BHARATHIDASAN UNIVERSITY, TIRUCHIRAPPALL – 620 024

B.Sc. STATISTICS - STUDENTS

(For the candidates admitted from the academic year 2016-17 onwards)

ALLIED COURSE I

CALCULUS, LAPLACE TRANSFORM AND FOURIER SERIES

Objects :

1. To train the students in basic calculus
2. To learn the basic ideas of Fourier Series

UNIT I

Maxima & Minima – Concavity , Convexity – Points of inflexion - Partial differentiation – Euler's Theorem - Total differential coefficients (proof not needed) –Simple problems only.

UNIT II

Evaluation of integrals of types

1] $\int \frac{px+q}{ax^2+bx+c}dx$ 2] $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ 3] $\int \frac{dx}{(x+p)\sqrt{ax^2+bx+c}}$

4] $\int \frac{dx}{a+b\cos x}$ 5] $\int \frac{dx}{a+b\sin x}$ 6] $\int \frac{(a\cos x+b\sin x+c)}{(p\cos x+q\sin x+r)}dx$

Evaluation using Integration by parts

Integration by trigonometric substitution and by parts of the integrals

1] $\int \sqrt{a^2-x^2}\,dx$ 2] $\int \sqrt{a^2+x^2}\,dx$ 3] $\int \sqrt{x^2-a^2}\,dx$

UNIT III

General properties of definite integrals – Evaluation of definite integrals of types

1] $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$ 2] $\int_a^b \sqrt{(x-a)(b-x)}\,dx$ 3] $\int_a^b \sqrt{\frac{x-a}{b-x}}\,dx$

Other simple problems. - Evaluation of Double and Triple integrals in simple cases Changing the order and evaluation of the double integration – Beta, Gamma functions.

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UNIT IV

Laplace Transforms – Inverse Laplace Transforms –Application of Laplace Transform in Solving second order Ordinary differential equation with constant coefficients.

UNIT V

Definition of Fourier Series – Fourier Coefficients for a given periodic function with period 2π and with period 2ℓ - Use of Odd & Even functions in evaluating Fourier Coefficients– Half range sine & cosine series.

TEXT BOOK(S)

1. S. Narayanan, T.K. Manichavasagam Pillai, Calculus, Vol. II, S. Viswanathan Pvt Limited, 2003
2. S. Narayanan, T.K. Manicavachagam Pillai, Calculus, Vol. III, S. Viswanathan Pvt Limited, and Vijay Nicole Imprints Pvt Ltd, 2004.

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CALCULUS, LAPLACE TRANSFORM AND FOURIER SERIES

UNIT – I

Maxima and Minima

Definition:-

If a continuous function decreases upto a certain value and then increases that value is called a minimum value of the function.

If a continuous function increases upto a certain value and then decreases that value is called a maximum value of the function.

Problem:-

Determine the maxima and minima of $x^5 - 5x^4 + 5x^3 + 10$

Solution:-

$$f(x) = x^5 - 5x^4 + 5x^3 + 10$$

$$f'(x) = 5x^4 - 20x^3 + 15x^2$$

$$= 5x^2(x^2 - 4x + 3)$$

to find the maximum (or) minimum $f'(x) = 0$

$$5x^2(x^2 - 4x + 3) = 0$$

$$x^2 - 4x + 3 = 0$$

$$(x-1)(x-3) = 0$$

$$x = 1, 3$$

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$x = 1, 3$ give maximum and minimum

$$f''(x) = 20x^3 - 60x^2 + 30x$$

$x = 1 \Rightarrow f''(1) = 20 - 60$

$x = 3 \Rightarrow f''(3) = 20(27) - 60(9) + 30(3)$

$$= 540 - 540 + 90$$

$$= 90 \ (+ve)$$

$f(1) = 1 - 5 + 5 + 10 = 11 \ (+ve)$

$f(3) = 243 - 5(81) + 5(27) + 10$

$$= 243 - 405 + 135 + 10$$

$$= 388 - 405$$

$$= -17 \ (-ve)$$

$x = 1$ gives the maximum value

$x = 3$ gives the minimum value

Maximum Value $= f(1) = 11$

Minimum Value $= f(3) = -17$

(2). Find the maximum value of $\frac{\log x}{x}$ for positive value of x.

Solution :-

$$f(x) = \frac{\log x}{x}$$

$$f'(x) = \log x\left(-\frac{1}{x^2}\right) + \frac{1}{x}\left(\frac{1}{x}\right)$$

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$$= \frac{-\log x}{x^2} + \frac{1}{x^2}$$

$$f'(x) = \frac{1-\log x}{x^2}$$

To find the maximum or minimum $f'(x) = 0$

$$\frac{1-\log x}{x^2} = 0$$

$$1-\log x = 0$$

$$-\log x = -1$$

$$\log x = 1 \qquad \log e' = 1$$

$$\log x = \log e'$$

$$x = e$$

$x=e$ give the minimum value

$$f''(x) = \frac{x^2(-\frac{1}{x}) - (1-\log x)(2x)}{x^4}$$

$$= \frac{-x - 2x(1-\log x)}{x^4}$$

$$= \frac{x\{-1-2(1-\log x)\}}{x^4}$$

$$= \frac{-1-2+2\log x}{x^3}$$

$$f''(x) = \frac{2\log x - 3}{x^3}$$

$$x = e \Rightarrow f''(e) = \frac{2\log - 3}{e^3}$$

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$$= \frac{2-3}{e^3}$$

$$f''(e) = \frac{-1}{e^3} \quad (-ve)$$

$\therefore$ $x = e$ gives the maximum values.

$$\therefore f(e) = \frac{\log e}{e}$$

$$= \frac{1}{e} \quad (+ve)$$

$\therefore$ the maximum value

$$f(e) = \frac{1}{e}$$

(3). Investigate the maximum and minimum value of the function $\frac{1+x+x^2}{1-x+x^2}$.

Solution:-

$$f(x) = \frac{1+x+x^2}{1-x+x^2}$$

$$f'(x) = \frac{(1-x+x^2)(1+2x) - (1+x+x^2)(-1+2x)}{(1-x+x^2)^2}$$

$$= \frac{1+2x-x-2x^2+x^2+2x^3-(-1+2x-x+2x^2-x^2+2x^3)}{(1-x+x^2)^2}$$

$$= \frac{1+x-2x^2+2x^3+1-x-2x^2-2x^3}{(1-x+x^2)^2}$$

$$f'(x) = \frac{2-2x^2}{(1-x+x^2)^2}$$

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$$= \frac{2(1-x^2)}{(1-x+x^2)^2}$$

$$f'(x) = \frac{2(1-x^2)}{(1-x+x^2)^2} = 0$$

$$2(1-x^2) = 0$$

$$1-x^2 = 0$$

$$-x^2 = -1$$

$$x^2 = 1$$

$$x = \pm 1$$

$$f''(x) = \frac{(1-x+x^2)^2(-4x) - (2-2x^2)\,2(1-x+x^2)(-1+2x)}{(1-x+x^2)^4}$$

$$= \frac{(1-x+x^2)\left[(1-x+x^2)(-4x) - (2-2x^2)^2\,2(-1+2x)\right]}{(1-x+x^2)^4}$$

$$= \frac{-4x+4x^2-4x^3-(2-2x^2)(-2+4x)}{(1-x+x^2)^3}$$

$$= \frac{-4x-4x^2-4x^3\,(-4+8x+4x^2-8x^3)}{(1-x+x^2)^3}$$

$$f''(x) = \frac{-4x+4x^2-4x^3+4-8x-4x^2+8x^3}{(1-x+x^2)^3}$$

$$= \frac{-12x+4x^3+4}{(1-x+x^2)^3}$$

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$$= \frac{4x^3 - 12x + 4}{(1 - x + x^2)^3}$$

$$f''(1) = \frac{4 - 12 + 4}{(1 - 1 + 1)^3}$$

$$= -4 \text{ (-ve)}$$

$$f''(-1) = \frac{-4 + 12 + 4}{(1 + 1 + 1)^3}$$

$$= \frac{12}{27}$$

$$= \frac{4}{9} \text{ (+ ve)}$$

$x = 1$ give maximum value

$x = -1$ give minimum value

$$f(1) = \frac{1 + 1 + 1}{1 - 1 + 1}$$

$$= 3 \text{ (+ve)}$$

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Partial differentiation:

Problems :-

1) Find the partial differential coefficients of $u = \sin(ax+by+cz)$.

Soln:-

$$u = \sin(ax+by+cz)$$

$$\frac{\partial u}{\partial x} = \cos(ax+by+cz)\cdot a = a\cos(ax+bx+c)$$ //

Illy $$\frac{\partial u}{\partial y} = b\cdot\cos(ax+by+cz)$$ //

$$\frac{\partial u}{\partial z} = c\cdot\cos(ax+by+cz)$$ //

2) If $u = \frac{xy}{x+y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

proof:-

$$u = \frac{xy}{x+y}$$

$$\frac{\partial u}{\partial x} = \frac{(x+y)(y) - xy(1)}{(x+y)^2} = \frac{\cancel{xy} + y^2 - \cancel{xy}}{(x+y)^2} = \frac{y^2}{(x+y)^2}$$ //

$$\frac{\partial u}{\partial y} = \frac{(x+y)(x) - xy(1)}{(x+y)^2} = \frac{x^2 + \cancel{xy} - \cancel{xy}}{(x+y)^2} = \frac{x^2}{(x+y)^2}$$ //

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{xy^2}{(x+y)^2} + \frac{x^2y}{(x+y)^2} = \frac{xy^2 + x^2y}{(x+y)^2}$$

$$= \frac{xy(y+x)}{(x+y)^2} = \frac{xy}{x+y} = u$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

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3. If $u = \tan^{-1} \frac{x^3+y^3}{x-y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

Proof:-

$$u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$$

$$\tan u = \frac{x^3+y^3}{x-y} \rightarrow ①$$

Differentiating w.r.to 'x' alone,

$$\sec^2 u \frac{\partial u}{\partial x} = \frac{(x-y)(3x^2) - (x^3+y^3)(1)}{(x-y)^2}$$

$$= \frac{3x^3 - 3x^2y - x^3 - y^3}{(x-y)^2}$$

$$= \frac{2x^3 - 3x^2y - y^3}{(x-y)^2} \rightarrow ②$$

Differentiating eqn ① w.r.to 'x', alone,

$$\sec^2 u \frac{\partial u}{\partial y} = \frac{(x-y)(3y^2) - (x^3+y^3)(-1)}{(x-y)^2}$$

$$= \frac{3xy^2 - 3y^3 + x^3 + y^3}{(x-y)^2}$$

$$= \frac{-2y^3 + 3xy^2 + x^3}{(x-y)^2} \rightarrow ③$$

$(x \times \text{eqn} ②) + (y \times \text{eqn} ③) \Rightarrow$

$$\sec^2 u \cdot x\frac{\partial u}{\partial x} + \sec^2 u \cdot y \frac{\partial u}{\partial y} = \frac{x(2x^3 - 3x^2y - y^3)}{(x-y)^2} + \frac{y(-2y^3 + 3xy^2 + x^3)}{(x-y)^2}$$

$$\sec^2 u \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \frac{2x^4 - 3x^3y - xy^3 - 2y^4 + 3xy^3 + x^3y}{(x-y)^2}$$

$$= \frac{2x^4 - 2x^3y - 2y^4 + 2xy^3}{(x-y)^2}$$

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$$\therefore \sec^2 u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=\frac{2x^3[x-y]+2y^3[x-y]}{(x-y)^2}$$

$$=\frac{(x-y)(2x^3+2y^3)}{(x-y)^2}$$

$$=\frac{2(x^3+y^3)}{(x-y)}$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=2\tan u \qquad \text{(by eqn. ①)}$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\tan u}{\sec^2 u}$$

$$\left[\because \tan u=\frac{\sin u}{\cos u},\quad \sec^2 u=\frac{1}{\cos^2 u}\right.$$

$$=2\frac{\sin u}{\cos u}\times\cos^2 u$$

$$=2\sin u\cos u$$

$$\therefore \boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u} \qquad (\because 2\sin\theta\cos\theta=\sin 2\theta)$$

4. $V=(x^2+y^2+z^2)^{-1/2}$, prove that $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0.$

Proof:-

$$V=(x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial V}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)$$

$$=-x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 V}{\partial x^2}=-\left[x\cdot -\frac{3}{2}(x^2+y^2+z^2)^{-5/2}\cdot(2x)+(x^2+y^2+z^2)^{-3/2}(1)\right]$$

$$=3x^2(x^2+y^2+z^2)^{-5/2}-(x^2+y^2+z^2)^{-3/2} \qquad \left(\because -\tfrac{3}{2}=-\tfrac{5}{2}+1\right)$$

$$=3x^2(x^2+y^2+z^2)^{-5/2}-(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2)$$

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$$\frac{\partial^2 V}{\partial x^2} = (x^2+y^2+z^2)^{-5/2}\left[3x^2-(x^2+y^2+z^2)\right]$$

$$= V^5\,(3x^2-x^2-y^2-z^2)$$

$$= V^5\,(2x^2-y^2-z^2) \qquad \because V^5=\left[(x^2+y^2+z^2)^{-1/2}\right]^5$$

IIIrly $\dfrac{\partial^2 V}{\partial y^2} = V^5\,(2y^2-x^2-z^2)$

$$\frac{\partial^2 V}{\partial z^2} = V^5\,(2z^2-x^2-y^2)$$

$$\therefore \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = V^5(2x^2-y^2-z^2)+V^5(2y^2-x^2-z^2)+V^5(2z^2-x^2-y^2)$$

$$= V^5\left(2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2\right)$$

$$= V^5\left(2x^2+2y^2+2z^2-2x^2-2y^2-2z^2\right)$$

$$= V^5\,(0)$$

$$= 0$$

Hence the proof.

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Euler's theorem: (proof not needed) (*) 2marks

If $f(x,y)$ is a homogeneous function of degree 'n',

then $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$.

This is known as Euler's theorem on homogeneous function.

Problems:-

1. If $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$.

soln:-

$$u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$$

$$\sin u = \frac{x^2+y^2}{x+y} = \frac{x^2\left(1+y^2/x^2\right)}{\cancel{x}\left(1+y/x\right)}$$

$$=\frac{x\left(1+y^2/x^2\right)}{\left(1+y/x\right)}$$

$=x^1 f(y/x)$ which is homogeneous function of degree 1.

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By Euler's theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=f \qquad (f=\sin u$$

$$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=\sin u$$

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\sin u$$

$$\cos u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\sin u}{\cos u}$$

$$\Rightarrow \boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u}$$

2. Find $\frac{dy}{dx}$, when $x=a(\cos t+\log\tan t/2)$, and $y=a\sin t$.

Soln:-

$$x=a(\cos t+\log\tan t/2)$$

$$\frac{dx}{dt}=a\left[-\sin t+\frac{1}{\tan t/2}\sec^2 t/2\cdot\frac{1}{2}\right]$$

$$=a\left[-\sin t+\frac{\cos t/2}{\sin t/2}\frac{1}{\cos^2 t/2}\cdot\frac{1}{2}\right]$$

$$=a\left[-\sin t+\frac{1}{2\sin t/2\,\cos t/2}\right]$$

$$=a\left[-\sin t+\frac{1}{\sin t}\right]$$

$2\sin\theta\cos\theta=\sin 2\theta$

$\therefore 2\sin\theta/2\cos\theta/2=\sin\theta$

$$=a\left[\frac{-\sin^2 t+1}{\sin t}\right]$$

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$$\frac{dx}{dt} = a\left(\frac{\cos^2 t}{\sin t}\right)$$

$$= a\,\frac{\cos t}{\sin t}\cdot \cos t$$

$$\frac{dx}{dt} = a \cot t \cos t$$ //

$$y = a\sin t \Rightarrow \frac{dy}{dt} = a\cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{a\cos t}{a\cot t\cos t}$$

$$\frac{dy}{dx} = \frac{1}{\cot t}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \tan t}$$

3. If $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Proof:- $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}}\left(\frac{1}{y}\right) - \frac{1}{\frac{x^2+y^2}{x^2}}\left(\frac{y}{x^2}\right)$$

$$= \frac{1}{\frac{1}{y}\sqrt{y^2-x^2}}\left(\frac{1}{y}\right) - \frac{y}{x^2+y^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$ //

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-x^2/y^2}}\left(-x/y^2\right) + \frac{1}{1+y^2/x^2}\left(1/x\right)$$

$$= \frac{-x}{\cancel{1/y}\sqrt{y^2-x^2}}\left(1/y^{\cancel{2}}\right) + \frac{1}{\frac{x^2+y^2}{x^2}}\left(1/\cancel{x}\right)$$

$$\frac{\partial u}{\partial y} = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$ //

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left(\frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}\right) + y\left(\frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}\right)$$

$$= \frac{x}{\sqrt{y^2-x^2}} - \cancel{\frac{xy}{x^2+y^2}} - \frac{xy}{y\sqrt{y^2-x^2}} + \cancel{\frac{xy}{x^2+y^2}}$$

$$= \cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{x}{\sqrt{y^2-x^2}}}$$

$$= 0.$$

Hence the proof

4. If $f(x,y) = \log\sqrt{x^2+y^2}$, find the value of $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.

Soln:-

$$f(x,y) = \log\sqrt{x^2+y^2}$$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x^2+y^2}}\ \frac{1}{\cancel{2}\sqrt{x^2+y^2}}\left(\cancel{2}x\right)$$

$$\frac{\partial f}{\partial x} = \frac{x}{\left(\sqrt{x^2+y^2}\right)^2}$$

$$\therefore \frac{\partial f}{\partial x} = \frac{x}{x^2+y^2} \,//$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 f}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \,//$$

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2\sqrt{x^2+y^2}}(2y)$$

$$= \frac{y}{x^2+y^2} \,//$$

$$\therefore \frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 f}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \,//$$

$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{\cancel{y^2}-\cancel{x^2}+\cancel{x^2}-\cancel{y^2}}{(x^2+y^2)^2}$$

$$\therefore \boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0}$$

5. Find $\frac{dy}{dx}$, when $x = at^2$, $y = at$. Ans:- $\frac{dy}{dx} = \cot \theta/2$.

6. If $e^{-z/(x^2+y^2)} = x-y$, prove that $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x^2-y^2$.

Proof:-

Given that $e^{-z/(x^2+y^2)} = x-y$

Take 'log' on both sides, we get

$$\log e^{-z/(x^2-y^2)} = \log(x-y)$$

$$\Rightarrow \frac{-z}{x^2-y^2} = \log(x-y)$$

$$\Rightarrow -z = (x^2-y^2)\log(x-y)$$

$$\Rightarrow z = -(x^2-y^2)\log(x-y)$$

$$\frac{\partial z}{\partial x} = -\left[(x^2-y^2)\frac{1}{x-y} + \log(x-y)\,(2x)\right]$$

$$\frac{\partial z}{\partial x} = -\frac{(x^2-y^2)}{x-y} - 2x\log(x-y) \,//$$

$$\Rightarrow \frac{\partial z}{\partial y} = -\left[(x^2-y^2)\frac{1}{x-y}(-1) + \log(x-y)\,(-2y)\right]$$

$$= \frac{x^2-y^2}{x-y} + 2y\log(x-y) \,//$$

$$\therefore y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = y\left[-\frac{(x^2-y^2)}{x-y} - 2x\log(x-y)\right] + x\left[\frac{x^2-y^2}{x-y} + 2y\log(x-y)\right]$$

$$= -y\frac{(x^2-y^2)}{x-y} - \cancel{2xy\log(x-y)} + x\frac{(x^2-y^2)}{x-y} + \cancel{2xy\log(x-y)}$$

$$= \frac{-y(x^2-y^2) + x(x^2-y^2)}{x-y}$$

$$= \frac{(x^2-y^2)\cancel{(x-y)}}{\cancel{(x-y)}} = x^2-y^2 \,//$$

Hence the proof.

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7. Verify Euler's Theorem, when $u = x^3 + y^3 + z^3 + 3xyz$.

Soln. :-

$$u = x^3 + y^3 + z^3 + 3xyz$$

$$\frac{\partial u}{\partial x} = 3x^2 + 3yz$$

$$\frac{\partial u}{\partial y} = 3y^2 + 3xz$$

$$\frac{\partial u}{\partial z} = 3z^2 + 3xy$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x(3x^2 + 3yz) + y(3y^2 + 3zx) + z(3z^2 + 3xy)$$

$$= 3x^3 + 3xyz + 3y^3 + 3xyz + 3z^3 + 3xyz$$

$$= 3x^3 + 3y^3 + 3z^2 + 9xyz$$

$$= 3(x^3 + y^3 + z^3 + 3xyz)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$$

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Total differential coefficients:

1. Find $\frac{du}{dt}$, where $u=x^2+y^2+z^2$, $x=e^t$, $y=e^t \sin t$ and $z=e^t \cos t$.

Soln:-

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \rightarrow (1)$$

Given that $u=x^2+y^2+z^2$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial u}{\partial z} = 2z$$

$x=e^t$

$\frac{\partial x}{\partial t} = e^t$

$y=e^t \sin t$

$\frac{\partial y}{\partial t} = e^t \cos t + \sin t\, e^t$

$= e^t(\cos t + \sin t)$

$z=e^t \cos t$

$\frac{\partial z}{\partial t} = e^t(-\sin t) + \cos t\, e^t$

$= e^t(-\sin t + \cos t)$

$\therefore (1) \Rightarrow \frac{du}{dt} = (2x)(e^t) + (2y)\, e^t(\cos t + \sin t) + (2z) e^t (-\sin t + \cos t)$

$= 2e^t\left[x + y(\cos t + \sin t) + z(\cos t - \sin t)\right]$

$= 2e^t\left[e^t + e^t \sin t(\cos t + \sin t) + e^t \cos t(\cos t - \sin t)\right]$

$= 2e^t \cdot e^t\left[1 + \cancel{\sin t \cos t} + \sin^2 t + \cos^2 t - \cancel{\sin t \cos t}\right]$

$= 2e^{2t}[1+1]$ $\quad (\because \sin^2 t + \cos^2 t = 1)$

$$\boxed{\frac{du}{dt} = 4e^{2t}}$$

2. Find $\frac{du}{dx}$, where $u=x^2+y^2$ and $y=\frac{1-x}{x}$.

Soln:-

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

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$u = x^2 + y^2$ & $y = \frac{1-x}{x}$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$

$\frac{dy}{dx} = \frac{x(-1) - (1-x)(1)}{x^2}$ $\quad d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$

$= \frac{-x - 1 + x}{x^2}$

$\frac{dy}{dx} = -\frac{1}{x^2}$

$\therefore \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$

$= 2x + 2y\left(-\frac{1}{x^2}\right)$

$= 2x + 2\left(\frac{1-x}{x}\right)\left(-\frac{1}{x^2}\right)$

$= 2\left[x - \frac{1-x}{x^3}\right]$

$= 2\left[\frac{x^4 - 1 + x}{x^3}\right]$

$\therefore \boxed{\frac{du}{dx} = \frac{2}{x^3}(x^4 + x - 1)}$

3. If $x^3 + y^3 + 3axy$, find $\frac{dy}{dx}$.

Soln.:-

Let $f(x,y) = x^3 + y^3 + 3axy$.

$\frac{\partial f}{\partial x} = 3x^2 + 3ay$

& $\frac{\partial f}{\partial y} = 3y^2 + 3ax$.

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(7)

$$\frac{dy}{dx} = \frac{-\partial f/\partial y}{\partial f/\partial x} \rightarrow \text{formula}$$

$$= \frac{-(3y^2+3ax)}{(3x^2+3ay)}$$

$$= \frac{-\cancel{3}(y^2+ax)}{\cancel{3}(x^2+ay)}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{-(y^2+ax)}{x^2+ay}}$$

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UNIT-II

Evaluation of Integrals of following types:

Type: 1 $\int \frac{px+q}{ax^2+bx+c}\,dx$.

Main Formulas:

1. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)$

2. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$

3. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$

Problems:-

1. Evaluate: $\int \frac{2x+3}{x^2+x+1}\,dx \rightarrow (1)$

$px+q = A\frac{d}{dx}(ax^2+bx+c)+B$

Soln:-

$2x+3 = A\frac{d}{dx}(x^2+x+1)+B$

$2x+1=0$

$2x=-1$

$x=-\frac{1}{2}$

$2x+3 = A(2x+1)+B \rightarrow (2)$

put $x=-\frac{1}{2}$, we get

$2(-\frac{1}{2})+3 = A\left[2(-\frac{1}{2})+1\right]+B$

$-1+3 = A(0)+B$

$\therefore \boxed{B=2}$

put $x=0$, we get

$2(0)+3 = A[2(0)+1]+B$

$3 = A+B$

$\therefore A+2=3$

$A=3-2$

$\boxed{A=1}$

$\therefore (2) \Rightarrow 2x+3 = 1(2x+1)+2$

$\therefore 2x+3 = (2x+1)+2$

$\int \frac{f'(x)}{f(x)}dx = \log(f(x))$

$\therefore (1) \Rightarrow \int \frac{2x+3}{x^2+x+1}dx = \int \frac{(2x+1)+2}{x^2+x+1}dx$

$= \int \frac{2x+1}{x^2+x+1}dx + 2\int \frac{dx}{x^2+x+1}$

$= \log(x^2+x+1) + 2\int \frac{dx}{x^2+x+1+\frac{1}{4}-\frac{1}{4}}$

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$$\int \frac{2x+3}{x^2+x+1}dx = \log(x^2+x+1) + 2\int \frac{dx}{x^2+x+\frac{1}{4}+1-\frac{1}{4}}$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$= \log(x^2+x+1) + 2\frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)$$

$$= \log(x^2+x+1) + 2\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\frac{2x+1}{\cancel{2}}}{\frac{\sqrt{3}}{\cancel{2}}}\right)$$

$$\boxed{\therefore \int \frac{2x+3}{x^2+x+1}dx = \log(x^2+x+1) + \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)}$$

=x=

2. Evaluate: $\int \frac{x+4}{6x-7-x^2}dx \rightarrow (1)$

Soln:-

$$x+4 = A\frac{d}{dx}(6x-7-x^2)+B$$

$$x+4 = A(6-2x)+B \rightarrow (2)$$

$6-2x=0$
$\cancel{-}2x=\cancel{-}6$
$x=6/2$
$x=3$

put $x=3$, we get

$3+4 = A(6-6)+B$

$7 = A(0)+B$

$\therefore \boxed{B=7}$

put $x=0$, we get

$4 = A(6)+B$

$6A+7=4$

$6A = 4-7$

$A = \frac{-3}{6}$

$\boxed{A=-\frac{1}{2}}$

$(2) \Rightarrow x+4 = -\frac{1}{2}(6-2x)+7$

$$\therefore (1) \Rightarrow \int \frac{x+4}{6x-7-x^2}dx = \int \frac{-\frac{1}{2}(6-2x)+7}{6x-7-x^2}dx$$

$$= -\frac{1}{2}\int \frac{6-2x}{6x-7-x^2}dx + 7\int \frac{dx}{6x-7-x^2}$$

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$$\int \frac{x+4}{6x-7-x^2}\,dx = \frac{-1}{2}\log(6x-7-x^2) + 7\int \frac{dx}{-(x^2-6x+7)}$$

$$= -\frac{1}{2}\log(6x-7-x^2) - 7\int \frac{dx}{x^2-6x+7+9-9}$$

$$= -\frac{1}{2}\log(6x-7-x^2) - 7\int \frac{dx}{(x^2-6x+9)+(7-9)}$$

$$= -\frac{1}{2}\log(6x-7-x^2) - 7\int \frac{dx}{(x-3)^2-2}$$

$$= -\frac{1}{2}\log(6x-7-x^2) - 7\int \frac{dx}{(x-3)^2-(\sqrt{2})^2}$$

$$= -\frac{1}{2}\log(6x-7-x^2) - 7\,\frac{1}{2\sqrt{2}}\log\left(\frac{x-3-\sqrt{2}}{x-3+\sqrt{2}}\right)$$

$$\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$$

$$\boxed{\therefore \int \frac{x+4}{6x-7-x^2}\,dx = -\frac{1}{2}\log(6x-7-x^2) - \frac{7}{2\sqrt{2}}\log\left(\frac{x-3-\sqrt{2}}{x-3+\sqrt{2}}\right)}$$

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3. Evaluate: $\int \frac{3x+1}{2x^2+x+6}\,dx \rightarrow ①$

Soln:

$$3x+1 = A\frac{d}{dx}(2x^2+x+6)+B$$

$$3x+1 = A(4x+1)+B \rightarrow ②$$

put $x=-\frac{1}{4}$, we get

$$3\left(-\tfrac{1}{4}\right)+1 = A\left[4\left(-\tfrac{1}{4}\right)+1\right]+B$$

$$-\frac{3}{4}+1 = A(0)+B$$

$$\boxed{B = \tfrac{1}{4}}$$

put $x=0$, we get

$$1 = A+B$$

$$A+\tfrac{1}{4} = 1$$

$$A = 1-\tfrac{1}{4}$$

$$\boxed{A = \tfrac{3}{4}}$$

$$② \Rightarrow 3x+1 = \frac{3}{4}(4x+1)+\tfrac{1}{4}$$

$$① \Rightarrow \int \frac{3x+1}{2x^2+x+6}\,dx = \int \frac{\frac{3}{4}(4x+1)+\frac{1}{4}}{2x^2+x+6}\,dx$$

$$= \frac{3}{4}\int \frac{4x+1}{2x^2+x+6}\,dx + \tfrac{1}{4}\int \frac{dx}{2x^2+x+6}$$

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$$\int \frac{3x+1}{2x^2+x+6}\,dx = \frac{3}{4}\log(2x^2+x+6) + \frac{1}{4}\int \frac{dx}{2(x^2+\frac{x}{2}+3)}$$

$$= \frac{3}{4}\log(2x^2+x+6) + \frac{1}{8}\int \frac{dx}{x^2+\frac{x}{2}+3+\frac{1}{16}-\frac{1}{16}}$$

$$= \frac{3}{4}\log(2x^2+x+6) + \frac{1}{8}\int \frac{dx}{(x^2+\frac{x}{2}+\frac{1}{16})+(3-\frac{1}{16})}$$

$$= \frac{3}{4}\log(2x^2+x+6) + \frac{1}{8}\int \frac{dx}{(x+\frac{1}{4})^2+(\frac{48-1}{16})}$$

$$= \frac{3}{4}\log(2x^2+x+6) + \frac{1}{8}\int \frac{dx}{(x+\frac{1}{4})^2+(\frac{47}{16})}$$

$$= \frac{3}{4}\log(2x^2+x+6) + \frac{1}{8}\int \frac{dx}{(x+\frac{1}{4})^2+(\frac{\sqrt{47}}{4})^2}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)$$

$$= \frac{3}{4}\log(2x^2+x+6) + \frac{1}{8}\,\frac{1}{\sqrt{47}/4}\tan^{-1}\left(\frac{x+\frac{1}{4}}{\sqrt{47}/4}\right)$$

$$= \frac{3}{4}\log(2x^2+x+6) + \frac{1}{\cancel{8}_{2}}\,\frac{\cancel{4}}{\sqrt{47}}\tan^{-1}\left(\frac{\frac{4x+1}{\cancel{4}}}{\sqrt{47}/\cancel{4}}\right)$$

$$\therefore \boxed{\int \frac{3x+1}{2x^2+x+6}\,dx = \frac{3}{4}\log(2x^2+x+6) + \frac{1}{2\sqrt{47}}\tan^{-1}\left(\frac{4x+1}{\sqrt{47}}\right)}$$

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Try this Problems:-

1. $\int \frac{3x+5}{x^2+4x+7}\,dx = \frac{3}{2}\log(x^2+4x+7) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right)$

2. $\int \frac{3x+1}{2x^2-x+5}\,dx = \frac{3}{4}\log(2x^2-x+5) - \frac{20}{\sqrt{429}}\log\left(\frac{2x+21-\sqrt{429}}{2x+21+\sqrt{429}}\right)$

3. $\int \frac{2x+3}{x^2+2x+5}\,dx = \log(x^2+2x+5) + \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)$

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Type: 2 $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

Main Formulas:-

1. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a)$

2. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}(x/a)$ (or) $\log\left(x+\sqrt{x^2+a^2}\right)$

3. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a)$ (or) $\log\left(x+\sqrt{x^2-a^2}\right)$

Problems:-

1. Evaluate: $\int \frac{x}{\sqrt{x^2+x+1}}\,dx \rightarrow (1)$

Soln:-

$$x = A\frac{d}{dx}(x^2+x+1)+B$$

$$x = A(2x+1)+B \rightarrow (2)$$

put $x=-1/2$, we get

$$-\frac{1}{2} = A(0)+B$$

$$\boxed{B=-1/2}$$

put $x=0$, we get

$$0 = A+B$$

$$A-1/2=0$$

$$\boxed{A=1/2}$$

$\therefore (2) \Rightarrow \quad x = \frac{1}{2}(2x+1)-\frac{1}{2}$

$\therefore (1) \Rightarrow \quad \int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{\frac{1}{2}(2x+1)-\frac{1}{2}}{\sqrt{x^2+x+1}}\,dx$

$$= \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$$

$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \frac{1}{\cancel{2}}\int \frac{\cancel{2}y\,dy}{y} - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+1+\frac{1}{4}-\frac{1}{4}}}$$

put $y=\sqrt{x^2+x+1}$

$y^2 = x^2+x+1$

$2y\frac{dy}{dx} = 2x+1$

$2y\,dy = (2x+1)dx$

$$= \int dy - \frac{1}{2}\int \frac{dx}{\sqrt{(x^2+x+\frac{1}{4})+(1-\frac{1}{4})}}$$

$$= y - \frac{1}{2}\int \frac{dx}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}$$

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$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx = y - \frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= y - \frac{1}{2}\sinh^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$\because \int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}(x/a)$

$$= y - \frac{1}{2}\sinh^{-1}\left(\frac{\frac{2x+1}{\cancel{2}}}{\sqrt{3}/\cancel{2}}\right)$$

$$= y - \frac{1}{2}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$\therefore \boxed{\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \sqrt{x^2+x+1} - \frac{1}{2}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).}$$

Try this Problems:-

1. $\int \frac{6x+5}{\sqrt{6+x-2x^2}}\,dx = -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)$

2. $\int \frac{3x-2}{\sqrt{4x^2-4x-5}}\,dx = \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\cosh^{-1}\left(\frac{2x-1}{\sqrt{6}}\right)$

3. $\int \left(\frac{3-2x}{1-x}\right)^{1/2} dx = -\sqrt{3-5x+2x^2} + \frac{1}{2\sqrt{2}}\cosh^{-1}(4x-5)$

2. Evaluate: $\int \left(\frac{x-1}{2x+3}\right)^{1/2} dx$

Soln.:-

$$\int \left(\frac{x-1}{2x+3}\right)^{1/2} dx = \int \frac{\sqrt{x-1}}{\sqrt{2x+3}}\,dx$$

$$= \int \frac{\sqrt{x-1}}{\sqrt{2x+3}} \times \frac{\sqrt{x-1}}{\sqrt{x-1}}\,dx$$

$\sqrt{x} \times \sqrt{x} = x$

$$= \int \frac{(x-1)}{\sqrt{(2x+3)(x-1)}}\,dx$$

$$= \int \frac{x-1}{\sqrt{2x^2-2x+3x-3}}\,dx$$

$$\therefore \int \left(\frac{x-1}{2x+3}\right)^{1/2} dx = \int \frac{x-1}{\sqrt{2x^2+x-3}}\,dx \longrightarrow (1)$$

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$$x-1 = A\frac{d}{dx}(2x^2+x-3)+B$$

$$x-1 = A(4x+1)+B \rightarrow ②$$

put $x=-\frac{1}{4}$, we get

$$-\frac{1}{4}-1 = A[4(-\tfrac{1}{4})+1]+B$$

$$-\frac{5}{4} = A(0)+B$$

$$\therefore \boxed{B=-\frac{5}{4}}$$

put $x=0$, we get

$$-1 = A+B$$

$$\therefore A-\frac{5}{4} = -1$$

$$A = -1+\frac{5}{4}$$

$$\therefore \boxed{A=\frac{1}{4}}$$

$$\therefore ② \Rightarrow x-1 = \frac{1}{4}(4x+1)-\frac{5}{4}$$

$$\therefore ① \Rightarrow \int\frac{(x-1)}{\sqrt{2x^2+x-3}}\,dx = \frac{1}{4}\int\frac{4x+1}{\sqrt{2x^2+x-3}}\,dx - \frac{5}{4}\int\frac{dx}{\sqrt{2x^2+x-3}}$$

$$\int\frac{x-1}{\sqrt{2x^2+x-3}}\,dx = \frac{1}{\cancel{4}_2}\int\frac{\cancel{2}y\,dy}{y} - \frac{5}{4}\int\frac{dx}{\sqrt{2(x^2+\frac{x}{2}-\frac{3}{2})}}$$

Let $y=\sqrt{2x^2+x-3}$

$$y^2 = 2x^2+x-3$$

$$2y\frac{dy}{dx} = 4x+1$$

$$2y\,dy = (4x+1)\,dx$$

$$= \frac{1}{2}\int dy - \frac{5}{4}\int\frac{dx}{\sqrt{2}\times\sqrt{x^2+\frac{x}{2}-\frac{3}{2}+\frac{1}{16}-\frac{1}{16}}}$$

$$= \frac{1}{2}y - \frac{5}{4\sqrt{2}}\int\frac{dx}{\sqrt{(x^2+\frac{x}{2}+\frac{1}{16})-\frac{3}{2}-\frac{1}{16}}}$$

$$= \frac{1}{2}y - \frac{5}{4\sqrt{2}}\int\frac{dx}{\sqrt{(x+\frac{1}{4})^2-\frac{25}{16}}}$$

$$= \frac{1}{2}y - \frac{5}{4\sqrt{2}}\int\frac{dx}{\sqrt{(x+\frac{1}{4})^2-(\frac{5}{4})^2}}$$

$$= \frac{1}{2}y - \frac{5}{4\sqrt{2}}\cosh^{-1}\left(\frac{x+\frac{1}{4}}{\frac{5}{4}}\right)$$

$$\because \int\frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a)$$

$$= \frac{1}{2}y - \frac{5}{4\sqrt{2}}\cosh^{-1}\left(\frac{\frac{4x+1}{\cancel{4}}}{5/\cancel{4}}\right)$$

$$\therefore \int\frac{x-1}{\sqrt{2x^2+x-3}}\,dx = \frac{1}{2}\sqrt{2x^2+x-3} - \frac{5}{4\sqrt{2}}\cosh^{-1}\left(\frac{4x+1}{5}\right)$$

ie., $$\boxed{\int\left(\frac{x-1}{2x+3}\right)^{1/2}dx = \frac{1}{2}\sqrt{2x^2+x-3} - \frac{5}{4\sqrt{2}}\cosh^{-1}\left(\frac{4x+1}{5}\right)}$$

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Type: 3 $\int \frac{dx}{a+b\cos x}$ and Type: 4 $\int \frac{dx}{a+b\sin x}$

Problems:-

1. Evaluate: $\int \frac{dx}{4+5\cos x}$

Soln:- put $t = \tan x/2$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2 x/2$$

$$dt = \frac{1}{2}\sec^2 x/2 \, dx$$

$$dt = \frac{1}{2}(1+\tan^2 x/2)\, dx$$

$$\therefore dt = \frac{1}{2}(1+t^2)\, dx$$

$$\Rightarrow \boxed{dx = \frac{2dt}{1+t^2}}$$

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\Rightarrow \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$$

$$\therefore \boxed{\cos x = \frac{1-t^2}{1+t^2}}$$

$$\therefore \int \frac{dx}{4+5\cos x} = \int \frac{2dt/1+t^2}{4+5\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= 2\int \frac{dt/\cancel{1+t^2}}{\frac{4(1+t^2)+5(1-t^2)}{(\cancel{1+t^2})}}$$

$$= 2\int \frac{dt}{4(1+t^2)+5(1-t^2)}$$

$$= 2\int \frac{dt}{4+4t^2+5-5t^2}$$

$$= 2\int \frac{dt}{9-t^2}$$

$$= 2\int \frac{dt}{3^2-t^2}$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$$

$$= \cancel{2}\frac{1}{\cancel{2}\times 3}\log\left(\frac{3+t}{3-t}\right)$$

$$= \frac{1}{3}\log\left(\frac{3+t}{3-t}\right)$$

$$\boxed{\int \frac{dx}{4+5\cos x} = \frac{1}{3}\log\left(\frac{3+\tan x/2}{3-\tan x/2}\right)}$$

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2. Evaluate: $\int \frac{dx}{3\sin x + 4\cos x}$

Soln:-

put $t = \tan x/2$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2 x/2$$

$$\frac{dt}{dx} = \frac{1}{2}(1+\tan^2 x/2)$$

$$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$$

$$\boxed{dx = \frac{2dt}{1+t^2}}$$

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\therefore \sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$$

$$\Rightarrow \boxed{\sin x = \frac{2t}{1+t^2}}$$

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\Rightarrow \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$$

$$\Rightarrow \boxed{\cos x = \frac{1-t^2}{1+t^2}}$$

$$\therefore \int \frac{dx}{3\sin x + 4\cos x} = \int \frac{2dt/1+t^2}{3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= 2\int \frac{dt/\cancel{1+t^2}}{\frac{6t+4(1-t^2)}{\cancel{1+t^2}}}$$

$$= 2\int \frac{dt}{6t+4-4t^2}$$

$$= 2\int \frac{dt}{4\left(\frac{3}{2}t+1-t^2\right)} = \frac{2}{4}\int \frac{dt}{-t^2+\frac{3}{2}t+1+\frac{9}{16}-\frac{9}{16}}$$

$$= \frac{1}{2}\int \frac{dt}{\left(-t^2+\frac{3}{2}t-\frac{9}{16}\right)+\left(1+\frac{9}{16}\right)}$$

$$= \frac{1}{2}\int \frac{dt}{-\left(t^2-\frac{3}{2}t+\frac{9}{16}\right)+\frac{25}{16}} = \frac{1}{2}\int \frac{dt}{-\left(t-\frac{3}{4}\right)^2+\left(\frac{5}{4}\right)^2}$$

$$= \frac{1}{\cancel{2}}\,\frac{1}{\cancel{2}\frac{5}{4}}\log\left(\frac{\frac{5}{4}+\left(t-\frac{3}{4}\right)}{\frac{5}{4}-\left(t-\frac{3}{4}\right)}\right) \qquad \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$$

$$= \frac{1}{5}\log\left(\frac{\frac{5}{4}+t-\frac{3}{4}}{\frac{5}{4}-t+\frac{3}{4}}\right)$$

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$$\int \frac{dx}{3\sin x + 4\cos x} = \frac{1}{5}\log\left(\frac{\frac{5-3}{4}+t}{\frac{5+3}{4}+t}\right)$$

$$= \frac{1}{5}\log\left(\frac{\frac{2}{4}+t}{\frac{8}{4}+t}\right)$$

$$= \frac{1}{5}\log\left(\frac{\frac{1}{2}+t}{2+t}\right)$$

i.e. $$\boxed{\int \frac{dx}{3\sin x + 4\cos x} = \frac{1}{5}\log\left(\frac{\frac{1}{2}+\tan x/2}{2+\tan x/2}\right)}$$

3. Evaluate: $\int \frac{dx}{1+3\sin x + 4\cos x}$

Soln:-

put $t = \tan x/2$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2 x/2$$

$$\Rightarrow dx = \frac{2dt}{\sec^2 x/2}$$

$$dx = \frac{2dt}{1+\tan^2 x/2}$$

$$\boxed{dx = \frac{2dt}{1+t^2}}$$

$$\boxed{\sin x = \frac{2t}{1+t^2}}$$

$$\boxed{\cos x = \frac{1-t^2}{1+t^2}}$$

$$\therefore \int \frac{dx}{1+3\sin x+4\cos x} = \int \frac{2dt/1+t^2}{1+3\left(\frac{2t}{1+t^2}\right)+4\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= 2\int \frac{dt/\cancel{1+t^2}}{\frac{(1+t^2)+3(2t)+4(1-t^2)}{\cancel{1+t^2}}} = 2\int \frac{dt}{1+t^2+6t+4-4t^2}$$

$$= 2\int \frac{dt}{5+6t-3t^2} = \frac{2}{3}\int \frac{dt}{5/3+2t-t^2} = \frac{2}{3}\int \frac{dt}{-t^2+2t+5/3+1-1}$$

$$= \frac{2}{3}\int \frac{dt}{(-t^2+2t-1)+\frac{5}{3}+1} = \frac{2}{3}\int \frac{dt}{-(t^2-2t+1)+\frac{8}{3}}$$

$$= \frac{2}{3}\int \frac{dt}{-(t-1)^2+\left(\frac{2\sqrt{2}}{\sqrt{3}}\right)^2} = \frac{\cancel{2}}{3}\frac{1}{\cancel{2}\left(\frac{2\sqrt{2}}{\sqrt{3}}\right)}\log\left(\frac{\frac{2\sqrt{2}}{\sqrt{3}}+(t-1)}{\frac{2\sqrt{2}}{\sqrt{3}}-(t-1)}\right)$$

$$= \frac{1}{3}\frac{\sqrt{3}}{2\sqrt{2}}\log\left(\frac{2\sqrt{2}+\sqrt{3}(t-1)/\cancel{\sqrt{3}}}{2\sqrt{2}-\sqrt{3}(t-1)/\cancel{\sqrt{3}}}\right)$$

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$$\int \frac{dx}{1+3\sin x+4\cos x} = \frac{\sqrt{3}}{\sqrt{3}\sqrt{3}\cdot 2\sqrt{2}} \log\left(\frac{2\sqrt{2}+\sqrt{3}(t-1)}{2\sqrt{2}-\sqrt{3}(t-1)}\right)$$

$$= \frac{1}{2\sqrt{3\times 2}} \log\left(\frac{2\sqrt{2}+\sqrt{3}t-\sqrt{3}}{2\sqrt{2}-\sqrt{3}t+\sqrt{3}}\right)$$

$$= \frac{1}{2\sqrt{6}} \log\left(\frac{2\sqrt{2}+\sqrt{3}t-\sqrt{3}}{2\sqrt{2}-\sqrt{3}t+\sqrt{3}}\right)$$

ie.,
$$\boxed{\int \frac{dx}{1+3\sin x+4\cos x} = \frac{1}{2\sqrt{6}} \log\left(\frac{2\sqrt{2}+\sqrt{3}\tan x/2-\sqrt{3}}{2\sqrt{2}-\sqrt{3}\tan x/2+\sqrt{3}}\right)}$$

Try This Problems:-

1. $$\int \frac{dx}{12+13\cos x} = \frac{1}{5}\log\left(\frac{5+\tan x/2}{5-\tan x/2}\right)$$

2. $$\int \frac{dx}{1+\sin x+\cos x} = \log(1+\tan x/2)$$

3. $$\int \frac{dx}{\sin x+\sqrt{3}\cos x} = \frac{1}{2}\log\left(\frac{1+\sqrt{3}\tan x/2}{\sqrt{3}-\tan x/2}\right)$$

Type: 5 $\int \frac{dx}{(x+p)\sqrt{ax^2+bx+c}}$

Problems:-

1. Evaluate: $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$

Soln:-

put $x+1 = \frac{1}{t}$

$$\frac{dx}{dt} = -\frac{1}{t^2}$$

$$\boxed{dx = -\frac{dt}{t^2}}$$

$x+1 = \frac{1}{t}$

$x = \frac{1}{t}-1$

$x^2 = \left(\frac{1}{t}-1\right)^2$

$x^2 = \frac{1}{t^2}-\frac{2}{t}+1$

$x^2+x+1 = \frac{1}{t^2}-\frac{2}{t}+1+1+\frac{1}{t}$

$$\boxed{x^2+x+1 = \frac{1}{t^2}-\frac{1}{t}+1}$$

$$\therefore \int \frac{dx}{(x+1)\sqrt{x^2+x+1}} = \int \frac{-dt/t^2}{\frac{1}{t}\sqrt{\frac{1}{t^2}-\frac{1}{t}+1}} = -\int \frac{dt/t^2}{\frac{1}{t}\sqrt{\frac{1}{t^2}(1-t+t^2)}}$$

$$= -\int \frac{dt/t^2}{\frac{1}{t}\cdot\frac{1}{t}\sqrt{t^2-t+1}} = -\int \frac{dt/t^2}{\frac{1}{t^2}\sqrt{t^2-t+1+\frac{1}{4}-\frac{1}{4}}}$$

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$$\int \frac{dx}{(x+1)\sqrt{x^2+x+1}} = -\int \frac{dt}{\sqrt{(t^2-t+1/4)+(1-1/4)}}$$

$$= -\int \frac{dt}{\sqrt{(t-1/2)^2+3/4}} = -\int \frac{dt}{\sqrt{(t-1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= -\sinh^{-1}\left(\frac{t-\frac{1}{2}}{\sqrt{3}/2}\right) \quad \because \int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}(x/a)$$

$$= -\sinh^{-1}\left(\frac{2t-1/\cancel{2}}{\sqrt{3}/\cancel{2}}\right)$$

ie., $$\int \frac{dx}{(x+1)\sqrt{x^2+x+1}} = -\sinh^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)$$

$$= -\sinh^{-1}\left(\frac{2\left(\frac{1}{x+1}\right)-1}{\sqrt{3}}\right) \qquad x+1 = \frac{1}{t} \Rightarrow t = \frac{1}{x+1}$$

$$= -\sinh^{-1}\left(\frac{\frac{2-(x+1)}{x+1}}{\sqrt{3}}\right)$$

$$= -\sinh^{-1}\left(\frac{2-x-1}{\sqrt{3}\,(x+1)}\right)$$

ie., $$\boxed{\int \frac{dx}{(x+1)\sqrt{x^2+x+1}} = -\sinh^{-1}\left(\frac{1-x}{\sqrt{3}\,(x+1)}\right)}$$

2. Evaluate: $\int \frac{dx}{(3+x)\sqrt{x}}$.

Soln.:- put $3+x = \frac{1}{t}$

$$\frac{dx}{dt} = -\frac{1}{t^2}$$

$$\boxed{dx = -\frac{dt}{t^2}}$$

$$3+x = \frac{1}{t}$$

$$\boxed{x = \frac{1}{t}-3}$$

$$\therefore \int \frac{dx}{(3+x)\sqrt{x}} = \int \frac{-dt/t^2}{\frac{1}{t}\sqrt{\frac{1}{t}-3}} = -\int \frac{dt/t^2}{\frac{1}{t}\cdot\frac{1}{t}\sqrt{t-3t^2}} = -\int \frac{dt/\cancel{t^2}}{\frac{1}{\cancel{t^2}}\sqrt{t-3t^2}}$$

$$= -\int \frac{dt}{\sqrt{t-3t^2}} = -\int \frac{dt}{\sqrt{3}\sqrt{\frac{t}{3}-t^2}} = -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{-t^2+\frac{t}{3}+\frac{1}{36}-\frac{1}{36}}}$$

$$\int \frac{dx}{(3+x)\sqrt{x}} = -\frac{1}{\sqrt{3}} \int \frac{dt}{\sqrt{-(t^2 - t/3 + \frac{1}{36}) + \frac{1}{36}}}$$

$$= -\frac{1}{\sqrt{3}} \int \frac{dt}{\sqrt{-(t-1/6)^2 + (\frac{1}{6})^2}}$$

$$= -\frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{t-\frac{1}{6}}{1/6}\right)$$

$$\left(\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a)\right)$$

$$= -\frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{\frac{6t-1}{6}}{1/6}\right)$$

$$3+x = \frac{1}{t}$$

$$\Rightarrow t = \frac{1}{3+x}$$

$$= -\frac{1}{\sqrt{3}} \sin^{-1}(6t-1)$$

$$= -\frac{1}{\sqrt{3}} \sin^{-1}\left(6\frac{1}{3+x} - 1\right) = -\frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{6-(3+x)}{3+x}\right)$$

$$= -\frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{6-3-x}{3+x}\right)$$

ie., $\boxed{\int \frac{dx}{(3+x)\sqrt{x}} = -\frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{3-x}{3+x}\right)}$

=====X=====

Integration by trigonometric substitution and by parts of integrals:-

1. $\int \sqrt{a^2-x^2}\,.dx$

Soln:-

put $x = a\sin\theta$

$$\frac{dx}{d\theta} = a\cos\theta$$

$$dx = a\cos\theta\, d\theta$$

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta}$$

$$= \sqrt{a^2(1-\sin^2\theta)}$$

$$= a\sqrt{1-\sin^2\theta}$$

$$= a\sqrt{\cos^2\theta}$$

$$\sqrt{a^2-x^2} = a\cos\theta$$

$$\therefore \int \sqrt{a^2-x^2}\,.dx = \int a\cos\theta \,.\, a\cos\theta\, d\theta$$

$$= a^2 \int \cos^2\theta\, d\theta$$

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$$\therefore \int \sqrt{a^2-x^2}\,dx = a^2 \int \frac{1+\cos 2\theta}{2}\,d\theta$$

$\left(\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right)$

$$= \frac{a^2}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$$

$\int \cos 2\theta\, d\theta = \frac{\sin 2\theta}{2}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{a^2}{2}\left[\theta + \sin\theta\cos\theta\right]$$

$a\cos\theta = \sqrt{a^2-x^2}$

$a\cos\theta = a\sqrt{1-\frac{x^2}{a^2}}$

$\cos\theta = \sqrt{1-\frac{x^2}{a^2}}$

$$= \frac{a^2}{2}\left[\sin^{-1}(x/a) + \left(\frac{x}{a}\right)\sqrt{1-\frac{x^2}{a^2}}\right]$$

$$= \frac{a^2}{2}\sin^{-1}(x/a) + \frac{a^2}{2}\frac{x}{\cancel{a}}\sqrt{\frac{a^2-x^2}{a^2}}$$

$x = a\sin\theta$

$\sin\theta = x/a$

$$= \frac{a^2}{2}\sin^{-1}(x/a) + \frac{\cancel{a}}{2}\frac{x}{\cancel{a}}\sqrt{a^2-x^2}$$

$$\boxed{\therefore \int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}(x/a) + \frac{x}{2}\sqrt{a^2-x^2}}$$

=x=

2. $\int \sqrt{a^2+x^2}\,.\,dx$

Soln.

Put $x = a\sinh\theta$

$$\frac{dx}{d\theta} = a.\cosh\theta$$

$$\therefore \boxed{dx = a\cosh\theta\, d\theta}$$

$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\sinh^2\theta}$$

$$= \sqrt{a^2(1+\sinh^2\theta)}$$

$$= \sqrt{a^2\cosh^2\theta}$$

$$\boxed{\sqrt{a^2+x^2} = a\cosh\theta}$$

$$\therefore \int \sqrt{a^2+x^2}\,dx$$

$$= \int a\cosh\theta \,.\, a\cosh\theta\, d\theta$$

$$= \int a^2\cosh^2\theta\, d\theta$$

$$= a^2\int \cosh^2\theta\, d\theta$$

$$= a^2\int \frac{1+\cosh 2\theta}{2}\, d\theta$$

$$= \frac{a^2}{2}\int (1+\cosh 2\theta)\, d\theta$$

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$$\int\sqrt{a^2+x^2}\,.\,dx = \frac{a^2}{2}\left[\theta+\frac{\sinh 2\theta}{2}\right]$$

$a\cosh\theta=\sqrt{a^2+x^2}$

$a\cosh\theta = a\sqrt{1+x^2/a^2}$

$\cosh\theta=\sqrt{1+x^2/a^2}$

$$=\frac{a^2}{2}\theta+\frac{a^2}{2}\left(\frac{\sinh 2\theta}{2}\right)$$

$$=\frac{a^2}{2}\theta+\frac{a^2}{2}\sinh\theta\cosh\theta$$

$$=\frac{a^2}{2}\sinh^{-1}(x/a)+\frac{a^2}{2}\left(\frac{x}{a}\right)\sqrt{1+\frac{x^2}{a^2}}$$

$$=\frac{a^2}{2}\sinh^{-1}(x/a)+\frac{a}{2}(x)\sqrt{\frac{a^2+x^2}{a^2}}$$

$$=\frac{a^2}{2}\sinh^{-1}(x/a)+\frac{\cancel{a}}{2}\frac{x}{\cancel{a}}\sqrt{a^2+x^2}$$

ie., $\boxed{\int\sqrt{a^2+x^2}\,dx = \frac{a^2}{2}\sinh^{-1}(x/a)+\frac{x}{2}\sqrt{a^2+x^2}}$

3. $\int\sqrt{x^2-a^2}\,.\,dx$

Soln:-

Put $x=a\cosh\theta$

$\frac{dx}{d\theta}=a\sinh\theta$

$\therefore \boxed{dx=a\sinh\theta\,d\theta}$

$\sqrt{x^2-a^2}=\sqrt{a^2\cosh^2\theta-a^2}$

$=\sqrt{a^2(\cosh^2\theta-1)}$

$=\sqrt{a^2\sinh^2\theta}$

$\boxed{\sqrt{x^2-a^2}=a\sinh\theta}$

$$\therefore \int\sqrt{x^2-a^2}\,dx = \int a\sinh\theta\,.\,a\sinh\theta\,d\theta$$

$$=\int a^2\sinh^2\theta\,d\theta$$

$\sinh^2 x=\frac{\cosh 2x-1}{2}$

$$=a^2\int\frac{\cosh 2\theta-1}{2}\,d\theta$$

$$=\frac{a^2}{2}\int(\cosh 2\theta-1)\,d\theta$$

$$=\frac{a^2}{2}\left[\frac{\sinh 2\theta}{2}-\theta\right]$$

$$=\frac{a^2}{2}\left(\frac{\sinh 2\theta}{2}\right)-\frac{a^2}{2}\theta$$

$$=\frac{a^2}{2}(\sinh\theta\cosh\theta)-\frac{a^2}{2}\theta$$

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$$\int \sqrt{x^2-a^2}\,dx = \frac{a^2}{2}\left(\frac{x}{a}\right)\left(\sqrt{\frac{x^2}{a^2}-1}\right) - \frac{a^2}{2}\cosh^{-1}(x/a)$$

$$= \frac{a^2}{2}\,\frac{x}{a^2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}(x/a)$$

ie., $$\boxed{\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}(x/a)}$$

=====X=====

Examples:

1. Evaluate: $\int \sqrt{x^2+2x+10}\,dx$

Soln.:-

$$\int \sqrt{x^2+2x+10}\,dx = \int \sqrt{x^2+2x+10+1-1}$$

$$= \int \sqrt{(x^2+2x+1)+(10-1)}$$

$$= \int \sqrt{(x+1)^2+9}$$

$$= \int \sqrt{(x+1)^2+3^2}$$

$$= \cancel{\frac{1}{2}(x+1)\sqrt{\ }}$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{a^2}{2}\sinh^{-1}(x/a) + \frac{x}{2}\sqrt{x^2+a^2}$$

$$\therefore \int \sqrt{x^2+2x+10}\,dx = \frac{3^2}{2}\sinh^{-1}\left(\frac{x+1}{3}\right) + \frac{x+1}{2}\sqrt{x^2+2x+10}$$

$$\boxed{\int \sqrt{x^2+2x+10}\,dx = \frac{9}{2}\sinh^{-1}\left(\frac{x+1}{3}\right) + \frac{x+1}{2}\sqrt{x^2+2x+10}}$$

=====X=====

2. Evaluate: $\int \sqrt{1+x-2x^2}\,dx$

Soln.:-

$$\int \sqrt{1+x-2x^2}\,dx = \int \sqrt{2\left(\tfrac{1}{2}+\tfrac{x}{2}-x^2\right)}$$

$$= \sqrt{2}\int \sqrt{-x^2+x/2+1/2+\tfrac{1}{16}-1/16}$$

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$$\int \sqrt{1+x-2x^2}\,dx = \sqrt{2}\int \sqrt{\left(-x^2+\tfrac{x}{2}-\tfrac{1}{16}\right)+\left(\tfrac{1}{2}+\tfrac{1}{16}\right)}$$

$$= \sqrt{2}\int \sqrt{-\left(x^2-\tfrac{x}{2}+\tfrac{1}{16}\right)+\left(\tfrac{8+1}{16}\right)}$$

$$= \sqrt{2}\int \sqrt{-\left(x-\tfrac{1}{4}\right)^2+\tfrac{9}{16}}$$

$$= \sqrt{2}\int \sqrt{-\left(x-\tfrac{1}{4}\right)^2+\left(\tfrac{3}{4}\right)^2}$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\left(\tfrac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}$$

$$\therefore \int \sqrt{1+x-2x^2}\,dx = \sqrt{2}\left[\frac{(3/4)^2}{2}\sin^{-1}\left(\frac{x-\frac{1}{4}}{3/4}\right)+\frac{x-\frac{1}{4}}{2}\sqrt{-\left(x-\tfrac{1}{4}\right)^2+\left(\tfrac{3}{4}\right)^2}\right]$$

$$= \sqrt{2}\,\frac{9/16}{2}\sin^{-1}\left(\frac{4x-1/\cancel{4}}{3/\cancel{4}}\right)+\frac{\frac{4x-1}{\cancel{4}}}{2}\sqrt{2}\sqrt{-\left(x-\tfrac{1}{4}\right)^2+\left(\tfrac{3}{4}\right)^2}$$

$$= \sqrt{2}\,\frac{9}{32}\sin^{-1}\left(\frac{4x-1}{3}\right)+\frac{4x-1}{8}\sqrt{1+x-2x^2}$$

$$= \sqrt{2}\,\frac{9}{2\times16}\sin^{-1}\left(\frac{4x-1}{3}\right)+\frac{4x-1}{8}\sqrt{1+x-2x^2}$$

i.e. $\boxed{\int \sqrt{1+x-2x^2}\,dx = \frac{9}{\sqrt{2}\times16}\sin^{-1}\left(\frac{4x-1}{3}\right)+\frac{4x-1}{8}\sqrt{1+x-2x^2}}$

=== X ===

Unit-2 is over

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