilon < \left|\frac{a_n}{a_{n+1}}\right| < l + \varepsilon \ \forall \ n \geq m.$$

Choosing, $\varepsilon = l - K$ we obtain,

$$\left|\frac{a_n}{a_{n+1}}\right| > K \ \forall \ n \geq m.$$

Now, fixing $n \geq m$.

Then,

$$\left|\frac{a_m}{a_{m+1}}\right| > K;$$

$$\left|\frac{a_{m+1}}{a_{m+2}}\right| > K, \ldots\ldots, \left|\frac{a_{n-1}}{a_n}\right| > K,$$

Multiplying the above inequalities, we get

$$\left|\frac{a_m}{a_n}\right| > k^{n-m}.$$

$$\therefore \left|\frac{a_n}{a_m}\right| < k^m \left(\frac{1}{k}\right)^n.$$

$$\therefore |a_n| < k^m |a_m| \left(\frac{1}{k}\right)^n$$

$$\therefore |a_n| < A r^n$$

where $A = |a_m| k^m$ is a constant.

And,

$$r = 1/k.$$

Now,

$$k > 1 \Rightarrow 0 < r < 1.$$

$$\therefore (r^n) \to 0.$$

$$\therefore \boxed{(a_n) \to 0}$$

//.

Thank you!

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⑤. Theorem : 5

let (a_n) be any sequence of positive terms and $\lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}}\right) = l$. If $l < 1$ then $(a_n) \to \infty$.

Proof :-

let k be any real number such that,

$$1 > k > l.$$

Since,

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = l.$$

There exists $m \in N$ such that,

$$l - \varepsilon > \left|\frac{a_n}{a_{n+1}}\right| > l + \varepsilon \ \forall\ n \geq m.$$

Choosing, $\varepsilon = l - k$ we obtain,

$$\left|\frac{a_n}{a_{n+1}}\right| < k \ \forall\ n \geq m.$$

Now, fixing $n \geq m$.

Then,

$$\left|\frac{a_m}{a_{m+1}}\right| < K ;$$

$$\left|\frac{a_{m+1}}{a_{m+2}}\right| < K, \ldots\ldots, \left|\frac{a_{n-1}}{a_n}\right| < K,$$

Multiplying the above inequalities, we get

$$\left|\frac{a_m}{a_n}\right| < K^{n-m} .$$

$$\therefore \left|\frac{a_n}{a_m}\right| > K^m \left(\frac{1}{K}\right)^n .$$

$$\therefore |a_n| > K^m |a_m| \left(\frac{1}{K}\right)^n .$$

$$\therefore |a_n| > A r^n .$$

Where, $A = |a_m| K^m$ is a Constant. And

$$r = 1/K .$$

Now,

$$K < 1 \Rightarrow 0 > r > 1 .$$

$$\therefore (r^n) \to \infty$$

$$\therefore \boxed{(a_n) \to \infty} .$$

//.

(b). **Theorem : 6**

If the sequence (a_n) and (b_n) converge to 0, and (b_n) is strictly monotonic decreasing then $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{a_n - a_{n+1}}{b_n - b_{n+1}}\right)$ provided the limit on the right hand side exists whether finite (or) infinite.

Proof:-

Case(i).-

Let $\lim_{n\to\infty}\left(\frac{a_n - a_{n+1}}{b_n - b_{n+1}}\right) = l$, finite.

Let $\varepsilon > 0$ be given.

Then there exists $m \in N$ such that,

$$l - \varepsilon < \frac{a_n - a_{n+1}}{b_n - b_{n+1}} < l + \varepsilon \;\; \forall \; n \geq m.$$

Since, $b_n - b_{n+1} > 0$, we get

$$(b_n - b_{n+1})(l - \varepsilon) < a_n - a_{n+1}$$

$$< (b_n - b_{n+1})(l + \varepsilon) \;\; \forall \; n \geq m.$$

Let $n > p \geq m$.

Then,

$$(b_p - b_{p+1})(l-\varepsilon) < a_p - a_{p+1} < (b_p - b_{p+1})(l+\varepsilon)$$

$$(b_{p+1} - b_{p+2})(l-\varepsilon) < a_{p+1} - a_{p+2} < (b_{p+1} - b_{p+2})(l+\varepsilon)$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

$$(b_{n-1} - b_n)(l-\varepsilon) < a_{n-1} - a_n < (b_{n-1} - b_n)(l+\varepsilon).$$

Adding the above inequalities, we get

$$(b_p - b_n)(l-\varepsilon) < a_p - a_n < (b_p - b_n)(l+\varepsilon).$$

Taking limit as $n \to \infty$, we get.

$$b_p(l-\varepsilon) < a_p < b_p(l+\varepsilon)$$

Since, (a_n) and $(b_n) \to 0$

$$\therefore \; l-\varepsilon < \frac{a_p}{b_p} < l+\varepsilon$$

Since, $b_p > 0$.

$$\therefore \left| \frac{a_p}{b_p} - l \right| < \varepsilon \;\; \forall \; p \geq m.$$

$$\therefore \boxed{\lim_{n \to \infty} \frac{a_n}{b_n} = l.}$$

Case (ii) :-

$$\lim_{n\to\infty}\left(\frac{a_n - a_{n+1}}{b_n - b_{n+1}}\right) = \infty.$$

Let $K > 0$ be any real number.

Then there exists $m \in \mathbb{N}$ such that

$$\frac{a_n - a_{n+1}}{b_n - b_{n+1}} > K \quad \forall\ n \geq m.$$

$$\therefore\ a_n - a_{n+1} > (b_n - b_{n+1})\,K \quad \forall\ n \geq m.$$

Let $n > p \geq m$.

Writing the inequalities for $n = p, p+1, \ldots, n$ and adding we get

$$a_p - a_n > K\,(b_p - b_n).$$

Taking limit as $n \to \infty$, we get $a_p \geq K b_p$.

$$\therefore\ \frac{a_p}{b_p} \geq K \quad \forall\ p \geq m.$$

$\therefore$ $\boxed{\left(\frac{a_n}{b_n}\right) \text{ diverges to } \infty.}$ //

$==*==$

Thank you!

$==*==$

04/09/20 2. ①

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Problems:-

①. Show that $\lim_{n\to\infty} \frac{1}{n}\left(1+\frac{1}{2}+\cdots\cdots+\frac{1}{n}\right)=0$.

Solution:-

Let $a_n = \frac{1}{n}$.

We know that $(a_n) \to 0$.

Hence by Cauchy's first limit theorem, we get

$$\left(\frac{a_1+a_2+\cdots\cdots+a_n}{n}\right) \to 0.$$

$$\therefore \left(\frac{1}{n}\left(1+\frac{1}{2}+\cdots\cdots+\frac{1}{n}\right)\right) \to 0. //$$

②. Show that $\lim_{n\to\infty} n^{1/n} = 1$.

Solution:-

Let $a_n = n$.

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)$$

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1$$

$\therefore$ By Cauchy's second limit theorem, we get

$$\boxed{\lim_{n \to \infty} n^{1/n} = 1}$$

(3). Prove that $\frac{1}{n}\left((n+1)(n+2)\ldots(n+n)\right)^{1/n} \to 4/e$

Solution:

Let $a_n = \frac{1}{n}\left[(n+1)(n+2)\ldots\ldots(n+n)\right]^{1/n}$

$$= \left[\frac{(n+1)(n+2)\ldots\ldots(n+n)}{n^n}\right]^{1/n}$$

$$= \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}$$

Let $b_n = \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\left(1+\frac{n}{n}\right)$

So that $a_n = b_n^{1/n}$.

Now,

$$\frac{b_{n+1}}{b_n} = \frac{\left(1+\frac{1}{n+1}\right)\left(1+\frac{2}{n+1}\right)\ldots\ldots\left(1+\frac{n+1}{n+1}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\left(1+\frac{n}{n}\right)}.$$

$$= (2n+1)(2n+2)\frac{n^n}{(n+1)^{n+2}}$$

$$= \frac{2(2n+1)}{n+1}\frac{n^n}{(n+1)^{n}}$$

$$= 2\left(\frac{2+1/n}{1+1/n}\right)\frac{1}{(1+1/n)^n}$$

$$\therefore \left(\frac{b_{n+1}}{b_n}\right) \to \frac{4}{e}.$$

$$(b_n^{1/n}) \to \frac{4}{e}.$$

$$\therefore \boxed{(a_n) \to 4/e}.$$

//.

---

(4). Prove that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

Solution:-

Let $a_n = \frac{x^n}{n!}$.

$$\therefore \frac{a_n}{a_{n+1}} = \frac{x^n}{n!}\frac{(n+1)!}{x^{n+1}}$$

$$= \frac{n+1}{x}.$$

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \infty .$$

$\therefore \boxed{(a_n) \to 0}$ & Hence $\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0}$ //.

---

5. Show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

__Solution:-__

Let $a_n = \frac{n!}{n^n}$.

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{n!}{n^n} \frac{(n+1)^{n+1}}{(n+1)!} .$$

$$\left(\frac{n+1}{n} \right)^n = \left(1+\frac{1}{n}\right)^n .$$

$$\therefore \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n .$$

$$= e$$

$$> 1 .$$

$\therefore \boxed{(a_n) \to 0}$.

//.

= * =

Thank you!

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Unit - III of Continue notes

3.9 SUBSEQUENCES

(*) **Definition:-**

Let (a_n) be a sequence. Let (n_k) be a strictly increasing sequence of natural number. Then (a_{n_k}) is called a subsequence of (a_n).

Note:-

The terms of a subsequence occur in the same order in which they occur in the original sequence.

Example:- 1

Let (a_{2n}) is a subsequence of any sequence (a_n). The interval between any two terms of the subsequence is the same, (ie), $n_1 = 2,\ n_2 = 4,\ n_3 = 6, \ldots\ n_k = 2k$.

Example: 2

Let (a_{n^2}) is a subsequence of any sequence (a_n). Hence $a_{n_1} = a_1,\ a_{n_2} = a_4,\ a_{n_3} = a_9 \ldots\ldots$

Here the interval between two successive terms of the subsequence goes on increasing as k becomes large. Thus the interval between various terms of a subsequence need not be regular.

Any sequence (a_n) is a subsequence of itself.

Example: 3

Consider the sequence (a_n) given by 1, 0, 1, 0, ...
Now, (b_n) given by 1, 1, 1, ... is a subsequence of (a_n).
Here (a_n) is not convergent whereas the subsequence (b_n) converges to 1.

Thus a subsequence of non-convergent sequence can be a convergent sequence.

Note :-

A subsequence of a given subsequence (a_{nk}) of a sequence (a_n) is again a subsequence of (a_n).

Theorem: 1

If a sequence (a_n) converges to l, then every subsequence (a_{nk}) of (a_n) also converges to l.

Proof :-

Let $\varepsilon > 0$ be given.

Since $(a_n) \rightarrow l$,

There exists $m \in N$ such that,

$$|a_n - l| < \varepsilon \;\forall\; n \geq m.$$

Now, choose $n_{k0} \geq m$.

Then,
$$k \geq k_0$$
$$\Rightarrow n_k \geq n_{k_0}$$

Since (n_k) is a monotonic increasing.

$$\Rightarrow n_k \geq m.$$
$$\Rightarrow |a_{n_k} - l| < \varepsilon \quad [\text{by } ①], \text{ we get}$$

Thus $|a_{n_k} - l| < \varepsilon \ \forall\ k \geq k_0$.

$$\therefore \boxed{(a_{n_k}) \rightarrow l}.$$

//.

Note: 1

If a subsequence of a sequence converges, then the original sequence need not converge.

Note: 2

If a sequence (a_n) has two subsequence converging to two different limits, then (a_n) does not converge.

Ex:-

Consider the sequence (a_n) given by,

$$a_n = \begin{cases} 1/n & \text{if } n \text{ is even} \\ 1+1/n & \text{if } n \text{ is odd} \end{cases}.$$

Here the subsequence $(a_{2n}) \to 0$ and the subsequence $(a_{2n-1}) \to 1$.

Hence the given sequence (a_n) does not converge.

Theorem: 2

If the subsequence (a_{2n-1}) and (a_{2n}) of a sequence (a_n) converge to the same limit l then (a_n) also converges to l.

Proof:-

let $\varepsilon > 0$ be given.

Since $(a_{2n-1}) \to l$.

There exists $n_1 \in N$ such that

$$|a_{2n-1} - l| < \varepsilon \ \forall \ 2n-1 \geq n_1.$$

|||ly,

There exists $n_2 \in N$ such that,

$$|a_{2n} - l| < \varepsilon \ \forall \ 2n \geq n_2.$$

let $m = \max\{n_1, n_2\}$.

Clearly $|a_n - l| < \varepsilon \ \forall \ n \geq m$.

$$\therefore \boxed{(a_n) \to l}$$ //.

Note:- The above result is true even if we have $l = \infty$ (or) $-\infty$.

Thank You!

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PEAK POINT

Definition:

Let (a_n) be a sequence. A natural number m is called a peak point of the sequence (a_n) if $a_n < a_m \; \forall \; n > m$.

Example:- 1

For the sequence $(1/n)$, every natural number is a peak point and hence the sequence has infinite number of peak points.

In general for a strictly monotonic decreasing sequence every natural number is a peak point.

Example: 2

Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, -1, -1, \ldots$

Here 1, 2, 3 are the peak points of the sequence.

Example: 3

The sequence 1, 2, 3, ... has no peak point.

In general a monotonic increasing sequence has no peak point.

(1). Theorem: 1

Every sequence (a_n) has a monotonic subsequence.

Proof:-

Case (i) :-

Let (a_n) has infinite number of peak point.

Let the peak point be $n_1 < n_2 < \ldots < n_k < \ldots$

Then,

$$a_{n_1} > a_{n_2} > \ldots > a_{nk} > \ldots$$

$\therefore$ (a_{nk}) is a monotonic decreasing subsequence of (a_n).

Case (ii) :-

Let (a_n) has only a finite number of peak point (or) no peak point.

Choose a natural number n_1 such that,

There is no peak point $\geq n_1$.

Since n_1 is not a peak point of (a_n),

There exists $n_2 > n_1$ such that $a_n \geq a_{n_1}$.

Again,

Since n_2 is not a peak point of (a_n)

There exists $n_3 > n_2$ such that $a_{n_1} \geq a_{n_2}$.

$\therefore$ Repeating this process we get a monotonic increasing subsequence (a_{n_1}) of (a_n).

Hence the proof. //.

(2) Theorem : 2

Every bounded sequence has a convergent subsequence.

Proof :-

Let (a_n) be a bounded sequence.
Let (a_{n_1}) be a monotonic subsequence of (a_n).
Since (a_n) is bounded (a_{n_1}) is also bounded.

$\therefore$ (a_{n_1}) is a bounded monotonic sequence and hence converges.

$\therefore$ (a_{n_1}) is a convergent subsequence of (a_n).

=== * ===

Thank you!

=== * ===

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3.10. LIMIT POINTS

Definition:-

Let (a_n) be a sequence of real numbers 'a' is called a limit point (or) a cluster point of the sequence (a_n) if given $\varepsilon > 0$, there exists infinite number of terms of the sequence in $(a-\varepsilon, a+\varepsilon)$.

If the sequence (a_n) is not bounded above then ∞ is a limit point of the sequence.

If (a_n) is not bounded below then $-\infty$ is a limit point of the sequence.

Example: 1

Consider the sequence $1, 0, 1, 0, \ldots$ for this sequence 1 is a limit point.

Since,

$\varepsilon > 0$ be given

The interval $(1-\varepsilon, 1+\varepsilon)$ contains infinitely many terms $a_1, a_3, a_5, \ldots$ of this sequence.

∞,

Let 0 is also a limit point of the sequence.

Example : 2

If a sequence (a_n) converges to l then l is a point of the sequence.

Since,

$\varepsilon > 0$ be given,

There exists $m \in N$ such that,

$a_n \in (l-\varepsilon, l+\varepsilon)$ $\forall$ $n \geq m$.

$\therefore$ $(l-\varepsilon, l+\varepsilon)$ contains infinitely many terms of the sequence.

Example : 3

The sequence $(a_n) = 1, 2, 3, \dots n \dots$ is not bounded above and hence ∞ is a limit point.

Example : 4

The sequence $(a_n) = 1, -1, 2, -2, \dots n, -n \dots$ is neither bounded above nor bounded below.

Hence ∞ and $-\infty$ are limit points of the sequence.

1. <u>Theorem: 1</u>

Let (a_n) be a sequence. A real number 'a' is a limit point of (a_n) iff there exists a subsequence (a_{n_k}) of (a_n) converging to a.

<u>Proof:-</u>

Suppose there exists a subsequence (a_{n_k}) of (a_n) converging to a.

Let $\varepsilon > 0$ be given.

Then there exists $k_0 \in \mathbb{N}$ such that,

$$a_{n_k} \in (a-\varepsilon, a+\varepsilon) \ \forall \ k \geq k_0 .$$

$\therefore$ $(a-\varepsilon, a+\varepsilon)$ contains infinitely many terms of the sequence (a_n).

$\therefore$ 'a' is a limit point of the sequence (a_n).

Conversely suppose 'a' is a limit point of (a_n).

Then,

For each $\varepsilon > 0$ the interval,

$(a-\varepsilon, a+\varepsilon)$ contains infinitely many terms of the sequence.

find $n_1 \in N$ such that $a_{n_1} \in (a-1, a+1)$.

Also,

$n_2 > n_1$ such that, $a_{n_2} \in (a-\frac{1}{2}, a+\frac{1}{2})$

Let. In particular we can find $n_1 \in N$ such that

$$a_{n_1} \in (a-1, a+1).$$

Also,

find $n_2 > n_1$ such that,

$$a_{n_2} \in (a-\frac{1}{2}, a+\frac{1}{2}).$$

Then,

natural numbers $n_1 < n_2 < n_3 \ldots\ldots$ such that

$$a_{n_K} \in (a-1/K, a+1/K).$$

Clearly,

(a_{n_K}) is a subsequence of (a_n) and

$$|a_{n_K} - a| < 1/K$$

for any $\varepsilon > 0$ be given.

$$|a_{n_K} - a| < \varepsilon$$

If $K > 1/\varepsilon$.

$\therefore \boxed{a_{n_K} \to a}$ //.

(2). Theorem: 2

Every bounded sequence has at least one limit point.

Proof:-

Let (a_n) be a bounded sequence.

Let (a_{nk}) be a monotonic subsequence of (a_n).

Since (a_n) is bounded (a_{nk}) is also bounded.

$\therefore$ (a_{nk}) is a bounded monotonic sequence and Convergent.

$\therefore$ (a_{nk}) is a convergent subsequence of (a_n).

Then there exists a convergent subsequence (a_{nk}) of (a_n) converging to l.

$\therefore$ Hence l is a limit point of (a_n).

Note:-

In general every sequence (a_n) has at least one limit point in finite (or) infinite.

③. Theorem: 3

A sequence (a_n) converges to l iff (a_n) is bounded and l is the only limit point of the sequence.

Proof:-

Let $(a_n) \to l$.

Then (a_n) is bounded

Also, l is a limit point of the sequence (a_n)

Now,

Suppose l_1 is any other limit point of (a_n).

Then there exists a subsequence (a_{nk}) of (a_n) such that,

$$(a_{nk}) \to l_1 .$$

Now,

Since $(a_n) \to l$,

We have $(a_{nk}) \to l$

$$\therefore \quad l = l_1 .$$

Thus, l is the only limit point of the sequence.

Conversely, Suppose l is the only limit point of (a_n).

Aid, (a_n) does not converge to l.

Then there exists at least one $\varepsilon > 0$ such that infinitely many terms of the sequence lie outside

$$(l-\varepsilon,\ l+\varepsilon).$$

Hence,

find a subsequence (a_{n_k}) of (a_n) such that

$$a_{n_k} \notin (l-\varepsilon,\ l+\varepsilon)\ \forall\ k.$$

Since (a_n) is a bounded sequence,

(a_{n_k}) is also a bounded sequence.

Hence,

(a_{n_k}) has also a limit point l' and $l' \neq l$.

$\therefore$ (a_n) has two limit points l and l.

Which is a contradiction.

Hence, $\boxed{(a_n) \to l}$. //.

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Thank you!

$=$ * $=$

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Unit - III of Continue notes.

3.11. CAUCHY SEQUENCES

Definition:-

A sequence (a_n) is said to be a Cauchy sequence if given $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$|a_n - a_m| < \varepsilon \ \forall \ n, m \geq n_0.$$

Note:-

The condition $|a_n - a_m| < \varepsilon \ \forall \ n, m \geq n_0$ Can be written in the following equivalent form, namely, $|a_{n+p} - a_n| < \varepsilon \ \forall \ n \geq n_0$ and positive integers p.

Example:- 1

The sequence $(1/n)$ is a Cauchy sequence.

Proof:-

Let $(a_n) = (1/n)$.

Let $\varepsilon > 0$ be given.

Now, $|a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right|$.

$\therefore$ If n_0 to be any positive integer $> \frac{1}{\varepsilon}$, we get

$$|a_n - a_m| < \varepsilon \ \forall \ n, m \geq n_0.$$

$\therefore$ $(1/n)$ is a Cauchy sequence.

Example: 2

The sequence $((-1)^n)$ is not a Cauchy sequence.

Proof:-

Let $(a_n) = ((-1)^n)$.

$$\therefore \quad |a_n - a_{n+1}| = 2.$$

$\therefore$ If $\varepsilon < 2$ be given.

we cannot find n_0 such that

$$|a_n - a_{n+1}| < \varepsilon \ \forall \ n \geq n_0.$$

$\therefore$ $((-1)^n)$ is not a Cauchy sequence.

Example: 3

~~Hm~~ The sequence (n) is not a Cauchy sequence.

<u>Proof:-</u>

let $(a_n) = (n)$.

$\therefore |a_n - a_m| \geq 1$ if $n \neq m$.

$\therefore$ Of $\varepsilon < 1$ be given.

we cannot find n_0 such that,

$$|a_n - a_m| < \varepsilon \ \forall\ n, m \geq n_0.$$

$\therefore (n)$ is not a Cauchy sequence. //.

---

(1). <u>Theorem: 1</u>

(Ex.) Any Convergent sequence is a Cauchy sequence.

<u>Proof:-</u>

let $(a_n) \rightarrow l$.

Then given $\varepsilon > 0$, there exists $n_0 \in N$ such that,

$$|a_n - l| < \frac{1}{2}\varepsilon \ \forall\ n \geq n_0.$$

$$\begin{aligned} \therefore |a_n - a_m| &= |a_n - l + l - a_m| \\ &\leq |a_n - l| + |l - a_m| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \ \forall\ n, m \geq n_0. \end{aligned}$$

$\therefore (a_n)$ is a Cauchy sequence. //.

(2). **Theorem: 2**

Any Cauchy sequence is a bounded sequence.

Proof:-

Let (a_n) be a Cauchy sequence.

Let $\varepsilon > 0$ be given.

Then there exists $n_0 \in N$ such that,

$$|a_n - a_m| < \varepsilon \ \forall \ n, m \geq n_0.$$

$$\therefore \ |a_n| < |a_{n_0}| + \varepsilon \ \forall \ n \geq n_0.$$

Now,

Let $K = \max\{|a_1|, |a_2|, \ldots |a_{n_0}| + \varepsilon\}$.

Then $|a_n| \leq K \ \forall \ n$.

Hence,

(a_n) is a bounded sequence.

---

(3). **Theorem: 3**

(X) Let (a_n) be a Cauchy sequence.

If (a_n) has a subsequence (a_{n_k}) converging to l.

then $(a_n) \longrightarrow l$.

Proof:

Let $\varepsilon > 0$ be given.

Then there exists $n_0 \in \mathbb{N}$ such that,

$$|a_n - a_m| < \frac{1}{2}\varepsilon \ \forall\ n, m \geq n_0 \longrightarrow (1)$$

Also,

Since $(a_{n_k}) \to l$,

There exists $k_0 \in \mathbb{N}$ such that,

$$|a_{n_k} - l| < \frac{1}{2}\varepsilon \ \forall\ k \geq k_0. \longrightarrow (2)$$

Choose n_k such that, $n_k \geq n_k$ and n_0.

Then,

$$|a_n - l| = |a_n - a_{n_k} + a_{n_k} - l|$$ [By (1) & (2) we get].

$$\leq |a_n - a_{n_k}| + |a_{n_k} - l|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$$

$$= \varepsilon \ \forall\ n \geq n_0.$$

Hence, $\boxed{(a_n) \to l.}$ //.

Unit - III is Completed

Thank you!

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Unit - IV of notes.

4.1. INFINITE SERIES

Definition :-

Let $(a_n) = a_1, a_2, \ldots, a_n, \ldots$ be a sequence of real numbers. Then the formal expression $a_1 + a_2 + \ldots + a_n + \ldots$ is called an infinite series of real numbers and is denoted by $\sum_1^{\infty} a_n$ (or) $\sum a_n$.

III^{ly}, Let $s_1 = a_1$; $s_2 = a_1 + a_2$; $s_3 = a_1 + a_2 + a_3, \ldots$

$$s_n = a_1 + a_2 + \ldots + a_n .$$

Then (s_n) is called the sequence of partial sums of the given series $\sum a_n$.

The series $\sum a_n$ is said to converge, diverge (or) oscillate according as the sequence of partial sums (s_n) converges, diverges (or) oscillates.

If $(s_n) \to s$, the series $\sum a_n$ converges to the sum s.

Then, the behaviour of a series does not change if a finite number of terms are added (or) altered.

Example: 1

Consider the series $1+1+1+1\ldots\ldots$

Here, $S_n = n$.

Clearly, the sequence (S_n) diverges to ∞.

Hence, the given series diverges to ∞.

Example: 2

Consider the geometric series $1+r+r^2+\ldots\ldots+r^n+\ldots$

Here, $S_n = 1+r+r^2+\ldots\ldots+r^{n-1}$

$$= \frac{1-r^n}{1-r}.$$

Case(i) :-

Let $0 \leq r < 1$.

Then $(r^n) \to 0$

$$\therefore (S_n) \to \frac{1}{1-r}.$$

$\therefore$ The given series converges to the sum $\frac{1}{(1-r)}$ //.

Case (ii) :-

Let $r > 1$.

Then, $$S_n = \frac{r^n - 1}{r-1}.$$

Also, $(r^n) \to \infty$.

when $r > 1$.

Hence,

the series diverges to ∞. //.

<u>Case (iii)</u> :-

let $r = 1$.

Then the series becomes $1 + 1 + \ldots\ldots$

$\therefore (S_n) = (n)$

which diverges to ∞. //.

<u>Case (iv)</u> :-

let $r = -1$.

Then the series becomes $1 - 1 + 1 - 1 + \ldots\ldots$

$$S_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

$\therefore (S_n)$ oscillates finitely.

Hence, the given series oscillates finitely. //.

<u>Case (v)</u> :-

let $r < -1$.

$\therefore (r^n)$ oscillates infinitely.

$\therefore (S_n)$ oscillates infinitely.

Hence,

the given series oscillates infinitely. //.

Example: 3

Consider the series, $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots$

Then,

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!}$$

The sequence $(s_n) \to e$.

$\therefore$ The given series converges to the sum e. //.

Example:- 4

Consider the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$

Then, $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Here, $(s_n) \to \infty$

$\therefore$ The given series diverges to ∞.

//.

Thank you!

12/09/20

(1)

Class: II B.Sc, Maths, & Sub: Sequences and Series

Unit - IV of Continue notes

Note :- 1

Let $\sum a_n$ be a series of positive terms. Then (S_n) is a monotonic increasing sequence. Hence (S_n) converges (or) diverges to ∞ according a (S_n) is bounded (or) unbounded.

Hence the series $\sum a_n$ converges (or) diverges to ∞. Thus a series of positive terms cannot oscillate.

Note : 2

Let $\sum a_n$ be a convergent series of positive terms converging to the sum S.

Then 'S' is the l.u.b of (S_n).

Hence, $S_n \leq S \ \forall n$.

Also, given $\epsilon > 0$, there exists $m \in N$ such that,

$$S - \epsilon < S_n \ \forall \ n \geq m.$$

Hence, $S - \epsilon < S_n \leq S \ \forall \ n \geq m$.

Theorem : 1

Let $\sum a_n$ be a convergent series converging to the sum S. The $\lim\limits_{n \to \infty} a_n = 0$.

Proof:-

let $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1})$

$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$

$= S - S = 0.$

Note: 3

The Converse of the above theorem is not true.

(ie) If $\lim a_n = 0$, then Σa_n need not Converge.

Ex:-

Consider the series $\Sigma \frac{1}{n}$.

Here, $\lim_{n\to\infty} \frac{1}{n} = 0$.

However the series $\Sigma \frac{1}{n}$ diverges

Note: 4

If $\lim a_n \neq 0$ then the series Σa_n is not Convergent.

If further Σa_n is a series of positive terms then the series Cannot oscillate and hence the diverges.

(X) **Theorem: 2**

Let Σa_n Converge to 'a' and Σb_n Converge to 'b'. Then $\Sigma (a_n \pm b_n)$ Converges to $a \pm b$ and $\Sigma k a_n$ Converges to ka.

Proof:-

Let $s_n = a_1 + a_2 + \ldots\ldots + a_n$ and

$$t_n = b_1 + b_2 + \ldots\ldots + b_n.$$

Then,

$$(s_n) \to a \quad \text{and}$$

$$(t_n) \to b \quad .$$

$$\therefore (s_n \pm t_n) \to a \pm b$$

Also, $(s_n \pm t_n)$ is the sequence of Partial sums of $\Sigma (a_n \pm b_n)$.

$\therefore \Sigma (a_n \pm b_n)$ Converges to $a \pm b$.

|||ly,

$\Sigma k a_n$ converges to ka. //.

(X) **Theorem: 3** (Cauchy's general principle of convergence)

The series Σa_n is convergent iff given $\epsilon > 0$ there exists $n_0 \in N$ such that $|a_{n+1} + a_{n+2} + \ldots + a_{n+p}| < \epsilon$ $\forall\ n \geq n_0$, Positive integers p.

Proof:-

Let Σa_n be a Convergent sequences.

Then, $S_n = a_1 + a_2 + \dots + a_n$.

$\therefore (S_n)$ is a Convergent sequence.

Also,

Any Convergent sequence is a Cauchy's sequence.

Then, there exists $n_0 \in N$ such that,

$\therefore |S_{n+p} - S_n| < \epsilon \; \forall \; n \geq m$.

$\therefore |a_{n+1} + a_{n+2} \dots + a_{n+p}| < \epsilon \; \forall \; n \geq n_0$ and $\forall \; p \in N$.

(ii), A sequence (a_n) in R is convergent iff (a_n) is a Cauchy sequence.

$\therefore (S_n)$ is a Cauchy sequence in R.

Hence

(S_n) is Convergent.

$\therefore$ Then the series Σa_n is Converges. //.

Thank you!

14/09/20

(1)

Class: II. B.Sc, Maths, & Sub: Sequences and Series

Unit - IV of Continue notes

4.2 Comparison Test

Theorem: 1 (Comparison Test).

(i) let Σc_n be a convergent series of positive terms. let Σa_n another series of positive terms. If there exists $m \in N$ such that $a_n \leq c_n \; \forall \; n \geq m$ then, Σa_n is also convergent.

Proof: (i), Comparison test of proof (i)

Since the convergence or divergence of a series is not altered by the removal of a finite number of terms.

We may assume without loss of generality that

$$a_n \leq c_n \; \forall \; n.$$

let $s_n = c_1 + c_2 + \ldots + c_k$ and

$$t_n = a_1 + a_2 + \ldots + a_n.$$

Since $a_n \leq c_n$

we have $t_n \leq s_n$.

Now,

Since Σc_n is Convergent,

(S_n) is a Convergent sequence.

$\therefore$ (S_n) is a bounded sequence,

$\therefore$ There exists a real positive number k such that

$$S_n \leq k \ \forall n.$$

$\therefore t_n \leq k \ \forall n$

Hence, (t_n) is bounded above.

Also, (t_n) is a monotonic increasing sequence,

$\therefore$ (t_n) Converges

$\therefore$ $\boxed{\Sigma a_n \text{ Converges}}$. //

~~Proof~~ (ii). Let Σd_n be a divergent series of positive terms. Let Σa_n be another series of positive terms. If there exists $m \in N$ such that $a_n \geq d_n \ \forall \ n \geq m$, then Σa_n is also divergent.

Proof (ii), Comparison test of proof (ii)

Let Σd_n diverge and ~~a~~.

$a_n \geq d_n \ \forall \ n$.

$\therefore t_n \geq S_n$.

Now,

(S_n) diverges to ∞.

$\therefore (S_n)$ is not bounded above.

$\therefore (t_n)$ is not bounded above.

Further,

(t_n) is monotonic increasing,

Hence,

(t_n) diverges to ∞.

$\therefore$ $\boxed{\Sigma a_n \text{ diverges to } \infty}$.

Hence the proof.

Theorem: 2

Comparison test of ~~proof (ii)~~ Theorem: 2.

(i) If ΣC_n Converges and if $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{C_n}\right)$ exists and is finite then Σa_n also converges.

Proof (i), Comparison test of proof (i)

Let $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{C_n}\right) = k$.

Let $\varepsilon > 0$ be given.

Then there exists $n_1 \in N$ such that

$$\frac{a_n}{c_n} < k + \varepsilon \;\forall\; n \geq n_1 .$$

$\therefore\; a_n < (k+\varepsilon)\, c_n \;\forall\; n \geq n_1$.

Also,

since Σc_n is a Convergent series,

$\Sigma (k+\varepsilon)\, c_n$ is also a Convergent series.

$\therefore$ By Comparison test Σa_n is Convergent.

(ii), $\boxed{\Sigma a_n \text{ is Convergent}}$ //.

(ii). If Σd_n diverges and if $\lim\limits_{n\to\infty}\left(\frac{a_n}{d_n}\right)$ exists and is greater than zero then Σa_n diverges.

Proof: (ii), Comparison test of Proof (ii)

let $\lim\limits_{n\to\infty}\left(\frac{a_n}{d_n}\right) = k > 0$.

Choose $\varepsilon = \frac{1}{2} k$.

Then there exists $n_1 \in N$ such that,

$$k - \frac{1}{2}k < \frac{a_n}{d_n} < k + \frac{1}{2}k \quad \forall\ n \geq n_1 .$$

$$\therefore \frac{a_n}{d_n} > \frac{1}{2}k \quad \forall\ n \geq n_1 .$$

$$\therefore a_n > \frac{1}{2}k\, d_n \quad \forall\ n \geq n_1 .$$

Since,

Σd_n is a divergent series,

$\therefore$ $\Sigma \frac{1}{2} k\, d_n$ is also a divergent series.

$\therefore$ By Comparison test,

(i), $\boxed{\Sigma a_n \text{ diverges.}}$ //.

Hence the proof.

Thank you!

Class : II B.Sc, Maths, & Sub : Sequences and series

Unit - IV of Continuous notes

Theorem : 3

Comparison test (1)

(i). Let $\sum c_n$ be a Convergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in N$ such that $\frac{a_{n+1}}{a_n} \leq \frac{c_{n+1}}{c_n} \forall n \geq m$, then $\sum a_n$ is Convergent.

Proof :- Comparison test of proof (1)

let $$\frac{a_{n+1}}{c_{n+1}} \leq \frac{a_n}{c_n}$$

Since, $$\frac{a_{n+1}}{a_n} \leq \frac{c_{n+1}}{c_n}$$

$$\therefore \frac{a_n}{c_n} \leq k \quad \forall n.$$

Where $k = \frac{a_1}{c_1}$.

$$\therefore a_n \leq k\, c_n \quad \forall n \in N.$$

Now,

$\sum c_n$ is Convergent.

Hence,

$\sum k c_n$ is also a Convergent series of positive terms.

$\therefore$ $\boxed{\sum a_n \text{ is also Convergent}}$.

//.

Comparison test (ii)

(ii) Let $\sum d_n$ be a divergent series of positive terms. Let $\sum a_n$ be another series of positive terms. If there exists $m \in N$ such that $\frac{a_{n+1}}{a_n} \geq \frac{d_{n+1}}{d_n} \ \forall n \geq m$. then $\sum a_n$ is divergent.

Proof: (ii)

Comparison test of Proof (ii)

Let
$$\frac{a_{n+1}}{d_{n+1}} \geq \frac{a_n}{d_n}.$$

Since,
$$\frac{a_{n+1}}{a_n} \geq \frac{d_{n+1}}{d_n}.$$

$\therefore$ $\left(\frac{a_n}{d_n}\right)$ is a monotonic increasing sequence.

$$\therefore \quad \frac{a_n}{d_n} \geq k \ \forall n.$$

③

where, $k = \frac{a_1}{d_1}$

$\therefore \; a_n \geq k d_n \;\forall\; n \in N$.

Now,

Σd_n is ~~convergent~~ divergent.

Hence

$\Sigma k d_n$ is also a ~~convergent~~ divergent series of

positive terms.

$\therefore$ $\boxed{\Sigma a_n \text{ is also } \sout{\text{convergent}} \text{ divergent}}$. //.

Note:

A geometric series Σr^n converges if $0 \leq r < 1$

and diverges if $r \geq 1$.

Theorem: 4

The harmonic series $\Sigma \frac{1}{n^p}$ converges if $p > 1$

and diverges if $p \leq 1$.

Proof:-

let $\Sigma \frac{1}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p}$

$S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p}$.

Case(i) :-

let $P = 1$.

Then the series $\Sigma \frac{1}{n^P} = \Sigma \frac{1}{n}$.

$\therefore S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{n}$.

$\therefore (S_n) \to \infty$.

$\therefore$ $\boxed{\Sigma \left(\frac{1}{n}\right) \text{ is divergent}}$. //.

Case (ii) :-

let $P < 1$.

Then the series $n^P < n \ \forall n$

$\therefore \frac{1}{n^P} > \frac{1}{n} \ \forall n$.

$\therefore$ By Comparison test is $\Sigma \frac{1}{n^P}$ diverges.

$\therefore$ $\boxed{\Sigma \frac{1}{n^P} \text{ is diverges}}$. //.

Case (iii) :-

let $P > 1$.

let $S_n = 1 + \frac{1}{2^P} + \frac{1}{3^P} + \cdots\cdots + \frac{1}{n^P}$.

$$S_{2^{n+1}-1} = 1 + \frac{1}{2^p} + \cdots\cdots + \frac{1}{(2^{n+1}-1)^p}$$

$$= 1 + \left(\frac{1}{2^p} \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots\cdots$$

$$\cdots\cdots + \left(\frac{1}{(2^n)^p} + \frac{1}{(2^{n+1})^p} + \cdots\cdots + \frac{1}{(2^{n+1}-1)^p}\right).$$

$$< 1 + 2\left(\frac{1}{2^p}\right) + 4\left(\frac{1}{4^p}\right) + \cdots\cdots + 2^n \frac{1}{(2^n)^p}.$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{2^{2p-2}} + \cdots\cdots + \frac{1}{2^{(p-1)n}}.$$

$$\therefore S_{2^{n+1}-1} < 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \cdots\cdots + \left(\frac{1}{2^{p-1}}\right)^n.$$

Now,

Since $p > 1$, $p-1 > 0$.

Hence,

$$\frac{1}{2^{p-1}} < 1,$$

$$\therefore 1 + \left(\frac{1}{2^{p-1}}\right) + \left(\frac{1}{2^{p-1}}\right)^2 + \cdots\cdots + \left(\frac{1}{2^{p-1}}\right)^n.$$

$$< \frac{1}{1 - \frac{1}{2^{p-1}}} = K,$$

$\therefore S_{2^{m+1}-1} < K$.

Now,

let n be any positive integer.

Choose $m \in \mathbb{N}$ such that $n \leq 2^{m+1} - 1$.

Since,

(S_n) is a monotonic increasing sequence,

$$S_n \leq S_{2^{m+1}} .$$

Hence,

$$S_n < K \ \forall n .$$

Thus, (S_n) is a monotonic increasing sequence and is bounded above.

$\therefore (S_n)$ is convergent.

$\therefore$ $\boxed{\sum \frac{1}{n^p} \text{ is Convergent}}$.

*

Thank you!

*

16/09/20

Class: II B.Sc, Maths, & Sub: Sequences and series.

Unit - IV of Continue notes.

Problems :-

(1). Discuss the Convergence of the series $\sum \frac{1}{\sqrt{(n^3+1)}}$.

Solution:-

let $\sum \frac{1}{\sqrt{(n^3+1)}}$ be given,

$$\therefore \frac{1}{\sqrt{(n^3+1)}} = \frac{1}{n^{3/2}\sqrt{1+1/n^3}}$$

(ie), $$\frac{1}{\sqrt{(n^3+1)}} < \frac{1}{n^{3/2}}$$

$\therefore$ The harmonic series $\sum \frac{1}{n^P}$ Converges if $P>1$ and diverges if $P \le 1$.

$\therefore \sum \frac{1}{n^{3/2}}$ is a Convergent.

$\therefore$ By Comparison test, $\sum \frac{1}{\sqrt{(n^3+1)}}$ is Convergent

$$\therefore \boxed{\sum \frac{1}{\sqrt{(n^3+1)}} \text{ is Convergent}}$$

(2) Discuss the convergence of the series $\sum \frac{\sqrt{(n+1)} - \sqrt{n}}{n^p}$

Solution:-

let, $a_n = \frac{\sqrt{(n+1)} - \sqrt{n}}{n^p}$

$$= \frac{n+1-n}{n^p\left(\sqrt{(n+1)} + \sqrt{n}\right)}$$

$$= \frac{1}{n^p\left(\sqrt{(n+1)} + \sqrt{n}\right)}$$

Now,

let $b_n = \frac{1}{n^{p+1/2}}$.

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{p+1/2}}{n^p\left(\sqrt{(n+1)} + \sqrt{n}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\sqrt{(1+1/n)} + 1\right)}$$

$$= \frac{1}{2}.$$

Also,

$\sum b_n$ is Convergent if $p + \frac{1}{2} > 1$, and

$\sum b_n$ is divergent if $p + \frac{1}{2} \leq 1$

Then,

$\therefore \sum a_n$ is Convergent if $p > \frac{1}{2}$, and

$\sum a_n$ is divergent if $p \leq \frac{1}{2}$. //.

(3) Discuss the Convergence of the series $\sum \frac{1^2+2^2+\dots+n^2}{n^4+1}$.

Solution:-

let $$a_n = \frac{1^2+2^2+\dots+n^2}{n^4+1}$$

$$= \frac{n(n+1)(2n+1)}{6(n^4+1)}.$$

Now,

let $b_n = \frac{1}{n}$.

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(n+1)(2n+1)}{6(n^4+1)}.$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6\left(1+\frac{1}{n^4}\right)} \Rightarrow \lim_{n\to\infty} \frac{\cancel{n^4}(1+1/n)(2+1/n)}{6\cancel{n^4}(1+1/n^4)}$$

$$= \frac{(1+1/\infty)(2+1/\infty)}{6(1+1/\infty)} = 2/6$$

$$= \frac{1}{3} \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{3}$$

Also,

$\therefore \sum b_n$ is divergent

Hence

$\therefore \sum a_n$ is divergent //.

---

(4). Discuss the Convergence of the series $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\dots$

Solution:-

let $$a_n = \frac{n^n}{(n+1)^{n+1}}$$

$$\frac{n^n}{n^{n+1}(1+1/n)^{n+1}} = \frac{n^n}{n(1+1/n)^{n+1}}$$

let $b_n = 1/n$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{n+1}}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{(1+1/n)^{n+1}}$$

$$= \frac{1}{e} > 0.$$

$\therefore \Sigma b_n$ is divergent.

Hence,

Σa_n is divergent. //.

---

(5). Discuss the convergence of the series $\sum_{3}^{\infty} (\log \log n)^{-\log n}$.

Solution:-

let $a_n = (\log \log n)^{-\log n}$.

$\therefore a_n = n^{-\theta_n}$,

Where $\theta_n = \log(\log \log n)$.

Since

$$\lim_{n\to\infty} \log \log \log n = \infty,$$

There exists $m \in N$ such that

$$\theta_n \geq 2 \ \forall \ n \geq m.$$

$$\therefore n^{-\theta_n} \leq n^{-2} \ \forall \ n \geq m.$$

$$\therefore a_n \leq n^{-2} \ \forall \ n \geq m$$

Also,

Σn^{-2} is convergent.

$\therefore$ By Comparison test the given series is Convergent. //.

---

(6). Show that $\Sigma \dfrac{1}{4n^2-1} = \dfrac{1}{2}$.

Solution:-

Let $a_n = \dfrac{1}{4n^2-1}$

Clearly, $a_n < \dfrac{1}{n^2}$

WKT $\Sigma \dfrac{1}{n^p}$ is Convergent if $p \geq 1$

$\therefore \Sigma \dfrac{1}{n^2}$ is Convergent

$\therefore$ By Comparison test Σa_n is convergent to prove that

$$\frac{1}{4n^2-1} = \frac{1}{2}$$

$$a_n = \frac{1}{4n^2-1}$$

(b)

By partial fraction is $\frac{1}{4n^2-1} = \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$

$\therefore S_n = a_1 + a_2 + \dots\dots + a_n.$

$$= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \dots\dots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right).$$

$$= \frac{1}{2}\left[1 - \frac{1}{2n+1}\right]$$

$$\therefore \lim_{n\to\infty} S_n = \frac{1}{2}.$$

$$\therefore \boxed{\sum \frac{1}{4n^2-1} = \frac{1}{2}}$$

Home work :- Discuss the convergence of the following series whose n^{th} terms are given below.

(i) $\frac{(n+1)^3}{n^k + (n+2)^k}$, (ii) $\frac{n(n+1)}{(n+2)(n+3)(n+4)}$.

Unit - IV is Completed

Thank you!

Class: II. B.Sc, Maths, Sub: Sequences and Series

Unit - V

Corollary: 1 (D' Alembert's ratio test).

Let Σa_n be a series of positive terms. Then Σa_n Converges if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} > 1$ and diverges if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$.

Proof:-

The series $1+1+1\ldots$ is divergent.

$\therefore$ put $d_n = 1$ we get,

Then $d_n = 1$ is also,

$$d_n \frac{a_n}{a_{n+1}} - d_{n+1} = \frac{a_n}{a_{n+1}} - 1$$

$\therefore$ Σa_n converges if $\lim_{n\to\infty} \left(\frac{a_n}{a_{n+1}} - 1 \right) > 0$.

$\therefore$ Σa_n Converges if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} > 1$.

III^{ly},

Σa_n diverges if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$.

Corollary : 2

(Raabe's Test)

Let Σa_n be a series of positive terms. Then Σa_n Converges if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1$ and

diverges if $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1$.

Proof:

The series $\Sigma \frac{1}{n}$ is divergent.

$\therefore$ put $d_n = n$ we get,

Then
$$d_n \frac{a_n}{a_{n+1}} - d_{n+1} = n \frac{a_n}{a_{n+1}} - (n+1)$$
$$= n\left(\frac{a_n}{a_{n+1}} - 1\right) 1.$$

$\therefore \Sigma a_n$ Converges if,

$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1$ and

$\therefore$ diverges if,

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) < 1.$$

//.

Corollary: 3

(De Morgan and Bertrand's test).

Let Σa_n be a series of positive terms. Then Σa_n is convergent if $\lim_{n\to\infty} \log n\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right] > 1$

and is divergent if $\lim_{n\to\infty} \log n\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right] < 1$.

Proof:-

The series $\Sigma \frac{1}{n\log n}$ is divergent.

$\therefore$ Put $d_n = n\log n$ we get,

Then $d_n \frac{a_n}{a_{n+1}} - d_{n+1} = (n\log n)\frac{a_n}{a_{n+1}} - (n+1)\log(n+1)$

$= \log n\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right] + (n+1)\log n - (n+1)\log(n+1)$

(4)

$$= \log n\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]-(n+1)\log\left(\frac{n+1}{n}\right).$$

$$\therefore \lim_{n\to\infty}\left(d_n\frac{a_n}{a_{n+1}}-d_{n+1}\right)$$

$$= \lim_{n\to\infty}(\log n)\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]-\lim_{n\to\infty}\log\left(1+\frac{1}{n}\right)^{n+1}.$$

$$= \lim_{n\to\infty}(\log n)\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]-1.$$

//.

~~Problems~~ Solved problems:-

(1). Test the convergence of the series $\frac{1}{3}+\frac{1\cdot2}{3\cdot5}+\frac{1\cdot2\cdot3}{3\cdot5\cdot7}+\ldots\ldots$

Solution:-

$$\text{let } a_n = \frac{1\cdot2\cdot3\ldots\ldots\ldots n}{3\cdot5\cdot7\ldots\ldots\ldots(2n+1)}$$

$$a_{n+1} = \frac{1\cdot2\cdot3\ldots\ldots n(n+1)}{3\cdot5\cdot7\ldots\ldots 2(n+1)+1}$$

$$\frac{a_n}{a_{n+1}} = \frac{1\cdot2\cdot3\ldots n}{3\cdot5\cdot7\ldots(2n+1)} \Rightarrow \frac{3\cdot5\cdot7\ldots(2n+1)(2n+3)}{1\cdot2\cdot3\ldots\ldots n(n+1)}$$

$$= \frac{2n+3}{n+1}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\cancel{n}(2+3/n)}{\cancel{n}(1+1/n)}$$

$$= \lim_{n\to\infty} \frac{2+3/n}{1+1/n}$$

$$= 2 > 1$$

$\therefore$ By D' Alembert's Ratio test,

$\therefore$ $\boxed{\Sigma\, a_n \text{ is Convergent}}$. //.

Thank you!

21/09/2020

Class: II. B.Sc, Maths, & Sub: Sequences and Series.

Unit - V of Continue notes.

② Test the Convergence of $\sum \frac{n^n}{n!}$.

Solution:

$$\text{let } a_n = \frac{n^n}{n!}$$

$$a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$\frac{a_n}{a_{n+1}} = \frac{n^n}{n!} \cdot \frac{(n+1)!}{(n+1)^{n+1}}$$

$$= \frac{n^n \; n!\,(n+1)}{n!\;(n+1)^{n+1}} \Rightarrow \frac{n^n}{(n+1)^n}$$

$$\therefore = \frac{n^n}{n^n(1+1/n)} \Rightarrow \frac{1}{(1+1/n)^n}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n}$$

W.K.T,

$$\lim_{n\to\infty} (1+1/n)^n = e$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{e} < 1$$

$\therefore$ By D'Alembert's ratio test $\sum \frac{n^n}{n!}$ is divergent.

$\therefore$ $\boxed{\sum \frac{n^n}{n!} \text{ is divergent}}$ //

(3) Test the Convergence of the series $\Sigma \frac{2^n n!}{n^n}$.

Solution:-

$$\text{let } a_n = \frac{2^n n!}{n^n}.$$

$$\therefore a_{n+1} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}$$

$$\frac{a_n}{a_{n+1}} = \frac{2^n n!}{n^n} \cdot \frac{(n+1)^{n+1}}{2^{n+1}(n+1)!}$$

$$= \frac{(n+1)^{n+1}}{2(n+1)n^n} \Rightarrow \frac{1}{2}\left(1+\frac{1}{n}\right)^n.$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{e}{2} > 1.$$

$\therefore$ By ratio test the series is converged. //.

---

(4). Test the Convergence of the series $\Sigma \frac{3^n n!}{n^n}$.

Solution:-

$$\text{let } a_n = \Sigma \frac{3^n n!}{n^n}.$$

$$\therefore a_{n+1} = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}}.$$

$$\frac{a_n}{a_{n+1}} = \frac{3^n n!}{n^n} \cdot \frac{(n+1)^{n+1}}{3^{n+1}(n+1)!}.$$

(3)

$$= \frac{3^n n!}{n^n} \frac{(n+1)^{n+1}}{3^n \cdot 3^1 n!(n+1)}$$

$$= \frac{(n+1)^{n+1}}{n^n 3n(n+1)} \Rightarrow \frac{(n+1)^n}{3n^n}$$

$$= \frac{(1+1/n)^n}{3}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{(1+1/n)}{3}$$

$$\therefore \boxed{\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{e}{3} < 1.}$$

$\therefore$ By ratio test the series is diverges. //.

---

(5). Test the Convergence of the series $\sum \sqrt{\frac{n}{n+1}}\, x^n$, where x is any positive real number.

Solution:-

$$\text{Let } \sum \sqrt{\frac{n}{n+1}}\, x^n.$$

Since,

'x' is positive the given series is a series of positive terms.

$$a_n = \sqrt{\frac{n}{n+1}}\, x^n \quad \text{and} \quad a_{n+1} = \sqrt{\frac{n+1}{n+2}}\, x^{n+1}.$$

$$\therefore \frac{a_n}{a_{n+1}} = \sqrt{\frac{n(n+2)}{n+1}}\, \frac{1}{x}.$$

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$$= \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{\sqrt{n+2}}{\sqrt{n+1}} \cdot x^{n+1}$$

$$= \frac{\sqrt{n}\sqrt{n+2}}{(n+1)} \frac{1}{x} \Rightarrow \frac{n\sqrt{1+2/n}}{n(1+1/n)} \frac{1}{x}.$$

$$\therefore = \frac{\sqrt{1+2/n}}{1+1/n} \, 1/x.$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\sqrt{1+2/n}}{1+1/n} \frac{1}{x} \Rightarrow \frac{1}{x}.$$

W.K.T,

Ratio test $\sum a_n$ Converges if $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} > 1$,

Converges. and diverges $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$.

If $x > 1 \Rightarrow \frac{1}{x} < 1$

If $x < 1 \Rightarrow \frac{1}{x} > 1$

$\therefore$ $\sum a_n$ Converges is $x < 1$. & diverges is $x > 1$.

If $x = 1$ the test is fails.

Now,

when, $x = 1$, $a_n = \sqrt{\frac{n}{n+1}} = \frac{1}{\sqrt{1+1/n}} = 1 \neq 0$

W.K.T,

$\sum a_n$ Converges if $\lim_{n\to\infty} a_n = 0$ & diverges if $\lim_{n\to\infty} a_n \neq 0$.

$\therefore$ $\sum a_n$ diverges if $n = 1$.

$\sum a_n$ Converges if $x < 1$ & diverges if $x > 1$. //

Thank you!

22/09/20

2.

(1)

Class: II. B.Sc, Maths., & Sub: Sequences and Series.

Unit - V of Continue notes.

(6). Test the Convergence of the series $1+\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\cdots\cdots$

10. Mark where x is any positive real numbers.

(14)

Solution:-

$$\text{Let } 1+\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\cdots\cdots$$

'x' is positive real number the given series is a series of positive terms.

$$\Sigma a_n = \Sigma \frac{x^{2n-2}}{2n-2} \Rightarrow (n>1)$$

$$a_n = \frac{x^{2n-2}}{2n-2}$$

$$a_{n+1} = \frac{x^{2(n+1)-2}}{2(n+1)-2} \Rightarrow \frac{x^{2n}}{2n}$$

$$\frac{a_n}{a_{n+1}} = \frac{x^{2n-2}}{2n-2} \cdot \frac{2n}{x^{2n}}$$

$$= \frac{x^{2n}}{x^2} \cdot \frac{2n}{(2n-2)\cdot x^{2n}}$$

$$= \frac{2n}{x^2(2n-2)} = \frac{2n}{2(n-1)n^2}$$

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$$= \frac{n}{n-1} \, {}^{1}/_{x^2}$$

$$= \frac{n}{n(1-1/n)} \, {}^{1}/_{x^2} \Rightarrow \frac{1}{x^2(1-1/n)}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{1}{x^2(1-1/n)}$$

$$= 1/x^2$$

WKT,

By ratio test, Σa_n is convergence.

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} > 1 \text{ and } \lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1.$$

Of $x > 1 \Rightarrow 1/x^2 < 1$

Of $x < 1 \Rightarrow 1/x^2 > 1$

$\therefore$ Σa_n converges is $x^2 < 1$

$\therefore$ Σa_n diverges is $x^2 > 1$.

Of $x = 1$, the test is fails.

NOW,

let $a_n = \frac{1}{2n-2} = \frac{1}{2(n-1)}$

let $b_n = 1/n$.

WKT,

$\Sigma \frac{1}{n^p}$ converges is $p>1$ and diverges is $p<1$

$\therefore$ Σb_n is divergent.

By Comparison test,

Hence,

Σa_n diverges if ~~$x^2 \leq 1$~~ $x^2 > 1$

Σa_n converges if $x^2 < 1$ //.

---

(7). Test the converges of the $\Sigma \frac{n^2+1}{5n}$.

Solution:-

Given $\frac{n^2+1}{5n}$.

$$a_n = \frac{n^2+1}{5n}$$

$$a_{n+1} = \frac{(n+1)^2+1}{5^{n+1}}$$

$$\frac{a_n}{a_{n+1}} = \frac{n^2+1}{5^n} \cdot \frac{5^{n+1}}{(n+1)^2+1}$$

$$= \frac{n^2+1}{5^n} \cdot \frac{5^n.5}{n^2+1+2n+1}$$

$$= \frac{n^2+1}{5^n} \cdot \frac{5^n.5}{n^2+2n+2}$$

$$= \frac{n^2(1 + 1/n^2) \cdot 5}{n^2(1 + 2/n + 2/n^2)}$$

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \lim_{n \to \infty} \frac{5(1 + 1/n^2)}{(1 + 2/n + 2/n^2)}$$

$$= 5$$

WKT,

By ratio test, Σa_n is convergent if.

$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} > 1$ and diverges if $\lim_{n \to \infty} \frac{a_n}{a_{n+1}} < 1$.

$$\Sigma a_n = 5$$

$\therefore$ Σa_n is convergence. //.

Thank you!

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Class: II B.Sc, Maths, & Sub: Sequences and Series

Unit - V of Continue notes

(8). Test the Convergence of the series $\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\ldots\ldots$

Solution:-

$$\text{Let } a_n = \frac{1}{2^n}+\frac{1}{3^n}$$

$$a_{n+1} = \frac{1}{2^{n+1}}+\frac{1}{3^{n+1}}$$

$$\frac{a_n}{a_{n+1}} = \frac{1}{2^n}+\frac{1}{3^n} \cdot \frac{2^{n+1}}{1}+\frac{3^{n+1}}{1}$$

$$= \frac{1}{2^n}+\frac{1}{3^n} \cdot \frac{2^n \cdot 2}{1}+\frac{3^n \cdot 3}{1}$$

(or)

$$\frac{\frac{1}{2^n}+\frac{1}{3^n}}{\frac{1}{2^{n+1}}+\frac{1}{3^{n+1}}} \Rightarrow \frac{\frac{3^n+2^n}{2^n \cdot 3^n}}{\frac{3^{n+1}+2^{n+1}}{2^{n+1}\cdot 3^{n+1}}}$$

$$= \frac{3^n+2^n}{2^n \cdot 3^n} \cdot \frac{2^{n+1}\cdot 3^{n+1}}{2^{n+1}\cdot 3^{n+1}}$$

$$= \frac{6\left(3^n+2^n\right)}{3^{n+1}+2^{n+1}} \Rightarrow \frac{6\cdot 2^n\left(1+(3/2)^n\right)}{2^{n+1}\left(1+(3/2)^{n+1}\right)}.$$

(2)

$$\frac{\overset{3}{\cancel{6}} \cdot \cancel{2^n} \left(1+(3/2)^n\right)}{\cancel{2^n} \cdot \cancel{2} \; (1+3/2)^{n+1}}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{3\left(1+(3/2)^n\right)}{\left(1+(3/2)^{n+1}\right)} \Rightarrow > 1$$

$\therefore$ By ratio test, Σa_n is convergent

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} > 1 \quad //.$$

---

(9). Test the converges series $\Sigma \frac{x^n}{n}$.

Solution:-

let $\Sigma \frac{x^n}{n}$.

$$a_n = \frac{x^n}{n} \quad \text{and} \quad a_{n+1} = \frac{x^{n+1}}{n+1}.$$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{\frac{x^n}{n}}{\frac{x^{n+1}}{n+1}} \Rightarrow \frac{x^n}{n} \cdot \frac{n+1}{x^{n+1}} = \frac{\cancel{x^n}}{n} \cdot \frac{n+1}{\cancel{x^n} \cdot x^1}$$

$$= \frac{n+1}{n} \cdot \frac{1}{x}$$

$$= \frac{\cancel{n}\,(1+1/n)}{\cancel{n}} \Rightarrow (1+1/x)\; 1/x.$$

$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1/x$

W.K.T,

By ratio test, Σa_n is Convergent

If $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} > 1$, and if diverges

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$.

If $x > 1 \Rightarrow 1/x < 1$ if Σa_n is Convergent

If $x < 1 \Rightarrow 1/x > 1$ if Σa_n is divergent

$\therefore x = 1$, $a_n = (1 + 1/x)$ then $b_n = 1/p$ is

Σb_n is divergent.

$\therefore \Sigma a_n$ is also divergent.

//.

---

(10). Test the converges of the series $\Sigma \frac{n^p}{n!}$ $(p>0)$.

Solution:-

Given $\frac{n^p}{n!}$ $(p>0)$

$a_n = \frac{n^p}{n!}$ and $a_{n+1} = \frac{(n+1)^p}{(n+1)!}$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{n^p}{n!} \cdot \frac{(n+1)!}{(n+1)^p}$$

$$= \frac{n^p}{n!} \cdot \frac{n!(n+1)}{(n+1)^p}$$

$$= \frac{(n+1)}{(1+1/n)^p}$$

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)}{(1+1/n)^p}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{n+1}{(1+1/n)^p} \Rightarrow \infty .$$

$\therefore$ By the ratio test if Σa_n is Convergence. //.

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \infty .$$

$\therefore$ The ratio test Σa_n is convergent. //.

Thank you!

24/09/20

(1)

Class: II B.Sc, Maths, & Sub: Sequences and Series.

Unit - V of Continue notes.

Problem: 11

Test the convergence of the series

$$\frac{1}{3}x + \frac{1.2}{3\,5}x^2 + \frac{1}{3}\frac{2}{5}\frac{3}{7}x^3 + \ldots\ldots$$

Solution:-

$$\text{let } a_n = \frac{1.2.3,\ldots n}{3.5.7\ldots(2n+1)}\,x^n.$$

$$a_{n+1} = \frac{1.2.3\ldots\ldots n(n+1)}{3.5.7\ldots.2n+3}\,\frac{1}{x^{n+1}}\left(x^{n+1}\right)$$

$$\frac{a_n}{a_{n+1}} = \frac{1.2.3\ldots n}{3.5.7\ldots(2n+1)} \cdot \frac{3.5.7\ldots.(2n+2)}{1.2.3,\ldots n(n+1)}\,\frac{1}{x}$$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{2n+3}{n+1}\left(\frac{1}{x}\right).$$

$$= \frac{2+3/n}{1+1/n}\left(\frac{1}{x}\right).$$

$$\therefore \lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \frac{2}{x}.$$

$\therefore$ By ratio test the series converges if $\frac{2}{x} > 1$.

$\therefore$ The series converges, if $x < 2$ and

diverges if $x > 2$.

If $x = 2$, the ratio test fails.

$$\therefore \frac{a_n}{a_{n+1}} - 1 = \frac{1}{2n+2}.$$

$$\therefore n\left(\frac{a_n}{a_{n+1}} - 1\right)$$

$$= \frac{n}{2n+2} = \frac{1}{2 + 2/n}.$$

$$\therefore \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{1}{2}.$$

$\therefore$ By Raabe's test the series diverges. //.

4.4. Root Test and Condensation test

Theorem:- 1 (Cauchy's root test)

Let Σa_n be a series of positive terms. Then Σa_n is convergent if $\lim_{n\to\infty} a_n^{1/n} < 1$ and divergent if $\lim_{n\to\infty} a_n^{1/n} > 1$.

Proof:-

Case(i):- Let $\lim_{n\to\infty} a_n^{1/n} = l < 1$

Choose $\epsilon > 0$ such that $l + \epsilon < 1$.

Then there exists $m \in N$ such that

$$a_n^{1/n} < l + \varepsilon \quad \forall \quad n \geq m.$$

$\therefore a_n < (l+\varepsilon)^n \ \forall \ n \geq m.$

Now,

since $l + \varepsilon < 1$,

$\Sigma (l+\varepsilon)^n$ is convergent.

$\therefore$ By Comparison test Σa_n is Convergent.

Case (ii).

$$\text{let} \lim_{n \to \infty} a_n^{1/n} = l > 1.$$

Choose $\varepsilon > 0$ such that $l - \varepsilon > 1$.

Then there exists $m \in N$ such that,

$$a_n^{1/n} > l - \varepsilon \ \forall \ n \geq m.$$

$\therefore a_n > (l-\varepsilon)^n \ \forall \ n \geq m.$

Now,

since $l - \varepsilon > 1$,

$\Sigma (l-\varepsilon)^n$ is divergent

$\therefore$ By Comparison test, Σa_n is divergent //

Note:-

The following is a more general form of Cauchy's test.

Let Σa_n be a series of positive terms. Then Σa_n is convergent if $\lim\sup a_n^{1/n} < 1$, and divergent if $\lim\sup a_n^{1/n} > 1$.

$\therefore$ Σa_n is convergent if $\lim\sup a_n^{1/n} < 1$,

$\therefore$ Σa_n is divergent if $\lim\sup a_n^{1/n} > 1$.

//.

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Thank you!

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25/09/20

(1)

Class: II. B.Sc, Maths, & Sub: Sequences and Series.

Unit – V of Continue notes.

Theorem: 2 (Cauchy's condensation test).

Let $a_1 + a_2 + a_3 + \dots + a_n + \dots \rightarrow (1)$
be a series of positive terms and whose terms are monotonic decreasing. Then this series converges (or) diverges according as the series.

$$g a_g + g^2 a_{g^2} + \dots + g^n a_{g^n} + \dots \rightarrow (2)$$

Converges (or) diverges, where g is any positive integer > 1.

Proof:-

Let $s_n = a_1 + a_2 + \dots + a_n$ and

$$t_n = g a_g + g^2 a_{g^2} + \dots + g^n a_{g^n}.$$

Then,

$$s_{g^n} = (a_1 + a_2 + \dots + a_g) + \dots$$
$$+ (a_{g+1} + a_{g+2} + \dots + a_{g^2}) + \dots$$
$$+ (a_{g^{n-1}+1} + a_{g^{n-1}+2} + \dots + a_{g^n})$$

$$\leq g a_1 + (g^2 - g) a_g + \dots + (g^n - g^{n-1}) a_{g^{n-1}}$$

Since, the terms of the series are monotonic decreasing.

$$= g a_1 + g(g-1) a_g + g^2(g-1) a_{g^2} + \dots + g^{n-1}(g-1) a_{g^{n-1}}$$
$$= g a_1 + (g-1)(g a_g + g^2 a_{g^2} + \dots + g^{n-1} a_{g^{n-1}})$$
$$= g a_1 + (g-1) t_{n-1}.$$

$\therefore S_{g^n} \leq g a_1 + (g-1) t_{n-1}$.

$\therefore$ If the series (2) converges, then (1) converges.

Now, $S_{g^n} \geq g a_g + (g^2-g) a_{g^2} + \dots + (g^n - g^{n-1}) a_{g^n}$.

$= g a_g + \frac{g-1}{g} (g^2 a_{g^2} + \dots + g^n a_{g^n})$

$= g a_g + \frac{g-1}{g} (t_n - g a_g)$.

$= a_g + \frac{g-1}{g} t_n$.

$\therefore$ If the series (2) diverges, then (1) diverges.

Problems:-

1. Test the convergence of $\Sigma \frac{1}{(\log n)^n}$.

Solution:-

let $a_n = \frac{1}{(\log n)^n}$.

$a_n^{1/n} = \left[\frac{1}{(\log n)}\right]^{1/n}$ $\quad$ $a_n^{1/n} = \sqrt[n]{a_n}$

$\therefore \sqrt[n]{a_n} = \frac{1}{\log n}$

$\therefore \lim_{n\to\infty} \sqrt[n]{a_n} = 1/\infty$

$\therefore \lim_{n\to\infty} \sqrt[n]{a_n} = 0 < 1$.

$\therefore$ By Cauchy's root test $\sum \frac{1}{(\log n)^n}$ is convergence.

//.

---

(2). Test the Convergence of $\sum \left(1+\frac{1}{n}\right)^{-n^2}$.

Solution:-

$$\text{let } a_n = \left(1+\frac{1}{n}\right)^{-n^2}.$$

$\because a^{1/n} = \sqrt[n]{a_n}$

$$a_n^{1/n} = \left(\left(1+\frac{1}{n}\right)^{-n^2}\right)^{1/n}.$$

$$\therefore \sqrt[n]{a_n} = \left(1+\frac{1}{n}\right)^{-n}$$

$$\therefore \lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{e} < 1.$$

$\therefore$ By Cauchy's root test the series Converges.

//.

---

(3). Prove that the series $\sum e^{-\sqrt{n}} x^n$ Converges if $0 < x < 1$ and diverges if $x > 1$.

Solution:-

$$\text{let } a_n = e^{-\sqrt{n}} x^n.$$

$\because a^{1/n} = \sqrt[n]{a_n}$

$$a_n^{1/n} = \left(e^{-\sqrt{n}}\right)^{1/n} \left(x^n\right)^{1/n}.$$

$\therefore \; a_n^{1/n} = e^{-1/\sqrt{n}} \cdot x.$

$\therefore \lim_{n\to\infty} a_n^{1/n} = x.$

$\therefore$ By Cauchy's root test the given series Converges $0 < x < 1$ and diverges if $x > 1$.

Hence the proof

Thank you!

26/09/20

Class: II. B.Sc, Maths, & Sub: Sequences and Series.

Unit - V of Continue notes.

Probelm: 4

Test the convergence of $\sum \frac{n^3+a}{2^n+a}$.

Solution:-

Let $a_n = \frac{n^3+a}{2^n+a}$ and

$$b_n = \frac{n^3}{2^n}.$$

$$\therefore \frac{a_n}{b_n} = \left(\frac{n^3+a}{2^n+a}\right)\left(\frac{2^n}{n^3}\right)$$

$$= \left(\frac{n^3+a}{n^3}\right)\left(\frac{2^n}{2^n+a}\right)$$

$$= \left(1+\frac{a}{n^3}\right)\left(\frac{1}{1+\left(\frac{a}{2^n}\right)}\right)$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

$\therefore$ By Comparison test, the given series is Convergent (or) divergent according as $\sum \frac{n^3}{2^n}$ is Convergent (or) divergent.

Now, $\sqrt[n]{b_n} = \left(\frac{n^3}{2^n}\right)^{1/n}$.

(2)

$$= \frac{n^{3/n}}{2}.$$

Also,

$$\lim_{n\to\infty} n^{3/n} = 1.$$

$$\therefore \lim_{n\to\infty} \sqrt[n]{b_n} = \frac{1}{2}.$$

$\therefore \Sigma b_n$ is Convergent.

$\therefore \Sigma a_n$ is Convergent.

//.

---

(5). Test the Convergence of $\Sigma \frac{1}{n \log n}$.

Solution:-

By Cauchy's Condensation test,

$\Sigma \frac{1}{n \log n}$ converges (or) diverges with the series.

$$\Sigma \frac{2^n}{2^n \log 2^n} = \Sigma \frac{1}{n \log 2}$$

$$= \Sigma \frac{1}{n}.$$

Now, the series $\Sigma \frac{1}{n}$ diverges.

$\therefore$ The given series also diverges.

//.

6. Test the Convergence of the series $\sum \frac{1}{n(\log n)^p}$.

Solution:-

The given series Converges (or) diverges with the series.

$$\sum \frac{2^n}{2^n (\log 2^n)^p} = \sum \frac{1}{(\log 2)^p n^p}$$

$$= \frac{1}{(\log 2)^p} \Rightarrow \sum \frac{1}{n^p}.$$

The series $\sum \frac{1}{n^p}$ Converges if $p > 1$ and diverges if $p \leq 1$.

$\therefore$ The given series converges if $p > 1$ and diverges if $p \leq 1$. //.

---

7. Test the Convergence of the series.

$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \dots$

Solution:-

We have $a_n^{1/n} = \begin{cases} \left(\frac{1}{3^{n/2}}\right)^{1/n} & \text{if } n \text{ is even} \\ \left(\frac{1}{2^{(n+1)/2}}\right)^{1/n} & \text{if } n \text{ is odd} \end{cases}$

$$a_n^{1/n} = \begin{cases} \dfrac{1}{\sqrt{3}} & \text{if } n \text{ is even} \\[2ex] \dfrac{1}{2^{\frac{1}{2}(1+1/n)}} & \text{if } n \text{ is odd} \end{cases}.$$

Now, the sequence $\left(\dfrac{1}{2^{\frac{1}{2}(1+1/n)}}\right)$ Converges to $(1/\sqrt{2})$ as $n \to \infty$.

$\therefore$ $\dfrac{1}{\sqrt{2}}$ and $\dfrac{1}{\sqrt{3}}$ are the only limit point of the given sequence,

$$\limsup a_n^{1/n} = \frac{1}{\sqrt{2}} < 1.$$

$\therefore$ By Cauchy's root test the given series is Convergent.

//.

Note :-

In this case the limit of $a_n^{1/n}$ does not exist.

Since, $\liminf a_n^{1/n} \neq \limsup a_n^{1/n}$.

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Thank you !

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28/09/20

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Class: II. B.Sc., Maths, & Sub: Sequences and Series.

Unit - V of continue notes.

5.1. ALTERNATING SERIES

Definition:

A Series whose terms are alternatively positive and negative is called an alternating series.

Thus an alternating series is of the form

$$a_1 - a_2 + a_3 - a_4 + \cdots\cdots = \sum (-1)^{n+1} a_n \; \forall n,$$

Where $a_n \geq 0$.

For example,

(i). $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum (-1)^{n+1}\left(\frac{1}{n}\right)$ is an Alternating series.

(ii). $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots = \sum (-1)^{n+1}\left(\frac{n+1}{n}\right)$ is an Alternating series.

We now prove a test for convergence of an alternating series.

(X) **Theorem :- 1 (Leibnitz's test).**

(10. marks) Let $\sum (-1)^{n+1} a_n$ be an alternating series whose

(V.I) terms a_n satisfy the following conditions,

(i). (a_n) is a monotonic decreasing sequence.

(ii). $\lim_{n \to \infty} a_n = 0$.

Then the given alternating series converges.

Proof :-

Let (S_n) denote the sequence of partial sums of the given series.

Then, $S_{2n} = a_1 - a_2 + a_3 - a_4 + \dots\dots + a_{2n-1} - a_{2n}$.

$$S_{2n+2} = S_{2n} + a_{2n+1} - a_{2n+2}.$$

$\therefore S_{2n+2} = S_{2n} + a_{2n+1} - a_{2n+2}$.

$\therefore S_{2n+2} - S_{2n} = (a_{2n+1} - a_{2n+2}) \geq 0$

$\because$ (by (i)).

$\therefore S_{2n+2} \geq S_{2n}$.

$\therefore (S_{2n})$ is a monotonic increasing sequence.

Also,

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots\ldots (a_{2n-2} - a_{2n-1}) - a_{2n}.$$

$$\leq a_1 \quad , \quad (\text{by (i)}).$$

$\therefore (S_{2n})$ is bounded above.

$\therefore (S_{2n})$ is a convergent sequence.

Let $(S_{2n}) \to S$.

Now,

$$S_{2n+1} = S_{2n} + a_{2n+1}.$$

$$\therefore \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1}$$

$$= S + 0 = S \quad . \quad \because (\text{by (ii)}).$$

$\therefore (S_{2n+1}) \to S$

Thus the subsequences (S_{2n}) and (S_{2n+1}) converge to the same limits.

$\therefore (S_n) \to S$.

$\therefore$ The given series converges. //.

Hence the proof

Note :-

In the above theorem if $\lim_{n\to\infty} a_n = a \neq 0$,

then $\lim_{n\to\infty} S_{2n} = S$ and $\lim_{n\to\infty} S_{2n+1} = S + a$.

Hence the sequence (S_n) Cannot Converge.

Further, (S_n) is a bounded sequence.

~~Hen~~ Hence,

(S_n) is oscillates.

$\therefore$ The given series oscillates. //.

$=\!=\; * \;=\!=$

Thank you!

$=\!=\; * \;=\!=$

30/09/20

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Class: II. B.Sc., Maths., & Sub: Sequences and Series.

Unit - V of Continue notes.

Problem : 1

Show that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ Converges.

Solution:

The given series is $\sum (-1)^{n+1} a_n$,

where $a_n = 1/n$ and $a_{n+1} = 1/n+1$.

Clearly,

$$a_n > a_{n+1} \ \forall n.$$

Hence,

(a_n) is monotonic decreasing sequence.

Also,

$$\therefore \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0.$$

$$\therefore = 0.$$

∴ By Leibnitz's test the given series Converges. //.

Hence the solution.

(2). Show that the series $\sum \frac{(-1)^{n+1}}{\log(n+1)}$ converges.

Solution:

$$\text{let } a_n = \frac{1}{\log(n+1)} \text{ and}$$

$$a_{n+1} = \frac{1}{\log(n+2)}$$

Clearly

$$(a_n) \to 0 \quad , \text{ as } n \to \infty.$$

Also,

$$\frac{1}{\log n} > \frac{1}{\log(n+1)} \quad \forall\, n \geq 2 .$$

$$\because (a_n > a_{n+1}).$$

Hence,

(log n) is monotonic decreasing sequence.

$\therefore$ By Leibnitz's test the given series converges.

//.

Hence the solution.

3. Show that the series $\sum (-1)^{n+1} \frac{n}{3n-2}$ oscillates.

Solution:-

Given $\sum (-1)^{n+1} \frac{n}{3n-2}$.

let

$$a_n = \frac{n}{3n-2} \Rightarrow \frac{1}{3-\frac{2}{n}}.$$

And

$$a_{n+1} = \frac{1}{3-\frac{2}{n+1}}$$

$\therefore \quad a_n < a_{n+1}$

$\therefore \quad a_n < a_{n+1} \ \forall \ n$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{3n-2} = \lim_{n\to\infty} \frac{1}{3-2/n}$$

$$\therefore \lim_{n\to\infty} a_n = \frac{1}{3} \neq 0.$$

$\therefore$ The given series oscillates.

//.

Hence the solution

(4) Show that the following series Converges.

$$\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \dots$$

Solution:-

$$\text{let } a_n = \frac{1+2+3+\dots+n}{(n+1)^3}$$

$$\therefore a_n = \frac{n}{2(n+1)^2} \quad \text{and} \quad a_{n+1} = \frac{n+1}{2(n+2)^2}.$$

$$\frac{n(n+1)}{2(n+1)^3} = \frac{n}{2(n+1)^2}.$$

Clearly,

$$a_n > a_{n+1} \ \forall n.$$

$\therefore a_n$ is monotonic decreasing sequence.

Also,

$$\lim_{n\to\infty} a_n = \lim \frac{n}{2(n+1)^2}.$$

$$= \lim_{n\to\infty} \frac{1}{2n\left(1+\frac{1}{n}\right)^2} \Rightarrow 1/\infty$$

$$= 0.$$

$\therefore$ By Leibnitz's test the given series Converges. //.

Thank you!

01/10/20

(1)

Class: II B.Sc, Maths, Sub: Sequence and Series.

Unit - V of continue notes.

5.2 ABSOLUTE CONVERGENCE

Definition:-

A series $\sum a_n$ is said to be absolutely convergent if the series $\sum |a_n|$ is convergent.

Examples:-

(i). The series $\sum \frac{(-1)^n}{n^2}$ is absolutely convergent, for $\sum \left|\frac{(-1)^n}{n^2}\right| = \sum \frac{1}{n^2}$ which is convergent.

(ii). The series $\sum \frac{(-1)^n}{n}$ is not absolutely convergent, for $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ is divergent.

However the given series is convergent.

Note:-

If $\sum a_n$ is a convergent series of positive terms $\sum a_n$ is absolutely convergent.

Theorem: 1 Any absolutely convergent series is convergent.

Proof:- Let Σa_n be absolutely convergent.

$\therefore \Sigma |a_n|$ is convergent.

Let $s_n = a_1 + a_2 + \dots\dots + a_n$, and

$t_n = |a_1| + |a_2| + \dots\dots + |a_n|$.

By hypothesis (t_n) is convergent,

Hence convergent is a Cauchy sequence.

Then, given $\varepsilon > 0$, there exists $n_1 \in N$. such that

$$|t_n - t_m| < \varepsilon \ \forall \ n, m, \geq n_1 \longrightarrow (1)$$

Now, let $m > n$.

Then,
$$|s_n - s_m| = |a_{n+1} + a_{n+2} + \dots\dots + a_m|$$
$$\leq |a_{n+1}| + |a_{n+2}| + \dots\dots + |a_m|$$

$$= |t_n - t_m|$$

$$< \varepsilon \ \forall \ n, m \geq n_1 \quad \Big| \because (\text{by } (1)) \text{ we get}$$

$\therefore (S_n)$ is a Cauchy sequence in R and (S_n) is hence convergent.

$\therefore \sum a_n$ is a convergent series. //.

Note : 1

The converse of the above theorem is not true. For example, the series $\sum (-1)^n \frac{1}{n}$ is convergent. However $\sum \frac{1}{n}$ is divergent. So that the series is not absolutely convergent.

Note : 2

Since $\sum |a_n|$ is a series of positive terms, the test developed is condition for series of positive terms can be used to test the absolute convergence of a given series.

---

(4)

Definition:-

A series $\sum a_n$ is said to be conditionally convergent if it is convergent but not absolutely convergent.

Example:-

The series $\sum \frac{(-1)^n}{n}$ is conditionally convergent.

Thank you!

03/10/2020

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(1)

Class: II. B.Sc, Maths, & Sub: Sequences and Series.

Unit – V of continue notes.

Theorem: 1

(v) In an absolutely convergent series, the series formed by its positive terms alone is convergent and the series formed by its negative terms alone is convergent and conversely.

Proof:- Let Σa_n be the given absolutely convergent series.

We define $P_n = \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{if } a_n \leq 0 \end{cases}$ and

$$q_n = \begin{cases} 0 & \text{if } a_n \geq 0 \\ -a_n & \text{if } a_n < 0 \end{cases}$$

(ie), P_n is a positive term of the given series and q_n is the modulus of a negative term.

$\therefore$ ΣP_n is the series formed with the positive terms of the given series and

Σq_n is the series formed with the moduli of the negative terms of the given series.

Clearly, $P_n \leq |a_n|$, and

$q_n \leq |a_n| \ \forall\ n$.

Since, the given series is absolutely convergent, $\Sigma |a_n|$ is ~~the~~ a convergent series of positive terms.

Hence,

By comparison test ΣP_n and

Σq_n are convergent.

$\therefore$ Conversely ΣP_n and Σq_n are converge to P and q respectively.

We claim that Σa_n is a absolutely convergent.

$\therefore$ ΣP_n is also converge to P. And

$\therefore$ Σq_n is also converge to q.

$\therefore$ Σa_n is a absolutely convergent.

We have $|a_n| = P_n + q_n$

$\therefore \Sigma |a_n|$ is also absolutely convergent

Then,

$$\Sigma |a_n| = P_n + q_n .$$

$$\therefore \Sigma |a_n| = \Sigma (P_n + q_n)$$

$$= \Sigma P_n + \Sigma q_n$$

$$= P + q$$

$\therefore \Sigma a_n$ is a absolutely convergent.

//.

Hence the proof.

Theorem: 2

If Σa_n is an absolutely convergent series and (b_n) is a bounded sequence, then the series $\Sigma a_n b_n$ is an absolutely convergent series.

Proof:-

Since, (b_n) is a bounded sequence,

Then there exists a real number $K > 0$ such that

$$|b_n| \leq K \ \forall n.$$

$$\therefore |a_n b_n| = |a_n| \, |b_n|$$

$$\leq K |a_n| \ \forall n.$$

~~Since~~,

Since, $\sum a_n$ is absolutely Convergent

$\therefore \sum |a_n|$ is Convergent.

$\therefore \sum K |a_n|$ is Convergent.

$\therefore$ By Comparison test is

$\sum |a_n b_n|$ is Convergent.

$\therefore \sum a_n b_n$ is absolutely Convergent. //.

Hence the proof.

Thank you!

Class: II. B.Sc, Maths, & Sub: Sequences and Series.

Unit - V of Continue notes.

Problems:-

(1). Test for convergence of the series $\sum \frac{(-1)^n}{n^p}$.

Solution:-

Case (i)

Let $p > 1$.

Then $\sum \left| \frac{(-1)^n}{n^p} \right| = \sum \frac{1}{n^p}$ is Convergent.

$\therefore$ The given series is absolutely convergent and hence convergent.

Case (ii)

Let $0 < p \leq 1$.

Then $\left(\frac{1}{n^p}\right)$ is a monotonic decreasing sequence.

Converging to 0.

$\therefore$ By Leibnitz's test the given series is Converges.

(2)

In this case the convergence is not absolute series

$\Sigma \frac{1}{n^p}$ diverges.

When $0 < p \leq 1$.

<u>Case (iii)</u>,

let $p = 0$.

Then the series reduces to $-1 + 1 - 1 + \ldots$

Which oscillates finitely.

<u>Case (iv)</u>,

let $p < 0$.

Then the ~~series~~ sequence $\left(\frac{1}{n^p}\right)$ is unbounded.

Hence the given series oscillates infinitely //.

Hence the solution.

---

(2). Show that the series $\sum (-1)^n \left| \sqrt{(n^2+1)} - n \right|$ is conditionally convergent.

<u>Solution</u>:-

let $a_n = \sqrt{(n^2+1)} - n$

$$= \frac{1}{\sqrt{(n^2+1)} + n}$$

Clearly, (a_n) is a monotonic decreasing sequence converging to 0.

$\therefore$ By Leibnitz's test the given series converges.

Now,

we prove that $\sum \left|(-1)^n \left(\sqrt{(n^2+1)} - n\right)\right|$ is divergent.

$$\therefore \left|(-1)^n \left(\sqrt{(n^2+1)} - n\right)\right| = a_n .$$

$$\therefore a_n = \frac{1}{\sqrt{(n^2+1)} + n} .$$

let $b_n = 1/n$.

$$\therefore \frac{a_n}{b_n} = \frac{n}{\sqrt{(n^2+1)} + n}$$

$$= \frac{1}{\sqrt{\left(1 + \frac{1}{n^2}\right)} + 1}$$

(7)

$\therefore \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{1}{2}$.

$\therefore$ By Comparison test Σa_n is divergent.

$\therefore$ The given series is not absolutely convergent.

$\therefore$ The given series is conditionally convergent. //.

Hence the solution.

Thank you!

08/10/20

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Class: II. B.Sc, Maths; Sub: Sequences and series.

Unit - V of continue notes.

(3). Show that the series $\sum \frac{x^{n-1}}{(n-1)!}$ Converges absolutely for all values of x.

Solution:-

$$\text{Let } a_n = \frac{x^{n-1}}{(n-1)!}$$

$$a_{n+1} = \frac{x^{n+1-1}}{(n+1-1)!} = \frac{x^n}{n!}$$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{x^{n-1}}{(n-1)!} \cdot \frac{n!}{x^n}$$

$$= \frac{x^{n\cancel{-1}}}{\cancel{(n-1)}!} \cdot \frac{n\cancel{(n-1)}!}{x^n}$$

$$= x^{-1} n.$$

$$= \frac{n}{x}$$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{n}{x}.$$

$\therefore \left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{n}{x} \right|$

$\Rightarrow \frac{n}{|x|}$

$\therefore \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{n}{|x|}$.

$\therefore \lim_{n\to\infty} \frac{n}{|x|} = \infty \;\forall\; x \neq 0$.

$\therefore$ By ratio test $\sum \left| \frac{x^{n-1}}{(n-1)!} \right|$ is Convergent

for all $x \neq 0$. And

If $x = 0$,

Then, $\sum \left| \frac{x^{n-1}}{(n-1)!} \right|$ is Convergent.

$\therefore$ The series converge absolutely for all x. //.

Hence the solution.

(4). Test the convergence of $\sum \frac{(-1)^n \sin n\alpha}{n^3}$.

Solution:-

$$\text{Given } \sum \frac{(-1)^n \sin n\alpha}{n^3}.$$

$$a_n = \frac{(-1)^n \sin n\alpha}{n^3}.$$

$$|a_n| = \left| \frac{(-1)^n \sin n\alpha}{n^3} \right|.$$

Since $|\sin\theta| \le 1$.

Then, $\left| \frac{(-1)^n \sin n\alpha}{n^3} \right| \le \frac{1}{n^3}$ is convergent series.

$\therefore$ By Comparison test the given series is absolutely Convergent. //.

Hence the solution.

Important Notes:-

V.V.I (1). Every bounded sequences has a convergent.
2. mark

Proof:-

let (a_n) be a bounded sequences. And

let (a_{nk}) be a monotonic subsequence of (a_n).

Since (a_n) is bounded. And (a_{nk}) is also bounded.

$\therefore$ (a_{nk}) is bounded monotonic sequence

Hence,

(a_{nk}) is convergence.

$\therefore$ (a_{nk}) is convergence subsequence of (a_n). //.

Thank you!

04/10/20

(1)

Class : II B.Sc, Maths, Sub : Sequences and Series.

Unit - V end of the notes.

Important Notes :- 2

(x):

Any convergent sequence is a Cauchy sequence.

Proof:-

Let $(a_n) \to l$

Then,

Given $\varepsilon > 0$, such that $n_0 \in N$.

$\therefore |a_n - l| \; \varepsilon/2 \;\forall\; n \geq n_0$.

(ii), $|a_n - a_m| < \varepsilon$

$$|a_n - a_m| = |a_n - l + l - a_m|$$

$$\leq |a_n - l| + |a_m - l|$$

$$< \varepsilon/2 + \varepsilon/2$$

$$< \varepsilon \quad , n, m \geq n_0$$

(a_n) is a Cauchy sequence. //.

(N.) Important Notes :- 3

Any Cauchy sequence is a bounded sequence.

Proof:-

Let (a_n) be a Cauchy sequence.

And $\varepsilon > 0$ be given

Then,

There exists $n_0 \in N$ such that, $|a_n - a_m| < \varepsilon$

$\therefore |a_n - a_m| < \varepsilon \; \forall \; n, m \geq n_0$

$\therefore |a_n| < |a_{n_0}| + \varepsilon \; \forall \; n \geq n_0$.

Now,

Let $k = \max\{|a_1|, |a_2|, \dots |a_{n_0}|\}$.

Then,

$|a_n| \leq k \; \forall \; n$.

Hence,

(a_n) is a bounded sequence. //.

Every sequence (a_n) has a monotonic sub sequence.

Proof:-

Case (i)

Let (a_n) has infinite number of peak point

Then, the peak point be $n_1 < n_2 < n_3, \ldots < n_k < \ldots$

And
$$a_{n_1} > a_{n_2} > \ldots > a_{n_k} > \ldots$$

$\therefore$ (a_{n_k}) is a monotonic decreasing sub sequence of (a_n).

Case (ii)

Let (a_n) has a finite number of peak point (or) no peak point.

Choose $n_1 \in N$ such that there is no peak point greater than (or) equal to n_1.

Since, n_1 is not peak point of (a_n) there exists.
$n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$.

Again,
Since n_2 is not peak point of (a_n),
there exists $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$.

Repeating the above process, we have

$$a_{n_1} \leq a_{n_2} \leq a_{n_3}, \ldots < a_{n_k} < \ldots$$

$\therefore (a_{n_k})$ is a monotonic ~~s~~ Increasing sequence of (a_n)

By case (i) & (ii) Every sequence (a_n) has a monotonic sub sequence.

//.

Unit - V is Completed notes.

Thank you!

(x) [Since the next online class in subject on "sequences and series" start will be "revision" at only.]. So Pls kindly come and attend in your online class to all the students. Thank you to all.