BHARATHIDASAN UNIVERSITY, TIRUCHIRAPPALL – 620 024

B.Sc. Physics / Chemistry / Industrial Electronics / Geology - Students

(For the candidates admitted from the academic year 2016-17 onwards)

ALLIED MATHEMATICS

ALLIED COURSE I

CALCULUS AND FOURIER SERIES

Objects :

1. To learn the basic need for their major concepts
2. To train the students in the basic Integrations

UNIT I

Successive Differentiation – n^{th} derivative of standard functions (Derivation not needed) simple problems only-Leibnitz Theorem (proof not needed) and its applications- Curvature and radius of curvature in Cartesian only (proof not needed)–Total differential coefficients (proof not needed) - Jacobians of two & three variables –Simple problems in all these.

UNIT II

Evaluation of integrals of types

1] $\int \frac{px+q}{ax^2+bx+c}dx$ 2] $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ 3] $\int \frac{dx}{(x+p)\sqrt{ax^2+bx+c}}$

4] $\int \frac{dx}{a+b\cos x}$ 5] $\int \frac{dx}{a+b\sin x}$ 6] $\int \frac{(a\cos x+b\sin x+c)}{(p\cos x+q\sin x+r)}dx$

Integration by trigonometric substitution and by parts of the integrals

1] $\int \sqrt{a^2-x^2}\,dx$ 2] $\int \sqrt{a^2+x^2}\,dx$ 3] $\int \sqrt{x^2-a^2}\,dx$

UNIT III

General properties of definite integrals – Evaluation of definite integrals of types

1] $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$ 2] $\int_a^b \sqrt{(x-a)(b-x)}\,dx$ 3] $\int_a^b \sqrt{\frac{x-a}{b-x}}\,dx$

Reduction formula (When n is a positive integer) for

1] $\int_a^b e^{ax}x^n dx$ 2] $\int_a^b \sin^n x\,dx$ 3] $\int_a^b \cos^n x\,dx$

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4] $\int_0^x e^{ax} x^n dx$ 5] $\int_0^{\frac{\pi}{2}} \sin^n x dx$

6] Without proof $\int_0^{\frac{\pi}{2}} \sin^n x \cos^m x dx$ - and illustrations

UNIT IV

Evaluation of Double and Triple integrals in simple cases –Changing the order and evaluating of the double integration. (Cartesian only)

UNIT V

Definition of Fourier Series – Finding Fourier Coefficients for a given periodic function with period 2π and with period 2ℓ - Use of Odd & Even functions in evaluating Fourier Coefficients - Half range sine & cosine series.

TEXT BOOK(S)

1. S. Narayanan, T.K. Manichavasagam Pillai, Calculus, Vol. I, S. Viswanathan Pvt Limited, 2003
2. S. Arumugam, Isaac and Somasundaram, Trigonometry & Fourier Series, New Gamma Publishers, Hosur, 1999.

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Allied-Mathematics-I
(Physics & Chemistry)
Calculus and Fourier Series
Unit-I

Chapter-III - Successive Differentiation:

If a function $y = f(x)$ of x. Differentiate this function with respect to 'x', we get $\frac{dy}{dx} = y' = f'(x)$. This is called first derivative. Again differentiate w.r.t. 'x', we get $\frac{d^2y}{dx^2} = y'' = f''(x)$. This is called Second derivative and so on upto n^{th} derivative. This is called successive differentiation.

$d(x^n) = n x^{n-1}$

$d(x^{n-1}) = (n-1)x^{n-2}$

Example:

$$y = 4x^5$$

$y = f(x)$

$y' = f'(x)$

$y'' = f''(x)$

I^{st} derivative is $\frac{dy}{dx} = y' = 20x^4$

2^{nd} derivative is $\frac{d^2y}{dx^2} = y'' = 80x^3$ $4(5x^4)$

3^{rd} derivative is $\frac{d^3y}{dx^3} = y''' = 240x^2$ $d(x^4) = 4x^3$

4^{th} derivative is $\frac{d^4y}{dx^4} = y^{iv} = 480x$

5^{th} derivative is $\frac{d^5y}{dx^5} = y^{v} = 480$.

Note:

1) The symbols of the successive derivatives are usually abbreviated as follows:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2y$$

$\frac{d}{dx} = D$

$\frac{d^2}{dx^2} = D^2$

$$\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{d^3y}{dx^3} = D^3y$$ (2)

|||^{ly} $$\frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \frac{d^4y}{dx^4} = D^4y$$

$$\vdots$$

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = D^ny$$

2) If $y=f(x)$, the successive derivatives are also denoted by

(i) $f'(x), f''(x), \dots, f^n(x)$.

(ii) $y', y'', \dots, y^n$ $\qquad y^{iv} \qquad y^n = y_n$

(iii) $y_1, y_2, \dots, y_n$ $\qquad y_4$

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The n^{th} derivative:

1. Find n^{th} derivative, if $y = e^{ax}$.

Soln:-

Given that $y = e^{ax}$

e^{ax} $\quad d(e^a) = e^a$

$$y' = \frac{dy}{dx} = ae^{ax}$$

$e^{ax} = e^{ax} \cdot a(1)$

$= ae^{ax}$

$$y'' = \frac{d^2y}{dx^2} = a^2e^{ax}$$ $\quad a(ae^{ax})$

$d(e^{3x}) = 3e^{3x}$

$$y''' = \frac{d^3y}{dx^3} = a^3e^{ax}$$ $\quad a^2(ae^{ax})$

$e^{3x} \cdot 3$

$3e^{3x}$

$$\vdots$$

$$y^n = \frac{d^ny}{dx^n} = a^ne^{ax}.$$

$\therefore$ The n^{th} derivative is $\boxed{y^n = a^ne^{ax}}$.

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2) If $y = \frac{1}{x^2}$, define y^n (or) y_n.

Soln :-

$$y = \frac{1}{x^2} = x^{-2}$$

$$y' = -2x^{-3} = -(2!)\,x^{-3}$$

$$y'' = 6x^{-4} = 3!\,x^{-4}$$

$$y''' = -24x^{-5} = -(4!)\,x^{-5}$$

$\vdots$

$$y^n = (-1)^n (n+1)!\, x^{-(n+2)}$$

$$\therefore \boxed{y^n = \frac{(-1)^n (n+1)!}{x^{n+2}}}$$

$2! = 2\cdot 1 = 2$

$3! = 3\cdot 2\cdot 1 = 6$

$4! = 4\times 3\times 2\times 1 = 24$

Standard Results :-

1. If $y = (ax+b)^{-1}$, then $D^n (ax+b)^{-1} = \frac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}$.
   define y_n

2. If $y = \log(ax+b)$, then $y_n = \frac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$.

3. If $y = \sin(ax+b)$, then $D^n(\sin(ax+b)) = a^n \sin\left(\frac{n\pi}{2} + ax+b\right)$.

4. If $y = \cos(ax+b)$, then $D^n(\cos(ax+b)) = a^n \cos\left(\frac{n\pi}{2} + ax+b\right)$.

5. If $y = e^{ax}\sin(bx+c)$, then $D^n\{e^{ax}\sin(bx+c)\} = r^n e^{ax}\sin(bx+c+n\phi)$
   where $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}(b/a)$.

6. If $y = e^{ax}\cos(bx+c)$, then $D^n\{e^{ax}\cos(bx+c)\} = r^n e^{ax}\cos(bx+c+n\phi)$
   where $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}(b/a)$.

Problems:-

1. Find y_n, where $y = \dfrac{3}{(x+1)(2x-1)}$.

Soln.:-

Given that,

$$y = \frac{3}{(x+1)(2x-1)} = \frac{A}{(x+1)} + \frac{B}{(2x-1)} \rightarrow (*)$$

$$\Rightarrow \frac{3}{(x+1)(2x-1)} = \frac{A(2x-1) + B(x+1)}{(x+1)(2x-1)}$$

$$\Rightarrow 3 = A(2x-1) + B(x+1) \rightarrow (1)$$

put $x = \frac{1}{2}$ in (1), we get

$3 = A(2\frac{1}{2} - 1) + B(\frac{1}{2} + 1)$

$3 = A(1-1) + B(\frac{3}{2})$

$3 = A(0) + B\frac{3}{2}$

$\therefore B = \cancel{3} \times \frac{2}{\cancel{3}}$

$\boxed{B = 2}$

put $x = -1$ in (1), we get

$3 = A(2(-1)-1) + B(-1+1)$

$3 = A(-2-1) + B(0)$

$3 = A(-3)$

$\therefore A = \cancel{3}(-\frac{1}{\cancel{3}})$

$\boxed{A = -1}$

$2x-1=0$
$2x=1$
$x=\frac{1}{2}$

$x+1=0$
$x=-1$

$\therefore (*) \Rightarrow$

$$y = \frac{3}{(x+1)(2x-1)} = -\frac{1}{(x+1)} + \frac{2}{(2x-1)} = -(x+1)^{-1} + 2(2x-1)^{-1}$$

$$= 2\underbrace{(2x-1)^{-1}}_{}^{ax+b} - (x+1)^{-1}$$

$x^a x^b = x^{a+b}$

$$\therefore y_n = 2^1\left(\frac{(-1)^n n!\, 2^n}{(2x-1)^{n+1}}\right) - \frac{(-1)^n n!\, 1^n}{(x+1)^{n+1}}$$

$D^n((ax+b)^{-1}) = \dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

$$= \frac{(-1)^n n!\, 2^{n+1}}{(2x-1)^{n+1}} - \frac{(-1)^n n!}{(x+1)^{n+1}}$$

by standard result (1)

$$\boxed{y_n = (-1)^n n! \left[\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}}\right]}$$

—x—

2. Find y_n, when $y = \frac{x^2}{(x-1)^2(x+2)}$.

Soln:-

Given that $y = \frac{x^2}{(x-1)^2(x+2)}$

$$\therefore \frac{x^2}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)} \rightarrow (1)$$

$$\Rightarrow \frac{x^2}{(x-1)^2(x+2)} = \frac{A(x-1)(x+2)+B(x+2)+C(x-1)^2}{(x-1)^2(x+2)}$$

$$\Rightarrow \quad x^2 = A(x-1)(x+2)+B(x+2)+C(x-1)^2 \rightarrow (2)$$

put $x=1$, we get

$1 = A(0)+B(1+2)+C(0)$

$1 = 3B$

$\boxed{B = 1/3}$

put $x=-2$, we get

$(-2)^2 = A(0)+B(0)+C(-3)^2$

$4 = 9C$

$\boxed{C = 4/9}$

put $x=0$, we get

$0 = A(-1)(2)+B(2)+C(-1)^2$

$0 = -2A+2B+C$

Sub. B and C values, we get

$-2A+2(1/3)+4/9 = 0$

$-2A+\frac{2}{3}+\frac{4}{9} = 0$

$-2A+\left(\frac{6+4}{9}\right) = 0$

$-2A+\frac{10}{9} = 0$

$\cancel{-}2A = \cancel{-}\frac{10}{9} \Rightarrow A = \frac{\overset{5}{\cancel{10}}}{9} \times \frac{1}{\cancel{2}}$

$\therefore \boxed{A = \frac{5}{9}}$

Sub. A, B and C values in eqn. (1), we get

$$\frac{x^2}{(x-1)^2(x+2)} = \frac{5}{9}\frac{1}{(x-1)} + \frac{1}{3}\frac{1}{(x-1)^2} + \frac{4}{9}\frac{1}{(x+2)}$$

$$= \frac{5}{9}(x-1)^{-1} + \frac{1}{3}(x-1)^{-2} + \frac{4}{9}(x+2)^{-1}$$

$$\therefore \underset{y_n =}{D^n}\left(\frac{x^2}{(x-1)^2(x+2)}\right) = \frac{5}{9}D^n\left[(x-1)^{-1}\right] + \frac{1}{3}D^n\left[(x-1)^{-2}\right] + \frac{4}{9}D^n\left[(x+2)^{-1}\right]$$

$D = \frac{d}{dx}$

$$\therefore y_n = \frac{5}{9}\left[\frac{n!(-1)^n 1^n}{(x-1)^{n+1}}\right] + \frac{1}{3}\left[\frac{(n+1)!(-1)^n 1^n}{(x-1)^{n+2}}\right] + \frac{4}{9}\left[\frac{(-1)^n n! 1^n}{(x+2)^{n+1}}\right]$$

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$$\because D^n[(ax+b)^{-1}] = \frac{n!(-1)^n a^n}{(ax+b)^{n+1}}$$

&

$$D^n[(ax+b)^{-2}] = \frac{(n+1)!(-1)^n a^n}{(ax+b)^{n+2}}$$

$$\therefore y_n = \frac{5}{9}\left[\frac{n!(-1)^n}{(x-1)^{n+1}}\right] + \frac{1}{3}\left[\frac{(n+1)!(-1)^n}{(x-1)^{n+2}}\right] + \frac{4}{9}\left[\frac{n!(-1)^n}{(x+2)^{n+1}}\right]$$

i.e., $$\boxed{y_n = (-1)^n n!\left[\frac{5}{9(x-1)^{n+1}} + \frac{n+1}{3(x-1)^{n+2}} + \frac{4}{9(x+2)^{n+1}}\right]}$$

$\because (n+1)! = (n+1)n!$

3. Find y_n, when $y = \frac{1}{x^2+a^2}$.

$a^2-b^2=(a+b)(a-b)$

$(n+1)\,n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

$(n+1)\,n!$

Soln:-

Let $y = \frac{1}{x^2+a^2} = \frac{1}{x^2-(ia)^2} = \frac{1}{(x+ia)(x-ia)} = \frac{A}{(x+ia)} + \frac{B}{(x-ia)} \rightarrow (1)$

$x^2-(-a^2)$ $\quad -a^2$

$\Rightarrow 1 = A(x-ia) + B(x+ia)$

put $x = ia$, we get

$1 = A(0) + B(2ia)$

$\therefore \boxed{B = \frac{1}{2ia}}$

put $x=-ia$, we get

$1 = A(-2ia) + B(0)$

$\therefore \boxed{A = -\frac{1}{2ia}}$

Sub. A & B values, in eqn. (1), we get

$$\frac{1}{x^2+a^2} = -\frac{1}{2ia}\frac{1}{(x+ia)} + \frac{1}{2ia}\frac{1}{(x-ia)}$$

$$= \frac{1}{2ia}\left[\frac{1}{(x-ia)} - \frac{1}{(x+ia)}\right]$$

$$= \frac{1}{2ia}\left[(x-ia)^{-1} - (x+ia)^{-1}\right]$$

$D^n[(ax+b)^{-1}] = \frac{n!(-1)^n a^n}{(ax+b)^{n+1}}$

$$\therefore D^n\left(\frac{1}{x^2+a^2}\right) = \frac{1}{2ia}\left\{D^n[(x-ia)^{-1}] - D^n[(x+ia)^{-1}]\right\}$$

$$= \frac{1}{2ia}\left\{\left[\frac{n!(-1)^n 1^n}{(x-ia)^{n+1}}\right] - \left[\frac{n!(-1)^n 1^n}{(x+ia)^{n+1}}\right]\right\}$$

$$\therefore \boxed{y_n = \frac{n!(-1)^n}{2ia}\left[\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right]}$$

4. Find n^{th} differential coefficients of $\cos x . \cos 2x . \cos 3x$.

Soln:-

$$= \cos 2x \cdot \cos x \cdot \cos 3x$$
$$= \frac{1}{2}[\cos(2x+x) + \cos(2x-x)] \cos 3x$$

$$\cos x . \cos 2x . \cos 3x = \frac{1}{2}[\cos 3x + \cos x] . \cos 3x$$

$$= \frac{1}{2}[\cos^2 3x + \cos x \cos 3x] \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$= \frac{1}{2}\left[\frac{1}{2}(\cos 6x + 1) + \frac{1}{2}(\cos 4x + \cos 2x)\right]$$

$$= \frac{1}{4}(\cos 6x + 1) + \frac{1}{4}(\cos 4x + \cos 2x)$$
$$= \frac{1}{4}\cos 6x + \frac{1}{4} + \frac{1}{4}(\cos 4x + \cos 2x)$$
$$= \frac{1}{4} + \frac{1}{4}[\cos 6x + \cos 4x + \cos 2x]$$

$$\therefore D^n[\cos x . \cos 2x . \cos 3x] = D^n\left[\frac{1}{4}\right] + \frac{1}{4} D^n[\cos 6x + \cos 4x + \cos 2x]$$

$D^n[\cos(ax+b)] = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

$$= 0 + \frac{1}{4}\left\{6^n \cos\left(\frac{n\pi}{2} + \cancel{\cos} 6x\right) + 4^n \cos\left(\frac{n\pi}{2} + \cancel{\cos} 4x\right) + 2^n \cos\left(\frac{n\pi}{2} + \cancel{\cos} 2x\right)\right\}$$

$$\therefore \boxed{y_n = \frac{1}{4}\left[6^n \cos\left(\frac{n\pi}{2} + 6x\right) + 4^n \cos\left(\frac{n\pi}{2} + 4x\right) + 2^n \cos\left(\frac{n\pi}{2} + 2x\right)\right]}$$

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5. Find the n^{th} differential coefficient of $\cos^5\theta . \sin^7\theta$.

Soln:-

Let $x = \cos\theta + i\sin\theta$

then, $\frac{1}{x} = \cos\theta - i\sin\theta$

$x + \frac{1}{x} = 2\cos\theta$

$x - \frac{1}{x} = 2i\sin\theta$

Also, from De Moivre's theorem,

$x^n = \cos n\theta + i\sin n\theta$

$\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$x^n + \frac{1}{x^n} = 2\cos n\theta$

$x^n - \frac{1}{x^n} = 2i\sin n\theta$

$\therefore$ ~~$2^7\ldots$~~

$$(2\cos\theta)^5 (2i\sin\theta)^7 = 2^5 \cos^5\theta . 2^7 i^7 \sin^7\theta$$

$$\Rightarrow (-i)\, 2^{12} \cos^5\theta \sin^7\theta =$$

$$\therefore (2\cos\theta)^5 (2i\sin\theta)^7 = \left(x+\frac{1}{x}\right)^5 \cdot \left(x-\frac{1}{x}\right)^7$$

$$\Rightarrow 2^5 \cos^5\theta \; 2^7 i^7 \sin^7\theta = \left(x+\frac{1}{x}\right)^5 \cdot \left(x-\frac{1}{x}\right)^5 \cdot \left(x-\frac{1}{x}\right)^2$$

$$\Rightarrow (-i)\, 2^{12} \cos^5\theta \sin^7\theta = \left(x^2-\frac{1}{x^2}\right)^5 \left(x-\frac{1}{x}\right)^2$$

$i^2=-1$

$$= \left[(x^2)^5 - 5C_1 (x^2)^4\left(\frac{1}{x^2}\right) + 5C_2 (x^2)^3 \left(\frac{1}{x^2}\right)^2 - 5C_3 (x^2)^2\left(\frac{1}{x^2}\right)^3 + 5C_4 (x^2)^1 \left(\frac{1}{x^2}\right)^4 - 5C_5 (x^2)^0 \left(\frac{1}{x^2}\right)^5\right] \left(x-\frac{1}{x}\right)^2$$

$(x+y)^n = x^n + nC_1 x^{n-1} y^1 + nC_2 x^{n-2} y^2 + nC_3 x^{n-3} y^3 + \dots$

$5C_2 = \frac{5\times4}{2\times1}$, $5C_3 = \frac{5\times4\times3}{3\times2\times1}$, $5C_4 = \frac{5\times4\times3\times2}{4\times3\times2\times1}$

$$= \left[x^{10} - 5x^8 \frac{1}{x^2} + 10 x^6 \frac{1}{x^4} - 10 x^4 \frac{1}{x^6} + 5(x^2)\frac{1}{x^8} - \frac{1}{x^{10}}\right]\left(x-\frac{1}{x}\right)^2$$

$$= \left[x^{10} - 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} - \frac{1}{x^{10}}\right]\left(x^2 - 2 + \frac{1}{x^2}\right)$$

$$= x^{12} - 5x^8 + 10x^4 - 10 + \frac{5}{x^4} - \frac{1}{x^8} - 2x^{10} + 10x^6 - 20x^2 + \frac{20}{x^2} - \frac{10}{x^6} + \frac{2}{x^{10}} + x^8 - 5x^4 + 10 - \frac{10}{x^4} + \frac{5}{x^8} - \frac{1}{x^{12}}.$$

$$= x^{12} - 2x^{10} - 4x^8 + 10x^6 + 5x^4 - 20x^2 + \frac{20}{x^2} + \frac{5}{x^4} + \frac{10}{x^6} + \frac{4}{x^8} + \frac{2}{x^{10}} - \frac{1}{x^{12}}$$

$$= \left(x^{12} - \frac{1}{x^{12}}\right) - 2\left(x^{10} - \frac{1}{x^{10}}\right) - 4\left(x^8 - \frac{1}{x^8}\right) + 10\left(x^6 - \frac{1}{x^6}\right) + 5\left(x^4 - \frac{1}{x^4}\right) - 20\left(x^2 - \frac{1}{x^2}\right)$$

$x^n - \frac{1}{x^n} = 2i\sin n\theta$

$$= 2i\sin 12\theta - 2(2i\sin 10\theta) - 4(2i\sin 8\theta) + 10(2i\sin 6\theta) + 5(2i\sin 4\theta) - 20(2i\sin 2\theta)$$

$$= 2i\left[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta - 5\sin 4\theta - 20\sin 2\theta\right]$$

$(-i)\, 2^{13} \cos^5\theta \sin^7\theta$

$$= 2i\left[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta - 5\sin 4\theta - 20\sin 2\theta\right]$$

$$\therefore \cos^5\theta \sin^7\theta = \frac{2i}{-i\,2^{13}}\left[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta - 5\sin 4\theta - 20\sin 2\theta\right]$$

$D^n[\sin(ax+b)] = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$

$$= -\frac{1}{2^{11}}\left[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta - 5\sin 4\theta - 20\sin 2\theta\right]$$

$$\boxed{\therefore D^n(\cos^5\theta \cdot \sin^7\theta) = -\frac{1}{2^{11}}\left[12^n \sin\left(\frac{n\pi}{2}+12\theta\right) - 2\cdot 10^n \sin\left(\frac{n\pi}{2}+10\theta\right) - 4\cdot 8^n \sin\left(\frac{n\pi}{2}+8\theta\right) + 10\cdot 6^n \sin\left(\frac{n\pi}{2}+6\theta\right) - 5\cdot 4^n \sin\left(\frac{n\pi}{2}+4\theta\right) - 20\cdot 2^n \sin\left(\frac{n\pi}{2}+2\theta\right)\right].}$$

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6. Find the n^{th} differential coefficient of $e^{5x}\cdot \sin^3 ax$.

Soln.:-

$$e^{5x}\sin^3 ax = e^{5x}\left[\frac{3}{4}\sin ax - \frac{1}{4}\sin 3ax\right]$$

$e^{ax}\sin(ax+b)$

$\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

$$e^{5x}\sin^3 ax = \frac{3}{4}e^{5x}\sin ax - \frac{1}{4}e^{5x}\sin 3ax$$

$$\therefore D^n\left[e^{5x}\sin^3 ax\right] = \frac{3}{4}D^n\left[e^{5x}\sin ax\right] - \frac{1}{4}D^n\left[e^{5x}\sin 3ax\right]$$

$D^n\left[e^{ax}\sin(bx+c)\right] = r^n e^{ax}\sin(bx+c+n\phi)$

where, $r = \sqrt{a^2+b^2}$, $\phi = \tan^{-1}(b/a)$

$$= \frac{3}{4}r^n e^{5x}\sin(ax+n\phi) - \frac{1}{4}r^n e^{5x}\sin(3ax+n\phi)$$

$r = \sqrt{5^2+a^2}$

$r = (25+a^2)^{1/2}$

$\phi = \tan^{-1}(a/5)$

$r = \sqrt{5^2+(3a)^2}$

$r = (25+9a^2)^{1/2}$

$\phi = \tan^{-1}\left(\frac{3a}{5}\right)$

$$\therefore D^n(e^{5x}\sin^3 ax) = \frac{3}{4}(25+a^2)^{n/2}\sin(ax+n\phi) + \frac{1}{4}(25+9a^2)^{n/2}\sin(3ax+n\phi)$$

7. Find the n^{th} differential coefficient of $\log(4-x^2)$

Soln:-

$$\log(4-x^2) = \log(2^2-x^2)$$
$$= \log[(2-x)(2+x)]$$
$$\log(4-x^2) = \log(2-x)+\log(2+x)$$

$\log xy = \log x + \log y$

$$\therefore D^n(\log(4-x^2)) = D^n(\log(2-x)) + D^n(\log(2+x))$$

$$D^n(\log(ax+b)) = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$$

$$= \frac{(-1)^{n-1}(n-1)!\,(-1)^n}{(2-x)^n} + \frac{(-1)^{n-1}(n-1)!\,1^n}{(2+x)^n}$$

$$= \frac{(-1)^{n-1+n}(n-1)!}{(2-x)^n} + \frac{(-1)^{n-1}(n-1)!}{(2+x)^n}$$

$$= \frac{(-1)^{2n-1}(n-1)!}{(2-x)^n} + \frac{(-1)^{n-1}(n-1)!}{(2+x)^n}$$

$$\boxed{D^n(\log(4-x^2)) = \frac{-(n-1)!}{(2-x)^n} + \frac{(-1)^{n-1}(n-1)!}{(2+x)^n}}$$

$(-1)^{2n-1}$, $(-1)^3$, $(-1)^1$, $(-1)^9 = -1$

Home Works:-

Find the n^{th} differential coefficient of:

(a) $\frac{x^4}{(x-1)(x-2)}$ (b) $\frac{1}{4x^2-1}$ (c) $\sin^3 x$ (d) $\sin^3 x\cos^5 x$

(e) $\sin x\cdot\sin 2x\cdot\sin 3x$ (f) $e^{4x}\sin^2 x$

① $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

② $\cos 2\theta = 1-2\sin^2\theta$

$\Rightarrow 2\sin^2\theta = 1-\cos 2\theta$

$\Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$

③ $\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

1) If $y^3-3ax^2+x^3=0$, Prove that $\frac{d^2y}{dx^2}+\frac{2a^2x^2}{y^5}=0$.

proof:- Given that

$$y^3-3ax^2+x^3=0 \rightarrow (1)$$

$y^3=3ax^2-x^3$

$$3y^2\frac{dy}{dx}-3a(2x)+3x^2=0$$

$d(x^2)=2x(1)=2x$

$$\Rightarrow 3y^2y_1-6ax+3x^2=0$$

$d(x^n)=nx^{n-1}(1)$

$d(y)=\frac{dy}{dx}$

$d(y^3)=3y^2\frac{dy}{dx}$

$$\Rightarrow 3y^2y_1=6ax-3x^2$$

$$\Rightarrow y_1=\frac{6ax-3x^2}{3y^2}$$

$$\Rightarrow y_1=\frac{\cancel{3}(2ax-x^2)}{\cancel{3}y^2}$$

$$\Rightarrow y_1=\frac{2ax-x^2}{y^2} \rightarrow (2)$$

$$\boxed{d\left(\frac{u}{v}\right)=\frac{v\,du-u\,dv}{v^2}}$$

Diff. eqn (2), w.r.to 'x', we get

$$y_2=\frac{y^2(2a-2x)-(2ax-x^2)(2yy_1)}{(y^2)^2}$$

$$y_2=\frac{2y^2(a-x)-2yy_1(2ax-x^2)}{y^4}$$

$$=\frac{2\cancel{y}[y(a-x)-y_1(2ax-x^2)]}{y^{\cancel{4}3}}$$

$$=\frac{2}{y^3}\left[y(a-x)-\frac{2ax-x^2}{y^2}(2ax-x^2)\right] \text{ (by eqn (2))}$$

$$=\frac{2}{y^3}\left[\frac{y^3(a-x)-(2ax-x^2)^2}{y^2}\right]$$

$$=\frac{2}{y^5}\left[y^3(a-x)-(4a^2x^2-4ax^3+x^4)\right]$$

$$=\frac{2}{y^5}\left[(3ax^2-x^3)(a-x)-4a^2x^2+4ax^3-x^4\right] \text{ (by eqn (1))}$$

$$=\frac{2}{y^5}\left[3a^2x^2-\cancel{x^3a}-\cancel{3a}x^3+\cancel{x^4}-4a^2x^2+\cancel{4ax^3}-\cancel{x^4}\right]$$

$$y_2=\frac{2}{y^5}\left[3a^2x^2-4a^2x^2\right]$$

$\therefore y_3 = \frac{2}{y^5}(-a^2x^2)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2a^2x^2}{y^5}$

i.e., $\boxed{\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0}$

Hence the proof.

———X———

2) Find $\frac{d^2y}{dx^2}$, if $x = a(t-\sin t)$ and $y = a(1+\cos t)$.

Proof:-

Given that

$x = a(t - \sin t)$

Diff. w.r.to 't', we get

$\frac{dx}{dt} = a(1-\cos t)$

$\frac{dx}{dt} = a(1-\cos t)$ $\rightarrow$ ①

$y = a(1+\cos t)$

Diff. w.r.to 't', we get

$\frac{dy}{dt} = a(-\sin t)$

$\frac{dy}{dt} = -a\sin t$ $\rightarrow$ ②

$d(\sin\theta) = \cos\theta$

$d(\cos\theta) = -\sin\theta$

$\frac{dy}{dt} \times \frac{dt}{dx}$

i.e., $\frac{dy/dt}{dx/dt} = \frac{-a\sin t}{a(1-\cos t)}$

$\Rightarrow \frac{dy}{dx} = \frac{-\sin t}{1-\cos t}$

$y_1 = \frac{-2\sin t/2 \cos t/2}{2\sin^2 t/2}$

$y_1 = -\frac{\cos t/2}{\sin t/2}$

$\therefore y_1 = -\cot t/2 \rightarrow$ ③

$\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow \sin\theta = 2\sin\theta/2 \cos\theta/2$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$2\sin^2\theta = 1-\cos 2\theta$

$\Rightarrow 2\sin^2\theta/2 = 1-\cos\theta$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$

$d(\cot\theta) = -\csc^2\theta$

Diff. this eqn. w.r.to 'x', we get

$y_2 = +\csc^2 t/2 \left(\frac{1}{2}\frac{dt}{dx}\right)$

$$y_2 = \operatorname{Cosec}^2 t/2 \; \frac{1}{2}\left(\frac{1}{a(1-\cos t)}\right) \quad \text{(by eqn. 1)}$$

$$= \frac{1}{2a} \operatorname{cosec}^2 t/2 \; \frac{1}{2\sin^2 t/2}$$

$$= \frac{1}{4a} \operatorname{cosec}^2 t/2 \; \frac{1}{\sin^2 t/2}$$

$$= \frac{1}{4a} \; \frac{1}{\sin^2 t/2} \; \frac{1}{\sin^2 t/2}$$

$$y_2 = \frac{1}{4a} \; \frac{1}{\sin^4 t/2}$$

$$\therefore \boxed{\frac{d^2y}{dx^2} = \frac{1}{4a\sin^2 t/2}}$$

$\dfrac{dt}{dx}$

=====X=====

3) If $x = \sqrt{\sin 2t}$ and $y = \sqrt{\cos 2t}$, then find $\dfrac{d^2y}{dx^2}$.

Soln.:- Given that

$\frac{d}{d\theta}(\sin\theta) = \cos\theta\,(1)$, $\quad d(x^n) = nx^{n-1}(1)$, $\quad \frac{1}{2}-1 = -\frac{1}{2}$

$$x = \sqrt{\sin 2t} = (\sin 2t)^{1/2}$$

$$\frac{dx}{dt} = \frac{1}{2}(\sin 2t)^{-1/2}(\cos 2t)(2)$$

$$\frac{dx}{dt} = (\sin 2t)^{-1/2}\cos 2t$$

$$\frac{dx}{dt} = \frac{\cos 2t}{(\sin 2t)^{1/2}} \rightarrow (1)$$

$$y = \sqrt{\cos 2t} = (\cos 2t)^{1/2}$$

$$\frac{dy}{dt} = \frac{1}{2}(\cos 2t)^{-1/2}(-\sin 2t)(2)$$

$$= \frac{-\sin 2t}{(\cos 2t)^{1/2}} \rightarrow (2)$$

$\frac{d}{d\theta}(\sin 2\theta) = \cos 2\theta \cdot 2 = 2\cos 2\theta$

$$\frac{dy/dt}{dx/dt} = \frac{\dfrac{-\sin 2t}{(\cos 2t)^{1/2}}}{\dfrac{\cos 2t}{(\sin 2t)^{1/2}}}$$

$x^a \cdot x^b = x^{a+b}$

$$\therefore \frac{dy}{dx} = \frac{-\sin 2t}{(\cos 2t)^{1/2}} \times \frac{(\sin 2t)^{1/2}}{\cos 2t}$$

$1 + \frac{1}{2} = \frac{3}{2}$

$$y_1 = \frac{-(\sin 2t)^{3/2}}{(\cos 2t)^{3/2}} = -(\tan 2t)^{3/2} \rightarrow (3)$$

Diff. eqn. ③, w.r.to 'x', we get

$$y_2 = -\left[\frac{3}{2}(\tan 2t)^{\frac{3}{2}-1}(\sec^2 2t)\left(2\frac{dt}{dx}\right)\right] \quad \frac{d}{dt}(t)=1 \quad \frac{dt}{dx}(t)=\frac{dt}{dx} \quad d(\tan\theta)=s$$

$$= -\left[3(\tan 2t)^{\frac{1}{2}}(\sec^2 2t)\frac{(\sin 2t)^{\frac{1}{2}}}{\cos 2t}\right] \text{ (by ①}$$

$$= -\left[3\frac{(\sin 2t)^{\frac{1}{2}}}{(\cos 2t)^{\frac{1}{2}}}\frac{1}{\cos^2 2t}\frac{(\sin 2t)^{\frac{1}{2}}}{\cos 2t}\right]$$

$$= -\left\{3\frac{(\sin 2t)^{\frac{1}{2}}}{(\cos 2t)^{\frac{1}{2}}}\frac{1}{(\cos 2t)^2}\frac{(\sin 2t)^{\frac{1}{2}}}{(\cos 2t)^1}\right\}$$

$$y_2 = -3\frac{\sin 2t}{(\cos 2t)^{\frac{7}{2}}}$$

$$\frac{1}{2}+2+1=\frac{1+4+2}{2}$$

$$\therefore \boxed{\frac{d^2y}{dx^2} = \frac{-3\sin 2t}{(\cos 2t)^{\frac{7}{2}}}}$$

$$\frac{1}{2}+\frac{1}{2}=\frac{2}{2}=1$$

4. If $x^3+y^3-3axy=0$, Prove that $D^2y = \dfrac{2a^3xy}{(ax-y^2)^3}$.

proof:-

Given that $\frac{d^2y}{dx^2}$

$$x^3+y^3-3axy=0 \rightarrow ①$$

$3x^2(1)$

Diff. eqn. ① w.r.to 'x', we get $\quad d(uv)=udv+vdu$

$$3x^2+3y^2\frac{dy}{dx}-3a\left(x\frac{dy}{dx}+y(1)\right)=0$$

$$\Rightarrow 3x^2+3y^2y_1-3a(xy_1+y)=0$$

$$\Rightarrow 3x^2+3y^2y_1-3axy_1-3ay=0 \quad (3\div)$$

$$\Rightarrow x^2+y^2y_1-axy_1-ay=0$$

$$\Rightarrow x^2+y_1(y^2-ax)-ay=0$$

~~$y_1(y^2-ax) = ay-x^2$~~

$$\Rightarrow y_1(y^2-ax) = ay-x^2$$

$$\Rightarrow y_1 = \frac{ay-x^2}{y^2-ax} = \frac{\cancel{-}(x^2-ay)}{\cancel{-}(ax-y^2)}$$

$$\therefore \boxed{y_1 = \frac{x^2-ay}{ax-y^2}} \rightarrow (2)$$

Again diff. eqn (2) w.r.to 'x', we get

$$y_2 = \frac{(ax-y^2)(2x-ay_1)-(x^2-ay)(a-2yy_1)}{(ax-y^2)^2}$$

$$\because d\left(\frac{u}{v}\right) = \frac{vdu-udv}{v^2}$$

$$\Rightarrow y_2 = \frac{1}{(ax-y^2)^2}\left[(ax-y^2)\left(2x-a\frac{x^2-ay}{ax-y^2}\right)-(x^2-ay)\left(a-2y\frac{x^2-ay}{ax-y^2}\right)\right]$$

$$\Rightarrow y_2 = \frac{1}{(ax-y^2)^2}\left[(ax-y^2)\left(\frac{2x(ax-y^2)-a(x^2-ay)}{ax-y^2}\right)-(x^2-ay)\left(\frac{a(ax-y^2)-2y(x^2-ay)}{(ax-y^2)}\right)\right]$$

$\because$ by eqn (2)

$$= \frac{1}{(ax-y^2)^3}\left[(ax-y^2)(2ax^2-2xy^2-ax^2+a^2y) - (x^2-ay)(a^2x-ay^2-2x^2y+2ay^2)\right]$$

$$= \frac{1}{(ax-y^2)^3}\left[2a^2x^3-2ax^2y^2-a^2x^3+a^3xy-2ax^2y^2+2xy^4+a^2x^2y^2-a^2y^3 - (a^2x^3-ax^2y^2-2x^4y+2ax^2y^2-a^3xy+a^2y^3+2ax^2y^2-2a^2y^3)\right]$$

$$= \frac{1}{(ax-y^2)^3}\left[2a^2x^3-2ax^2y^2-a^2x^3+a^3xy-2ax^2y^2+2xy^4+a^2x^2y^2-a^2y^3 -a^2x^3+ax^2y^2+2x^4y-2ax^2y^2+a^3xy-a^2y^3-2ax^2y^2+2a^2y^3\right]$$

$$= \frac{1}{(ax-y^2)^3}(-6ax^2y^2+2a^3xy+2xy^4+2x^4y)$$

$\because$ by (1)

$$= \frac{1}{(ax-y^2)^3}(2a^3xy+2xy(y^3+x^3-3axy)) = \frac{1}{(ax-y^2)^3}(2a^3xy+2xy(0))$$

$$\therefore \boxed{D^2y = \frac{2a^3xy}{(ax-y^2)^3}}$$

Hence the proof.

Leibnitz formula for the n^{th} derivative of a product

Leibnitz Theorem: (statement only proof not needed) (2 marks)

If u and v are functions of x, and 'n' is positive real number, then

$$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \cdots + nc_{r-1} u_{n-r+1} v_{r-1} + nc_r u_{n-r} v_r + \cdots + u v_n .$$

Problems:-

Find the n^{th} differential coefficient of

1) $x e^x$ (H.W), 2) $x^2 e^{3x}$, 3) $x \sin x$, 4) $x^2 \cos x$, 5) $e^x \log x$, 6) $x^3 \sin^3 x$ 7) $x^n a^x$ 8) $x^2 \sin 3x$.

2) $x^2 e^{3x}$

$d(e^x) = e^x$, $d^2(e^x) = e^x$, $d(e^{ax}) = ae^{ax} = a.ae^{ax}$, $a^n e^{ax}$

Leibnitz theorem

$$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \cdots + u v_n .$$

$$\therefore D^n(e^{3x} x^2) = D^n(e^{3x})\, x^2 + nc_1 D^{n-1}(e^{3x})\, D(x^2) + nc_2 D^{n-2}(e^{3x})\, D^2(x^2) + \ldots$$

$$= 3^n e^{3x} x^2 + \frac{n}{1!} 3^{n-1} e^{3x} (2x) + \frac{n(n-1)}{2!} 3^{n-2} e^{3x} (2)$$

$$= 3^n e^{3x} x^2 + n\, 3^{n-1} e^{3x} (2x) + \frac{n(n-1)}{2} 3^{n-2} e^{3x} (2)$$

$$= 3^n e^{3x} x^2 + 2n\, 3^{n-1} e^{3x} . x + n(n-1)\, 3^{n-2} e^{3x}$$

$$\cancel{= 3^{n-2} e^{3x} [3^2 x^2 + 2n.3.x + n(n-1)]}$$

$$\cancel{= 3^{n-2} e^{3x} [9x^2 + 6nx + n(n-1)]}$$

$$\boxed{D^n(e^{3x} x^2) = e^{3x}\left[3^n x^2 + 3^{n-1}\, 2nx + 3^{n-2}\, n(n-1)\right]}$$

4) $x^2 \cos x$

Leibnitz theorem

$$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \cdots + u v_n$$

$$\therefore D^n(\cos x \cdot x^2) = D^n(\cos x)\cdot x^2 + nc_1 D^{n-1}(\cos x)(2x) + nc_2 D^{n-2}(\cos x)(2)$$

$$= 1^n \cos\left(\frac{n\pi}{2}+x\right)\cdot x^2 + \frac{n}{1!} 1^{n-1} \cos\left(\frac{(n-1)\pi}{2}+x\right)(2x)$$

$$+ \frac{n(n-1)}{2!} 1^{n-2} \cos\left(\frac{(n-2)\pi}{2}+x\right)\cdot 2$$

$$= \cos\left(\frac{n\pi}{2}+x\right)x^2 + n\cos\left(\frac{(n-1)\pi}{2}+x\right)2x + n(n-1)\cos\left(\frac{(n-2)\pi}{2}+x\right)$$

$$\boxed{\therefore D^n(\cos x \cdot x^2) = x^2\cos\left(\frac{n\pi}{2}+x\right) + 2nx\cos\left(\frac{(n-1)\pi}{2}+x\right) + n(n-1)\cos\left(\frac{(n-2)\pi}{2}+x\right)}$$

5) $e^x \log x$

$$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \cdots + u v_n$$

$$\therefore D^n(e^x \log x) = D^n(e^x)\log x + nc_1 D^{n-1}(e^x) D(\log x) + \cdots + e^x D^n(\log x)$$

$$= 1^n e^x \log x + \frac{n}{1!} 1^{n-1} e^x \frac{1}{x} + \cdots + e^x \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$= e^x \log x + n\frac{e^x}{x} + \cdots + e^x \frac{(n-1)!\,(-1)^{n-1}}{x^n}$$

$D^n(\log(ax+b)) = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$

$$\boxed{\therefore D^n(e^x \log x) = e^x\left(\log x + \frac{n}{x} + \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n}\right)}$$

6) $x^3 \sin^3 x$

$$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \cdots + u v_n$$

$$\therefore D^n(\sin^3 x \cdot x^3) = D^n\left[\left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right)\cdot x^3\right]$$

$\because \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

$$= D^n\left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right)\cdot x^3 + nc_1 D^{n-1}\left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right)(3x^2)$$

$$+ nc_2 D^{n-2}\left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right)(6x) + nc_3 D^{n-3}\left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right)\cdot(6)$$

$D^n[\sin^3 x \cdot x^3]$

$$= \left[\frac{3}{4} 1^n \sin\left(\frac{n\pi}{2}+x\right) - \frac{1}{4} 3^n \sin\left(\frac{n\pi}{2}+3x\right)\right] \cdot x^3$$

$D^n(\sin(ax+b)) = a^n \sin(\frac{n\pi}{2}+ax+b)$

$$+ \frac{n}{1!}\left[\frac{3}{4} 1^{n-1} \sin\left(\frac{(n-1)\pi}{2}+x\right) - \frac{1}{4} 3^{n-1} \sin\left(\frac{(n-1)\pi}{2}+3x\right)\right] \cdot (3x^2)$$

$$+ \frac{n(n-1)}{2!}\left[\frac{3}{4} 1^{n-2} \sin\left(\frac{(n-2)\pi}{2}+x\right) - \frac{1}{4} 3^{n-2} \sin\left(\frac{(n-2)\pi}{2}+3x\right)\right] (6x)$$

$$+ \frac{n(n-1)(n-2)}{3!}\left[\frac{3}{4} 1^{n-3} \sin\left(\frac{(n-3)\pi}{2}+x\right) - \frac{1}{4} 3^{n-3} \sin\left(\frac{(n-3)\pi}{2}+3x\right)\right] (6)$$

$$= \left[\frac{3}{4} \sin\left(\frac{n\pi}{2}+x\right) - \frac{1}{4} 3^n \sin\left(\frac{n\pi}{2}+3x\right)\right] \cdot x^3$$

$$+ 3nx^2\left[\frac{3}{4} \sin\left(\frac{(n-1)\pi}{2}+x\right) - \frac{3^{n-1}}{4} \sin\left(\frac{(n-1)\pi}{2}+3x\right)\right]$$

$$+ \frac{n(n-1)}{\cancel{2}}(\cancel{6}^{3}x)\left[\frac{3}{4} \sin\left(\frac{(n-2)\pi}{2}+x\right) - \frac{3^{n-2}}{4} \sin\left(\frac{(n-2)\pi}{2}+3x\right)\right]$$

$3 \times 2 \times 1$

$$+ \frac{n(n-1)(n-2)}{\cancel{6}} (\cancel{6})\left[\frac{3}{4} \sin\left(\frac{(n-3)\pi}{2}+x\right) - \frac{3^{n-3}}{4} \sin\left(\frac{(n-3)\pi}{2}+3x\right)\right]$$

$$\therefore D^n[\sin^3 x \cdot x^3] = x^3\left[\frac{3}{4} \sin\left(\frac{n\pi}{2}+x\right) - \frac{3^n}{4} \sin\left(\frac{n\pi}{2}+3x\right)\right]$$
$$+ 3nx^2\left[\frac{3}{4} \sin\left(\frac{(n-1)\pi}{2}+x\right) - \frac{3^{n-1}}{4} \sin\left(\frac{(n-1)\pi}{2}+3x\right)\right]$$
$$+ 3n(n-1)x\left[\frac{3}{4} \sin\left(\frac{(n-2)\pi}{2}+x\right) - \frac{3^{n-2}}{4} \sin\left(\frac{(n-2)\pi}{2}+3x\right)\right]$$
$$+ n(n-1)(n-2)\left[\frac{3}{4} \sin\left(\frac{(n-3)\pi}{2}+x\right) - \frac{3^{n-3}}{4} \sin\left(\frac{(n-3)\pi}{2}+3x\right)\right]$$

—x—

7) $x^n a^x$

$$D^n(uv) = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots + u v_n.$$

$$\therefore D^n(x^n a^x) = D^n(x^n) \cdot a^x + {}^nC_1 D^{n-1}(x^{n-1}) D(a^x) + {}^nC_2 D^{n-2}(x^{n-2}) D^2(a^x)$$
$$+ \dots + x^n D^n(a^x)$$

$d(x^n) = nx^{n-1}$

$d^2(x^n) = n(n-1)x^{n-2}$

$d^3(x^n) = n(n-1)(n-2)x^{n-3}$

$\vdots$

$d^{n-1}(x^n) = n(n-1)(n-2)\dots 3 \cdot 2 \cdot x^1$

$d^n(x^n) = n(n-1)(n-2)\dots 3.2.1 = n!$

$d(a^x) = a^x \log a$

$x^{n-(n-1)}$

x^{n-n+1}

x^{n-n}

x^0

$\therefore D^n(x^n \cdot a^x)$

$= n!\,a^x + \frac{n}{1!}\, n(n-1)(n-2)\cdots 3\cdot 2\cdot x\, a^x \log a + \frac{n(n-1)}{2!}\, n(n-1)(n-2)\cdots 4\cdot 3\cdot x^2 \cdot a^x (\log a)^2 + \cdots + x^n a^x (\log a)^n$

$= n!\,a^x + n\frac{n!}{1!}\, x a^x \log a + \frac{n(n-1)}{2!}\, n(n-1)(n-2)\cdots 4\cdot 3\cdot 2\cdot \frac{1}{2} x^2\, a^x (\log a)^2 + \cdots + x^n a^x (\log a)^n.$

$d(a^x) = a^x \log a$

$d^2(a^x) = \log a\,(a^x \log a) = (\log a)^2 a^x$

$d^3(a^x) = (\log a)^3 a^x$

$d^n(a^x) = (\log a)^n a^x$

$= n!\,a^x + n\frac{n!}{1!}\, x a^x \log a + \frac{n(n-1)}{2}\,\frac{n!}{2!}\, x^2 a^x (\log a)^2 + \cdots + x^n a^x \frac{n!}{n!} (\log a)^n$

$\therefore D^n(x^n a^x)$

$= n!\,a^x \left[1 + n\,\frac{x \log a}{1!} + \frac{n(n-1)}{2}\,\frac{x^2 (\log a)^2}{2!} + \cdots + \frac{x^n (\log a)^n}{n!}\right]$

1) Find the n^{th} differential coefficient of $x^2 \log x$.

Soln:-

Take $u = \log x$ and $v = x^2$

$d(a^x) = a^x \log a$

By Leibnitz theorem,

$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \cdots + u v_n$

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$\therefore D^n[\log x \cdot x^2] = D^n(\log x)(x^2) + nc_1 D^{n-1}(\log x)(2x) + nc_2 D^{n-2}(\log x)(2)$

$= \frac{(-1)^{n-1}(n-1)!\,1^n}{x^n}(x^2) + \frac{n}{1!}\,\frac{(-1)^{n-2}(n-2)!\,1^{n-1}}{x^{n-1}}(2x) + \frac{n(n-1)}{2!}\,\frac{(-1)^{n-3}(n-3)!\,1^{n-2}}{x^{n-2}}(2)$

$= \frac{(-1)^{n-1}(n-1)!}{x^{n-2}} + 2n\,\frac{(-1)^{n-2}(n-2)!}{x^{n-2}} + n(n-1)\,\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}$

$= \frac{1}{x^{n-2}}\left[(-1)^{n-1}(n-1)(n-2)(n-3)! + 2n(-1)^{n-2}(n-2)(n-3)! + n(n-1)(n-3)!\,(-1)^{n-3}\right]$

$= \frac{(n-3)!}{x^{n-2}}\left[(-1)^{n-1}(-1)^{2-2}(n-1)(n-2) + 2n(-1)^{n-2}(-1)^{1-1}(n-2) + n(n-1)(-1)^{n-3}\right]$

$$= \frac{(n-3)!}{x^{n-2}} \Big[(-1)^{n-3} (-1)^2 (n-1)(n-2) + 2n (-1)^{n-3} (-1)^1 (n-2) + n(n-1)(-1)^{n-3} \Big]$$

$$= \frac{(n-3)!\,(-1)^{n-3}}{x^{n-2}} \left[(n-1)(n-2) - 2n(n-2) + n(n-1) \right]$$

$$= \frac{(n-3)!\,(-1)^{n-3}}{x^{n-2}} \left[n^2 - 2n - n + 2 - 2n^2 + 4n + n^2 - n \right]$$

$$= \frac{(n-3)!\,(-1)^{n-3}}{x^{n-2}} [2]$$

$$\therefore \boxed{D^n[\log x \cdot x^2] = \frac{2(n-3)!\,(-1)^{n-3}}{x^{n-2}}}$$

2) If $y = \sin(m \sin^{-1} x)$, Prove that (i) $(1-x^2)y_2 - xy_1 + m^2 y = 0$ and (ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$.

Proof:- Given that

$$y = \sin(m \sin^{-1} x)$$

$$\Rightarrow \sin^{-1} y = m \sin^{-1} x$$

Diff. this eqn. w.r.to 'x', we get

$$\Rightarrow \frac{1}{\sqrt{1-y^2}}\, y_1 = m \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\,.\, y_1 = m\sqrt{1-y^2}$$

Squaring on both sides, we get

$$(1-x^2)\, y_1^2 = m^2 (1-y^2)$$

Diff. this equation w.r.to 'x', we get

$$(1-x^2)\, 2y_1 y_2 + (-2x)\, y_1^2 = m^2(-2yy_1)$$

$$\cancel{2y_1}\left[(1-x^2)y_2 - xy_1\right] = -m^2 y\, (\cancel{2y_1})$$

$$\Rightarrow \boxed{(1-x^2)y_2 - xy_1 + m^2 y = 0}$$

Hence (i) is proved

Using the Leibnitz theorem in (i), we get

$$D^n[y_2(1-x^2)] - D^n[y_1 x] + m^2 D^n(y) = 0$$

$$D^n(y_2)(1-x^2) + nc_1 D^{n-1}(y_2)(-2x) + nc_2 D^{n-2}(y_2)(-2)$$
$$-\left(D^n(y_1)\,x + nc_1 D^{n-1}(y_1)(1)\right) + m^2 D^n(y) = 0$$

$$y_{n+2}(1-x^2) + \frac{n}{1!} y_{n+1}(-2x) + \frac{n(n-1)}{\cancel{2}!} y_n(-\cancel{2})$$
$$-\left(y_{n+1}x + \frac{n}{1!} y_n\right) + m^2 y_n = 0$$

$$(1-x^2)y_{n+2} - 2nx\,y_{n+1} - n(n-1)y_n - x y_{n+1} - n y_n + m^2 y_n = 0$$

$$(1-x^2)y_{n+2} - x y_{n+1}(2n+1) + y_n(-n(n-1) - n + m^2) = 0$$

$$(1-x^2)y_{n+2} - (2n+1)x y_{n+1} + y_n(-n^2 + \cancel{n} - \cancel{n} + m^2) = 0$$

$$\boxed{(1-x^2)y_{n+2} - (2n+1)x y_{n+1} + (m^2 - n^2)y_n = 0}$$

Hence (ii) is proved

3) If $y = \sin^{-1}x$, Prove that $(1-x^2)y_2 - xy_1 = 0$ and $(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - n^2 y_n = 0$.

proof:- $y = \sin^{-1}x$

Diff. w.r.to 'x', we get

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

$\Rightarrow \sqrt{1-x^2}\,.\,y_1 = 1$ squaring on both sides, we get

$\Rightarrow (1-x^2)y_1^2 = 1$

Again diff. w.r.to 'x', we get

$$(1-x^2)2y_1y_2 + (-2x)y_1^2 = 0$$

$$2y_1[(1-x^2)y_2 - xy_1] = 0$$

$\therefore \boxed{(1-x^2)y_2 - xy_1 = 0}$ Hence the proof.

Using Leibnitz theorem, we get

$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \cdots + u v_n$

$\therefore D^n[(1-x^2)y_2] - D^n[xy_1] = 0$

$\Rightarrow D^n[y_2](1-x^2) + nc_1 D^{n-1}(y_2)(-2x) + nc_2 D^{n-2}(y_2)(-2)$

$- [D^n(y_1)(x) + nc_1 D^{n-1}(y_1)(1)] = 0$

$\Rightarrow y_{n+2}(1-x^2) + \frac{n}{1!} y_{n+1}(-2x) + \frac{n(n-1)}{2!} y_n(-2) - (y_{n+1} x + \frac{n}{1!} y_n) = 0$

$\Rightarrow (1-x^2) y_{n+2} + n y_{n+1}(-2x) - n(n-1) y_n - x y_{n+1} - n y_n = 0$

$\Rightarrow (1-x^2) y_{n+2} - x y_{n+1}(2n+1) - y_n(n(n-1)+n) = 0$

$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - y_n(n^2 - \cancel{n} + \cancel{n}) = 0$

$\therefore \boxed{(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0}$

Hence the proof.

4) If $y = e^{a\sin^{-1}x}$, Prove that $(1-x^2)y_2 - xy_1 - a^2 y = 0$

Hence show that $(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 + a^2) y_n = 0$.

proof:-

$y = e^{a\sin^{-1}x} \rightarrow (1)$

Diff. w.r.to 'x', we get

$y_1 = e^{a\sin^{-1}x}\left(a\frac{1}{\sqrt{1-x^2}}\right)$

$\sqrt{1-x^2} \cdot y_1 = a e^{a\sin^{-1}x}$

$\therefore \sqrt{1-x^2} \cdot y_1 = ay$ (by eqn: (1)

Squaring on both sides, weget

$(1-x^2) y_1^2 = a^2 y^2$

Again diff. w.r.to 'x', we get

$$(1-x^2)\, 2y_1 y_2 + (-2x)\, y_1^2 = a^2\,(2yy_1)$$

$$2y_1\left[(1-x^2)\, y_2 - y_1 x\right] = a^2 y\,(2y_1)$$

$$\boxed{(1-x^2)\, y_2 - x y_1 - a^2 y = 0}$$

Hence the proof

(ii) Using Leibnitz theorem in this eqn; we get

$$D^n(uv) = u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \cdots + u v_n.$$

$$\Rightarrow D^n\left[(1-x^2)\, y_2\right] - D^n\left[x y_1\right] - a^2 D^n(y) = 0$$

$$\Rightarrow \left[D^n(y_2)(1-x^2) + {}^nC_1\, D^{n-1}(y_2)(-2x) + {}^nC_2\, D^{n-2}(y_2)(-2)\right]$$
$$- \left[D^n(y_1)(x) + {}^nC_1\, D^{n-1}(y_1)(1)\right] - a^2 D^n(y) = 0$$

$$\Rightarrow \left[y_{n+2}(1-x^2) + \frac{n}{1!}\, y_{n+1}(-2x) + \frac{n(n-1)}{\cancel{2}!}\, y_n(-\cancel{2})\right]$$
$$- \left[y_{n+1}\cdot x + \frac{n}{1!}\, y_n\right] - a^2 y_n = 0$$

$$\Rightarrow (1-x^2)\, y_{n+2} - 2nx\, y_{n+1} - n(n-1)\, y_n - x y_{n+1} - n y_n - a^2 y_n = 0$$

$$\Rightarrow (1-x^2)\, y_{n+2} - x y_{n+1}(2n+1) - y_n\left(n(n-1) + n + a^2\right) = 0$$

$$\Rightarrow (1-x^2)\, y_{n+2} - (2n+1)\, y_{n+1} - (n^2 - \cancel{n} + \cancel{n} + a^2)\, y_n = 0$$

$$\therefore \boxed{(1-x^2)\, y_{n+2} - (2n+1)\, y_{n+1} - (n^2 + a^2)\, y_n = 0}$$

Hence the proof

5) If $y = a\cos(\log x) + b\sin(\log x)$, show that $x^2 y_{n+2} + (2n+1)\, x y_{n+1} + (n^2+1)\, y_n = 0$.

Proof:- Given that

$$y = a\cos(\log x) + b\sin(\log x) \rightarrow ①$$

Diff. eqn. ① w.r.to 'x', we get

$\Rightarrow y_1 = a\left(-\sin(\log x)\right)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$

$\Rightarrow y_1 = \frac{1}{x}\left[-a\sin(\log x) + b\cos(\log x)\right] \rightarrow ②$

Diff. w.r.to 'x', we get

$\Rightarrow y_2 = \frac{1}{x}\left[-a\cos(\log x)\frac{1}{x} + b\left(-\sin(\log x)\right)\cdot\frac{1}{x}\right]$
$+\left[-a\sin(\log x) + b\cos(\log x)\right]\left(-\frac{1}{x^2}\right)$

$\Rightarrow y_2 = -\frac{1}{x^2}\left[a\cos(\log x) + b\sin(\log x)\right] - \frac{1}{x^2}\left[-a\sin(\log x) + b\cos(\log x)\right]$

$\Rightarrow y_2 = -\frac{1}{x^2}y - \frac{1}{x}y_1$ $\quad$ $\because$ by eqns. ① & ②

$\Rightarrow y_2 = \frac{-y - xy_1}{x^2}$

$\Rightarrow x^2y_2 = -y - xy_1$

$\Rightarrow \boxed{x^2y_2 + xy_1 + y = 0}$

Using Leibnitz theorem in this eqn., we get

$D^n[y_2 x^2] + D^n[y_1 x] + D^n(y) = 0$

$\Rightarrow D^n(y_2)\cdot x^2 + nC_1 D^{n-1}(y_2)(2x) + nC_2 D^{n-2}(y_2)(2)$
$+\left[D^n(y_1)\cdot x + nC_1 D^{n-1}(y_1)(1)\right] + D^n(y) = 0$

$\Rightarrow y_{n+2}x^2 + \frac{n}{1!}y_{n+1}2x + \frac{n(n-1)}{\cancel{2}!}y_n(\cancel{2}) + y_{n+1}x + ny_n + y_n = 0$

$\Rightarrow y_{n+2}x^2 + 2ny_{n+1}x + n(n-1)y_n + xy_{n+1} + ny_n + y_n = 0$

$\Rightarrow x^2y_{n+2} + xy_{n+1}(2n+1) + [n(n-1) + n + 1]y_n = 0$

$\Rightarrow x^2y_{n+2} + xy_{n+1}(2n+1) + (n^2 - \cancel{n} + \cancel{n} + 1)y_n = 0$

$\therefore \boxed{x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0}$

Hence the proof.

$=\!=\!\times\!=\!=$

6. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, prove that

$x^2 y_{n+2} + (2n+1) x y_{n+1} + 2n^2 y_n = 0$.

proof:- Given that

$$\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n \rightarrow ①$$

$$d\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$$

$$\log x^a = a \log x$$

Diff. eqn. ①, w.r.to 'x', we get

$$\Rightarrow \frac{-1}{\sqrt{1-(y/b)^2}}\left(\frac{y_1}{b}\right) = d\left[n \log\left(\frac{x}{n}\right)\right]$$

$$\Rightarrow \frac{-y_1}{b\sqrt{\frac{b^2-y^2}{b^2}}} = n \frac{1}{x/n} \frac{1}{n}$$

$$\Rightarrow \frac{-y_1}{\frac{\cancel{b}}{\cancel{b}}\sqrt{b^2-y^2}} = n \frac{\cancel{n}}{x} \frac{1}{\cancel{n}} \Rightarrow \frac{-y_1}{\sqrt{b^2-y^2}} = \frac{n}{x}$$

$$\Rightarrow -x y_1 = n\sqrt{b^2-y^2}$$

Squaring on both sides, we get

$$\Rightarrow x^2 y_1^2 = n^2(b^2-y^2)$$

$$\Rightarrow x^2 y_1^2 = n^2 b^2 - n^2 y^2$$

Diff. w.r.to 'x', we get

$$x^2(2y_1 y_2) + 2x \cdot y_1^2 = 0 - n^2(2y y_1)$$

$$2y_1\left[x^2 y_2 + x y_1\right] = 2y_1(-n^2 y)$$

$$\Rightarrow \boxed{x^2 y_2 + x y_1 + n^2 y = 0}$$

Using Leibnitz theorem, we get

$$D^n(uv) = u_n v + nC_1 u_{n-1} v_1 + nC_2 u_{n-2} v_2 + \cdots + u v_n$$

$$\therefore D^n\left[y_2 x^2\right] + D^n\left[y_1 x\right] + n^2 D^n(y) = 0$$

$$\left[D^n(y_2) x^2 + nC_1 D^{n-1}(y_2)(2x) + nC_2 D^{n-2}(y_2)(2)\right]$$
$$+\left[D^n(y_1)(x) + nC_1 D^{n-1}(y_1)(1)\right] + n^2 D^n(y) = 0$$

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$$\left[y_{n+2}\,x^2 + \frac{n}{1!}\,y_{n+1}\,(2x) + \frac{n(n-1)}{2!}\,y_n\,(2)\right]$$

$$+\left[y_{n+1}\,x + \frac{n}{1!}\,y_n\right] + n^2 y_n = 0$$

$$x^2 y_{n+2} + 2nx\,y_{n+1} + n(n-1)y_n + x\,y_{n+1} + ny_n + n^2 y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + x\,y_{n+1}(2n+1) + y_n\left[n(n-1) + n + n^2\right] = 0$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)x\,y_{n+1} + y_n\left[n^2 - \cancel{n} + \cancel{n} + n^2\right] = 0$$

$$\Rightarrow \boxed{x^2 y_{n+2} + (2n+1)x\,y_{n+2} + 2n^2 y_n = 0}$$

Hence the proof.

7. If $y = (x+\sqrt{1+x^2})^m$, Prove that $(1+x^2)\,y_{n+2} + (2n+1)x\,y_{n+1} + (n^2-m^2)$

proof:-

Given that

$$y = (x+\sqrt{1+x^2})^m \longrightarrow (1)$$

$$d(\sqrt{x}) = \frac{1}{2\sqrt{x}} \quad (1)$$

$$d(\sqrt{1+x^2}) = \frac{1}{\cancel{2}\sqrt{1+x^2}} \quad (\cancel{2}x)$$

Diff. w.r.to 'x', we get

$$\Rightarrow y_1 = m\,(x+\sqrt{1+x^2})^{m-1}\left(1+\frac{1}{\sqrt{1+x^2}}\cdot x\right)$$

$$\Rightarrow y_1 = m\,(x+\sqrt{1+x^2})^{m-1}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \sqrt{1+x^2}\cdot y_1 = m\,(x+\sqrt{1+x^2})^{m-1}\,(x+\sqrt{1+x^2})^1$$

$$\Rightarrow \sqrt{1+x^2}\;y_1 = m\,(x+\sqrt{1+x^2})^m$$

$$\Rightarrow \sqrt{1+x^2}\;y_1 = my \qquad \text{(by eqn. (1))}$$

Squaring on both sides, we get

$$\Rightarrow (1+x^2)\,y_1^2 = m^2 y^2$$

$$\Rightarrow (1+x^2)\,y_1^2 - m^2 y^2 = 0$$

Diff. w.r.to 'x', we get

$$(1+x^2)\,2y_1 y_2 + (2x)y_1^2 - m^2(2y\,y_1) = 0 \quad (2y_1 \div)$$

$$\Rightarrow \boxed{(1+x^2)\,y_2 + x\,y_1 - m^2 y = 0}$$

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Leibnitz theorem,

$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + \cdots + u v_n$.

$\therefore D^n[y_2(1+x^2)] + D^n(y_1 x) - m^2 D^n(y) = 0$.

$$\left[D^n(y_2)(1+x^2) + nc_1 D^{n-1}(y_2)(2x) + nc_2 D^{n-2}(y_2)(2)\right]$$
$$+\left[D^n(y_1)(x) + nc_1 D^{n-1}(y_1)(1)\right] - m^2 D^n(y) = 0$$

$$y_{n+2}(1+x^2) + n\, y_{n+1}(2x) + \frac{n(n-1)}{\cancel{2}} y_n(\cancel{2}) + y_{n+1}x + n y_n - m^2 y_n = 0$$

$$(1+x^2)y_{n+2} + x y_{n+1}(2n+1) + y_n[n(n-1) + n - m^2] = 0$$

$$(1+x^2)y_{n+2} + x y_{n+1}(2n+1) + y_n[n^2 - \cancel{n} + \cancel{n} - m^2] = 0$$

$$\boxed{(1+x^2)y_{n+2} + (2n+1)x y_{n+1} + (n^2 - m^2)y_n = 0}$$

Hence the proof.

Curvature :-

Let P be a given point on a given curve and Q any other point on it. Let the normals at P and Q intersect in N. If N tend to a definite position C as Q tends to P, then C is called the centre of curvature of the curve at P.

The reciprocal of the distance CP is called the curvature of the curve at P.

The circle with its centre at C and radius CP is called the circle of curvature of the curve at P.

The distance CP is called the radius of curvature of the curve at P. The radius of curvature is usually denoted by the Greek letter 'ρ'.

Formula for the radius of curvature :-

Radius of curvature $\boxed{\rho = \dfrac{\partial s}{\partial \psi}}$

Examples :

1. Find 'ρ' for the catenary whose intrinsic equation is $s = a \tan \psi$.

Soln :- Given that $s = a \tan \psi$

$$\text{Radius } \rho = \frac{\partial s}{\partial \psi} = \frac{d}{d\psi}(a \tan \psi)$$

$$\boxed{\rho = a \sec^2 \psi}$$

Cartesian formula for Radius of Curvature:

$$\boxed{\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}}$$

Examples:-

1. Find the radius of curvature for the curve $\sqrt{x}+\sqrt{y}=1$ at $\left(\frac{1}{4},\frac{1}{4}\right)$.

Soln:-

Given that $\sqrt{x}+\sqrt{y}=1$

diff. this eqn. w.r.to 'x', we get

$$\Rightarrow \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{1}{2\sqrt{y}}\frac{dy}{dx}=-\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{\cancel{2}\sqrt{x}}\times\frac{\cancel{2}\sqrt{y}}{1}$$

$$\Rightarrow \boxed{\frac{dy}{dx}=-\sqrt{y/x}} \rightarrow (1)$$

$$\therefore \left(\frac{dy}{dx}\right)_{(1/4,1/4)} = -\sqrt{\frac{1/4}{1/4}}$$

$$=-\sqrt{1}$$

$$=-1 \rightarrow (2)$$

$$d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}$$

Diff. eqn. (1), w.r.to 'x', we get

$$\frac{d^2y}{dx^2}=-\left[\frac{\sqrt{x}\,\frac{1}{2\sqrt{y}}\frac{dy}{dx}-\sqrt{y}\,\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}\right]$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x}\left[+\frac{1}{2}\frac{\sqrt{x}}{\sqrt{y}}\frac{dy}{dx} - \frac{\sqrt{y}}{2\sqrt{x}}\right]$$

$$= -\frac{1}{x}\left[\frac{1}{2}\sqrt{\frac{x}{y}}\frac{dy}{dx} - \frac{1}{2}\sqrt{y/x}\right]$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{(1/4,1/4)} = -\frac{1}{1/4}\left\{\frac{1}{2}\sqrt{\frac{1/4}{1/4}}\left(\frac{dy}{dx}\right)_{(1/4,1/4)} - \frac{1}{2}\sqrt{\frac{1/4}{1/4}}\right\}$$

$$= -4\left[\frac{1}{2}\sqrt{1}\,(-1) - \frac{1}{2}\sqrt{1}\right]$$

$$= -4\left[\frac{1}{2}(1)(-1) - \frac{1}{2}(1)\right]$$

$$= -4\left[-\tfrac{1}{2} - \tfrac{1}{2}\right]$$

$$= -4(-1)$$

$$= 4 \rightarrow (3)$$

From (2) & (3), we get, The radius of curvature

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left(1+(-1)^2\right)^{3/2}}{4} = \frac{(1+1)^{3/2}}{4} = \frac{(2)^{3/2}}{4} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\therefore \boxed{\rho = \frac{1}{\sqrt{2}}}$$

—x—

2. Find the radius of curvature at (x, y) for the curve $a^2y = x^3 - a^3$.

Soln:- Given $a^2y = x^3 - a^3$

diff. w.r.to 'x', we get

$$a^2\frac{dy}{dx} = 3x^2 \Rightarrow \boxed{\frac{dy}{dx} = \frac{3x^2}{a^2}} \rightarrow (1)$$

Again diff. eqn. ① w.r.to 'x', we get

$$\frac{d^2y}{dx^2} = \frac{3}{a^2}(2x)$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{6x}{a^2}}$$

∴ Radius of curvature

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left\{1+\left(\frac{3x^2}{a^2}\right)^2\right\}^{3/2}}{\frac{6x}{a^2}} = \frac{\left\{1+\frac{9x^4}{a^4}\right\}^{3/2}}{6x/a^2}$$

$$= \frac{\left\{\frac{a^4+9x^4}{a^4}\right\}^{3/2}}{\frac{6x}{a^2}} = \frac{(a^4+9x^4)^{3/2}}{(a^4)^{3/2}} \times \frac{a^2}{6x}$$

$$= \frac{(a^4+9x^4)^{3/2}}{a^6} \times \frac{a^2}{6x}$$

$$\boxed{\rho = \frac{(a^4+9x^4)^{3/2}}{6a^4x}}$$

$x^{3/2} = x\sqrt{x}$

$(a^4)^{3/2} = \ldots = a^6$

---

3. Prove that at the point $x = \frac{\pi}{2}$ of the curve $y = 4\sin x - \sin 2x$, $\rho = \frac{5\sqrt{5}}{4}$.

Proof :- Given $y = 4\sin x - \sin 2x$

$$\frac{dy}{dx} = 4\cos x - 2\cos 2x \rightarrow ①$$

$$\frac{d^2y}{dx^2} = -4\sin x + 4\sin 2x \rightarrow ②$$

$d(\sin 2\theta) = 2\cos 2\theta$

$d(\cos 2\theta) = -2\sin 2\theta$

$$① \Rightarrow \left(\frac{dy}{dx}\right)_{x=\pi/2} = 4\cos\left(\frac{\pi}{2}\right) - 2\cos 2\left(\frac{\pi}{2}\right)$$

$$= 4(0) - 2(-1)$$

$$= 0 + 2$$

$$\boxed{\frac{dy}{dx} = 2}$$

$$② \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=\pi/2} = -4\sin\left(\frac{\pi}{2}\right) + 4\sin 2\left(\frac{\pi}{2}\right)$$

$$= -4(1) + 4(0)$$

$$\boxed{\frac{d^2y}{dx^2} = -4}$$

∴ Radius of Curvature

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left(1 + (2)^2\right)^{3/2}}{-4} = \frac{(1+4)^{3/2}}{-4} = \frac{(5)^{3/2}}{-4}$$

$$\rho = \frac{5\sqrt{5}}{-4}$$

Omitting -ve sign, we get

$$\boxed{\rho = \frac{5\sqrt{5}}{4}}$$

Hence the proof.

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Total differential Coefficients :-

Problems:-

1. Find $\frac{du}{dt}$, where $u = x^2+y^2+z^2$, $x = e^t$, $y = e^t \sin t$ & $z = e^t \cos t$.

Soln.:-

u, x, y, z given

1st formula

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}}$$

$\frac{\partial}{\partial x}$ means partial differentiation

$u = x^2+y^2+z^2 \Rightarrow$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$ and $\frac{\partial u}{\partial z} = 2z$

$x = e^t \Rightarrow \frac{dx}{dt} = e^t$

$y = e^t \sin t \Rightarrow \frac{dy}{dt} = e^t \cos t + \sin t\, e^t = e^t(\cos t + \sin t)$

$z = e^t \cos t \Rightarrow \frac{dz}{dt} = e^t(-\sin t) + e^t \cos t = e^t(\cos t - \sin t)$

$$\therefore \frac{du}{dt} = (2x)(e^t) + (2y)\left[e^t(\cos t+\sin t)\right] + (2z)\left[e^t(\cos t - \sin t)\right]$$

$$= \cancel{2xe^t + 2ye^t\cos t}$$

$$= 2xe^t + 2e^t y(\cos t+\sin t) + 2e^t z(\cos t - \sin t)$$

$$= 2e^t\left[x + y(\cos t+\sin t) + z(\cos t-\sin t)\right]$$

$$= 2e^t\left[e^t + e^t\sin t(\cos t+\sin t) + e^t\cos t(\cos t-\sin t)\right]$$

$$= 2e^t \cdot e^t\left[1 + \sin t(\cos t+\sin t) + \cos t(\cos t-\sin t)\right]$$

$$= 2e^{2t}\left[1 + \cancel{\sin t\cos t} + \underbrace{\sin^2 t + \cos^2 t}_{(1)} - \cancel{\sin t\cos t}\right]$$

$$= 2e^{2t}[1+1] = 2e^{2t}(2)$$

$$\therefore \boxed{\frac{du}{dt} = 4e^{2t}}$$

2. Find $\frac{du}{dx}$, where $u = x^2+y^2$ and $y = \frac{1-x}{x}$.   (35)

Soln.:-

$$\boxed{\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}}$$ 2nd formula

u & y given

$u = x^2+y^2$

$\Rightarrow \frac{\partial u}{\partial x} = 2x$ and $\frac{\partial u}{\partial y} = 2y$

$d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$

$y = \frac{1+x}{x} \Rightarrow \frac{dy}{dx} = \frac{x(-1) - (1-x)(1)}{x^2}$

$= \frac{\cancel{-x} - 1 + \cancel{x}}{x^2}$

$\therefore \frac{dy}{dx} = \left(-\frac{1}{x^2}\right)$

ie., $\frac{du}{dx} = (2x) + (2y)\left(-\frac{1}{x^2}\right)$

$= 2x - \frac{2y}{x^2} = 2x - \frac{2}{x^2}\left(\frac{1-x}{x}\right)$

$= 2x - \frac{2(1-x)}{x^3} = \frac{2x^4 - 2 + 2x}{x^3}$

$$\boxed{\frac{du}{dx} = \frac{2(x^4+x-1)}{x^3}}$$

=====X=====

3. If x^3+y^3+3axy, find $\frac{dy}{dx}$.

Soln.:-

Let $f(x) = x^3+y^3+3axy$.

$\frac{\partial f}{\partial x} = 3x^2+3ay$ and $\frac{\partial f}{\partial y} = 3y^2+3ax$

$$\therefore \boxed{\frac{dy}{dx} = \frac{-\partial f/\partial y}{\partial f/\partial x}}$$ 3rd formula (given eqn.)

$$\therefore \frac{dy}{dx} = \frac{-(3y^2+3ax)}{3x^2+3ay}$$

$$= \frac{-\cancel{3}(y^2+ax)}{\cancel{3}(x^2+ay)}$$

i.e., $\boxed{\dfrac{dy}{dx} = \dfrac{-(y^2+ax)}{x^2+ay}}$

Jacobians of two and three variables :-

(i) Jacobian of two variables formula is

$$\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix}$$

(ii) Jacobian of three variables formula is

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

Problems :-

1. If $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial(x,y)}{\partial(r,\theta)}$.

Soln. :-

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta)$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta) = r(1)$$

$$\therefore \boxed{\frac{\partial(x,y)}{\partial(r,\theta)} = r}$$

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2. If $u = \frac{1}{x}$, $v = \frac{x^2}{y}$ and $w = x + y + zy^2$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Soln:-

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{1}{x^2} & 0 & 0 \\ \frac{2x}{y} & -\frac{x^2}{y^2} & 0 \\ 1 & 1+2zy & y^2 \end{vmatrix}$$

$$= -\frac{1}{x^2}\left[\left(\frac{-x^2}{\cancel{y^2}}\right)\cancel{y^2} - (0)(1+2zy)\right]$$

$$- 0\left[\frac{2x}{y}(y^2) - (0)(1)\right]$$

$$+ 0\left[\left(\frac{2x}{y}\right)(1+2zy) - \left(-\frac{x^2}{y}\right)(1)\right]$$

$$= -\frac{1}{x^2}\left[(-x^2) - 0\right] - 0 + 0$$

$$= -\frac{1}{x^2}(-x^2) = 1$$

$$\therefore \boxed{\frac{\partial(u,v,w)}{\partial(x,y,z)} = 1}.$$

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3. If $x = u(1+v)$ and $y = v(1+u)$, find $\frac{\partial(x,y)}{\partial(u,v)}$

Soln :-

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv = 1+u+v+\cancel{uv}-\cancel{uv}$$

$$\therefore \boxed{\frac{\partial(x,y)}{\partial(u,v)} = 1+u+v}$$

4. If $u = \frac{y^2}{2x}$, $v = \frac{x^2+y^2}{2x}$, find $\frac{\partial(u,v)}{\partial(x,y)}$

Soln :-

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{-y^2}{2x^2} & \frac{y}{x} \\ \frac{x^2-y^2}{2x^2} & \frac{y}{x} \end{vmatrix}$$

$$= \left(\frac{-y^2}{2x^2}\right)\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)\left(\frac{x^2-y^2}{2x^2}\right)$$

$$= \frac{-y^3}{2x^3} - \frac{y(x^2-y^2)}{2x^3} = \frac{-\cancel{y^3}-x^2y+\cancel{y^3}}{2x^3} = \frac{-x^2y}{2x^{\cancel{3}}}$$

ie, $\boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{-y}{2x}}$

5. If $u = a\cosh x \cos y$, $v = a\sinh x \sin y$, then show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}a^2(\cosh 2x - \cos 2y)$.

proof :-

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} a\sinh x \cos y & -a\cosh x \sin y \\ a\cosh x \sin y & a\sinh x \cos y \end{vmatrix}$$

$$= a^2 \sinh^2 x \cos^2 y + a^2 \cosh^2 x \sin^2 y$$

$$= a^2(\sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y)$$

formula formula

$\cosh 2x = \sinh^2 x + \cosh^2 x$, $\sinh^2 x - \cosh^2 x = 1$

$$= a^2\left[\sinh^2 x\left(\frac{1+\cos 2y}{2}\right) + \cosh^2 x\left(\frac{1-\cos 2y}{2}\right)\right]$$

$$= \frac{a^2}{2}\left[\sinh^2 x + \sinh^2 x \cos 2y + \cosh^2 x - \cosh^2 x \cos 2y\right]$$

$$= \frac{a^2}{2}\left[(\sinh^2 x + \cosh^2 x) + \cos 2y(\sinh^2 x - \cosh^2 x)\right]$$

$$\therefore \boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{a^2}{2}\left[\cosh 2x + \cos 2y\right]}$$

Hence the proof

Unit - 1 is over

UNIT-II

Evaluation of Integrals of following types:

Type: 1 $\int \frac{px+q}{ax^2+bx+c}\, dx$.

Main Formulas:

1. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)$

2. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$

3. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$

Problems:-

1. Evaluate: $\int \frac{2x+3}{x^2+x+1}\, dx \rightarrow ①$

$px+q = A\frac{d}{dx}(ax^2+bx+c)+B$

Soln:-

$$2x+3 = A\frac{d}{dx}(x^2+x+1)+B$$

$$2x+3 = A(2x+1)+B \rightarrow ②$$

$2x+1=0$

$2x=-1$

$x=-\frac{1}{2}$

put $x=-\frac{1}{2}$, we get

$$2(-\tfrac{1}{2})+3 = A\left[2(-\tfrac{1}{2})+1\right]+B$$

$$-1+3 = A(0)+B$$

$$\therefore \boxed{B=2}$$

put $x=0$, we get

$$2(0)+3 = A[2(0)+1]+B$$

$$3 = A+B$$

$$\therefore A+2=3$$

$$A=3-2$$

$$\boxed{A=1}$$

$$\therefore ② \Rightarrow 2x+3 = 1(2x+1)+2$$

$$\therefore 2x+3 = (2x+1)+2$$

$\int \frac{f'(x)}{f(x)}dx = \log(f(x))$

$$\therefore ① \Rightarrow \int \frac{2x+3}{x^2+x+1}dx = \int \frac{(2x+1)+2}{x^2+x+1}dx$$

$$= \int \frac{2x+1}{x^2+x+1}dx + 2\int \frac{dx}{x^2+x+1}$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{x^2+x+1+\frac{1}{4}-\frac{1}{4}}$$

$$\int \frac{2x+3}{x^2+x+1}dx = \log(x^2+x+1) + 2\int \frac{dx}{x^2+x+\frac{1}{4}+1-\frac{1}{4}}$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$= \log(x^2+x+1) + 2\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)$$

$$\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)$$

$$= \log(x^2+x+1) + 2\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\frac{2x+1}{2}}{\sqrt{3}/2}\right)$$

$$\therefore \boxed{\int \frac{2x+3}{x^2+x+1}dx = \log(x^2+x+1) + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)}$$

=x=

2. Evaluate: $\int \frac{x+4}{6x-7-x^2}dx \rightarrow (1)$

Soln:-

$$x+4 = A\frac{d}{dx}(6x-7-x^2)+B$$

$$x+4 = A(6-2x)+B \rightarrow (2)$$

$6-2x=0$
$\cancel{-}2x=\cancel{-}6$
$x=6/2$
$x=3$

put $x=3$, we get

$3+4=A(6-6)+B$

$7=A(0)+B$

$\therefore \boxed{B=7}$

put $x=0$, we get

$4=A(6)+B$

$6A+7=4$

$6A=4-7$

$A=\frac{-3}{6}$

$\boxed{A=-\frac{1}{2}}$

$(2) \Rightarrow x+4 = -\frac{1}{2}(6-2x)+7$

$$\therefore (1) \Rightarrow \int \frac{x+4}{6x-7-x^2}dx = \int \frac{-\frac{1}{2}(6-2x)+7}{6x-7-x^2}dx$$

$$= -\frac{1}{2}\int \frac{6-2x}{6x-7-x^2}dx + 7\int \frac{dx}{6x-7-x^2}$$

③

$$\int \frac{x+4}{6x-7-x^2}\,dx = \frac{-1}{2}\log(6x-7-x^2) + 7\int \frac{dx}{-(x^2-6x+7)}$$

$$= -\frac{1}{2}\log(6x-7-x^2) - 7\int \frac{dx}{x^2-6x+7+9-9}$$

$$= -\frac{1}{2}\log(6x-7-x^2) - 7\int \frac{dx}{(x^2-6x+9)+(7-9)}$$

$$= -\frac{1}{2}\log(6x-7-x^2) - 7\int \frac{dx}{(x-3)^2-2}$$

$$= -\frac{1}{2}\log(6x-7-x^2) - 7\int \frac{dx}{(x-3)^2-(\sqrt{2})^2}$$

$$= -\frac{1}{2}\log(6x-7-x^2) - 7\,\frac{1}{2\sqrt{2}}\log\left(\frac{x-3-\sqrt{2}}{x-3+\sqrt{2}}\right)$$

$$\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$$

$$\therefore \boxed{\int \frac{x+4}{6x-7-x^2}\,dx = -\frac{1}{2}\log(6x-7-x^2) - \frac{7}{2\sqrt{2}}\log\left(\frac{x-3-\sqrt{2}}{x-3+\sqrt{2}}\right)}$$

3. Evaluate: $\int \frac{3x+1}{2x^2+x+6}\,dx \rightarrow ①$

Soln:

$$3x+1 = A\frac{d}{dx}(2x^2+x+6)+B$$

$$3x+1 = A(4x+1)+B \rightarrow ②$$

put $x=-\frac{1}{4}$, we get

$$3\left(-\tfrac{1}{4}\right)+1 = A\left[4\left(-\tfrac{1}{4}\right)+1\right]+B$$

$$-\frac{3}{4}+1 = A(0)+B$$

$$\boxed{B=\tfrac{1}{4}}$$

put $x=0$, we get

$$1 = A+B$$

$$A+\tfrac{1}{4} = 1$$

$$A = 1-\tfrac{1}{4}$$

$$\boxed{A=\tfrac{3}{4}}$$

$$② \Rightarrow 3x+1 = \frac{3}{4}(4x+1)+\tfrac{1}{4}$$

$$① \Rightarrow \int \frac{3x+1}{2x^2+x+6}\,dx = \int \frac{\frac{3}{4}(4x+1)+\frac{1}{4}}{2x^2+x+6}\,dx$$

$$= \frac{3}{4}\int \frac{4x+1}{2x^2+x+6}\,dx + \frac{1}{4}\int \frac{dx}{2x^2+x+6}$$

$$\int \frac{3x+1}{2x^2+x+6}dx = \frac{3}{4}\log(2x^2+x+6) + \frac{1}{4}\int \frac{dx}{2(x^2+\frac{x}{2}+3)}$$

$$= \frac{3}{4}\log(2x^2+x+6) + \frac{1}{8}\int \frac{dx}{x^2+\frac{x}{2}+3+\frac{1}{16}-\frac{1}{16}}$$

$$= \frac{3}{4}\log(2x^2+x+6) + \frac{1}{8}\int \frac{dx}{(x^2+\frac{x}{2}+\frac{1}{16})+(3-\frac{1}{16})}$$

$$= \frac{3}{4}\log(2x^2+x+6) + \frac{1}{8}\int \frac{dx}{(x+\frac{1}{4})^2+(\frac{48-1}{16})}$$

$$= \frac{3}{4}\log(2x^2+x+6) + \frac{1}{8}\int \frac{dx}{(x+\frac{1}{4})^2+(\frac{47}{16})}$$

$$= \frac{3}{4}\log(2x^2+x+6) + \frac{1}{8}\int \frac{dx}{(x+\frac{1}{4})^2+(\frac{\sqrt{47}}{4})^2}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)$$

$$= \frac{3}{4}\log(2x^2+x+6) + \frac{1}{8}\frac{1}{\sqrt{47}/4}\tan^{-1}\left(\frac{x+\frac{1}{4}}{\sqrt{47}/4}\right)$$

$$= \frac{3}{4}\log(2x^2+x+6) + \frac{1}{\cancel{8}_2}\frac{4}{\sqrt{47}}\tan^{-1}\left(\frac{\frac{4x+1}{\cancel{4}}}{\sqrt{47}/\cancel{4}}\right)$$

$$\boxed{\therefore \int \frac{3x+1}{2x^2+x+6}dx = \frac{3}{4}\log(2x^2+x+6) + \frac{1}{2\sqrt{47}}\tan^{-1}\left(\frac{4x+1}{\sqrt{47}}\right)}$$

==X==

Try this Problems:-

1. $\int \frac{3x+5}{x^2+4x+7}dx = \frac{3}{2}\log(x^2+4x+7) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right)$

2. $\int \frac{3x+1}{2x^2-x+5}dx = \frac{3}{4}\log(2x^2-x+5) - \frac{20}{\sqrt{429}}\log\left(\frac{2x+21-\sqrt{429}}{2x+21+\sqrt{429}}\right)$

3. $\int \frac{2x+3}{x^2+2x+5}dx = \log(x^2+2x+5) + \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)$

==X==

Type: 2 $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

Main Formulas :-

1. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a)$

2. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}(x/a)$ (or) $\log\left(x+\sqrt{x^2+a^2}\right)$

3. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a)$ (or) $\log\left(x+\sqrt{x^2-a^2}\right)$

Problems :-

1. Evaluate: $\int \frac{x}{\sqrt{x^2+x+1}}\,dx \rightarrow (1)$

Soln :-

$$x = A\frac{d}{dx}(x^2+x+1)+B$$

$$x = A(2x+1)+B \rightarrow (2)$$

put $x=-1/2$, we get

$-\frac{1}{2} = A(0)+B$

$\boxed{B=-1/2}$

put $x=0$, we get

$0 = A+B$

$A-1/2=0$

$\boxed{A=1/2}$

$\therefore (2) \Rightarrow \quad x = \frac{1}{2}(2x+1)-\frac{1}{2}$

$$\therefore (1) \Rightarrow \int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{\frac{1}{2}(2x+1)-\frac{1}{2}}{\sqrt{x^2+x+1}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$$

$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \frac{1}{\cancel{2}}\int \frac{\cancel{2}y\,dy}{y} - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+1+\frac{1}{4}-\frac{1}{4}}}$$

put $y=\sqrt{x^2+x+1}$

$y^2 = x^2+x+1$

$2y\frac{dy}{dx} = 2x+1$

$2y\,dy = (2x+1)dx$

$$= \int dy - \frac{1}{2}\int \frac{dx}{\sqrt{(x^2+x+\frac{1}{4})+(1-\frac{1}{4})}}$$

$$= y - \frac{1}{2}\int \frac{dx}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}$$

$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx = y - \frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$\because \int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}(x/a)$$

$$= y - \frac{1}{2}\sinh^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= y - \frac{1}{2}\sinh^{-1}\left(\frac{\frac{2x+1}{\cancel{2}}}{\sqrt{3}/\cancel{2}}\right)$$

$$= y - \frac{1}{2}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$\therefore \boxed{\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \sqrt{x^2+x+1} - \frac{1}{2}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).}$$

==X==

Try this Problems:-

1. $\int \frac{6x+5}{\sqrt{6+x-2x^2}}\,dx = -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)$

2. $\int \frac{3x-2}{\sqrt{4x^2-4x-5}}\,dx = \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\cosh^{-1}\left(\frac{2x-1}{\sqrt{6}}\right)$

3. $\int \left(\frac{3-2x}{1-x}\right)^{1/2} dx = -\sqrt{3-5x+2x^2} + \frac{1}{2\sqrt{2}}\cosh^{-1}(4x-5)$

2. Evaluate: $\int \left(\frac{x-1}{2x+3}\right)^{1/2} dx$

Soln:-

$$\int \left(\frac{x-1}{2x+3}\right)^{1/2} dx = \int \frac{\sqrt{x-1}}{\sqrt{2x+3}}\,dx$$

$$= \int \frac{\sqrt{x-1}}{\sqrt{2x+3}} \times \frac{\sqrt{x-1}}{\sqrt{x-1}}\,dx$$

$$\sqrt{x} \times \sqrt{x} = x$$

$$= \int \frac{(x-1)}{\sqrt{(2x+3)(x-1)}}\,dx$$

$$= \int \frac{x-1}{\sqrt{2x^2-2x+3x-3}}\,dx$$

$$\therefore \int \left(\frac{x-1}{2x+3}\right)^{1/2} dx = \int \frac{x-1}{\sqrt{2x^2+x-3}}\,dx \longrightarrow (1)$$

$$x-1 = A\frac{d}{dx}(2x^2+x-3)+B$$

$$x-1 = A(4x+1)+B \rightarrow ②$$

put $x=-\frac{1}{4}$, we get

$$-\frac{1}{4}-1 = A[4(-\tfrac{1}{4})+1]+B$$

$$-\frac{5}{4} = A(0)+B$$

$$\therefore \boxed{B = -\frac{5}{4}}$$

put $x=0$, we get

$$-1 = A+B$$

$$\therefore A-\frac{5}{4} = -1$$

$$A = -1+\frac{5}{4}$$

$$\therefore \boxed{A = \frac{1}{4}}$$

$$\therefore ② \Rightarrow x-1 = \frac{1}{4}(4x+1)-\frac{5}{4}$$

$$\therefore ① \Rightarrow \int\frac{(x-1)}{\sqrt{2x^2+x-3}}\,dx = \frac{1}{4}\int\frac{4x+1}{\sqrt{2x^2+x-3}}\,dx - \frac{5}{4}\int\frac{dx}{\sqrt{2x^2+x-3}}$$

$$\int\frac{x-1}{\sqrt{2x^2+x-3}}\,dx = \frac{1}{\cancel{4}_2}\int\frac{\cancel{2}y\,dy}{y} - \frac{5}{4}\int\frac{dx}{\sqrt{2(x^2+\frac{x}{2}-\frac{3}{2})}}$$

Let $y=\sqrt{2x^2+x-3}$

$y^2 = 2x^2+x-3$

$2y\frac{dy}{dx} = 4x+1$

$2y\,dy = (4x+1)\,dx$

$$= \frac{1}{2}\int dy - \frac{5}{4}\int\frac{dx}{\sqrt{2}\times\sqrt{x^2+\frac{x}{2}-\frac{3}{2}+\frac{1}{16}-\frac{1}{16}}}$$

$$= \frac{1}{2}y - \frac{5}{4\sqrt{2}}\int\frac{dx}{\sqrt{(x^2+\frac{x}{2}+\frac{1}{16})-\frac{3}{2}-\frac{1}{16}}}$$

$$= \frac{1}{2}y - \frac{5}{4\sqrt{2}}\int\frac{dx}{\sqrt{(x+\frac{1}{4})^2-\frac{25}{16}}}$$

$$= \frac{1}{2}y - \frac{5}{4\sqrt{2}}\int\frac{dx}{\sqrt{(x+\frac{1}{4})^2-(\frac{5}{4})^2}}$$

$$= \frac{1}{2}y - \frac{5}{4\sqrt{2}}\cosh^{-1}\left(\frac{x+\frac{1}{4}}{5/4}\right)$$

$$\because \int\frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a)$$

$$= \frac{1}{2}y - \frac{5}{4\sqrt{2}}\cosh^{-1}\left(\frac{\frac{4x+1}{\cancel{4}}}{5/\cancel{4}}\right)$$

$$\therefore \int\frac{x-1}{\sqrt{2x^2+x-3}}\,dx = \frac{1}{2}\sqrt{2x^2+x-3} - \frac{5}{4\sqrt{2}}\cosh^{-1}\left(\frac{4x+1}{5}\right)$$

ie., $$\boxed{\int\left(\frac{x-1}{2x+3}\right)^{1/2}dx = \frac{1}{2}\sqrt{2x^2+x-3} - \frac{5}{4\sqrt{2}}\cosh^{-1}\left(\frac{4x+1}{5}\right)}$$

=x=

Type : 3 $\int \frac{dx}{a+b\cos x}$ and **Type : 4** $\int \frac{dx}{a+b\sin x}$

Problems:-

1. Evaluate: $\int \frac{dx}{4+5\cos x}$

Soln:-

put $t = \tan x/2$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2 x/2$$

$$dt = \frac{1}{2}\sec^2 x/2 \, dx$$

$$dt = \frac{1}{2}(1+\tan^2 x/2)\, dx$$

$$\therefore dt = \frac{1}{2}(1+t^2)\, dx$$

$$\Rightarrow \boxed{dx = \frac{2dt}{1+t^2}}$$

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\Rightarrow \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$$

$$\therefore \boxed{\cos x = \frac{1-t^2}{1+t^2}}$$

$$\therefore \int \frac{dx}{4+5\cos x} = \int \frac{2dt/1+t^2}{4+5\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= 2\int \frac{dt/\cancel{1+t^2}}{\frac{4(1+t^2)+5(1-t^2)}{(\cancel{1+t^2})}}$$

$$= 2\int \frac{dt}{4(1+t^2)+5(1-t^2)}$$

$$= 2\int \frac{dt}{4+4t^2+5-5t^2}$$

$$= 2\int \frac{dt}{9-t^2}$$

$$= 2\int \frac{dt}{3^2-t^2}$$

$$\left(\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)\right)$$

$$= \cancel{2}\frac{1}{\cancel{2}\times 3}\log\left(\frac{3+t}{3-t}\right)$$

$$= \frac{1}{3}\log\left(\frac{3+t}{3-t}\right)$$

$$\boxed{\int \frac{dx}{4+5\cos x} = \frac{1}{3}\log\left(\frac{3+\tan x/2}{3-\tan x/2}\right)}$$

2. Evaluate: $\int \frac{dx}{3\sin x + 4\cos x}$ ⑨

Soln:-

put $t = \tan x/2$

$\frac{dt}{dx} = \frac{1}{2}\sec^2 x/2$

$\frac{dt}{dx} = \frac{1}{2}(1+\tan^2 x/2)$

$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$

$\boxed{dx = \frac{2dt}{1+t^2}}$

$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

$\therefore \sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$

$\Rightarrow \boxed{\sin x = \frac{2t}{1+t^2}}$

$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

$\Rightarrow \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$

$\Rightarrow \boxed{\cos x = \frac{1-t^2}{1+t^2}}$

$$\therefore \int \frac{dx}{3\sin x + 4\cos x} = \int \frac{2dt/1+t^2}{3\left(\frac{2t}{1+t^2}\right)+4\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= 2\int \frac{dt/\cancel{1+t^2}}{\frac{6t+4(1-t^2)}{\cancel{1+t^2}}}$$

$$= 2\int \frac{dt}{6t+4-4t^2}$$

$$= 2\int \frac{dt}{4\left(\frac{3}{2}t+1-t^2\right)} = \frac{2}{4}\int \frac{dt}{-t^2+\frac{3}{2}t+1+\frac{9}{16}-\frac{9}{16}}$$

$$= \frac{1}{2}\int \frac{dt}{\left(-t^2+\frac{3}{2}t-\frac{9}{16}\right)+\left(1+\frac{9}{16}\right)}$$

$$= \frac{1}{2}\int \frac{dt}{-\left(t^2-\frac{3}{2}t+\frac{9}{16}\right)+\frac{25}{16}} = \frac{1}{2}\int \frac{dt}{-\left(t-\frac{3}{4}\right)^2+\left(\frac{5}{4}\right)^2}$$

$$= \frac{1}{\cancel{2}}\,\frac{1}{\cancel{2}\frac{5}{4}}\log\left(\frac{\frac{5}{4}+\left(t-\frac{3}{4}\right)}{\frac{5}{4}-\left(t-\frac{3}{4}\right)}\right) \qquad \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$$

$$= \frac{1}{5}\log\left(\frac{\frac{5}{4}+t-\frac{3}{4}}{\frac{5}{4}-t+\frac{3}{4}}\right)$$

$$\int \frac{dx}{3\sin x + 4\cos x} = \frac{1}{5}\log\left(\frac{\frac{5-3}{4}+t}{\frac{5+3}{4}+t}\right)$$ (10)

$$= \frac{1}{5}\log\left(\frac{\frac{2}{4}+t}{\frac{8}{4}+t}\right)$$

$$= \frac{1}{5}\log\left(\frac{\frac{1}{2}+t}{2+t}\right)$$

ie. $$\boxed{\int \frac{dx}{3\sin x + 4\cos x} = \frac{1}{5}\log\left(\frac{\frac{1}{2}+\tan x/_2}{2+\tan x/_2}\right)}$$

3. Evaluate: $\int \frac{dx}{1+3\sin x + 4\cos x}$

Soln:-

put $t = \tan x/_2$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2 x/_2$$

$$\Rightarrow dx = \frac{2dt}{\sec^2 x/_2}$$

$$dx = \frac{2dt}{1+\tan^2 x/_2}$$

$$\boxed{dx = \frac{2dt}{1+t^2}}$$

$$\boxed{\sin x = \frac{2t}{1+t^2}}$$

$$\boxed{\cos x = \frac{1-t^2}{1+t^2}}$$

$$\therefore \int \frac{dx}{1+3\sin x+4\cos x} = \int \frac{2dt/1+t^2}{1+3\left(\frac{2t}{1+t^2}\right)+4\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= 2\int \frac{dt/\cancel{1+t^2}}{\frac{(1+t^2)+3(2t)+4(1-t^2)}{\cancel{1+t^2}}} = 2\int \frac{dt}{1+t^2+6t+4-4t^2}$$

$$= 2\int \frac{dt}{5+6t-3t^2} = \frac{2}{3}\int \frac{dt}{5/_3+2t-t^2} = \frac{2}{3}\int \frac{dt}{-t^2+2t+5/_3+1-1}$$

$$= \frac{2}{3}\int \frac{dt}{(-t^2+2t-1)+\frac{5}{3}+1} = \frac{2}{3}\int \frac{dt}{-(t^2-2t+1)+\frac{8}{3}}$$

$$= \frac{2}{3}\int \frac{dt}{-(t-1)^2+\left(\frac{2\sqrt{2}}{\sqrt{3}}\right)^2} = \frac{\cancel{2}}{3}\frac{1}{\cancel{2}\left(\frac{2\sqrt{2}}{\sqrt{3}}\right)}\log\left(\frac{\frac{2\sqrt{2}}{\sqrt{3}}+(t-1)}{\frac{2\sqrt{2}}{\sqrt{3}}-(t-1)}\right)$$

$$= \frac{1}{3}\frac{\sqrt{3}}{2\sqrt{2}}\log\left(\frac{2\sqrt{2}+\sqrt{3}(t-1)/\cancel{\sqrt{3}}}{2\sqrt{2}-\sqrt{3}(t-1)/\cancel{\sqrt{3}}}\right)$$

$$\int \frac{dx}{1+3\sin x+4\cos x} = \frac{\sqrt{3}}{\sqrt{3}\sqrt{3}\cdot 2\sqrt{2}} \log\left(\frac{2\sqrt{2}+\sqrt{3}(t-1)}{2\sqrt{2}-\sqrt{3}(t-1)}\right)$$

$$= \frac{1}{2\sqrt{3\times 2}} \log\left(\frac{2\sqrt{2}+\sqrt{3}t-\sqrt{3}}{2\sqrt{2}-\sqrt{3}t+\sqrt{3}}\right)$$

$$= \frac{1}{2\sqrt{6}} \log\left(\frac{2\sqrt{2}+\sqrt{3}t-\sqrt{3}}{2\sqrt{2}-\sqrt{3}t+\sqrt{3}}\right)$$

ie.,
$$\boxed{\int \frac{dx}{1+3\sin x+4\cos x} = \frac{1}{2\sqrt{6}} \log\left(\frac{2\sqrt{2}+\sqrt{3}\tan x/2-\sqrt{3}}{2\sqrt{2}-\sqrt{3}\tan x/2+\sqrt{3}}\right)}$$

=====×=====

Try This Problems:-

1. $\int \frac{dx}{12+13\cos x} = \frac{1}{5}\log\left(\frac{5+\tan x/2}{5-\tan x/2}\right)$

2. $\int \frac{dx}{1+\sin x+\cos x} = \log(1+\tan x/2)$

3. $\int \frac{dx}{\sin x+\sqrt{3}\cos x} = \frac{1}{2}\log\left(\frac{1+\sqrt{3}\tan x/2}{\sqrt{3}-\tan x/2}\right)$

=====×=====

Type: 5 $\int \frac{dx}{(x+p)\sqrt{ax^2+bx+c}}$

Problems:-

1. Evaluate: $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$

Soln:-

put $x+1 = \frac{1}{t}$

$\frac{dx}{dt} = -\frac{1}{t^2}$

$\boxed{dx = -\frac{dt}{t^2}}$

$x+1 = \frac{1}{t}$

$x = \frac{1}{t}-1$

$x^2 = \left(\frac{1}{t}-1\right)^2$

$x^2 = \frac{1}{t^2}-\frac{2}{t}+1$

$x^2+x+1 = \frac{1}{t^2}-\frac{2}{t}+1+1+\frac{1}{t}$

$\boxed{x^2+x+1 = \frac{1}{t^2}-\frac{1}{t}+1}$

$$\therefore \int \frac{dx}{(x+1)\sqrt{x^2+x+1}} = \int \frac{-dt/t^2}{\frac{1}{t}\sqrt{\frac{1}{t^2}-\frac{1}{t}+1}} = -\int \frac{dt/t^2}{\frac{1}{t}\sqrt{\frac{1}{t^2}(1-t+t^2)}}$$

$$= -\int \frac{dt/t^2}{\frac{1}{t}\cdot\frac{1}{t}\sqrt{t^2-t+1}} = -\int \frac{dt/t^2}{\frac{1}{t^2}\sqrt{t^2-t+1+\frac{1}{4}-\frac{1}{4}}}$$

$$\int\frac{dx}{(x+1)\sqrt{x^2+x+1}} = -\int\frac{dt}{\sqrt{(t^2-t+\frac{1}{4})+(1-\frac{1}{4})}}$$

$$= -\int\frac{dt}{\sqrt{(t-\frac{1}{2})^2+\frac{3}{4}}} = -\int\frac{dt}{\sqrt{(t-\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= -\sinh^{-1}\left(\frac{t-\frac{1}{2}}{\sqrt{3}/2}\right) \qquad \because \int\frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}(x/a)$$

$$= -\sinh^{-1}\left(\frac{2t-1/\cancel{2}}{\sqrt{3}/\cancel{2}}\right)$$

ie., $$\int\frac{dx}{(x+1)\sqrt{x^2+x+1}} = -\sinh^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)$$

$$= -\sinh^{-1}\left(\frac{2\left(\frac{1}{x+1}\right)-1}{\sqrt{3}}\right) \qquad x+1 = \frac{1}{t} \Rightarrow t = \frac{1}{x+1}$$

$$= -\sinh^{-1}\left(\frac{\frac{2-(x+1)}{x+1}}{\sqrt{3}}\right)$$

$$= -\sinh^{-1}\left(\frac{2-x-1}{\sqrt{3}\,(x+1)}\right)$$

ie., $$\boxed{\int\frac{dx}{(x+1)\sqrt{x^2+x+1}} = -\sinh^{-1}\left(\frac{1-x}{\sqrt{3}\,(x+1)}\right)}$$

2. Evaluate: $\int\frac{dx}{(3+x)\sqrt{x}}$.

Soln.:- put $3+x = \frac{1}{t}$

$$\frac{dx}{dt} = -\frac{1}{t^2}$$

$$\boxed{dx = -\frac{dt}{t^2}}$$

$$3+x = \frac{1}{t}$$

$$\boxed{x = \frac{1}{t}-3}$$

$$\therefore \int\frac{dx}{(3+x)\sqrt{x}} = \int\frac{-dt/t^2}{\frac{1}{t}\sqrt{\frac{1}{t}-3}} = -\int\frac{dt/t^2}{\frac{1}{t}\cdot\frac{1}{t}\sqrt{t-3t^2}} = -\int\frac{dt/\cancel{t^2}}{\frac{1}{\cancel{t^2}}\sqrt{t-3t^2}}$$

$$= -\int\frac{dt}{\sqrt{t-3t^2}} = -\int\frac{dt}{\sqrt{3}\sqrt{\frac{t}{3}-t^2}} = -\frac{1}{\sqrt{3}}\int\frac{dt}{\sqrt{-t^2+\frac{t}{3}+\frac{1}{36}-\frac{1}{36}}}$$

$$\int \frac{dx}{(3+x)\sqrt{x}} = -\frac{1}{\sqrt{3}} \int \frac{dt}{\sqrt{-(t^2 - t/3 + \frac{1}{36}) + \frac{1}{36}}}$$

$$= -\frac{1}{\sqrt{3}} \int \frac{dt}{\sqrt{-(t-1/6)^2 + (\frac{1}{6})^2}}$$

$$= -\frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{t-\frac{1}{6}}{1/6}\right)$$

$$\left(\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a)\right)$$

$$= -\frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{\frac{6t-1}{6}}{1/6}\right)$$

$$3+x = \frac{1}{t} \Rightarrow t = \frac{1}{3+x}$$

$$= -\frac{1}{\sqrt{3}} \sin^{-1}(6t-1)$$

$$= -\frac{1}{\sqrt{3}} \sin^{-1}\left(6\frac{1}{3+x} - 1\right) = -\frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{6-(3+x)}{3+x}\right)$$

$$= -\frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{6-3-x}{3+x}\right)$$

ie., $$\boxed{\int \frac{dx}{(3+x)\sqrt{x}} = -\frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{3-x}{3+x}\right)}$$

=====X=====

Integration by trigonometric substitution and by parts of integrals:-

1. $\int \sqrt{a^2-x^2}\, .dx$

Soln:-

put $x = a\sin\theta$

$$\frac{dx}{d\theta} = a\cos\theta$$

$$dx = a\cos\theta\, d\theta$$

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta}$$

$$= \sqrt{a^2(1-\sin^2\theta)}$$

$$= a\sqrt{1-\sin^2\theta}$$

$$= a\sqrt{\cos^2\theta}$$

$$\sqrt{a^2-x^2} = a\cos\theta$$

$$\therefore \int \sqrt{a^2-x^2}\, .dx = \int a\cos\theta \cdot a\cos\theta\, d\theta$$

$$= a^2 \int \cos^2\theta\, d\theta$$

$\therefore \int \sqrt{a^2-x^2}\, dx = a^2 \int \frac{1+\cos 2\theta}{2} d\theta$ $\qquad \left(\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right)$

$= \frac{a^2}{2} \int (1+\cos 2\theta)\, d\theta$

$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$ $\qquad \left(\int \cos 2\theta\, d\theta = \frac{\sin 2\theta}{2}\right.$

$\qquad \left.\sin 2\theta = 2\sin\theta\cos\theta\right)$

$= \frac{a^2}{2}\left[\theta + \sin\theta\cos\theta\right]$ $\qquad \left(a\cos\theta = \sqrt{a^2-x^2}\right.$

$\qquad a\cos\theta = a\sqrt{1-\frac{x^2}{a^2}}$

$= \frac{a^2}{2}\left[\sin^{-1}(x/a) + \left(\frac{x}{a}\right)\sqrt{1-\frac{x^2}{a^2}}\right]$ $\qquad \left.\cos\theta = \sqrt{1-\frac{x^2}{a^2}}\right)$

$= \frac{a^2}{2}\sin^{-1}(x/a) + \frac{a^2}{2}\frac{x}{\cancel{a}}\sqrt{\frac{a^2-x^2}{a^2}}$ $\qquad \left(x = a\sin\theta\right.$

$\qquad \left.\sin\theta = x/a\right)$

$= \frac{a^2}{2}\sin^{-1}(x/a) + \frac{\cancel{a}}{2}\frac{x}{\cancel{a}}\sqrt{a^2-x^2}$

$$\boxed{\therefore \int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2}\sin^{-1}(x/a) + \frac{x}{2}\sqrt{a^2-x^2}}$$

===X===

2. $\int \sqrt{a^2+x^2}\, . dx$

Soln.

Put $x = a\sinh\theta$

$\frac{dx}{d\theta} = a.\cosh\theta$

$\therefore \boxed{dx = a\cosh\theta\, d\theta}$

$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\sinh^2\theta}$

$= \sqrt{a^2(1+\sinh^2\theta)}$

$= \sqrt{a^2\cosh^2\theta}$

$\boxed{\sqrt{a^2+x^2} = a\cosh\theta}$

$\therefore \int \sqrt{a^2+x^2}\, dx$

$= \int a\cosh\theta . a\cosh\theta\, d\theta$

$= \int a^2\cosh^2\theta\, d\theta$

$= a^2 \int \cosh^2\theta\, d\theta$

$= a^2 \int \frac{1+\cosh 2\theta}{2} d\theta$

$= \frac{a^2}{2}\int (1+\cosh 2\theta)\, d\theta$

$$\int \sqrt{a^2+x^2}\, dx = \frac{a^2}{2}\left[\theta + \frac{\sinh 2\theta}{2}\right]$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\left(\frac{\sinh 2\theta}{2}\right)$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\sinh\theta \cosh\theta$$

$$= \frac{a^2}{2}\sinh^{-1}(x/a) + \frac{a^2}{2}\left(\frac{x}{a}\right)\sqrt{1+\frac{x^2}{a^2}}$$

$$= \frac{a^2}{2}\sinh^{-1}(x/a) + \frac{a}{2}(x)\sqrt{\frac{a^2+x^2}{a^2}}$$

$$= \frac{a^2}{2}\sinh^{-1}(x/a) + \frac{\cancel{a}}{2}\frac{x}{\cancel{a}}\sqrt{a^2+x^2}$$

$a\cosh\theta = \sqrt{a^2+x^2}$

$\cancel{a}\cosh\theta = \cancel{a}\sqrt{1+x^2/a^2}$

$\cosh\theta = \sqrt{1+x^2/a^2}$

ie., $$\boxed{\int \sqrt{a^2+x^2}\, dx = \frac{a^2}{2}\sinh^{-1}(x/a) + \frac{x}{2}\sqrt{a^2+x^2}}$$

3. $\int \sqrt{x^2-a^2}\,.\,dx$

Soln:-

Put $x = a\cosh\theta$

$$\frac{dx}{d\theta} = a\sinh\theta$$

$$\therefore \boxed{dx = a\sinh\theta\, d\theta}$$

$$\sqrt{x^2-a^2} = \sqrt{a^2\cosh^2\theta - a^2}$$

$$= \sqrt{a^2(\cosh^2\theta - 1)}$$

$$= \sqrt{a^2\sinh^2\theta}$$

$$\boxed{\sqrt{x^2-a^2} = a\sinh\theta}$$

$$\therefore \int \sqrt{x^2-a^2}\, dx = \int a\sinh\theta . a\sinh\theta\, d\theta$$

$$= \int a^2 \sinh^2\theta\, d\theta$$

$\sinh^2 x = \frac{\cosh 2x - 1}{2}$

$$= a^2\int \frac{\cosh 2\theta - 1}{2}\, d\theta$$

$$= \frac{a^2}{2}\int(\cosh 2\theta - 1)\, d\theta$$

$$= \frac{a^2}{2}\left[\frac{\sinh 2\theta}{2} - \theta\right]$$

$$= \frac{a^2}{2}\left(\frac{\sinh 2\theta}{2}\right) - \frac{a^2}{2}\theta$$

$$= \frac{a^2}{2}(\sinh\theta \cosh\theta) - \frac{a^2}{2}\theta$$

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$$\int\sqrt{x^2-a^2}\,dx = \frac{a^2}{2}\left(\frac{x}{a}\right)\left(\sqrt{\frac{x^2}{a^2}-1}\right) - \frac{a^2}{2}\cosh^{-1}(x/a)$$

$$= \frac{a^2}{2}\,\frac{x}{a^2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}(x/a)$$

ie., $$\boxed{\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}(x/a)}$$

—x—

Examples:

1. Evaluate: $\int\sqrt{x^2+2x+10}\,dx$

Soln.:-

$$\int\sqrt{x^2+2x+10}\,dx = \int\sqrt{x^2+2x+10+1-1}$$

$$= \int\sqrt{(x^2+2x+1)+(10-1)}$$

$$= \int\sqrt{(x+1)^2+9}$$

$$= \int\sqrt{(x+1)^2+3^2}$$

$$= \cancel{\frac{1}{2}(x+1)\sqrt{\ }}$$

$$\int\sqrt{x^2+a^2}\,dx = \frac{a^2}{2}\sinh^{-1}(x/a)+\frac{x}{2}\sqrt{x^2+a^2}$$

$$\therefore \int\sqrt{x^2+2x+10}\,dx = \frac{3^2}{2}\sinh^{-1}\left(\frac{x+1}{3}\right)+\frac{x+1}{2}\sqrt{x^2+2x+10}$$

$$\boxed{\int\sqrt{x^2+2x+10}\,dx = \frac{9}{2}\sinh^{-1}\left(\frac{x+1}{3}\right)+\frac{x+1}{2}\sqrt{x^2+2x+10}}$$

—x—

2. Evaluate: $\int\sqrt{1+x-2x^2}\,dx$

Soln.:-

$$\int\sqrt{1+x-2x^2}\,dx = \int\sqrt{2\left(\tfrac{1}{2}+\tfrac{x}{2}-x^2\right)}$$

$$= \sqrt{2}\int\sqrt{-x^2+x/2+1/2+\tfrac{1}{16}-1/16}$$

$$\int \sqrt{1+x-2x^2}\,dx = \sqrt{2}\int \sqrt{\left(-x^2+\tfrac{x}{2}-\tfrac{1}{16}\right)+\left(\tfrac{1}{2}+\tfrac{1}{16}\right)}$$

$$= \sqrt{2}\int \sqrt{-\left(x^2-\tfrac{x}{2}+\tfrac{1}{16}\right)+\left(\tfrac{8+1}{16}\right)}$$

$$= \sqrt{2}\int \sqrt{-\left(x-\tfrac{1}{4}\right)^2+\tfrac{9}{16}}$$

$$= \sqrt{2}\int \sqrt{-\left(x-\tfrac{1}{4}\right)^2+\left(\tfrac{3}{4}\right)^2}$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\left(\tfrac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}$$

$$\therefore \int \sqrt{1+x-2x^2}\,dx = \sqrt{2}\left[\frac{(3/4)^2}{2}\sin^{-1}\left(\frac{x-\frac{1}{4}}{3/4}\right)+\frac{x-\frac{1}{4}}{2}\sqrt{-\left(x-\tfrac{1}{4}\right)^2+\left(\tfrac{3}{4}\right)^2}\right]$$

$$= \sqrt{2}\ \frac{9/16}{2}\sin^{-1}\left(\frac{4x-1/\cancel{4}}{3/\cancel{4}}\right)+\frac{\frac{4x-1}{\cancel{4}}}{2}\ \sqrt{2}\sqrt{-\left(x-\tfrac{1}{4}\right)^2+\left(\tfrac{3}{4}\right)^2}$$

$$= \sqrt{2}\ \frac{9}{32}\sin^{-1}\left(\frac{4x-1}{3}\right)+\frac{4x-1}{8}\sqrt{1+x-2x^2}$$

$$= \sqrt{2}\ \frac{9}{2\times 16}\sin^{-1}\left(\frac{4x-1}{3}\right)+\frac{4x-1}{8}\sqrt{1+x-2x^2}$$

i.e. $\boxed{\int \sqrt{1+x-2x^2}\,dx = \frac{9}{\sqrt{2}\times 16}\sin^{-1}\left(\frac{4x-1}{3}\right)+\frac{4x-1}{8}\sqrt{1+x-2x^2}}$

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Unit-2 is over

Calculus and Fourier Series

(1)

Unit-III

Properties of definite Integrals :-

(X) 2 marks

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

3. $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(x)$ is an even function of x

4. $\int_{-a}^{a} f(x)\,dx = 0$, if $f(x)$ is odd function of x.

5. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. This result is very important (useful in evaluating many integrals).

6. $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$.

Problems :-

1. Prove that $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$.

proof :-

Let $f(x) = \sin^n x$, Here $a = \pi/2$

property five

$\therefore f(a-x) = \sin^n(\pi/2 - x) = \cos^n x$

$\because \sin(90-\theta) = \cos\theta$

ie., $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \sin^n(\pi/2 - x)\,dx$

$\Rightarrow \boxed{\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx}$

Hence the proof.

2. Prove that $\int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}\, dx = \frac{\pi}{4}$. 2

proof:-

Let $f(x) = \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}$, Here $a = \pi/2$

$$\therefore f(a-x) = f(\pi/2 - x)$$

$$= \frac{(\sin(\pi/2 - x))^{3/2}}{(\sin(\pi/2 - x))^{3/2} + (\cos(\pi/2 - x))^{3/2}}$$

$\sin(90-\theta) = \cos\theta$
$\cos(90-\theta) = \sin\theta$

$$= \frac{(\cos x)^{3/2}}{(\cos x)^{3/2} + (\sin x)^{3/2}} \rightarrow (1)$$

ie., $\cancel{\int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}\, dx \neq \int_0^{\pi/2} \frac{(\cos x)^{3/2}}{\cos}}$

ie., $$I = \int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}\, dx \rightarrow (2)$$

Also, $$I = \int_0^{\pi/2} f(a-x)\, dx$$

$$I = \int_0^{\pi/2} \frac{(\cos x)^{3/2}}{(\cos x)^{3/2} + (\sin x)^{3/2}}\, dx \rightarrow (3) \quad \text{(by (1)}$$

(2) + (3), we get

$$I + I = \int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}\, dx + \int_0^{\pi/2} \frac{(\cos x)^{3/2}}{(\cos x)^{3/2} + (\sin x)^{3/2}}\, dx$$

$$\therefore 2I = \int_0^{\pi/2} \frac{(\sin x)^{3/2} + (\cos x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}\, dx$$

$$2I = \int_0^{\pi/2} dx = (x)_0^{\pi/2} = (\pi/2 - 0) = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{2} \times \frac{1}{2} \Rightarrow I = \frac{\pi}{4} \Rightarrow \text{ie., } \boxed{\int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}\, dx = \pi/4}$$

Hence the proof.

3. Prove that $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \frac{\pi}{8}\log 2$.

proof:-

Let $f(\theta) = \log(1+\tan\theta)$, Here $a = \frac{\pi}{4}$.

$\therefore f(a-\theta) = \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)$

$= \log\left(1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)$

$= \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)$ $\quad \because \tan(A-B) = \frac{\tan A - \tan A}{1+\tan A \tan B}$

$\tan\frac{\pi}{4} = 1$

$= \log\left(\frac{1+\cancel{\tan\theta}+1-\cancel{\tan\theta}}{1+\tan\theta}\right)$

$f(a-\theta) = \log\left(\frac{2}{1+\tan\theta}\right) \rightarrow (1)$

$\therefore I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta \rightarrow (2)$

Also, $I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta \rightarrow (3)$ (by eqn. (1))

$(2)+(3) \Rightarrow I + I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta + \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$

$\therefore 2I = \int_0^{\pi/4} \left[\log(1+\tan\theta) + \log\left(\frac{2}{1+\tan\theta}\right)\right] d\theta$

$= \int_0^{\pi/4} \left\{\log(1+\tan\theta) + \left[\log 2 - \log(1+\tan\theta)\right]\right\} d\theta$

$\because \log\left(\frac{x}{y}\right) = \log x - \log y$

$= \int_0^{\pi/4} \log 2\, d\theta$

$\Rightarrow 2I = \log 2 \int_0^{\pi/4} d\theta = \log 2\, (\theta)_0^{\pi/4} = \log 2\left(\frac{\pi}{4}\right)$

$\Rightarrow I = \log 2 \times \frac{\pi}{4} \times \frac{1}{2} = \log 2 \times \frac{\pi}{8} = \frac{\pi}{8}\log 2$.

ie., $\boxed{\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \frac{\pi}{8}\log 2}$

Hence the proof.

4. Evaluate : $\int_0^{\pi/2} \log \sin x \, dx$

<u>Soln:-</u>

Let $I = \int_0^{\pi/2} \log \sin x \, dx \longrightarrow (1)$

Let $f(x) = \log \sin x$, Here $a = \pi/2$

$\therefore f(a-x) = \log\left(\sin\left(\frac{\pi}{2} - x\right)\right)$

$f(a-x) = \log(\cos x)$

$\therefore I = \int_0^{\pi/2} \log \cos x \, dx \longrightarrow (2)$

(1) + (2) , we get

$I + I = \int_0^{\pi/2} \log \sin x \, dx + \int_0^{\pi/2} \log \cos x \, dx$

$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x) \, dx$ $\quad$ ($\because \log a + \log b = \log ab$)

$= \int_0^{\pi/2} \log(\sin x \cdot \cos x) \, dx$

$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$ $\quad$ ($\because \sin 2\theta = 2 \sin\theta \cos\theta$)

$\Rightarrow 2I = \int_0^{\pi/2} (\log \sin 2x - \log 2) \, dx$ $\quad$ ($\because \log\left(\frac{a}{b}\right) = \log a - \log b$)

$2I = \int_0^{\pi/2} \log \sin 2x \, dx - \int_0^{\pi/2} \log 2 \, dx \longrightarrow (3)$

first we find,

$\int_0^{\pi/2} \log \sin 2x \, dx = \int_0^{\pi} \log \sin y \, \frac{dy}{2}$ $\quad$ (put $2x = y \Rightarrow 2dx = dy$; $x = 0 \Rightarrow y = 0$; $x = \pi \Rightarrow y = \pi/2$)

$= \frac{1}{2} \int_0^{\pi} \log \sin y \, dy$

$= \frac{1}{\cancel{2}} \cancel{2} \int_0^{\pi/2} \log \sin y \, dy$

$= \int_0^{\pi/2} \log \sin x \, dx$ $\quad$ ($\because \int_0^a f(x)dx = \int_0^a f(y) dy$)

$= I$ $\quad$ (by eqn. (1))

$$③ \Rightarrow 2I = I - \int_0^{\pi/2} \log 2 \, dx$$

$$2I - I = -\log 2 \int_0^{\pi/2} dx$$

$$\therefore I = \log 2^{-1} (x)_0^{\pi/2}$$

$$I = \log\left(\tfrac{1}{2}\right) \left(\tfrac{\pi}{2}\right)$$

$\because a \log x = \log x^a$

$\Rightarrow -\log 2 = \log 2^{-1}$

$2^{-1} = \frac{1}{2}$

ie.,
$$\boxed{\int_0^{\pi/2} \log \sin x \, dx = \frac{\pi}{2} \log\left(\tfrac{1}{2}\right)}$$

Home Work:

5. Prove that $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$.

6. Prove that $\int_0^{\pi/2} \frac{dx}{1+\tan x} = \frac{\pi}{4}$.

Integration by Parts

1. Evaluating $\int x e^x dx$

Soln.:-

$$\int u \, dv = uv - \int v \, du$$

$\therefore u = x$ | $\int dv = \int e^x$

$du = dx$ | $v = e^x$

$$\therefore \int x e^x dx = x e^x - \int e^x dx$$

$$= x e^x - e^x$$

$$\therefore \boxed{\int x e^x dx = e^x (x-1)}.$$

2. Evaluate: $\int \log x \, dx$.

Soln.:- $u = \log x$ | $\int dv = \int dx$

$\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x} dx$ | $v = x$

$$\therefore \int \log x \, dx = x \log x - \int x \left(\frac{1}{x} dx\right)$$
$$= x \log x - \int dx$$
$$= x \log x - x$$
$$\therefore \boxed{\int \log x \, dx = x(\log x - 1)}$$

Reduction formula:
(When n is a positive integer)

I) $\int x^n e^{ax} dx$

Soln:-

$u = x^n$, $\frac{du}{dx} = n x^{n-1}$, $du = n x^{n-1} dx$ | $\int dv = \int e^{ax} dx$, $v = \frac{e^{ax}}{a}$

$$\therefore I_n = \int x^n e^{ax} dx \qquad \left(\because \int u\,dv = uv - \int v\,du\right)$$
$$= x^n \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} n x^{n-1} dx$$
$$= \frac{e^{ax}}{a} x^n - \frac{n}{a} \int e^{ax} x^{n-1} dx$$
$$\therefore \boxed{I_n = \frac{e^{ax}}{a} x^n - \frac{n}{a} I_{n-1}} \quad \text{formula}$$

II) $\int \sin^n x \, dx \left(= \int \sin^{n-1} x \sin x \, dx\right)$

Soln:-

$u = \sin^{n-1} x$, $\frac{du}{dx} = (n-1)\sin^{n-2} x \cdot \cos x$ | $\int dv = \int \sin x \, dx$, $v = -\cos x$

$$du = (n-1)\sin^{n-2} x \cos x \, dx$$
$$\therefore \int u\,dv = uv - \int v\,du$$
$$\therefore I_n = \int \sin^n x \, dx$$

$$I_n = \sin^{n-1}x \cdot (-\cos x) - \int -\cos x \,(n-1)\sin^{n-2}x \cos x \, dx$$

$$= -\cos x \sin^{n-1}x + \int \cos^2 x \,(n-1)\sin^{n-2}x \, dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int \cos^2 x \sin^{n-2}x \, dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int (1-\sin^2 x)\sin^{n-2}x \, dx$$

$\because \sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$$= -\cos x \sin^{n-1}x + (n-1)\left[\int \sin^{n-2}x \, dx - \int \sin^2 x \sin^{n-2}x \, dx\right]$$

$$= -\cos x \sin^{n-1}x + (n-1)\left[\int \sin^{n-2}x \, dx - \int \sin^n x \, dx\right]$$

$$I_n = -\cos x \sin^{n-1}x + (n-1)\left[I_{n-2} - I_n\right]$$

$$= -\cos x \sin^{n-1}x + (n-1)I_{n-2} - (n-1)I_n$$

$$\Rightarrow I_n + (n-1)I_n = -\cos x \sin^{n-1}x + (n-1)I_{n-2}$$

$$\Rightarrow \cancel{I_n} + nI_n - \cancel{I_n} = -\cos x \sin^{n-1}x + (n-1)I_{n-2}$$

$$\Rightarrow nI_n = -\cos x \sin^{n-1}x + (n-1)I_{n-2}$$

ie., $$\boxed{I_n = \frac{-\sin^{n-1}x \cos x}{n} + \frac{n-1}{n} I_{n-2}}$$

If $I_n = \int_0^{\pi/2} \sin^n x \, dx \rightarrow$ Find the reduction formula for $\int_0^{\pi/2} \sin^n x \, dx$. X

5 marks

10 marks கேட்டா இரண்டையும் சேர்த்து எழுத வேண்டும்

Soln. :-

$$I_n = \int_0^{\pi/2} \sin^n x \, dx$$

$$= \left(-\frac{\sin^{n-1}x \cos x}{n}\right)_0^{\pi/2} + \frac{n-1}{n} I_{n-2}$$

$$= -\frac{1}{n}\left[\sin^{n-1}\tfrac{\pi}{2} \cos\tfrac{\pi}{2} - \sin^{n-1}0 \cdot \cos 0\right] + \frac{n-1}{n} I_{n-2}$$

$$= -\frac{1}{n}\left[0 - 0\right] + \frac{n-1}{n} I_{n-2}$$

$\because \sin 0 = 0$ & $\cos\frac{\pi}{2} = 0$

$$\therefore I_n = \frac{n-1}{n} I_{n-2}$$

$$= \frac{n-1}{n} \frac{n-3}{n-2} I_{n-4}$$

$$= \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} I_{n-6}$$

$$= \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \frac{n-7}{n-6} I_{n-8}$$

$\because I_0 = \int_0^{\pi/2} \sin^0 x\, dx$

$= \int_0^{\pi/2} dx$ $\quad (\sin^0 x = 1)$

$= (x)_0^{\pi/2}$

$= \frac{\pi}{2}$

Case: (i) n is an even number

$$I_n = \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \frac{n-7}{n-6} \cdots \frac{3}{4} \frac{1}{2} I_0$$

ie., $\boxed{I_n = \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \cdots \frac{3}{4} \frac{1}{2} \frac{\pi}{2}}$ formula

$I_1 = \int_0^{\pi/2} \sin^1 x\, dx$

$= (-\cos x)_0^{\pi/2}$

$= (-\cos \frac{\pi}{2} + \cos 0)$

$= -0 + 1$

$I_1 = 1$

Case: (ii) n is odd number

$$I_n = \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \frac{n-7}{n-6} \cdots \frac{2}{3} I_1$$

ie., $\boxed{I_n = \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot 1}$ formula

III $\quad I_n = \int \cos^n x\, dx$

Soln:-

$$I_n = \int \cos^n x\, dx$$

$$= \int \cos^{n-1} x \cos x\, dx$$

$u = \cos^{n-1} x$ $\qquad \int dv = \int \cos x\, dx$

$\frac{du}{dx} = (n-1) \cos^{n-2} x\, (-\sin x)$ $\qquad v = \sin x$

$\therefore du = -(n-1) \cos^{n-2} x \sin x\, dx$

$$\int u\, dv = uv - \int v\, du$$

$$\therefore I_n = \left[\cos^{n-1} x \sin x\right] - \int \sin x \left(-(n-1) \cos^{n-2} x \sin x\, dx\right)$$

$$\therefore I_n = \cos^{n-1}x \sin x + (n-1)\int \sin^2 x \cos^{n-2}x \, dx$$

$$= \cos^{n-1}x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x \, dx$$

$$= \cos^{n-1}x \sin x + (n-1)\int \left(\cos^{n-2}x - \cos^n x\right) dx$$

$$= \cos^{n-1}x \sin x + (n-1)\left[\int \cos^{n-2}x \, dx - \int \cos^n x \, dx\right]$$

$$I_n = \cos^{n-1}x \sin x + (n-1)\left[I_{n-2} - I_n\right]$$

$$I_n = \cos^{n-1}x \sin x + (n-1)I_{n-2} - (n-1)I_n$$

$$\Rightarrow I_n + (n-1)I_n = \cos^{n-1}x \sin x + (n-1)I_{n-2}$$

$$\Rightarrow \cancel{I_n} + nI_n - \cancel{I_n} = \cos^{n-1}x \sin x + (n-1)I_{n-2}$$

$$\therefore \boxed{nI_n = \cos^{n-1}x \sin x + (n-1)I_{n-2}}$$

If $I_n = \int_0^{\pi/2} \cos^n x \, dx$, where n is positive

Soln:-

$$I_n = \int_0^{\pi/2} \cos^n x \, dx$$

$$= \left(\frac{\cos^{n-1}x \sin x}{n}\right)_0^{\pi/2} + \frac{(n-1)}{n} I_{n-2}$$

$\sin 0 = 0$
$\cos \pi/2 = 0$

$$= \frac{1}{n}\left[\cos^{n-1}\frac{\pi}{2} \sin\frac{\pi}{2} - \cos^{n-1}0 \sin 0\right] + \frac{(n-1)}{n} I_{n-2}$$

$$= \frac{1}{n}\left[0 - 0\right] + \frac{n-1}{n} I_{n-2}$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\, I_{n-4}$$

$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\, I_{n-6}$$

$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\,\frac{n-7}{n-6}\, I_{n-8}$$

$\vdots$

Case:(i) n is an even number

$$I_n = \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{1}{2}\,I_0$$

ie., $$\boxed{I_n = \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{1}{2}\,\frac{\pi}{2}}$$ formula

$$I_0 = \int_0^{\pi/2} \cos^0 x\,dx = \int_0^{\pi/2} dx = (x)_0^{\pi/2} = \pi/2$$

$$I_1 = \int_0^{\pi/2} \cos x\,dx = (\sin x)_0^{\pi/2} = \sin\pi/2 - \sin 0 = 1-0 = 1$$

case:(ii) n is odd number

$$I_n = \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{2}{3}\,I_1$$

ie., $$\boxed{I_n = \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{2}{3}\cdot 1}$$ formula

Problems:-

1. Find $\int_0^{\pi/2} \sin^6 x\,dx$

Soln:- $n=6$ is even no.

$$\therefore I_n = \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{1}{2}\,\frac{\pi}{2}$$

$$= \frac{6-1}{6}\cdot\frac{6-3}{6-2}\cdot\frac{6-5}{6-4}\cdot\frac{\pi}{2}$$

$$= \frac{5}{6}\,\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}$$

$$I_n = \frac{5\pi}{32}$$

ie., $$\boxed{\int_0^{\pi/2} \sin^6 x\,dx = \frac{5\pi}{32}}$$

2. Find $\int_0^{\pi/2} \sin^7 x\,dx$.

Soln:- $n=7$ is odd no.

$$\therefore I_n = \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{2}{3}\cdot 1$$

$$= \frac{7-1}{7}\,\frac{7-3}{7-2}\,\frac{7-5}{7-4}\cdot 1$$

$$I_n = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

$$I_n = \frac{16}{35}$$

ie., $\boxed{\int_0^{\pi/2} \sin^7 x \, dx = \frac{16}{35}}$

IV.

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}$$

n is odd no.

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

n is even no.

Formula only

(Proof not needed)

Problems:-

1. Find $\int_0^{\pi/2} \sin^6 x \cos^5 x \, dx$

Soln.:- $n=5$ is odd no.

$$\therefore \int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}$$

$$\therefore \int_0^{\pi/2} \sin^6 x \cos^5 x \, dx = \frac{5-1}{6+5} \cdot \frac{5-3}{6+5-2} \cdot \frac{1}{6+1}$$

$$= \frac{4}{11} \cdot \frac{2}{9} \cdot \frac{1}{7} = \frac{8}{693}$$

$$\therefore \boxed{\int_0^{\pi/2} \sin^6 x \cos^5 x \, dx = \frac{8}{693}}$$

2. Find $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$

Soln:- $n = 4$ is even no.

$\therefore \int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{n-1}{m+n} \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \frac{m-1}{m} \frac{m-3}{m-2} \cdots \frac{1}{2} \frac{\pi}{2}$

$\therefore \int_0^{\pi/2} \sin^6 x \cos^4 x \, dx = \frac{4-1}{6+4} \frac{4-3}{6+4-2} \frac{6-1}{6} \frac{6-3}{4} \frac{1}{2} \frac{\pi}{2}$

$= \frac{3}{10} \frac{1}{8} \frac{5}{6} \frac{3}{4} \frac{1}{2} \frac{\pi}{2}$

$= \frac{3\pi}{512}$

ie., $\boxed{\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx = \frac{3\pi}{512}}$

3. Find $\int_0^{\pi/2} \cos^8 x \, dx$

Soln:- $n = 8$ is even no.

$\int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{1}{2} \frac{\pi}{2}$

$\therefore \int_0^{\pi/2} \cos^8 x \, dx = \frac{8-1}{8} \frac{8-3}{8-2} \frac{8-5}{8-4} \frac{8-7}{8-6} \frac{\pi}{2}$

$= \frac{7}{8} \frac{5}{6} \frac{3}{4} \frac{1}{2} \frac{\pi}{2}$

$= \frac{35\pi}{256}$

$\therefore \boxed{\int_0^{\pi/2} \cos^8 x \, dx = \frac{35\pi}{256}}$.

4. Find $\int_0^{\pi/2} \cos^5 x \, dx$

Soln:- $n = 5$ is odd no.

$\int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{2}{3} = \frac{5-1}{5} \frac{5-3}{5-2} = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$

$\therefore \boxed{\int_0^{\pi/2} \cos^5 x \, dx = \frac{8}{15}}$.