UNIT - I

Indefinite Integral.

Let, $f(x)$ is said to be an integral of $f'(x)$. if and only if

$$\frac{d}{dx}(f(x)) = f'(x).$$

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Integral of $f'(x) = f(x)$

$\nearrow$ Variable of Integral.

ie) $\int f'(x)\,dx = f(x) + c \rightarrow$ I. constant.

$\downarrow$ I. sign $\qquad \downarrow$ Integral

Note.

Integral is also known as antiderives or primities. Here, "c" is a constant of orbitarary. Which is not known [unknown] This type of Integraction is called "Indefinite Integral".

Problems.

Find it.

$$\int \frac{dx}{\sqrt[3]{x}}$$

Soln:

Given, $\int \frac{dx}{\sqrt[3]{x}}$

$$\Rightarrow \int \frac{dx}{(x)^{1/3}} = \int x^{-1/3}\,dx$$

$$= \frac{x^{-1/3+1}}{-1/3+1} + c$$

$$\left[\because \int x^n\,dx = \frac{x^{n+1}}{n+1}\,dx\right]$$

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$$= \frac{x^{2/3}}{2/3} + c$$

$$= 3/2\, x^{2/3} + c$$ //.

Find it.

$$\int \frac{k}{x^9}\, dx$$

Soln:

Given $\int \frac{k}{x^9}\, dx$

$$= k \int x^{-9}\, dx$$

$$= k\, \frac{x^{-9+1}}{-9+1} + c$$

$$= k\, \frac{x^{-8}}{-8} + c$$

$$= -k/8\, x^{-8} + c$$ //.

Find it.

$$\int 4^{5x}\, dx$$

Soln:

Given,

$$\int 4^{5x}\, dx$$

$$\int 4^{5x}\, dx = \frac{4^{5x}}{\log_4 5} + c$$

$$= \frac{4^{5x}}{5\log 4} + c$$ //.

$\left[\because \int a^x dx = \frac{a^x}{\log a} + c\right.$

$\log m^n = n \log m$

$\left.(a^m)^n = a^{mn}\right]$.

Find it.

$$\int (ax+b)^n dx$$

Soln : Given.

$$\int (ax+b)^n dx$$

$$= \frac{(ax+b)^{n+1}}{a(n+1)} + k$$ //.

Evaluate.

$$\int \cos x/2 \, dx$$

Given.

$$\int \cos x/2 \, dx$$

$$= \frac{\sin x/2 + c}{1/2}$$

$$= 2\sin x/2 + c$$ //.

Evaluate,

$$\int \cos^4 x \, dx$$

Soln:

Given,

$$\int \cos^4 x \, dx$$

$$= \int (\cos^2 x)^2 dx$$

$$= \int \left(\frac{1+\cos 2x}{2} \right)^2 dx$$

$$= \frac{1}{4} \int (1+\cos 2x)^2 dx$$

$$= \frac{1}{4} \int [1 + 2\cos 2x + \cos^2 2x] \, dx$$

$$= \frac{1}{4} \int \left[1 + 2\cos 2x + \left(\frac{1+\cos 4x}{2} \right) \right] dx$$

$$= \frac{1}{4} \left[\int dx + 2\int \cos 2x \, dx + \int \frac{1}{2} dx + \int \frac{\cos 4x}{2} dx \right]$$

$$= \frac{x}{4} + \frac{2}{4} \cdot \frac{\sin 2x}{2} + \frac{x}{8} + \frac{1}{8} \cdot \frac{\sin 4x}{4} + K$$

$$= \frac{3x}{8} + \frac{\sin 2x}{2} + \cancel{\frac{x}{8}} + \cancel{\frac{1}{8}} \cdot \frac{\sin 4x}{4} + K$$

$$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + K //.$$

Evaluate,

$$\int \left(x + \frac{1}{x}\right)^3 dx$$

Soln:

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Given,

$$\int (x + 1/x)^3 dx$$

$$= \int [x^3 + 3(x)^2 (1/x) + 3(x)(1/x)^2 + (1/x)^3] dx$$

$$= \int [x^3 + 3x^2 (1/x) + 3x \cdot (1/x^2) + 1/x^3] dx$$

$$= \int [x^3 + 3x + 3/x + 1/x^3] dx$$

$$= \frac{x^4}{4} + 3\frac{x^2}{2} + 3 + \frac{1}{2x^2}$$

$$= \frac{x^4}{4} + 3x^2/2 + 3 + 1/2x^2 \;//.$$

Evaluate,

$$\int \frac{dx}{\sqrt{x-1} - \sqrt{x-2}}$$

Soln:

Given, $\int \frac{dx}{\sqrt{x-1} - \sqrt{x-2}}$

$$= \int \frac{1}{\sqrt{x-1} - \sqrt{x-2}} \times \int \frac{\sqrt{x-1} + \sqrt{x-2}}{\sqrt{x-1} + \sqrt{x-2}} dx$$

$$= \int \frac{\sqrt{x-1} + \sqrt{x-2}}{(\sqrt{x-1})^2 - (\sqrt{x-2})^2} \cdot dx$$

$$= \int \frac{\sqrt{x-1} + \sqrt{x-2}}{(x-1)-(x-2)}\,dx$$

$$= \int \frac{(\sqrt{x-1}) + (\sqrt{x-2})}{(x-1)-(x-2)}\,dx$$

$$= \int \sqrt{x-1}\,dx + \int \sqrt{x-2}\,dx$$

$$= \int (x-1)^{1/2}\,dx + \int (x-2)^{1/2}\,dx$$

$$= \frac{(x-1)^{3/2}}{3/2} + \frac{(x-2)^{3/2}}{3/2} + c$$

$$= 2/3\left((x-1)^{3/2} + (x-2)^{3/2}\right) + c$$ //

Evaluate, $\int a^x b^x\,dx$

Soln :

Given. $\int a^x b^x\,dx$

$$= \int (ab)^x\,dx \qquad \left[\because \int a^x\,dx = \frac{a^x}{\log a} + c\right]$$

$$= \frac{(ab)^x}{\log ab} + k.$$

Note,

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Devisor}}$$

Evaluate,

$$\int \frac{x^2+1}{x+1}\, dx$$

Given,

$$\int \frac{x^2+1}{x+1}\, dx$$

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2+1 \\ & x^2+x \\ & (-)\ (-) \\ \hline & -x+1 \\ & -x-1 \\ & (+)\ (+) \\ \hline & 2 \\ \hline \end{array}$$

$\int dx/x = \log x$

$$= \int \left[x-1 + \frac{2}{x+1} \right] dx$$

$$= \int (x-1)\,dx + 2\int \frac{dx}{x+1}$$

$$= \frac{x^2}{2} - x + 2\log(x+1) + c$$

Evaluate,

$$\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$$

Soln:

Given,

$$\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$$

$$= \int \frac{1}{\sqrt{x+a}+\sqrt{x}} \times \frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}\, dx$$

$$= \int \frac{\sqrt{x+a}-\sqrt{x}}{(\sqrt{x+a})^2-(\sqrt{x})^2}\, dx$$

$$= \int \frac{\sqrt{x+a}-\sqrt{x}}{(x+a)-(x)}\, dx$$

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$$= \int \frac{\sqrt{x+a} - \sqrt{x}}{x+a-x}\, dx$$

$$= \int \frac{\sqrt{x+a} - \sqrt{x}}{a}\, dx$$

$$= \frac{1}{a}\int \sqrt{x+a}\, dx - \int \sqrt{x}\, dx$$

$$= \frac{1}{a}\int (x+a)^{1/2}\, dx - \int (x)^{1/2}\, dx$$

$$= \frac{1}{a}\left[\frac{2}{3} (x+a)^{3/2} - \frac{2}{3} (x)^{2/3} \right]$$

$$= \frac{2}{3a}\left[(x+a)^{3/2} - (x)^{3/2} \right] + c //.$$

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UNIT- III

Definite Integral.

Defn :

Let $f(x)$ be a continous function definite on $[a, b]$ ie) $a \leq x \leq b$ and $f'(x)$ is the integral of $f(x)$.

ie) $\int f'(x)\,dx = f(x)$

ie) $a \leq x \leq b$

ie) $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$

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it is called definite integral of $f(x)$ a to b.

Here, ② a and b are called upper and ① lower limits of $f(x)$ respectively.

Evaluate, $\int_0^{\pi/2} \sin x \, dx$

Soln:

Given, $\int_0^{\pi/2} \sin x \, dx$

$$\int_0^{\pi/2} \sin x \, dx = \left[-\cos x\right]_0^{\pi/2}$$

$$= -\cos(\pi/2 - \cos 0)$$

$$= -(0-1) \Rightarrow 1 //.$$

Evaluate, $\int_1^2 \frac{1}{x+1} dx$

Soln:

Given, $\int_1^2 \frac{1}{x+1} dx$

$$= \int_1^2 \frac{d(x+1)}{(x+1)} \qquad [\because dx = d(x+1)]$$

$$= [\log(x+1)]_1^2$$

$$= (\log 3 - \log 2)$$

$$= \log\left(\tfrac{3}{2}\right)$$

Evaluate,

$$\int_0^{\pi/2} \sqrt{1+\sin 2x}\, dx$$

Soln:

Given,

$$\int_0^{\pi/2} \sqrt{1+\sin 2x}\, dx$$

$$= \int_0^{\pi/2} \sqrt{1+2\sin x\cos x}\, dx \qquad [\because 1 = \sin^2 x + \cos^2 x]$$

$$= \int_0^{\pi/2} \sqrt{\sin^2 x+\cos^2 x + 2\sin x\cos x}\, dx$$

$$= \int_0^{\pi/2} \sqrt{(\sin x+\cos x)^2}\, dx$$

$$= \int_0^{\pi/2} \sin x+\cos x\, dx$$

$$= (-\cos x + \sin x)_0^{\pi/2}$$

$= 1 + 1 \Rightarrow 2$

Evaluate,

$$\int_0^1 \frac{1-x}{1+x}\,dx$$

Soln:

Given,

$$\int_0^1 \frac{1-x}{1+x}\,dx.$$

$$= \int_0^1 \frac{1+x-2x}{1+x}\,dx$$

$$= \int_0^1 dx - 2\int_0^1 \frac{x}{1+x}\,dx$$

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$$= (x)_0^1 - 2\int_0^1 \frac{x}{1+x}\,dx$$

$$= 1 - 2\int_0^1 \frac{x}{1+x}\,dx$$

Put, $1+x = t$ | limits,

$x = t-1$ | When $x=0$ then $t=1$

$dx = dt$ | When $x=1$ then $t=2$.

$$= 1 - 2\int_1^2 \left(\frac{t-1}{t}\right) dt$$

$$= 1 - 2\left[\int_1^2 dt - \int_1^2 dt/t\right]$$

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$$= 1 - 2\left[(t)_1^2 - (\log t)_1^2 \right]$$

$$= 1 - 2\left[1 - \log 2 + \log 1 \right]$$

$$= 1 - 2 + 2\log 2$$

$$= \log 4 - 1$$

Important results.

Result → ①

$$\int \tan x \, dx = \log \sec x + c$$

Soln: Given. $\int \tan x \, dx$

$$= \int \frac{\sin x}{\cos x} dx \qquad \left[\because \tfrac{1}{\cos x} = \sec x \right.$$

$$= -\int \frac{d(\cos x)}{\cos x} \qquad \left. \int \frac{dx}{x} = \log x + c \right].$$

$$= -\log(\cos x) + c$$

$$= \log(\cos x)^{-1} + c$$

$= \log \sec x + c$. Hence proved.

Results → ②

$$\int \cot x \, dx = \log(\sin x) + c$$

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Proof :

Given,

$$\int \cot x \, dx.$$

$$= \int \frac{\cos x}{\sin x} \, dx$$

$$= \int \frac{d(\sin x)}{\sin x} \, dx$$

$$= \log \sin x + C$$ //. Hence proved.

Result → 3

$$\int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x - \cot x) + C$$

Proof :

Given.

$$\int \operatorname{cosec} x \, dx$$

$$= \int \operatorname{cosec} x \times \frac{\operatorname{cosec} x - \cot x}{\operatorname{cosec} x - \cot x}$$

$$= \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{\operatorname{cosec} x - \cot x}$$

Put,

$$z = \operatorname{cosec} x - \cot x.$$

$$dz = (-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x) \, dx$$

$$dz = (\operatorname{cosec}^2 x - \operatorname{cosec} x \cdot \cot x) \, dx.$$

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$= \int \frac{dz}{z} = \log z + c$

$= \log(\operatorname{cosec} x - \cot x) + c$ // Hence proved

Result → (4)

$$\int \sec x\,dx = \log(\sec x + \tan x) + c$$

Proof :

Given. $\int \sec x\,dx$.

$$= \int \sec x \times \frac{(\sec x + \tan x)}{(\sec x + \tan x)}$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$$

Put, $z = \sec x + \tan x$

$dz = (\sec x \tan x + \sec^2 x)\,dx$

$dz = (\sec^2 x + \sec x \tan x)\,dx$

$$= \int \frac{dz}{z}$$

$= \log z + c$

$= \log(\sec x + \tan x) + c$

$\int \sec x\,dx = \log(\sec x + \tan x) + c$ // Hence prov

Evaluate,

$$\int 2x \sin(x^2+1)dx$$

Soln :

Given,

$$\int 2x\sin(x^2+1)\, dx$$

Put,

$x^2+1=\theta$

$2x\,dx = d\theta$

$\therefore \int \sin\theta\, d\theta = -\cos\theta + K$

$= -\cos(x^2+1) + K$ //.

Evaluate,

$$\int \frac{(\log x)^2}{x} dx.$$

Soln :

Given,

$$\int \frac{(\log x)^2}{x} dx$$

Put,

$\log x = u$

$1/x\, dx = du$.

$\int \frac{(\log x)^2}{x} dx = \int u^2 dx$

$= \frac{u^3}{3} + K$

$= \frac{(\log x)^3}{3} + K$ //.

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Evaluate,

$$\int \frac{x^{n-1}}{a+bx^n}\,dx.$$

Soln:

Given. $$\int \frac{x^{n-1}}{a+bx^n}\,dx$$

Put,

$$a + bx^n = t$$

$$bnx^{n-1}dx = dt$$

$$x^{n-1}dx = \frac{dt}{bn}$$

$$\therefore \int \frac{x^{n-1}dx}{a+bx^n}\,dx = \int \frac{dt/b.n}{t}$$

$$= \frac{1}{bn}\int \frac{dt}{dt}$$

$$= \frac{1}{bn}\log t + K.$$

$$= \frac{1}{bn}\log(a+bx^n) + K$$ //.

Evaluate,

$$\int \frac{10^x \log_e 10 + 10x^9}{10^x + x^{10}}\,dx$$

Soln:

Given. $$\int \frac{10^x \log_e 10 + 10x^9}{10^x + x^{10}}\,dx$$

Put,

$$10^x + x^{10} = t$$

$$(10^x \log_e 10 + 10x^9)\,dx = dt$$

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$$\therefore \int \frac{10^x \log_e 10 + 10x^9}{10^x + x^{10}}\,dx = \int \frac{dt}{t}$$

$$= \int \frac{dt}{t}$$

$$= \log t + c$$

$$= \log(10^x + x^{10}) + c$$

Evaluate,

$$\int \frac{x^3}{(x^2+1)^3}\,dx$$

Soln:

Given, $\int \frac{x^3}{(x^2+1)^3}\,dx$

Put, $x^2+1 = t$

$x^2 = t-1$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$$\therefore \int \frac{x^3}{(x^2+1)^3}\,dx = \int \frac{x^2\,x\,dx}{(x^2+1)^3}$$

$$= \int \frac{(t-1)\,dt/2}{t^3}$$

$$= \frac{1}{2}\int \left(\frac{t-1}{t^3}\right) dt$$

$$= \frac{1}{2}\left[\int \frac{dt}{t^2} - \int \frac{dt}{t^3}\right]$$

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$$= \frac{1}{2}\left[-\frac{1}{t} + \frac{1}{2}\,\frac{1}{t^2}\right]$$

$$= -\frac{1}{2}\left(\frac{1}{x^2+1}\right) + \frac{1}{4}\left(\frac{1}{(x^2+1)^2}\right) + c$$

$$\therefore \int \frac{x^3}{(x^2+1)^3}\,dx = -\frac{1}{2}\left(\frac{1}{(x^2+1)}\right) + \frac{1}{4(x^2+1)^2} + c$$

Evaluate,

$$\int_{\pi/2}^{\pi} e^{\cos x} . \sqrt{1-e^{\cos x}}\,\sin x\,dx$$

Soln:

Given,

$$\int_{\pi/2}^{\pi} e^{\cos x} \sqrt{1-e^{\cos x}}\,\sin x\,dx$$

Put.

$\cos x = t$ | limits,
$-\sin x\,dx = dt$ | when $x = \pi/2$ then $t = 0$
$\sin x\,dx = -dt$ | when $x = \pi$ then $t = -1$

$$\therefore \int_{\pi/2}^{\pi} e^{\cos x} \sqrt{1-e^{\cos x}}\,dx = -\int_{0}^{-1} e^{t}\sqrt{1-e^{t}}\,dt$$

$$= \int_{-1}^{0} e^{t}\sqrt{1-e^{t}}\,dt$$

Put.

$1-e^{t} = y$ | limits,
$-e^{t}\,dt = dy$ | when $t = -1$ then $y = 1-1/e$
| when $t = 0$ then $y = 0$

$$\therefore \int_{-1}^{0} e^t \sqrt{1-e^t}\, dt = -\int_{1-1/e}^{0} \sqrt{y}\, dy$$

$$= \int_{0}^{1-1/e} \sqrt{y}\, dy$$

$$= \left(\frac{y^{3/2}}{3/2} \right)_0^{1-1/e}$$

$$= \frac{(1-1/e)^{3/2}}{3/2} \Rightarrow 2/3\, (1-1/e)^{3/2} //.$$

Evaluate, $\int \frac{dx}{x^2+a^2}$

Soln: Given. $\int \frac{dx}{x^2+a^2}$

$$= \frac{1}{a^2} \int \frac{dx}{1+(x/a)^2}$$

$$= \frac{1}{a^2} \frac{\tan^{-1}(x/a)}{1/a} + c \quad (*)$$

$$= 1/a \tan^{-1}(x/a) + c //.$$

Evaluate, $\int \frac{dx}{(x+1)^2+4}$

Soln:

Given. $\int \frac{dx}{(x+1)^2+4}$

Put,

$x+1=t$

$dx=dt$

$\therefore \int \frac{dx}{(x+1)^2+4} = \int \frac{dt}{t^2+(2)^2}$

$= \frac{1}{2} \tan^{-1}\left(\frac{t}{2}\right)+c$

$= \frac{1}{2} \tan^{-1}\left(\frac{x+1}{2}\right)+c$ //.

Integrate by pass.

Consider. $\int u\,dv = uv - \int v\,du$

which is known as integraction by pass.

Note, (1) $\int u\,dv = uv - u'v_1 + u''v_2 - \dots$

(2) $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \dots$

Here $u' \rightarrow$ difference at once,

$v_1 \rightarrow$ Integration at once.

Evaluate,

$\int x e^x\,dx$

Soln:

Given.

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$$\int x e^x dx$$

$= \int x \, d(e^x)$ $\quad$ $[\because d/dx(e^x) = e^x$

$= xe^x - \int e^x (1) dx$ $\quad$ i.e., $d(e^x) = e^x dx$

$= xe^x - e^x + c$ $\quad$ $\int u dv = uv - \int v du]$

$= e^x (x-1) + c$ //.

Evaluate,

$$\int \log x \, dx.$$

Soln:

Given, $\quad$ $[\because \int u dv = uv - \int v du$

$\int \log x \, dx$ $\quad$ $u = \log x$

$dv = 1/x \, dx]$

Now, $\int \log x \, d(x) = x \log x - \int x \cdot 1/x \, dx$

$= x \log x - x + c$ //.

Evaluate,

$$\int x \log x \, dx$$

Soln:

Given, $\int x \log x \, dx$

$\int x \log x \, dx = 1/2 \int \log x \, d(x^2)$

$= 1/2 [x^2 \log x - \int x^2 \, 1/x \, dx]$

$$= \frac{1}{2}\left[x^2 \log x - \frac{x^2}{2}\right] + C$$

Evaluate,

$$\int_0^1 \log(1+x)\,dx$$

Soln:

Given, $\int_0^1 \log(1+x)\,dx$

Here,

$u = \log(1+x)$ | $dv = dx$

$du = \frac{1}{1+x}\,dx$ | $v = x.$

$$\left[\because \int u\,dv = uv - \int v\,du\right]$$

$$\therefore \int_0^1 \log(1+x)\,dx = \left[x \log(1+x)\right]_0^1 - \int_0^1 \frac{x\,dx}{1+x}$$

$$= \log 2 - \int_0^1 \frac{x}{1+x}\,dx$$

Put,

$1+x = t$ | limits,

$x = t-1$ | When $x=0$ then $t=1$

$dx = dt$ | When $x=1$ then $t=2$.

$$= \log 2 - \int_1^2 \frac{(t-1)}{t}\,dt$$

$$= \log 2 - \int_1^2 dt + \int_1^2 dt/t$$

$$= \log 2 - \int_1^2 dt + \int_1^2 dt/t$$

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$= \log 2 - (t)_1^2 + [\log(t)]_1^2$

$= \log 2 - (2-1) + \log 2 - \log 1$

$= \log 2 - (2-1) + \log 2 - \log 1$

$= 2 \log 2 - 1$

$= \log 4 - 1$ //

Prove that

$$\int_0^1 \frac{2x \tan^{-1}(x)^2}{1+x^4} dx = \frac{\pi^2}{32}.$$

Soln :

Given.

$$\int_0^1 \frac{2x \tan^{-1}(x)^2}{1+x^4} dx$$

put .

$\tan^{-1}(x^2) = t$

$\frac{1}{1+(x^2)^2} 2x\,dx = dt$

limits,

When $x = 0$ then $t = 0$

When $x = 1$ then $t = \pi/4$

$$\therefore \int_0^1 \frac{2x \tan^{-1}(x)^2}{1+x^4} dx = \int_0^{\pi/4} t\,dt$$

$$= \left(t^2/2\right)_0^{\pi/4}$$

$$= 1/2 \cdot \pi^2/16 \Rightarrow \pi^2/32$$ // Proved.

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Evaluate,
$$\int_0^2 \frac{5x+1}{x^2+1}\,dx.$$

Soln:

Given, $\int_0^2 \frac{5x+1}{x^2+1}\,dx.$

$$\left[\because \frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2},\ \int \frac{dx}{1+x^2} = \tan^{-1}(x)+c\right]$$

$$= 5\int_0^2 \frac{x\,dx}{x^2+1} + \int_0^2 \frac{dx}{1+x^2}$$

Put,
$x^2+1=t$
$2x\,dx = dt$

Limits,
When $x=0$ then $t=1$
When $x=2$ then $t=5$

$$= 5/2 \int_1^5 \frac{dt}{t} + \left[\tan^{-1}x\right]_0^2$$

$$= 5/2\,(\log t)_1^5 + \tan^{-1}(2) - \tan^{-1}(0)$$

$$= 5/2\,[\log 5] + \tan^{-1}(2)$$

$$= 5/2 \log 5 + \tan^{-1}(2)$$ //.

Evaluate.
$$\int_1^2 \frac{\log x}{(1+\log x)^2}\,dx$$

Soln:

Given, $\int_1^2 \frac{\log x}{(1+\log x)^2}\,dx$

Put,

$$\log x = y \qquad x = e^y$$

limits,
when $x = 1$ then $y = 0$
when $x = e$ then $y = 1$

$$\therefore \int_1^2 \frac{\log x}{(1+\log x)^2}\,dx = \int_0^1 \frac{y}{(1+y)^2} e^y\,dy.$$

$$= \int_0^1 e^y \left[\frac{1+y-1}{(1+y)^2}\right] dy.$$

$$= \int_0^1 e^y \left(\frac{1}{(1+y)} - \frac{1}{(1+y)^2}\right) dy$$

$$= \left[e^y \left(\frac{1}{1+y}\right)\right]_0^1$$

$$= \frac{e}{2} - 1 \,//.$$

Pratical Fraction.

Evaluate,

$$\int \frac{5x+1}{(2x-1)(x+2)}\,dx$$

Soln :

Given, $\displaystyle\int \frac{5x+1}{(2x-1)(x+2)}\,dx.$

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Consider,

$$\frac{5x+1}{(2x-1)(x+2)} = \frac{A}{(2x-1)} + \frac{B}{(x+2)}$$

$$5x+1 = A(x+2) + B(2x-1)$$

Put,

$$\boxed{x=-2}$$

$$5(-2)+1 = A(0) + B(2(-2)-1)$$

$$-10+1 = B(-4-1)$$

$$-9 = B(-5)$$

$$\cancel{+}5B = \cancel{+}9$$

$$\boxed{B = 9/5}$$

Put,

$$\boxed{x = 1/2}$$

$$5(1/2)+1 = A(1/2+2) + B(2(1/2)-1)$$

$$5/2 + 1 = A\left(\frac{1+4}{2}\right) + B(1-1)$$

$$\frac{5+2}{2} = A(5/2)$$

$$7/2 = A(5/2)$$

$$A(5/2) = 7/2$$

$$A = 7/2 \times 2/5$$

$$\boxed{A = 7/5}.$$

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$$\therefore \quad \frac{5x+1}{(2x-1)(x+2)} = \frac{7/5}{(2x-1)} + \frac{9/5}{(x+2)}$$

Hence,

$$\int \frac{5x+1}{(2x-1)(x+2)} = \int \left[\frac{7/5}{(2x-1)} + \frac{9/5}{(x+2)} \right] dx$$

$$= 7/5 \int \frac{dx}{(2x-1)} + 9/5 \int \frac{dx}{(x+2)}$$

$$= 7/5 \int \frac{dx}{(2x-1)} + 9/5 \int \frac{dx}{(x+2)}$$

$$= 7/5 \int \frac{d(2x-1)}{(2x-1)} + 9/5 \int \frac{d(x+2)}{(x+2)}$$

$$= 7/5 \log(2x-1) + 9/5 \log(x+2) + c$$

Evaluate,

$$\int \frac{7x-1}{(x-1)^2(x-2)} \cdot dx$$

Soln:

Given,

$$\int \frac{7x-1}{(x-1)^2(x-2)} \cdot dx$$

$$\frac{7x-1}{(x-1)^2(x-2)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x-2)}$$

$$= \frac{A}{(x-2)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$7x-1 = A(x-1)^2 + B(x-1)(x-2) + C(x-2)$$

Put, $\boxed{x=1}$

$$7-1 = C(1-2)$$

$$6 = -C$$

$$\boxed{C=-6}.$$

Put $\boxed{x=2}$

$$7(2)-1 = A(2-1)^2+0+0$$

$$14-1 = A(1)^2$$

$$\boxed{A=13}.$$

Put, $\boxed{x=0}$

$$-1 = A(-1)^2 + B(-1)(-2) + C(-2)$$

$$-1 = A(1) + B(2) - 2C$$

$$-1 = A + 2B - 2C$$

$$-1 = 13 + 2B - 2C$$

$$-1 = 13 + 2B - 2(-6)$$

$$-1 = 13 + 2B + 12$$

$$-1-13-12 = 2B$$

$$-14-12 = 2B$$

$$-26 = 2B$$

$$B = \frac{-26}{2}$$

$$\boxed{B=-12}.$$

$$\therefore \int \frac{7x-1}{(x-1)^2(x-2)}\,dx = \int \frac{13}{(x-2)}\,dx + \int \frac{-13}{(x-1)}\,dx + \int \frac{-6}{(x-1)^2}\,dx$$

$$= 13\int \frac{dx}{(x-2)} - 13\int \frac{dx}{(x-1)} - 6\int \frac{dx}{(x-1)^2}$$

$$= 13\log(x-2) - 13\log(x-1) - 6\log(x-1)^2$$

$$= 13\log(x-2) - 13\log(x-1) - 6\int \frac{dt}{t^2}$$

$$= 13\left[\log(x-2) - \log(x-1)\right] - 6\left(-\frac{1}{t}\right) + c$$

$$= 13\log\left(\frac{(x-2)}{(x-1)}\right) + \frac{6}{(x-1)} + c \;//.$$

Evaluate,

$$\int \frac{x^2+x+1}{x^3+x^2-6x}\, dx$$

Soln :

Given, $\displaystyle\int \frac{x^2+x-1}{x^3+x^2-6x}\, dx$

$$= \int \frac{x^2+x-1}{x(x^2+x-6)}\, dx$$

$$= \int \frac{x^2+x-1}{x(x+3)(x-2)} \cdot dx \longrightarrow (1)$$

Consider,

$$\frac{x^2+x-1}{x(x-2)(x+3)} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x+3)}$$

$$x^2+x-1 = A(x-2)(x+3) + B(x)(x+3) + C(x)(x-2)$$

Put, $\boxed{x=0}$.

$$-1 = A(-2)(3)+0+0$$
$$-1 = -6A$$
$$6A = 1$$
$$\boxed{A = 1/6}$$

Put, $\boxed{x=2}$

$$4+2-1 = B(2)(5)$$
$$5 = 10B$$
$$\boxed{B = 1/2}$$

$\boxed{x=-3}$

$$9-3-1 = C(-3)(-3-2)$$
$$5 = C(-3)(-5)$$
$$15C = 5$$
$$\boxed{C = 1/3}.$$

$$\therefore \int \frac{x^2+x-1}{(x)(x-2)(x+3)} dx = \int \frac{1/6}{x} dx + \int \frac{1/2\, dx}{(x-2)} + \int \frac{1/3\, dx}{x+3}$$

$$= 1/6 \log x + 1/2 \log(x-2) + 1/3 \log(x+3) + c$$

Evaluate,

$$\int \frac{\sin x}{\sin 4x} dx.$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 4x$
$= \sin 2(2x)$
$= 2\sin 2x \cos 2x$

Soln :

Given,

$$\int \frac{\sin x}{\sin 4x} dx$$

$$= \int \frac{\sin x}{2\sin 2x \cos 2x} \cdot dx$$

$$= \frac{1}{4}\int \frac{\cancel{\sin x}}{\cancel{\sin x}\cos x\cos 2x}\,dx$$

$$= \frac{1}{4}\int \frac{dx}{\cos x\cos 2x}$$

$$= \frac{1}{4}\int \frac{\cos x}{\cos^2 x\cos 2x}\,dx$$

$$= \frac{1}{4}\int \frac{\cos x}{(1-\sin^2 x)(1-2\sin^2 x)}\cdot dx$$

$\cos^2\theta = 1-\sin^2\theta$

$\cos 2\theta = 1-2\sin^2\theta$

Put, $\sin x = t$

$\cos x\,dx = dt$

$$= \frac{1}{4}\int \frac{dt}{(1^2-t^2)(1^2-2t^2)}$$

$2 = (\sqrt{2})^2$

$a^2-b^2 = (a+b)(a-b)$

$$= \frac{1}{4}\int \frac{dt}{(1^2-t^2)(1^2-t\sqrt{2})^2}$$

$$= \frac{1}{4}\int \frac{dt}{(1+t)(1-t)(1+t\sqrt{2})(1-t\sqrt{2})} \longrightarrow (1)$$

$$\frac{1}{(1+t)(1-t)(1+t\sqrt{2})(1-t\sqrt{2})} = \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(1+t\sqrt{2})} + \frac{D}{(1-t\sqrt{2})}$$

Put,

$t = 1 \Rightarrow \boxed{A = -\tfrac{1}{2}}$ $\quad t = \tfrac{1}{\sqrt{2}} \Rightarrow \boxed{C = 1}$

$t = -1 \Rightarrow \boxed{B = -\tfrac{1}{2}}$ $\quad t = -\tfrac{1}{\sqrt{2}} \Rightarrow \boxed{D = 1}$

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from (1) =>

$$= \frac{1}{4}\left[\int \frac{-\frac{1}{2}\,dt}{1-t} + \int \frac{-\frac{1}{2}\,dt}{1+t} + \int \frac{1\,dt}{1-t\sqrt{2}} + \int \frac{1\,dt}{1+t\sqrt{2}} \right]$$

$$= \frac{1}{4}\left[\frac{1}{2}\int \frac{dt}{t-1} - \frac{1}{2}\int \frac{dt}{t+1} - \frac{1}{\sqrt{2}}\int \frac{dt}{t-\frac{1}{\sqrt{2}}} + \frac{1}{\sqrt{2}}\int \frac{dt}{t+\frac{1}{\sqrt{2}}} \right]$$

$$= \frac{1}{4}\left[\frac{1}{2}\log(t-1) - \frac{1}{2}\log(t+1) - \frac{1}{\sqrt{2}}\log\left(t-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\log\left(t+\frac{1}{\sqrt{2}}\right) \right] + c$$

$$= \frac{1}{4}\left[\frac{1}{2}\log\left(\frac{t-1}{t+1}\right) + \frac{1}{\sqrt{2}}\log\left(\frac{t-\frac{1}{\sqrt{2}}}{t+\frac{1}{\sqrt{2}}}\right) \right] + c$$

$$= \frac{1}{8}\log\left(\frac{\sin x-1}{\sin x+1}\right) + \frac{1}{4\sqrt{2}}\log\left(\frac{\sin x+\frac{1}{\sqrt{2}}}{\sin x-\frac{1}{\sqrt{2}}}\right) + c$$

$$= \frac{1}{8}\log\left(\frac{\sin x-1}{\sin x+1}\right) + \frac{1}{4\sqrt{2}}\log\left(\frac{\sqrt{2}\sin x+1}{\sqrt{2}\sin x-1}\right) + c \,//$$

Evaluate,

$$\int_0^1 \frac{dx}{1+2x+2x^2+2x^3+x^4}$$

Soln:

Given,

$$\int_0^1 \frac{dx}{1+2x+2x^2+2x^3+x^4}$$

$$= \int_0^1 \frac{dx}{1+2x^2+x^4+2x(1+x^2)}$$

$$= \int_0^1 \frac{dx}{(1+x^2)^2+2x(1+x^2)}$$

$$= \int_0^1 \frac{dx}{(1+x^2)(1+x^2+2x)}$$

$$= \int_0^1 \frac{dx}{(1+x^2)(1+x)^2}$$

Consider,

$$\frac{1}{(1+x^2)(1+x^2)} = \frac{A}{(1+x)} + \frac{B}{(1+x)^2} + \frac{Cx+D}{1+x^2}$$

$$1 = A(1+x)(1+x^2) + B(1+x^2) + (x+D)(1+x^2)$$

Put, $\boxed{x=-1}$

$$1 = 2B \Rightarrow \boxed{B = 1/2}$$

Equating co-efficient of x^3

$$0 = A + C$$

$1+$

Put, $\boxed{x=0}$

$$1 = A+B+D$$

$$1-B = A+D$$

$$1-1/2 = A+D$$

$$\boxed{1/2 = A+D}$$

$2x$

$1+2x^2+2x^4+2x(1+x^2)$

$(1+x)^2 2x(1+x^2)$

$(1+x^2)(1+x^2+2x)$

$\overset{A}{(1+x^2)}\overset{B}{(1+x)^2}$ C

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Equating co-efficient of x^2

$0 = A + B + 2C + D$

$0 = A + D + B + 2C$

$0 = \frac{1}{2} + \frac{1}{2} + 2C$

$1 = -2C$

$\boxed{C = -\frac{1}{2}}$

Put,

$C = -\frac{1}{2}$ then $A = \frac{1}{2}$, $D = 0$

$$\therefore \int_0^1 \frac{dx}{1+2x+2x^2+2x^3+x^4} = \int_0^1 \frac{\frac{1}{2}\,dx}{1+x} + \int_0^1 \frac{\frac{1}{2}\,dx}{1+x^2} +$$

$$\int_0^1 \frac{(-\frac{1}{2})x+0}{1+x^2}\,dx$$

$$= \frac{1}{2}\left[\log(1+x)\right]_0^1 - \frac{1}{2}\left[\frac{1}{(1+x)}\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x\,dx}{1+x^2}$$

$$= \frac{1}{2}\log 2 + 0 - \frac{1}{2}\left[\frac{1}{2} - 1\right] - \frac{1}{2}\int_0^1 \frac{dt/2}{1+t}$$

$$= \frac{1}{4} + \frac{1}{2}\log 2 - \frac{1}{4}\left(\log(t+1)\right)_0^1$$

$$= \frac{1}{4} + \frac{1}{2}\log 2 - \frac{1}{4}\left[\log 2 - 0\right]$$

$$= \log 2\left[\frac{1}{2} - \frac{1}{4}\right] + \frac{1}{4}$$

$$= \frac{1}{4}\log 2 + \frac{1}{4}$$

$$= \frac{1}{4}\left[\log 2 + 1\right] //.$$

Evaluate,

$$\int \frac{x^2+1}{x^4+1}\,dx$$

soln:

Given,

$$\int \frac{x^2+1}{x^4+1}\,dx$$

$$= \int \frac{x^2(1+1/x^2)}{x^2(x^2+1/x^2)}\,dx$$

$$= \int \frac{(1+1/x^2)}{(x^2+1/x^2)}\,dx$$

$$a^2+b^2 = (a-b)^2+2ab$$
$$= a^2+b^2-\cancel{2ab}+\cancel{2ab}$$

$$= \int \frac{(1+1/x^2)}{(x-1/x)^2+2}\,dx$$

Put,

$$x - 1/x = z$$

$$(1+1/x^2)\,dx = dz.$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(x/a)+c$$

$$\Rightarrow \int \frac{dz}{z^2+2} = 1/\sqrt{2}\,\tan^{-1}\left(\frac{z}{\sqrt{2}}\right)+c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right)+c \qquad 2=(\sqrt{2})^2$$

$$\therefore \int \frac{x^2+1}{x^4+1}\,dx = 1/\sqrt{2}\,\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right)+c \;//$$

Reduction Formula

The Power of integral it is reduced and the process case condited till we get a power whose integral is known or which can easily be integrated.

Problems.

Obtained the Reduction Formula for $\int \sin^n x \, dx$. Evaluate, $\int_0^{\pi/2} \sin^n x \, dx$ for all positive odd and even integral power of n. Hence Evaluate $\int_0^{\pi/2} \sin^7 x \, dx$.

Soln:

Let

$$I_n = \int \sin^n x \, dx$$

$$= \int \sin^{n-1} x \sin x \, dx \qquad [\because \int u\,dv = uv - \int v\,du]$$

$$= -\int \sin^{n-1} x \, d(\cos x)$$

$$= -(\sin^{n-1} x)\cos x + \int (n-1)\sin^{n-2} x \cos x \cos x \, dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x (1-\sin^2 x)\, dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\left[\int \sin^{n-2} x \, dx - \int \sin^n x \, dx\right]$$

$$= -\cos x \sin^{n-1} x + (n-1)\left[\int \sin^{n-2} x \, dx - \int \sin^n x \, dx\right]$$

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$$= -\cos x \sin^{n-1} x + (n-1)[I_{n-2} - I_n]$$

$$\left[\because I_n = \int \sin^n x\, dx \right.$$

$$\left.\because \int \sin^{n-2} x\, dx = I_{n-2}\right]$$

$$\therefore I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

i.e) $I_n(1+n-1) = -\cos x \sin^{n-1} x + (n-1) I_{n-2}$

$$I_n = \frac{-\cos x \sin^{n-1} x + (n-1) I_{n-2}}{n}$$

Hence,

$$\int \sin^n x\, dx = 1/n\left[(n-1) I_{n-2} - \cos x \sin^{n-1} x\right]$$

Also,

$$I_n = \int_0^{\pi/2} \sin^n x\, dx.$$

$$= \left[\frac{(n-1) I_{n-2}}{n}\right]_0^{\pi/2} - \left[\cancel{\frac{\sin^{n-1} x \cos x}{n}}\right]_0^{\pi/2}$$

$$= \frac{(n-1) I_{n-2}}{n} - 0.$$

$$I_n = \frac{(n-1)}{n} I_{n-2}$$

Ill^ly

[Even numbers.]

$$I_{n-2} = \frac{(n-3)}{(n-2)} I_{n-4}$$

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Results → ①

$$\int_0^{\pi/2} \sin^n x \, dx$$

case (i) If "n" is even,

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1)(n-3)\cdots}{n(n-2)(n-4)\cdots} \times \frac{\pi}{2}$$

case (ii) If "n" is odd,

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1)(n-3)\cdots}{n(n-2)(n-4)\cdots} \times 1$$

Results → ②

$$\int_0^{\pi/2} \cos^n x \, dx .$$

case (i) If "n" is even,

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{(n-1)(n-3)\cdots}{n(n-2)(n-4)\cdots} \times \frac{\pi}{2}$$

case (ii) If "n" is odd,

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{(n-1)(n-3)\cdots}{n(n-2)(n-4)\cdots} \times 1$$

Results → ③

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

case (i) If m, n both are even,

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)(m+n-4)\cdots} \times \pi/2$$

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case (ii) otherwise,

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \times 1$$

Now,

$$\int_0^{\pi/2} \sin^7 x \, dx$$

$\left[\because \int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1)(n-3)\ldots}{n(n-2)(n-4)\ldots} \times 1 \text{ if "n" is odd}\right]$

$$= \frac{(7-1)(7-3)(7-5)}{7(7-2)(7-4)(7-6)} \times 1$$

$$= \frac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3 \cdot 1} \times 1$$

$$= \frac{16}{35} \text{ //.}$$

Evaluate,

$$\int_0^{\pi/6} \sin^7 3x \, dx$$

Soln:

Given, $\int_0^{\pi/6} \sin^7 3x \, dx.$

Put,

$3x = t$

$3 dx = dt$

$dx = \frac{dt}{3}$

limits,

when $x = 0$ then $t = 0$

when $x = \pi/6$ then $t = \pi/2$

$\left[\because \int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1)(n-3)\ldots}{n(n-2)(n-4)\ldots} \times 1 \text{ if "n" is odd}\right]$

$$\therefore \int_0^{\pi/6} \sin^7 3x \, dx = \int_0^{\pi/2} (\sin^7 t) \, dt/3$$

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$$= \frac{1}{3} \int_0^{\pi/2} \sin^7 t \, dt$$

$$= \frac{1}{3} \frac{6.4.2}{7.5.3.1} \times 1$$

$$= \frac{16}{105} //.$$

Evaluate,
$$\int_0^{\pi/4} (\cos 2x)^{3/2} \cos x \, dx$$

soln:

Given, $$\int_0^{\pi/4} (\cos 2x)^{3/2} \cos x \, dx$$

$$= \int_0^{\pi/4} (1-2\sin^2 x)^{3/2} \cos x \, dx$$

$$= \int_0^{\pi/4} [1-(\sqrt{2}\sin x)^2]^{3/2} \cos x \, dx.$$

Put,
$\sqrt{2} \sin x = \sin t$

$\cos x \, dx = \frac{1}{\sqrt{2}} \cos t \, dt$

limits,

When $x = 0$ then $t = 0$

When $x = \pi/4$ then $t = \pi/2$

[$\because \sqrt{2} \sin \pi/4 = \sin t$

$\sqrt{2} \cdot \frac{1}{\sqrt{2}} = \sin t$

$1 = \sin t$

$\boxed{t = \pi/2}$]:

$$= \int_0^{\pi/2} (1-\sin^2 t)^{3/2} \cos t \, \frac{dt}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} (\cos^2 t)^{3/2} \cos t \, dt$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \cos^4 t \, dt$$

$$= \frac{1}{\sqrt{2}} \times \frac{3.1}{4.2} \times \frac{\pi}{2}$$

$$= \frac{3\pi}{16\sqrt{2}} \Rightarrow \frac{3\pi}{\sqrt{512}} \; //.$$

Evaluate,

$$\int_0^{\pi/8} \cos^3 4t \, dt$$

Soln: Given. $\int_0^{\pi/8} \cos^3 4t \, dt$.

Put, $4t = y$ | limits,

$4\,dt = dy$ | When $t = 0$ then $y = 0$

$dt = dy/4$ | When $t = \pi/8$ then $y = \pi/2$

$$\therefore \int_0^{\pi/8} \cos^3 4t \, dt = \int_0^{\pi/2} \cos^3 y \, . \, dy/4$$

$$= \frac{1}{4} \int_0^{\pi/2} \cos^3 y \, dy$$

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$= \frac{1}{4} \cdot \frac{2}{3.1} \times 1$

$= \frac{2}{12} \Rightarrow 1/6$ //.

Evaluate,
$$\int_0^\infty \frac{dx}{(1+x^2)^2}$$

soln:

Given, $\int_0^\infty \frac{dx}{(1+x^2)^2}$

Put,

$x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

limits,

When $x = 0$ then $\theta = 0$

When $x = \infty$ then $\theta = \pi/2$

$$\therefore \int_0^\infty \frac{dx}{(1+x^2)^2} = \int_0^{\pi/2} \frac{\sec^2\theta \, d\theta}{(1+\tan^2\theta)^2}$$

$$= \int_0^{\pi/2} \frac{\sec^2\theta}{\sec^4\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\sec^2\theta} d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= 1/2 \times \pi/2 \Rightarrow \pi/4$$ //.

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If $I_n = \int_0^{\pi/4} \tan^n x \, dx$. show that $I_{n-2} + I_n = \frac{1}{(n-1)}$. Hence Evaluate I_5.

Soln :

Given.

$$I_n = \int_0^{\pi/4} \tan^n x \, dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \tan^2 x \, dx$$

$$[\because \sec^2\theta - \tan^2\theta = 1]$$

$$= \int_0^{\pi/4} \tan^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\pi/4} \tan^{n-2} x \, dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x \, dx - I_{n-2} \longrightarrow (1)$$

Now. $\int_0^{\pi/4} \tan^{n-2} x \sec^2 x \, dx$

Put, $\tan x = t$

$\sec^2 x \, dx = dt$

limits,

When $x = 0$ then $t = 0$

When $x = \pi/4$ then $t = 1$.

$$\therefore \int_0^{\pi/4} (\tan x)^{n-2} (\sec x)^2 \, dx = \int_0^1 t^{n-2} \, dt$$

$$= \left[\frac{t^{n-1}}{n-1} \right]_0^1$$

$$\Rightarrow \frac{1}{n-1}$$

from ①,

$$I_n = \frac{1}{n-1} - I_{n-2}$$

$$\therefore I_{n-2} + I_n = \frac{1}{n-1} \longrightarrow (2)$$

To find I_5,

$$I_n = \frac{1}{n-1} - I_{n-2}$$

$\boxed{n=5}$ $\quad I_5 = \frac{1}{4} - I_3$

$\boxed{n=3}$ $\quad I_3 = \frac{1}{n} - I_1$

Now,

$$I_1 = \int_0^{\pi/4} \tan x \, dx$$

$$= \int_0^{\pi/4} \tan x \, dx$$

$$= [\log \sec x]_0^{\pi/4}$$

$$= \log \sec \pi/4 - \log 1 \Rightarrow \log \sec \pi/4 .$$

Hence,

$$I_5 = 1/4 - 1/2 + \log \sec \pi/4$$

$$I_5 = 1/4 - 1/2 + \log \sqrt{2}$$

$$= -1/4 + 1/2 \log 2$$

$$= 1/2 \log 2 - 1/4 \; //.$$

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Abtain the Rotection formula for,

$\int x^n . e^{ax} dx$. Hence evaluate $\int x^3 e^{2x} dx$.

soln :

Let,

$$I_n = \int x^n e^{ax} dx$$

[$\because \int u dv = uv - \int v du$]

$u = x^n$ | $dv = e^{ax} dx$

$du = n x^{n-1}$ | $v = \frac{e^{ax}}{a}$

$$= x^n \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} n x^{n-1} dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int e^{ax} x^{n-1} dx$$

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1} \text{ //} \longrightarrow (1)$$

Evaluate, $\int x^3 e^{ax} dx$.

[$\because$ from (1)]

Put, $\boxed{n=3}$.

$$I_3 = \frac{x^3 e^{2x}}{2} - \frac{3}{2} I_2 \quad [\because a=2]$$

Also,

$$I_2 = \frac{x^2 e^{2x}}{2} - \frac{2}{2} I_1$$

Now,

$$I_1 = \int x^1 e^{2x} dx$$

$$= \int x e^{2x} dx$$

$$= \left(\frac{xe^{2x}}{2}\right) - 1\left(\frac{e^{2x}}{4}\right)$$

$$= e^{2x}\left(\frac{x}{2} - \frac{1}{4}\right).$$

$$\therefore \int x^3 e^{2x}\,dx = \frac{x^3 e^{2x}}{2} - \frac{3}{2}\left[\frac{x^2 e^{2x}}{2} - \left(\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right)\right]$$

$$= \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{3xe^{2x}}{4} - \frac{3}{8}e^{2x}.$$

$$= \frac{e^{2x}}{8}\left[x^3 - \frac{3x^2}{2} + \frac{3x}{2} - \frac{3}{4}\right] //.$$

If $I_{m,n} = \int \sin^m x \cos^n x\,dx$. This deduce that $I_{m,n} = \frac{-\sin^{m-1}x \cos^{n+1}x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$

Soln:

Given, $I_{m,n} = \int \sin^m x \cos^n x\,dx$

$$= \int \sin^{m-1}x \sin x \cos^n x\,dx.$$

$$\left[\because \int u\,dv = uv - \int v\,du\right]$$

$u = \sin^{m-1}x$	$dv = \cos^n x$
$du = (m-1)\sin^{m-2}x \cos x\,dx$	$v = \frac{\cos^{n+1}}{n+1}$.

$$= \left(-\sin^{m-1}x \frac{\cos^{n+1}}{n+1}\right) + \int \frac{\cos^{n+1}x}{n+1}(m-1)\sin^{m-2}\cos x\,dx$$

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$$= -\sin^{m-1}x \frac{\cos^{n+1}}{n+1} + \frac{m-1}{m+1}\int \cos^{n+2}x \sin^{m-2}x\,dx$$

$$= -\sin^{m-1}x \frac{\cos^{n+1}}{n+1} + \frac{m-1}{m+1}\int \cos^{n}x \sin^{m-2}\cos^2 x\,dx$$

$$= -\sin^{m-1}x \frac{\cos^{n+1}}{n+1} + \frac{m-1}{m+1}\int \sin^{m-2}\cos^{n}x(1-\sin^2 x)dx$$

$$= -\sin^{m-1}x \frac{\cos^{n+1}}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2}x \cos^{n}x\,dx -$$

$$\frac{m-1}{m+1}\int \sin^{m}x\cos^{n}x\,dx$$

$$= -\sin^{m-1}x \frac{\cos^{n+1}}{n+1} + \frac{m-1}{m+1} I_{m-2,n} - \frac{m-1}{m+1} I_{m,n}$$

$$= -\frac{\sin^{m-1}x\cos^{n+1}}{n+1} + \frac{m-1}{m+1} I_{m-2,n} - \frac{m-1}{m+1} I_{m,n}$$

ie) $I_{m,n}\left(1+\frac{m-1}{n+1}\right) = \frac{-\sin^{m-1}x\cos^{n+1}x}{n+1} +$

$$\frac{m-1}{m+1} I_{m-2,n}$$

$$\therefore I_{m,n} = \frac{n+1}{m+n}\left(-\frac{\sin^{m-1}x\cos^{n+1}x}{n+1}\right) +$$

$$\frac{n+1}{m+n} \times \frac{m-1}{n+1} I_{m-2,n}$$

Hence,

$$I_{m,n} = \frac{-\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} \quad //$$

Proved.

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Reduce the formula for $\int(a^2-x^2)^n dx, n>0$

Hence prove that $I_n = \int_0^a (a^2-x^2)^n dx = \frac{2na^2}{2n+1}$

$(I_{n-1})_0^a$.

soln: Given,

$$I_n = \int (a^2-x^2)^n dx$$

$u = (a^2-x^2)^n$ | $dv = dx$

$du = n(a^2-x^2)^{n-1}(-2x)dx$ | $v = x$.

$= (a^2-x^2)^n x + 2\int x^2 n(a^2-x^2)^{n-1} dx$

$= (a^2-x^2)^n x + 2n\int (a^2-x^2)^{n-1}(a^2-x^2-a^2)\,dx$

$= x(a^2-x^2)^n - 2n\int (a^2-x^2)^n dx + 2na^2\int (a^2-x^2)^{n-1} dx$

$= x(a^2-x^2)^n - 2nI_n + 2na^2 I_{n-1}$

$I_n(1-2n) = x(a^2-x^2)^n + 2na^2 I_{n-1}$

$$I_n = \frac{x(a^2-x^2)^n}{1+2n} + \frac{2na^2}{(1+2n)} I_{n-1}$$

Also, Let, $I_n = \int_0^a (a^2-x^2)^n dx$

$u = (a^2-x^2)^n$ | $dv = dx$

$du = n(a^2-x^2)^{n-1}(-2x)dx$ | $v = x$

$$= \left[x(a^2-x^2)^n\right]_0^a + 2n\int_0^a (a^2-x^2)^{n-1}x^2\,dx$$

$$= 2n\int_0^a (a^2-x^2)^{n-1}x^2\,dx$$

$$= \frac{2na^2}{1+2n}\left(I_{n-1}\right)_0^a \quad //$$

Evaluate, $\int_0^{\pi/2} \sin^5\theta\cos^7\theta\,d\theta$.

Soln :

Given, $\int_0^{\pi/2} \sin^5\theta\cos^7\theta\,d\theta$.

We know that,

$$\int_0^{\pi/2} \sin^m\theta\cos^n\theta\,d\theta =$$

$$\frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots} \times 1$$

$$\therefore \int_0^{\pi/2} \sin^5\theta\cos^7\theta\,d\theta = \frac{(5-1)(5-3)(7-1)(7-3)(7-5)}{(5+7)(5+7-2)(5+7-4)(5+7-6)(5+7-8)(5+7-10)} \times 1$$

$$= \frac{(4)(2)(6)(4)(2)}{(12)(10)(8)(6)(4)(2)} \times 1 \Rightarrow \frac{1}{120} //$$

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Evaluate, $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$.

soln: Given, $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$.

We know that,

$$\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta =$$

$$\frac{(m-1)(m-3)\ldots\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)\ldots\ldots\ldots} \times \pi/2 .$$

$$\therefore \int_0^{\pi/2} \sin^6 x \cos^4 x \, dx = \frac{(6-1)(6-3)(6-5)(4-1)(4-3)}{(6+4)(6+4-2)(6+4-4)(6+4-6)(6+4-8)} \times \pi/2$$

$$= \frac{(5)(3)(1)(3)(1)}{(10)(8)(6)(4)(2)} \times \pi/2$$

$$= \frac{15}{80 \times 16} \times \frac{\pi}{2}$$

$$= \frac{15\pi}{2560} \Rightarrow \frac{3\pi}{512} //$$

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$(1+1/x^2)\,dx = dz$

$$\Rightarrow \frac{dz}{z^2+2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}} + c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right) + c$$

$$\therefore \int \frac{x^2+1}{x^4+1}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) + c.$$

Reduction formula

The powes of integral it is reduced and the process care conducted till we get a power whose integral is known or which can easily be integrated.

Problems:

Obtained the reduction formulla for $\int \sin^n x\,dx$ Evaluate $\int_0^{\pi/2} \sin^n x\,dx$ for all positive odd and even integral powes of n Hence Evaluate $\int_0^{\pi/2} \sin^n x\,dx$

$I_n = \int n^{th}$ iterative

Soln:

Let $I_n = \int \sin^n x\,dx$ $\qquad \int u\,dv = uv - \int v\,du$

$= \int \sin^{n-1}x \sin x\,dx$

$= -\int \sin^{n-1}x \; d(\cos x)$

$= -(\sin^{n-1}x)\cos x + \int (n-1)\sin^{n-2}x \cos x$

$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,(1-\sin^2 x)\,dx$

$= -\cos x \sin^{n-1}x + (n-1)\left[\int \sin^{n-2}x\,dx - \int \sin^n x\,dx\right]$

$= -\cos x \sin^{n-1}x + (n-1)\left[\int \sin^{n-2}x\,dx - \int \sin^n x\,dx\right]$

$= -\cos x \sin^{n-1}x + (n-1)[I_{n-2} - I_n]$

$\therefore I_n = -\cos x \sin^{n-1}x + (n-1)I_{n-2} - (n-1)I_n$

ii) $I_n(1+n-1) = -\cos x \sin^{n-1}x + (n-1)I_{n-2}$

$$I_n = \frac{-\cos x \sin^{n-1}x + (n-1)I_{n-2}}{n}$$

Hence:

$$\int \sin^n x\,dx = \frac{1}{n}\left[(n-1)I_{n-2} - \cos x \cdot \sin^{n-1}x\right]$$

also

$I_n = \int_0^{\pi/2} \sin^n x\,dx$

$$= \left[\frac{(n-1)I_{n-2}}{n}\right]_0^{\pi/2} - \left[\frac{\sin^{n-1}x\cos x}{n}\right]_0^{\pi/2}$$

$$= \frac{(n-1)I_{n-2}}{n} - 0$$

$$I_n = \frac{(n-1)}{n} I_{n-2}$$

[Even numbers]

$$I_{n-2} = \frac{(n-3)}{(n-2)} I_{n-4}.$$

$$I_{n-4} = \frac{(n-5)}{(n-4)} I_{n-6}.$$

- - - - -

also odd numbers

$$I_{n-1} = \frac{(n-2)}{(n-1)} I_{n-3}$$

$$I_{n-3} = \frac{(n-4)}{(n-3)} I_{n-5}$$

- - - - - -

Now Let $I_n = \int_0^{\pi/2} \sin^n x\,dx$

put $\boxed{n=0}$

$I_0 = \int_0^{\pi/2} \sin^0 x\,dx$

$= \int_0^{\pi/2} 1\,dx$

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$= (x)_0^{\pi/2} = \pi/2$

$= \pi/2$

and $\boxed{n=1}$

$I_1 = \int_0^{\pi/2} \sin x \, dx$

$= (-\cos x)_0^{\pi/2}$

$= -0+1$

$= 1$

$\boxed{I_0 = \pi/2}$

Results - 1

$\int_0^{\pi/2} \sin^n x \, dx$

case i) If 'n' is even

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1)(n-3)\dots}{n(n-2)(n-4)} \times \frac{\pi}{2}$$

case ii) If 'n' is odd $I_1 = 1$

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1)(n-3)\dots}{n(n-2)(n-4)\dots} \times 1$$

Results - 2

$\int_0^{\pi/2} \cos^n x \, dx$

case i) If 'n' is even

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{(n-1)(n-3)\dots}{n(n-2)(n-4)\dots} \times$$

Results - 3

$\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

case i) If m, n both are even

$\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

$$= \frac{(m-1)(m-3)\dots(n-1)(n-3)}{(m+n)(m+n-2)(m+n-4)} \times \pi/2$$

Case (ii) otherwise

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{(m-1)(m-3)\dots(n-1)(n-3)}{(m+n)(m+n-2)(m+n-4)\dots} \times 1$$

Now

$\int_0^{\pi/2} \sin^7 x \, dx$ odd number

$$= \frac{(7-1)(7-3)(7-5)}{7(7-2)(7-4)(7-6)} \times 1 \quad so = 1.$$

$$= \frac{6\cdot 4\cdot 2}{7\cdot 5\cdot 3\cdot 1} \times 1$$

$$= \frac{16}{35} \quad //$$

[$\therefore \int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1)(n-3)}{n(n-2)(n-4)\dots} \times 1$ if n is odd)

Evaluate

$\int_0^{\pi/6} \sin^7 3x \, dx$

odd - ஒற்றைப்படை
even - இரட்டைப்படை

Soln:

Given $\int_0^{\pi/6} \sin^7 3x \, dx$

put $3x = t$, $3dx = dt$, $dx = \frac{dt}{3}$

Limits

when $x=0$ then $t=0$

when $x=\pi/6$ then $t=\pi/2$

$$\int_0^{\pi/6} \sin^7 3x \, dx = \int_0^{\pi/2} (\sin^7 t) \frac{dt}{3}$$

$$= \frac{1}{3} \int_0^{\pi/2} \sin^7 t \, dt$$

$$= \frac{1}{3} \frac{(n-1)(n-3)(n-5)}{(n)(n-2)(n-4)} \times 1$$

$$= \frac{1}{3} \frac{6\cdot 4\cdot 2}{7\cdot 5\cdot 3\cdot 1} \times 1$$

$$= \frac{16}{105} \quad //$$

(negative numbers, 0 zero, don't accepted)

(–, 0 எண்களை ஏற்காது)

Evaluate $\int_0^{\pi/4} (\cos 2x)^{3/2} \cos x \, dx$

Soln:

Given $\int_0^{\pi/4} (\cos 2x)^{3/2} \cos x \, dx$

$$= \int_0^{\pi/4} (1 - 2\sin^2 x)^{3/2} \cos x \, dx$$

$=$

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$= \int_0^{\pi/4} [1-(\sqrt{2}\sin x)^2]^{3/2} \cos x\, dx$

put

$\sqrt{2}\sin x = \sin t$ | limits
$\cos x\, dx = \frac{1}{\sqrt{2}} \cos t\, dt$ | when $x=0$ then $t=0$
when $x=\pi/4$ then $t=\pi/2$

$[\because \sqrt{2}\sin \pi/4 = \sin t$
$\sqrt{2}\cdot 1/\sqrt{2} = \sin t$
$1 = \sin t$
$t = \pi/2]$

$= \int_0^{\pi/2} (1-\sin^2 t)^{3/2} \cos t \frac{dt}{\sqrt{2}}$

$= 1/\sqrt{2} \int_0^{\pi/2} (\cos^2 t)^{3/2} \cos t\, dt$

$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \cos^4 t\, dt$

$= \frac{1}{\sqrt{2}} \times \frac{3\cdot 1}{4\cdot 2} \times \pi/2$

$= \frac{3\pi}{16\sqrt{2}}$

$= \frac{3\pi}{\sqrt{512}}$ //.

Evaluate $\int_0^{\pi/8} \cos^3 4t\, dt$

Soln:

Given $\int_0^{\pi/8} \cos^3 4t\, dt$

put $4t = y$ limits
$4\, dt = dy$ when $t=0$ then $y=0$
$dt = \frac{dy}{4}$ when $t=\pi/8$ then $y=\pi/2$

$\therefore \int_0^{\pi/8} \cos^3 4t\, dt = \int_0^{\pi/2} \cos^3 y \frac{dy}{4}$

$= 1/4 \int_0^{\pi/2} \cos^3 y\, dy$

$= \frac{1}{4} \cdot \frac{2}{3\cdot 1} \times 1$

Evaluate

$\int_0^{\infty} \frac{dx}{(1+x^2)^2}$

Soln:

Given $\int_0^{\infty} \frac{dx}{(1+x^2)^2}$

put

$x = \tan\theta$ limits
$dx = \sec^2\theta\, d\theta$ when $x=0$ then $\theta=0$
when $x=\infty$ then $\theta=\pi/2$

$\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^2}$

$= \int_0^{\pi/2} \frac{\sec^2\theta}{\sec^4\theta} d\theta$

$= \int_0^{\pi/4} \frac{1}{\sec^2\theta} d\theta$

$= \int_0^{\pi/4} \cos^2\theta\, d\theta$

$= \frac{1}{2} \times \frac{\pi}{2}$

$= \pi/4$ //.

Evaluate $\int_0^{\pi/2} \sin^5\theta \cos^7\theta\, d\theta$

Soln:

Given $\int_0^{\pi/2} \sin^5\theta \cos^7\theta\, d\theta$

We know that

$\int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta$

$= \frac{(m-1)(m-3)\ldots(n-1)(n-3)}{(m+n)(m+n-2)} \times 1$

$\int_0^{\pi/2} \sin^5\theta \cos^7\theta\, d\theta = \frac{(5-1)(5-3)(7-1)(7-3)(7-5)}{(5+7)(5+7-2)(5+7-4)(5+7-6)(5+7-8)(5+7-10)} \times 1$

$= \frac{(4)(2)(6)(4)(2)}{(12)(10)(8)(6)(4)(2)}$

$= \frac{1}{120}$ //

Evalu...

Soln:

Given $\int_0^{\pi/2}$ ...

we kn...

$\int_0^{\pi/2} \sin$ ...

$\int_0^{\pi/2} \sin^6$ ...

If

E...

Sol...

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Evaluate $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$

soln:

Given

$\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$

we know that

$$\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \frac{(m-1)(m-3)\cdots(n-1)(n-3)}{(m+n)(m+n-2)\cdots} \times \pi/2$$

$$\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx = \frac{(6-1)(6-3)(6-5)(4-1)(4-3)}{(6+4)(6+4-2)(6+4-4)(6+4-6)(6+4-8)} \times \pi/2$$

$$= \frac{5(3)(1)(3)(1)}{(10)(8)(6)(4)(2)} \times \pi/2$$

$$= \frac{15}{80 \times 16} \times \pi/2$$

$$= \frac{15\pi}{2560}$$

$$= \frac{3\pi}{512} \,//.$$

If $I_n = \int_0^{\pi/4} \tan^n x \, dx$ show that $I_{n-2} + I_n = \frac{1}{n-1}$ hence

Evaluate I_5

soln

Given

$$I_n = \int_0^{\pi/4} \tan^n x \, dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \tan^2 x \, dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\pi/4} \tan^{n-2} x \, dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x \, dx - I_{n-2} \rightarrow (1)$$

now $\int_0^{\pi/4} \tan^{n-2} x \sec^2 x \, dx$

(n-3)(n-5) x 1

(5+7-4)

(5+7-8)

put $\tan x = t$

$\sec^2 x \, dx = dt$

limits

when $x = 0$ then $t = 0$

when $x = \pi/4$, then $t = 1$

$$\therefore \int_0^{\pi/4} (\tan x)^{n-2} (\sec x)^2 dx = \int_0^1 t^{n-2} dt$$

$$= \left[\frac{t^{n-1}}{n-1}\right]_0^1 \Rightarrow \frac{1}{n-1}$$

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From ①

$$I_n = \frac{1}{n-1} - I_{n-2}$$

$$I_{n-2} + I_n - \frac{1}{n-1} \rightarrow ②$$

To find I_n,

$$I_n = \frac{1}{n-1} - I_{n-2}$$

$\boxed{n=5}$

$$I_5 = \frac{1}{4} - I_3$$

$\boxed{n=3}$

$$I_3 = \frac{1}{n} - I_1$$

Now

$$I_1 = \int_0^{\pi/4} \tan x \, dx$$

$$= \int_0^{\pi/4} \tan x \, dx$$

$$= [\log \sec x]_0^{\pi/4}$$

$$= \log \sec \pi/4 - \log 1$$

$$\Rightarrow \log \sec \pi/4$$

Hence

$$I_5 = 1/4 - 1/2 + \log \sec \pi/4$$

$$I_5 = 1/4 - 1/2 + \log \sqrt{2}$$

$$= -1/4 + 1/2 \log 2$$

$$= 1/2 \log 2 - 1/4.$$

Obtain the Reduction formula for $\int x^n e^{ax} dx$. Hence evaluate $\int x^5 e^{2x} dx$

Soln:

Let $I_n = \int x^n e^{ax} dx$

$u = x^n \qquad dv = e^{ax} dx$

$du = n x^{n-1} \qquad v = \frac{e^{ax}}{a}$

$$I = \frac{x^n e^{ax}}{a} - \int \frac{e^{ax}}{a} n x^{n-1} dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int e^{ax} x^{n-1} dx$$

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1} \rightarrow ①$$

Evaluate

$$\int x^3 e^{ax} dx$$

Put

$\boxed{n=3}$

$$I_3 = \frac{x^2 e^{2x}}{2} - \frac{3}{2} I_2$$

Also

$$I_2 = \frac{x^2 e^{2x}}{2} - \frac{2}{2} I_1$$

Now $I_1 = \int x^1 e^{2x} dx$

$$= \int x e^{2x} dx$$

$$= (x e^{2x/2}) - 1 (e^{2x}/4)$$

$$= e^{2x/2} (x/2 - 1/4)$$

$$\therefore \int x^3 e^{2x} dx = \frac{x^3 e^{2x}}{2} - \frac{3}{2}\left[\frac{x^2 e^{2x}}{2} - \left(\frac{x e^{2x}}{2} - e^{2x}/4\right)\right]$$

$$= \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{3x e^{2x}}{4} - \frac{3}{8} e^{2x}$$

$$= \frac{e^{2x}}{8}\left[x^5 - \frac{3x^2}{2} + \frac{3x}{2} - 3/4\right].$$

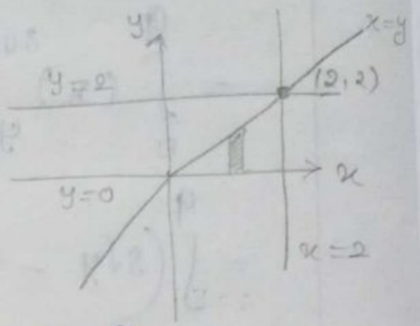
Change of order of Integraction.

Evaluate, $\int_0^2 \int_y^2 \frac{x}{x^2+y^2}\, dx\, dy$.

by changing the order of Integration.

Soln :-

Given $\int_0^2 \int_y^2 \frac{x}{x^2+y^2}\, dx\, dy$



limits → Given

$x = y$, $x = 2$ and $y = 0$, $y = 2$

by changing the order of Integration.

$$\therefore \int_0^2 \int_y^2 \frac{x}{x^2+y^2}\, dx\, dy = \int_0^2 \int_0^x \frac{x}{x^2+y^2}\, dy\, dx$$

[∵ Required limits, $x = 0$, $x = 2$ and $y = 0$, $y = x$].

$$= \int_0^2 \int_0^x \frac{x}{x^2+y^2}\, dy\, dx$$

$$= \int_0^2 \left(\int_0^x \frac{x}{x^2+y^2}\, dy \right) dx \qquad \left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a) + c\right]$$

$$= \int_0^2 x \left(\frac{1}{x} \tan^{-1}(y/x) \right)_0^x dx$$

$$= \int_0^2 \left[\tan^{-1}\left(\frac{x}{x}\right) - \tan^{-1}(0) \right] dx$$

$$= \int_0^2 \left(\frac{\pi}{4} - 0 \right) dx$$

$$= \frac{\pi}{4} (x)_0^2 = \frac{2\pi}{4} = \frac{\pi}{2} \,//.$$

Evaluate,
$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dx\, dy.$$

by using of change order of Integration.

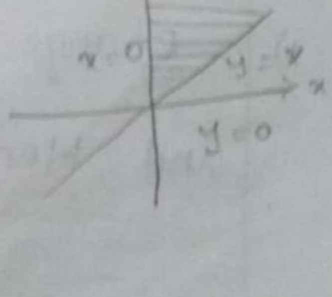
Soln :

Given, $$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dx\, dy.$$

[∵ It's does not exists].

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dx\, dy = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy\, dx.$$

Given limits,



$x = 0, \; x = \infty$

$y = x, \; y = \infty$

By changing the order of Integration

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy.$$

[∵ Required limits,

$y = 0$; $y = \infty$

$x = 0$, $x = y$].

$$= \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \left(\frac{e^{-y}}{y}\right)(x)_0^y\, dy$$

$$= \int_0^\infty \left(\frac{e^{-y}}{y} \cdot y\right) dy$$

$$= \left(\frac{e^{-y}}{-1}\right)_0^\infty = 1 //.$$

---

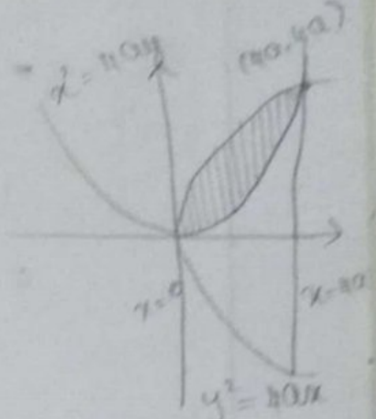
1) Change the order of Integration, and Hence find the value of

$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy\, dy\, dx.$$

Soln:

Given,

$$\int_0^{4a}\int_{x^2}^{2\sqrt{ax}} xy\,dy\,dx.$$



Given limits,

$y = x^2/4a \quad y = 2\sqrt{ax}$

$x = 0 \quad x = 4a.$

$x^2 = 4ay \;,\; y^2 = 4ax$ and $x = 0 , x = 4a$

by changing the order of Integraction,

$y^2 = 4ax$

$\boxed{y^2/4a = x}$

$x^2 = 4ay$

$\boxed{x = 2\sqrt{ay}}$

$$\therefore \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} xy\,dx\,dy$$

$$= \int_0^{4a} y\left(\frac{x^2}{2}\right)_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_0^{4a} y\left(4ay - y^4/16a^2\right) dy$$

$$= \frac{1}{2}\int_0^{4a} \left(4ay^2 - y^5/16a^2\right) dy$$

$$= \frac{1}{2}\left(4a\,y^3/3 - y^6/16\times6\times a^2\right)_0^{4a}$$

$$= \frac{1}{2}\left(\frac{4a}{3}\times 64a^3 - \frac{(4a)^6}{16\times6\times a^2}\right)$$

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$$= \frac{1}{2}\left(\frac{256a^4}{3} - \frac{4096a^6}{96a^4}\right)$$

$$= \frac{1}{2}\left(\frac{128}{3}\right)a^4$$

$$= \frac{64}{3}a^4 \,//.$$

Evaluate, $\int_0^a \int_0^{\sqrt{a^2-x^2}} xy \, dx \, dy$ by changing the Order of Integration.

soln:

Given, $\int_0^a \int_0^{\sqrt{a^2-x^2}} xy \, dx \, dy$

since, given integral does not exists,

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \, dx \, dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \, dy \, dx.$$

by changing the order of Integration,

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \, dy \, dx = \int_0^a \int_0^{\sqrt{a^2-y^2}} xy \, dx \, dy$$

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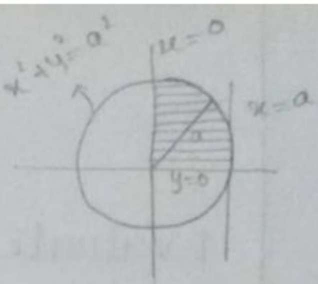
[$\because$ limits, $x=0$, $x=a$

and, $y=0$, $y=\sqrt{a^2-x^2}$



$y^2 = a^2 - x^2$

$x^2+y^2=a^2$.]

$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} xy \, dx \, dy.$$

$$= \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \frac{y}{2}\left(a^2-y^2-0\right) dy$$

$$= \frac{1}{2}\int_0^a \left(a^2y - y^3\right)dy$$

$$= \frac{1}{2}\left(\frac{a^2y^2}{2} - \frac{y^4}{4}\right)_0^a$$

$$= \frac{1}{2}\left(a^2\frac{a^2}{2} - \frac{a^4}{4}\right),$$

$$= \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$

$$= \frac{1}{2}\left(\frac{a^4}{4}\right) = \frac{a^4}{8} \;//.$$

Triple Integration.

Evaluate,

$$\int_0^4 \int_0^2 \int_0^1 (x+y+z)\, dz\, dy\, dx$$

Soln :-

Given,

$$\int_0^4 \int_0^2 \int_0^1 (x+y+z)\, dz\, dy\, dx$$

$$= \int_0^4 \int_0^2 \left(xz + yz + z^2/2\right)_0^1 dy\, dx$$

$$= \int_0^4 \int_0^2 \left[\left(x(1-0) + y(1-0) + \left(1^2/2 - 0^2/2\right)\right)\right] dy\, dx$$

$$= \int_0^4 \int_0^2 \left(x + y + 1/2\right) dy\, dx$$

$$= \int_0^4 \left[xy + y^2/2 + 1/2\, y\right]_0^2 dx$$

$$= \int_0^4 \left[x(2-0) + 2^2/2 + 1/2\; 2 - 0\right] dx$$

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$$= \int_0^4 (2x + 4/2 + 1)\,dx$$

$$= \left[2\cdot\left(\frac{x^2}{2}\right) + 4/2 + 1\right] dx$$

$$= \left[(4)^2 + 2(4) + 4\right] - 0$$

$$= 16 + 8 + 4 \Rightarrow 28 //.$$

Evaluate.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\,dy\,dx.$$

Soln:

Given.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\,dy\,dx.$$

$$\left[\because \int \frac{dx}{\sqrt{a^2-x^2}} = \cancel{\tfrac{1}{a}} \sin^{-1}(x/a) + C\right]$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{(\sqrt{1-x^2-y^2})^2 - z^2}}\, dz\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2-y^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} (\pi/2 - 0)\, dy\, dx$$

$$= \pi/2 \int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx$$

$$= \pi/2 \int_0^1 (y)_0^{\sqrt{1-x^2}} dx$$

$$= \pi/2 \int_0^1 \sqrt{1-x^2}\, dx$$

$$= \pi/2 \left[x/2 \sqrt{1-x^2} + 1/2 \sin^{-1}(x) \right]_0^1$$

$$= \pi/2 \left[1/2 \times \pi/2 \right]$$

$$= \pi^2/8 //.$$

Find the Area enclosed by the elipse $\rightarrow$ $x^2/a^2 + y^2/b^2 = 1$

Soln:

Given, $x^2/a^2 + y^2/b^2 = 1$

We know that,

Area,

$$A = \iint dx\,dy$$

$$A = \int_{-b}^{b} \int_{-a/b\sqrt{b^2-y^2}}^{a/b\sqrt{b^2-y^2}} dx\,dy.$$

$$= 2\int_{-b}^{b} \int_{0}^{a/b\sqrt{b^2-y^2}} dx\,dy$$

$$= 2\int_{-b}^{b} (x)_{0}^{a/b\sqrt{b^2-y^2}} dy$$

$$= 2\int_{-b}^{b} a/b\sqrt{b^2-y^2}\,dy$$

$$= \frac{2a}{b}\int_{-b}^{b} \sqrt{b^2-y^2}\,dy$$

$$= \frac{4a}{b}\int_{0}^{b} \sqrt{b^2-y^2}\,dy.$$

$$= \frac{4a}{b}\left[y/2\sqrt{b^2-y^2} + b^2/2 \sin^{-1}(y/b) \right]_0^b$$

$$= \frac{4a}{b}\left[b^2/2 \times \pi/2 \right] \Rightarrow \pi ab \text{ Squre units}$$

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Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z \, dz\, dy\, dx$.

Ans = ½

Find the volume of the ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Ans : $\frac{4\pi}{3}abc$ cubic units

soln:

Given.

ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

We know that,

$$\text{Volume, } V = \int_{-a}^{a}\int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}}\int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx.$$

$$\left[\because \frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right.$$

$$z^2 = c^2\left[1-x^2/a^2-y^2/b^2\right]$$

$$\left. z = c\sqrt{1-x^2/a^2-y^2/b^2}\right]$$

$$V = 8\int_0^a\int_0^{b/a\sqrt{a^2-x^2}}\int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx.$$

$$V = 8\int_0^a \int_0^{b/a} \left[z \right]_0^{c\sqrt{1-x^2/a - y^2/b^2}} dy\,dx$$

$$= 8\int_0^a \int_0^{b/a} \left[c\sqrt{1-x^2/a - y^2/b^2} \right] dy\,dx.$$

$$= 8c/b \int_0^a \int_0^{b/a\sqrt{a^2-x^2}} \sqrt{b^2(1-x^2/a^2) - y^2}\; dy\,dx.$$

$$= \frac{8c}{b}\int_0^a \left[y/2\sqrt{b^2(1-x^2/a^2)-y^2} + \frac{b^2(1-x^2/a^2)}{2} \sin^{-1}\left(y/b\sqrt{1-x^2/a} \right) \right]_0^{b/a\sqrt{a^2-x^2}} dx.$$

$$= \frac{8c}{b}\int_0^a \pi/2 \cdot b^2/2 \left(1 - x^2/a^2\right) dx$$

$$= \frac{8\pi c b}{4}\int_0^a \left(1 - x^2/a^2\right) dx.$$

$$= 2\pi c b \int_0^a \left(1 - x^2/a^2\right) dx$$

$$= 2\pi c b \left[x - x^3/3a^2 \right]_0^a$$

$$= 2\pi c b \left[a - a^3/3a^2 \right]$$

$$= 2\pi c b \left[a - a/3 \right] \Rightarrow 4/3\, \pi abc$$

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Hence,

Volume $V = \frac{4}{3} \pi abc$. Cubic units.

1. Double integral :

The integrant is a Function of $f(x,y)$ and if it is integrated with respect to x and y. we get the Double integral and donoted as $\iint f(x,y)\, dx\, dy$ (or) $\iint f(x,y)\, dy\, dx$.

2. Triple integral :

A function $f(x,y,z)$ integrated with respect to x, y and z respectively we get $\iiint f(x,y,z)\, dx\, dy\, dz$.

Evaluate.

1) $\int x^4\, dx$.

Soln:

Given,

$$\int x^4\, dx$$

$$= \frac{x^5}{5} + c$$

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$\int \sin\theta \cos 2\theta . d\theta$

Given, $\sin\theta \cos 2\theta d\theta$.

$= \int \cos 2\theta \sin\theta . d\theta$

$= \frac{1}{2} \int [\sin 3\theta - \sin\theta] d\theta$

$= \frac{1}{2} [\int \sin 3\theta d\theta - \int \sin\theta d\theta]$

$= \frac{1}{2} \left[\frac{-\cos 3\theta}{3} + \cos\theta \right] + c$

$= \frac{1}{2} \left[\cos\theta - \frac{\cos 3\theta}{3} \right] + c$ //.

$\int e^{5t} dt$

Given, $\int e^{5t} dt$

$= \frac{e^{5t}}{5} + c$ //.

$\int (x^2 + 2x - 4) dx$

Given, $\int (x^2 + 2x - 4) dx$

$= \int x^2 dx + 2 \int x dx - 4 \int dx$

$= \frac{x^3}{3} + \frac{2x^2}{2} - 4x + c$

$= \frac{x^3}{3} + x^2 - 4x + c$ //.

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$\int \sin^2\theta \, d\theta$

Given,

$$\int \sin^2\theta \, d\theta$$

$$= \int \left(\frac{1-\cos\theta}{2} \right) d\theta$$

$$= \frac{1}{2} \int (1-\cos 2\theta) \, d\theta$$

$$= \frac{1}{2} \left[\int d\theta - \int \cos 2\theta \, d\theta \right]$$

$$= \frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2} \right] + c$$

$$= \frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2} \right] + c$$

$\int \cos^3\theta \, d\theta$

Given,

$$\int \cos^3\theta \, d\theta$$

$$= \int \frac{1}{4} \left[\cos 3\theta + 3\cos\theta \right] d\theta$$

$$= \frac{1}{4} \left[\frac{\sin 3\theta}{3} + 3 \cdot \frac{\sin\theta}{1} \right] + c$$

$$= \frac{1}{12} \left[\sin 3\theta + 9 \sin\theta \right] + c.$$

$\int \cos^2\theta \, d\theta$

Given,

$$\int \cos^2\theta \, d\theta$$

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$$= \int \left(\frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= \frac{1}{2} \int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2} \left[\int d\theta + \int \cos 2\theta \cdot d\theta \right]$$

$$= \frac{1}{2} \left[\theta + \frac{\sin 2\theta}{2} \right] + c \,//.$$

$\int \sin^3\theta \, d\theta$

Given, $\int \sin^3\theta \, d\theta$

$$= \int \frac{1}{4} \left[3\sin\theta - \sin 3\theta \right]$$

$$= \frac{1}{4} \left[-\frac{3\cos\theta}{1} + \frac{\cos 3\theta}{3} \right] + c$$

$$= \frac{1}{12} \left[-9\cos\theta + \cos 3\theta \right] + c$$

$$= \frac{1}{12} \left[\cos 3\theta - 9\cos\theta \right] + c .$$

$\int_1^2 x^2 \, dx$

Given, $\int_1^2 x^2 \, dx$

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$$= \left[\frac{x^3}{3} \right]_1^2$$

$$= \frac{2^3}{3} - \frac{1^3}{3}$$

$$= \frac{8}{3} - \frac{1}{3} \Rightarrow 7/3$$

Evaluate

$$\int_0^1 (y - y^4)\, dy$$

$$= \left[\frac{y^2}{2} - \frac{y^5}{5} \right]_0^1$$

$$= 1/2 - 1/5 \Rightarrow \frac{5-2}{10} = 3/10$$

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

$m = 1$, $n = 1$.

$$\beta(1,1) = \int_0^1 x^{1-1} (1-x)^{1-1}\, dx$$

$$= \int_0^1 x^0 (1-x)^0\, dx$$

$$= \int_0^1 1.1\, dx$$

$$= \int_0^1 dx \Rightarrow (x)_0^1 \Rightarrow 1.$$

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definite integral.

$$\int_0^{\pi/2} \sqrt{1+\cos 2x}\, dx$$

Given,

$$\int_0^{\pi/2} \sqrt{1+\cos 2x}\, dx$$

$$= \int_0^{\pi/2} (2\cos^2 x)^{1/2}\, dx$$

$$= \sqrt{2} \int_0^{\pi/2} (\cos^2 x)^{1/2}\, dx$$

$$= \sqrt{2} \int_0^{\pi/2} \cos x\, dx$$

$$= \sqrt{2}\,(\sin x)_0^{\pi/2} = \sqrt{2}\,[\sin \pi/2 - \sin 0] = \sqrt{2}$$

(7) $$\int_0^{\pi/2} \cos^3 \theta\, d\theta.$$

We know that.

$$\int_0^{\pi/2} \cos^n \theta\, d\theta = \frac{(n-1)(n-3)\ldots\ldots}{n(n-2)(n-4)\ldots}$$ if 'n' is odd.

$$\therefore \int_0^{\pi/2} \cos^3\theta\, d\theta = \frac{2}{3.1} \times 1 = 2/3$$

(8) $$\int_0^{\pi/2} \sin^8\theta \cos^{12}\theta\, d\theta$$

Given, $\int_0^{\pi/2} \sin^8\theta \cos^{12}\theta \, d\theta$.

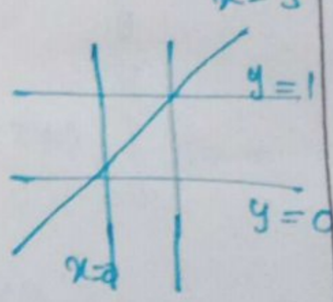
$$= \frac{7.5.3.1.11.9.7.5.3.1}{20.18.16.14.12.10.8.6.4.2} \times \pi/2$$

$$= \frac{6.7.11.\pi}{8\times 16\times 32\times 32\times 4} \Rightarrow \frac{77\pi}{524288} \text{ //.}$$

(7) Using the changing the order of Integration in Evaluate. If $\int_0^1 \int_0^y f(x,y)\,dx\,dy$

Given,

$$\int_0^1 \int_0^y f(x,y)\,dx\,dy.$$



limits,

When $x=0$, $x=y$.

$y=0$, $y=1$.

changing the order of Integration

$$\therefore \int_0^1 \int_0^y f(x,y)\,dx\,dy = \int_0^1 \int_x^1 f(x,y)\,dy\,dx \text{ //.}$$

UNIT - V

Beta and Grama functions.

Defn :- IA also

Beta functions.

If $m, n > 0$ than the Definite Integral, $\int_0^1 x^{m-1} (1-x)^{n-1} dx$ Denote by $\beta(m,n)$ is called Beta functions (or) First Enlerian Integral

ie) $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$.

Also,

$$\int_0^1 x^m (1-x)^n dx = \beta(m+1, n+1) //.$$

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Properties -> ①.

1) $\beta(m,n) = \beta(n,m) \Rightarrow$ prove that.

Proof :

We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}\,dx$$

$$[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx]$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1}\,dx$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx \Rightarrow \beta(n,m)$$

$\Rightarrow \beta(m,n) = \beta(n,m)$ //. Hence proved.

Problems.

1. Express $\int_0^1 x^5(1-x^3)^{1/2}\,dx$ in terms of Beta function.

Soln :

Given, $\int_0^1 x^5(1-x^3)^{1/2}\,dx$

We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

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Put.

$$x^3 = y \Rightarrow x = y^{1/3}$$

$$dx = \frac{1}{3} y^{1/3 - 1} dy$$

$$dx = \frac{1}{3} y^{-2/3} dy.$$

limits,

When $x=0$, then $y=0$

When $x=1$, then $y=1$

$$\therefore \int_0^1 x^5 (1-x^3)^{1/2} dx = \int_0^1 [(y)^{1/3}]^5 (1-y)^{1/2} \frac{1}{3} y^{-2/3} dy$$

$$= \frac{1}{3} \int_0^1 y^1 (1-y)^{1/2} dy$$

$[\because (a^m)^n = a^{m \times n}$

$(y^{5/3}) y^{-2/3} = y^{3/3} \Rightarrow y^1$

$$= \frac{1}{3} \beta(2, 3/2) //.$$

$m-1=1$ | $n-1 = \frac{1}{2}$

$\boxed{m=2}$ | $\boxed{n = 3/2}$

---

2) Express.

$$\int_0^2 (8-x^3)^{-1/3} dx \text{ in terms of}$$

Beta functions.

Soln :-

Given, $\int_0^2 (8-x^3)^{-1/3} dx$

Put,

$$x^3 = 8y$$

$$x = (8y)^{1/3}$$

$$dx = 2(1/3)\, y^{-2/3}\, dy.$$

limits,

When $x = 0$ then $y = 0$

When $x = 2$ then $y = 1$

$$\therefore \int_0^2 (8 - x^3)^{1/3}\, dx = \int_0^1 [8 - (8y)]^{-1/3}\, 2/3\, y^{-2/3}\, dy$$

$$= \frac{16}{3} \int_0^1 (1-y)^{-1/3}\, y^{2/3}\, dy$$

$[\because m - 1 = -2/3$

$m = 1 - 2/3$

$\boxed{m = 1/3}$

$$= \frac{16}{3}\, \beta(1/3, 2/3) \,//$$

$n - 1 = -1/3$

$n = -1/3 + 1$

$\boxed{n = 2/3}$ $]$

Properties $\rightarrow$ (2)

1) Proove that,

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \quad m, n > 0$$

Proof :

We know that,

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

Put,

$$x = \frac{y}{1+y}$$

$$x = (1-x)\, y$$

$$x(1+y) = y$$

Put,

$$x^3 = y \Rightarrow x = y^{1/3}$$

$$dx = \frac{1}{3} y^{1/3 - 1} dy$$

$$dx = \frac{1}{3} y^{-2/3} dy.$$

limits,

When $x = 0$, then $y = 0$

When $x = 1$, then $y = 1$

$$\therefore \int_0^1 x^5 (1-x^3)^{1/2} dx = \int_0^1 [(y)^{1/3}]^5 (1-y)^{1/2} \frac{1}{3} y^{-2/3} dy$$

$[\because (a^m)^n = a^{m \times n}$

$(y^{5/3}) y^{-2/3} = y^{3/3} \Rightarrow y^1$

$$= \frac{1}{3} \int_0^1 y^1 (1-y)^{1/2} dy$$

$m - 1 = 1$ | $n - 1 = \frac{1}{2}$

$\boxed{m = 2}$ | $\boxed{n = 3/2}$

$$= \frac{1}{3} \beta(2, 3/2) //.$$

---

2) Express.

$$\int_0^2 (8-x^3)^{-1/3} dx \text{ in terms of}$$

Beta functions.

Soln :-

Given, $\int_0^2 (8-x^3)^{-1/3} dx$

Put,

$$x^3 = 8y$$

$$x = (8y)^{1/3}$$

$$dx = 2(1/3)\, y^{-2/3}\, dy.$$

limits,

When $x = 0$ then $y = 0$

When $x = 2$ then $y = 1$

$$\therefore \int_0^2 (8 - x^3)^{1/3}\, dx = \int_0^1 [8 - (8y)]^{-1/3}\, 2/3\, y^{-2/3}\, dy$$

$$= \frac{16}{3} \int_0^1 (1-y)^{-1/3}\, y^{2/3}\, dy \qquad \left[\because m - 1 = -2/3 \right.$$

$m = 1 - 2/3$

$\boxed{m = 1/3}$

$$= \frac{16}{3}\, \beta(1/3, 2/3) \,//$$

$n - 1 = -1/3$

$n = -1/3 + 1$

$\left. \boxed{n = 2/3} \right]$

ProPerties → (2)

1) Proove that,

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \qquad m, n > 0$$

Proof :

We know that,

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

Put,

$$x = \frac{y}{1+y} \qquad \Bigg| \qquad x = (1-x)\, y$$

$$x(1+y) = y$$

$$x + yx = y \qquad \frac{x}{1-x} = y$$

$$x = y - yx \qquad x/y = 1-x.$$

$$\frac{y/_{1+y}}{y} = 1-x \qquad dx = \left(\frac{(1+x)(1-y)}{(1+y)^2}\right) dy$$

$$\frac{1}{1+y} = 1-x \qquad dx = \left(\frac{1}{(1+y)^2}\right) dy.$$

limits,

When $x = 0$ then $y = 0$

When $x = 1$ then $y = \infty$

that is,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1} \cdot \left(\frac{1}{1+y}\right)^{n-1} \left(\frac{1}{(1+y)^2}\right) dy$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy \quad //.$$

---

2) Proove that,

$$\int_0^\infty \frac{x^3}{(1+x)^7}\,dx = \frac{1}{60}$$

Proof :- We know that,

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)$$

$$\therefore \int_0^\infty \frac{x^3}{(1+x)^7}\,dx = \beta(4,3)$$

$$= \int_0^1 x^3(1-x)^2\,dx$$

$$= \int_0^1 x^3(1-2x+x^2)\,dx$$

$$= \int_0^1 (x^3-2x^4+x^5)\,dx$$

$$= \left(\frac{x^4}{4} - 2\frac{x^5}{5} + \frac{x^6}{6}\right)_0^1$$

$$= 1/4 - 2/5 + 1/6$$

$$= 1/60 \,//.$$

Hence, $\int_0^\infty \frac{x^3}{(1+x)^7}\,dx = 1/60$ //. Proved.

Gama functions.

Defn :- If $n>0$, then the definite

Integral, $\int_0^\infty e^{-x}\, x^{n-1}\,dx.$

$$x + yx = y \qquad \frac{x}{1-x} = y$$

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that is,

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$= \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1} \left(\frac{1}{1+y}\right)^{n-1} \left(\frac{1}{(1+y)^2}\right) dy$$

$$\beta(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy \,//.$$

---

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$$dx = \left(\frac{(1+x)(1-y)}{(1+y)^2}\right) dy$$

$$dx = \left(\frac{1}{(1+y)^2}\right) dy.$$

limits,

When $x=0$ then $y=0$

When $x=1$ then $y=\infty$

that is,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1} \left(\frac{1}{1+y}\right)^{n-1} \left(\frac{1}{(1+y)^2}\right) dy$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy \;//.$$

---

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Hence, $\displaystyle\int_0^\infty \frac{x^3}{(1+x)^7}\,dx = 1/60$ //. Proved.

Gama functions.

Defn :- If $n>0$, then the definite

Integral, $\displaystyle\int_0^\infty e^{-x}\,x^{n-1}\,dx$.

Denoted by Gama ($\sqrt{n}$) is called Gama functions (or) second Excerian integral,

i.e) $\sqrt{n} = \int_0^\infty e^{-x} x^{n-1} dx$.

Results — (1)

1) Prove that $\Gamma 1 = 1$

Proof :

We know that,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

[$\because e^{-\infty} = 0$

$e^{\infty} = \infty$

$e^{0} = 1$]

Put it, $\boxed{n = 1}$

$$\Gamma 1 = \int_0^\infty e^{-x} x^{0} dx$$

$$= \int_0^\infty e^{-x} dx = \left(\frac{e^{-x}}{-1}\right)_0^\infty$$

$$= \left(\frac{e^{-\infty}}{-1} + \frac{e^{-0}}{1}\right)$$

$$= 0 + 1 \Rightarrow 1.$$

Hence, $\Gamma 1 = 1$ Proved.

Results $\rightarrow$ (2)

(2) $\Gamma n = n-1 \Gamma n-1$, $n > 1 \Rightarrow$ Prove that.

Proof :-

We know that,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx$$

$$u = x^{n-1} \qquad \Big| \qquad dv = e^{-x} dx$$

$$du = (n-1) x^{n-2} dx \qquad \Big| \qquad v = \frac{e^{-x}}{-1}$$

$$= \left(x^{n-1} \frac{e^{-x}}{-1} \right)_0^\infty + \int_0^\infty e^{-x} (n-1) x^{n-2} \, dx .$$

$$= 0 + (n-1) \int_0^\infty e^{-x} x^{n-2} \, dx$$

$\Gamma n = (n-1) \Gamma n-1$ // Hence proved.

Note, $\Gamma n+1 = n \Gamma n = n!$

Results $\longrightarrow$ (3)

1) Prove that, $\Gamma n = (n-1)!$

B. Yamini.

Proof : We know that,

$$\Gamma n = (n-1) \Gamma n-1$$

$$= (n-1)(n-2) \Gamma n-2$$

$$= (n-1)(n-2)(n-3) \Gamma n-3$$

$$= \cdots\cdots\cdots\cdots$$

B. Yamini.

$$= (n-1)(n-2)(n-3)\cdots\cdots 3.2.1$$

$$= (n-1)!$$

$$[\because n! = n(n-1)(n-2)\cdots 2.1]$$

$$\Gamma n = (n-1)!$$

Hence proved.

Evaluate, $\int_0^\infty e^{-x} x^5 \, dx$

Soln: Given, $\int_0^\infty e^{-x} x^5 \, dx$

$n-1 = 5$

$n = 6$

$\Gamma n = (n-1)!$

We know that,

$$\int_0^\infty e^{-x} x^{n-1} \, dx = \Gamma n$$

$$\int_0^\infty e^{-x} x^5 \, dx = \Gamma 6 = 5! \Rightarrow 120$$

Find ~~of~~ $\int_0^\infty e^{-5x} x^2 \, dx$

Soln: Given, $\int_0^\infty e^{-5x} x^2 \, dx$

We know that, $\int_0^\infty e^{-x} x^{n-1} \, dx = \Gamma n$

Put,

$5x = y$

$x = y/5$

$dx = \frac{1}{5} dy$

limits,

When $x = 0$ then $y = 0$

When $x = \infty$ then $y = \infty$

$$\therefore \int_0^\infty e^{-5x} x^2 dx = \int_0^\infty e^{-y} (y/5)^2 (1/5)\, dy$$

$$= (1/5)^3 \int_0^\infty e^{-y} y^2 dy$$

$[\because \Gamma 3 = 2! \Rightarrow 2]$

$$= (1/5)^3 \Gamma 3$$

$n-1 = 2$
$n = 3$

$$= 2/125 \,//.$$

---

(7). Find ~~at~~. $\int_0^\infty e^{-x^2} dx$.

Soln :

Given $\int_0^\infty e^{-x^2} dx$

Put, $x^2 = t \Rightarrow x = t^{1/2}$

$dx = \frac{1}{2} t^{-1/2} dt$

$$\therefore \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-t} \tfrac{1}{2} t^{-1/2} dt$$

$$= \tfrac{1}{2} \int_0^\infty e^{-t} t^{-1/2} dt$$

$n-1 = -1/2$
$n = 1/2$

$= \frac{1}{2}\,\Gamma\frac{1}{2}$ $\quad [\because \Gamma\frac{1}{2} = \sqrt{\pi}]$

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Relationship between Beta and Gama functions.

1. Prove that, $\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$.

Proof :

We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Here, $u = (1-x)^{n-1}$ | $dv = x^{m-1}dx$

$u' = (n-1)(1-x)^{n-2}(-1)\,dx$ | $v = \frac{x^m}{m}$.

$$= \left[(1-x)^{n-1}\left(\frac{x^m}{m}\right)\right]_0^1 + \int_0^1 \frac{x^m}{m}(n-1)(1-x)^{n-2}\,dx$$

$$= 0 + \frac{(n-1)}{m}\int_0^1 x^m(1-x)^{(n-2)}\,dx.$$

Again by using integraction by ~~Parts~~, ~~Parts~~ we get,

$$= \frac{(n-1)(n-2)}{m(m+1)} \int_0^1 x^{m+1} (1-x)^{n-3}\, dx$$

Repeating, $(n-1)$ times, we get

$$\beta(m,n) = \frac{(n-1)(n-2)(n-3)\cdots 2.1}{m(m+1)(m+2)\cdots(m+n-3)(m+n-2)}$$

$$= \frac{(n-1)(n-2)(n-3)\cdots\cdots}{m(m+1)(m+2)(m+n-3)(m+n-2)} \left[\frac{x^{m+n-2+1}}{m+n-2+1} \right]_0^1$$

$$= \frac{(n-1)(n-2)(n-3)\cdots\cdots}{m(m+1)(m+2)\cdots(m+n-3)(m+n-2)} \cdot \frac{1}{m+n-1}$$

Multiply and devide by Γm then.

$$\beta(m,n) = \frac{(n-1)(n-2)(n-3)\cdots}{m(m+1)(m+2)\cdots(m+n-3)(m+n-2)} \cdot \frac{\Gamma 1}{(m+n-1)} \times \frac{\Gamma m}{\Gamma m}$$

$$\beta(m,n) = \frac{(n-1)!\,\Gamma m}{(m+1)(m+2)\cdots(m+n-2)(m+n-1)\,m\,\Gamma m}.$$

$$= \frac{\Gamma m\,\Gamma n}{(m+1)(m+2)\cdots(m+n-2)(m+n-1)\,m\,\Gamma m}$$

$$= \frac{\Gamma m\,\Gamma n}{(m+1)(m+2)\cdots(m+n-2)(m+n-1)\,\Gamma m+1}$$

$$= \frac{\Gamma m\,\Gamma n}{\Gamma m+n} \Rightarrow \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

$\left[\because \Gamma n = (n-1)!\right.$

$\Gamma m+1 = m\,\Gamma m$

$\Gamma m+n = (m+n-1)$

$\Gamma m+n-1$

$= (m+n-1)$

$\left.(m+n-2)\right]$

Evaluate,

$$\beta(15/2, 5/2)$$

Soln:

Given,

$$\beta(15/2, 5/2)$$

We know that,

$$\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$$

$$\beta(15/2, 5/2) = \frac{\Gamma 15/2 \, \Gamma 5/2}{\Gamma 20/2} \qquad [\because \Gamma n+1 = n\Gamma n$$

$$\Gamma n = n-1 \, \Gamma n-1]$$

$$\Gamma 15/2 = 13/2 \, \Gamma 13/2 \qquad \Gamma 1/2 = \sqrt{\pi}$$

$$= \frac{13}{2} \cdot \frac{11}{2} \Gamma \frac{11}{2}$$

$$\cancel{= 13/2 \cdot 11/2 \cdot 9/2 \cdot 7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \, \Gamma 1/2 / 9!}$$

$$\beta(15/2, 5/2) = \frac{13/2 \cdot 11/2 \cdot 9/2 \cdot 7/2 \cdot 5/2 \cdot \cancel{1/2 \cdot \Gamma 1/2} \cdot 3/2 \cdot 1/2 \cdot \Gamma 1/2 \times 3/2 \, 1/2 \, \Gamma 1/2}{1\times2\times3\times4\times5\times6\times7\times8\times9}$$

$$\Gamma 10 = 9!$$

$$= \frac{13 \times 11}{2^{16}} \times \pi$$

$$= \frac{143 \times \pi}{2^{16}} \Rightarrow \frac{143\pi}{2^{16}} \,//.$$

Prove that.

$$\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{\Gamma(p+1/2)\Gamma(q+1/2)}{2\Gamma\frac{p+q+2}{2}} \quad p, q > 1$$

Proof: $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$

Put, $\sin^2\theta = y \Rightarrow \sin\theta = \sqrt{y}$

$$\cos\theta \, d\theta = \frac{1}{2\sqrt{y}} \, dy.$$

limits,

When $\theta = 0$ then $y = 0$.

When $\theta = \pi/2$ then $y = 1$.

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \int_0^{\pi/2} \sin^p\theta \cos^{q-1}\theta \cos\theta \, d\theta.$$

$$= \int_0^1 y^{\frac{p-1}{2}} (1-y)^{\frac{q-1}{2}} \frac{1}{2\sqrt{y}} \, dy. \qquad \left[\because \sin\theta = \sqrt{y} = y^{1/2}, \quad \cos\theta = (1-y)^{1/2} = \sqrt{1-\sin^2\theta}\right]$$

We know that,

$$\left[\because \int_0^1 x^{m-1}(1-x)^{n-1} dx = \beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}\right]$$

$m-1 = p-\frac{1}{2}$	$n-1 = \frac{q-1}{2}$
$m = \frac{p-1}{2}+1$	$n = \frac{q-1}{2}+1$
$m = \frac{p+1}{2}$	$n = \frac{q+1}{2}$

$$= \frac{1}{2} \int_0^1 y^{\frac{p+1}{2}} (1-y)^{\frac{q+1}{2}} \, dy.$$

$$= \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

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$$= \frac{1}{2} \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}} \Rightarrow R.H.S$$

L.H.S = R.H.S.

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\frac{p+q+2}{2}} \text{ //.}$$

Prove that.

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} \frac{\Gamma m\,\Gamma n}{\Gamma m+n} = \frac{1}{2}\beta(m,n)$$

Proof :

L.H.S ⇒

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

We know that,

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}} \text{ ——— (1)}$$

$p = 2m-1$ | $q = 2n-1$

$\frac{p+1}{2} = m$ | $\frac{q+1}{2} = n$.

Also, $\frac{p+q+2}{2} = m+n$

$$\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma m\,\Gamma n}{2\,\Gamma m+n}$$

$= \frac{1}{2} \beta(m, n)$

$\Rightarrow$ R.H.S

$\therefore$ L.H.S = R.H.S.

Prove that.

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$

Proof :

We know that.

$$\beta(m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \qquad [\because \Gamma 1 = 1].$$

Put,

$$m = n = \tfrac{1}{2}$$

$$\beta(\tfrac{1}{2}, \tfrac{1}{2}) = \frac{\Gamma\tfrac{1}{2}\,\Gamma\tfrac{1}{2}}{\Gamma\tfrac{1}{2}+\tfrac{1}{2}} = \frac{\Gamma\tfrac{1}{2}\,\Gamma\tfrac{1}{2}}{\Gamma 1} = \left(\Gamma\tfrac{1}{2}\right)^2 \longrightarrow (1)$$

Also, We know that,

$$\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

$$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$$

Put,

$$m = n = \tfrac{1}{2}$$

$$2m-1 = 2\left(\tfrac{1}{2}\right) - 1 = 1 - 1 = 0.$$

$$2n-1 = 0.$$

$$\beta(\tfrac{1}{2}, \tfrac{1}{2}) = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta.$$

$$= 2\int_0^{\pi/2} d\theta = 2(\theta)_0^{\pi/2} \Rightarrow 2 \times \pi/2 = \pi \longrightarrow (2)$$

from (1) & (2)

$$\left(\Gamma \tfrac{1}{2}\right)^2 = \pi$$

Hence $\Gamma \tfrac{1}{2} = \sqrt{\pi}$ // Proved.

Prove that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$

Proof :

Given, $\int_0^\infty e^{-x^2} dx$

Put,

$x^2 = t$

$x = t^{1/2} = \sqrt{t}$.

$$\therefore \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-t} \frac{1}{2\sqrt{t}} dt$$

$$= \tfrac{1}{2} \int_0^\infty e^{-t} t^{-1/2} dt.$$

$$= \tfrac{1}{2} \Gamma \tfrac{1}{2} = \tfrac{1}{2}\sqrt{\pi} = \sqrt{\pi}/2 \; //$$

$\left[\because \Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \quad \Gamma \tfrac{1}{2} = \sqrt{\pi} \quad n-1 = -\tfrac{1}{2} \quad \boxed{n = \tfrac{1}{2}}\right]$.

Hence, $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ // Proved.

Prove that, $\int_{-\infty}^{0} e^{-x^2} dx = \sqrt{\pi}/2$

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Proof :

Given, $\int_{-\infty}^{0} e^{-x^2} dx$

Put,

$x = -y \Rightarrow dx = -dy$.

limits, When $x = 0$ then $y = 0$

When $x = -\infty$ then $y = \infty$

$\therefore \int_{-\infty}^{0} e^{-x^2} dx = \int_{\infty}^{0} e^{-(-y)^2} (-dy)$

$= -\int_{\infty}^{0} e^{-y^2} dy$. $\left[\because \int_a^b f\,dx = -\int_b^a f\,dx \right.$

$\left. \therefore \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \right]$

$= \int_0^{\infty} e^{-y^2} dy$.

Hence, $\int_{-\infty}^{0} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ //. Proved.

Evaluate. $\Gamma 7/2 = 7/2 \; 5/2 \; 3/2 \; 1/2 \; \Gamma 1/2$

Soln: $\Gamma(n+1) = n\Gamma n$ $[\because \Gamma 1/2 = \sqrt{\pi}]$

$\therefore \Gamma 7/2 = 5/2 \cdot 3/2 \cdot 1/2 \; \Gamma 1/2$

$15/8 \; \Gamma 1/2$

$= \frac{15}{8}\sqrt{\pi}$ //

Evaluate,

$$\int_0^{\pi/2} \sin^5 x \cos^4 x \, dx$$

Soln:

Given, $\int_0^{\pi/2} \sin^5 x \cos^4 x \, dx$

We know that,

$$\int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta = \frac{\Gamma\frac{p+1}{2} \Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}}$$

$$\Rightarrow \int_0^{\pi/2} \sin^5 x \cos^4 x \, dx = \frac{\Gamma 6/2 \, \Gamma 5/2}{2\Gamma 11/2}$$

$$= \frac{1}{2} \frac{\Gamma 3 \, \Gamma 5/2}{\Gamma 11/2}$$

$$= \frac{1}{2} \frac{2! \cdot 3/2 \cdot 1/2 \cdot \Gamma 1/2}{9/2 \cdot 7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \cdot \Gamma 1/2}$$

$$= \frac{1}{2} \frac{2}{9/2 \cdot 7/2 \cdot 5/2}$$

$$= \frac{8}{9 \times 7 \times 5} \Rightarrow \frac{8}{315}$$

Another method.

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m,n)$$

$$= \frac{\Gamma m \, \Gamma n}{2\,\Gamma m+n}$$

$$\int_0^{\pi/2} \sin^5 x \cos^4 x \, dx = \frac{1}{2}\beta(m,n)$$

$$= \frac{1}{2}\beta(3, 5/2)$$

$$= \frac{1}{2}\,\frac{\Gamma 3 \, \Gamma 5/2}{\Gamma 11/2} \Rightarrow \frac{8}{315} \,//.$$

Evaluate, $\int_0^{\pi/4} \sin^4 2x \, dx$.

Soln;

Given $\int_0^{\pi/4} \sin^4 2x \, dx$

Put,

$2x = t$

$x = t/2$

$dx = \frac{dt}{2}$.

limits, When $x = 0$ then $t = 0$

When $x = \pi/4$ then $t = \pi/2$

$$\therefore \int_0^{\pi/4} \sin^4 2x \, dx = \int_0^{\pi/2} \sin^4 t \, dt/2$$

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$$= \frac{1}{2}\int_0^{\pi/2} \sin^4 t \, dt \qquad [\because \sin^4 x = (\sin x)^4]$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^4 t \cos^0 t \, dt \qquad [x^0 = 1, \; \cos^0 t = 1]$$

$$= \frac{1}{2} \beta(5/2, 1/2)$$

$$= \frac{1}{2}\left(\frac{\Gamma 5/2 \; \Gamma 1/2}{2 \Gamma 3}\right)$$

$$= \frac{1}{2}\left(\frac{3/2 \cdot 1/2 \cdot \Gamma 1/2 \cdot \Gamma 1/2}{2 \cdot 2!}\right)$$

$$= \frac{1 \cdot 3 \sqrt{\pi} \sqrt{\pi}}{2 \times 2 \times 2 \times 2 \times 2}$$

$$= \frac{3\pi}{32} \,//.$$

Evaluate, $\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta$.

Soln :

Given, $\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta = \int_0^{\pi/2} (\tan\theta)^{1/2} d\theta$.

$$= \int_0^{\pi/2} \left(\frac{\sin\theta}{\cos\theta}\right)^{1/2} d\theta. \qquad \left[\because \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m, n)\right]$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta. \qquad [\Gamma 3/4 \, \Gamma 1/4 = \pi\sqrt{2}]$$

$= \frac{1}{2} \beta(3/4, 1/4)$ $\quad 2m-1 = 1/2 \quad 2m = 3/2 \quad \boxed{m = 3/4} \quad \Big| \quad 2n-1 = -1/2 \quad 2n = 1/2 \quad \boxed{n = 1/4}$

$= \frac{1}{2} \Gamma 3/4 \, \Gamma 1/4$

$= \frac{1}{2} \sqrt{2} . \pi$

$= \frac{1 \times \pi \times \sqrt{2}}{\sqrt{2}\sqrt{2}} \Rightarrow \frac{\pi}{\sqrt{2}}$ //.

Prove that,

$$\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta * \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi.$$

Proof :

Given,

$$\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta . \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \longrightarrow (1)$$

Consider, $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta \, d\theta$

$$= \frac{1}{2} \beta(3/4, 1/2)$$

$= \frac{1}{2} \frac{\Gamma 3/4 \, \Gamma 1/2}{\Gamma 5/4}$ $\quad [\because \Gamma 5/4 = (5/4 - 1)\Gamma 5/4 - 1$

$= 1/4 \Gamma 1/4$

$= \frac{1}{2} \frac{\Gamma 3/4 \, \Gamma 1/2}{1/4 \Gamma 1/4} \longrightarrow (2)$ $\quad \therefore \Gamma n = n-1 \Gamma n-1]$

Also, $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta \, d\theta.$

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$$= \frac{1}{2} \beta(1/4, 1/2)$$

$$2m-1 = -1/2$$

$$2m = -1/2 + 1 \quad | \quad 2n-1=0$$

$$2m = 1/2 \quad | \quad 2n = 1$$

$$\boxed{m = 1/4} \quad \boxed{n = 1/2}$$

$$= \frac{1}{2} \frac{\Gamma 1/4 \, \Gamma 1/2}{\Gamma 3/4} \longrightarrow (3)$$

Subtitude. (2) (3) in (1)

$$\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta - \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \frac{1}{2} \frac{\Gamma 3/4 \, \Gamma 1/2}{1/4 \, \Gamma 1/4} \times \frac{1}{2} \frac{\Gamma 1/4 \, \Gamma 1/2}{\Gamma 3/4}$$

$$= 4/4 \times \Gamma 1/2 \, \Gamma 1/2 \qquad [\because \Gamma 1/2 = \sqrt{\pi}]$$

Hence

$$= \sqrt{\pi} \times \sqrt{\pi} = \pi \,//. \text{ Proved.}$$

Proove that,

$$\beta(m+1, n) + \beta(m, n+1) = \beta(m, n)$$

Proof :

Given.

$$\beta(m+1, n) + \beta(m, n+1)$$

$$\left[\because \beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}\right]$$

$$= \frac{\Gamma m+1 \, \Gamma n}{\Gamma m+n+1} + \frac{\Gamma m \, \Gamma n+1}{\Gamma m+n+1}$$

$$[\because \Gamma n+1 = n \Gamma n]$$

$$= \frac{\Gamma m+1 \, \Gamma n + \Gamma m \, \Gamma n+1}{\Gamma m+n+1}$$

$$= \frac{m\Gamma m\Gamma n + \Gamma m\, n\Gamma n}{(m+n)\Gamma m+n}$$

$$= \frac{\Gamma m\Gamma n\,\cancel{(m+n)}}{\cancel{(m+n)}\,\Gamma m+n}$$

$$= \frac{\Gamma m\,\Gamma n}{\Gamma m+n} \Rightarrow \beta(m,n) \Rightarrow R.H.S.$$

Hence

L.H.S = R.H.S //. Proved.

Prove that

$$\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \cdot \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$$

Soln :

Consider

$$\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\, dx \text{ ——— } (1)$$

Put,

$x^2 = \sin\theta$

$(x^2)^2 \Rightarrow x^4 = \sin^2\theta$

$x = (\sin\theta)^{1/2} = \sqrt{\sin\theta}$

$dx = \frac{1}{2\sqrt{\sin\theta}} \cos\theta\, d\theta$

$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} dx$

Limits,

When $x = 0$ then $\theta = 0$

When $x = 1$ then $\theta = \pi/2$

$\sin 0 = 0$

$\sin \pi/2 = 1$

$\sin \pi = 0$

$$\therefore \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \cdot \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sin\theta}{\cancel{\cos\theta}} \cdot \frac{\cancel{\cos\theta}}{\sqrt{\sin\theta}}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\,d\theta \qquad \left[\because 2m-1 = \tfrac{1}{2}\right.$$

$$2m = 3/2$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\,d\theta \qquad m = 3/4,$$

$$2n-1 = 0$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot B(3/4, 1/2) \qquad 2n = 1$$

$$= \frac{1}{4} B(3/4, 1/2). \qquad \left. n = 1/2 \right]$$

$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$

$$= \frac{1}{4}\,\frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(3/4+1/2)} \Rightarrow \frac{1}{4}\,\frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(5/4)}$$

$$= \cancel{\frac{1}{4}}\,\frac{\Gamma(3/4)\,\sqrt{\pi}}{\cancel{\frac{1}{4}}\,\Gamma(1/4)} = \frac{\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(1/4)} \longrightarrow (2)$$

Also,

$$\int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

Put,

$$x^2 = \tan\theta \Rightarrow x^4 = \tan^2\theta$$

$$x = \sqrt{\tan\theta}$$

$$dx = \frac{1}{2\sqrt{\tan\theta}}\sec^2\theta\,d\theta$$

limits,

$$x = 0, \quad \theta = 0$$
$$x = 1, \quad \theta = \pi/4$$

$$\therefore \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \int_0^{\pi/4} \frac{\frac{\sec^2\theta}{2\sqrt{\tan\theta}} \cdot d\theta}{\sqrt{1+\tan^2\theta}}$$

$$= \int_0^{\pi/4} \frac{\sec^2\theta}{2\sqrt{\tan\theta}\,\cancel{\sec\theta}} \cdot d\theta$$

$$= \int_0^{\pi/4} \frac{\sec\theta}{2\sqrt{\tan\theta}} \, d\theta$$

$$= \int_0^{\pi/4} \frac{1/\cos\theta}{2\sqrt{\frac{\sin\theta}{\cos\theta}}} \, d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2\sqrt{\sin\theta\cos\theta}} \, d\theta. \qquad \left[\because \sin\theta\cos\theta = \frac{\sin 2\theta}{2}\right]$$

$$= \int_0^{\pi/4} \frac{1}{2\sqrt{\frac{\sin 2\theta}{2}}} \, d\theta$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{1}{\sqrt{\sin 2\theta}} \, d\theta.$$

Put,

$$2\theta = t$$
$$\theta = t/2$$
$$d\theta = dt/2$$

limits,

When $\theta = 0$ then $t = 0$

When $\theta = \pi/4$ then $t = \pi/2$

$$\therefore \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin 2\theta}} = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dt/2}{\sqrt{\sin t}}$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2} t \, dt$$

$2n-1=0$
$n=1/2$

$$= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2} t \cos^{0} t \, dt$$

$2m-1=-1/2$
$2m=1/2$
$m=1/4$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{1}{2} \beta(1/4, 1/2).$$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{1}{2} \cdot \frac{\Gamma 1/4 \, \Gamma 1/2}{\Gamma 1/4 + 1/2}$$

$$= \frac{1}{4\sqrt{2}} \cdot \frac{\Gamma 1/4 \sqrt{\pi}}{\Gamma 3/4} \longrightarrow (3)$$

Substitude (2) (3) in (1)

$$\int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx \cdot \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\sqrt{\pi}\,\Gamma 3/4}{\Gamma 1/4} \times$$

Hence

$$\frac{1}{4\sqrt{2}} \cdot \frac{\Gamma 1/4 \sqrt{\pi}}{\Gamma 3/4} \Rightarrow \frac{\pi}{4\sqrt{2}}$$ //. Proved.

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Prove that,

$$\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{\beta(m,n)}{a^n b^m} \quad \text{were}$$

$a, b, m, n > 0$

Soln :

L.H.S $\Rightarrow$

$$\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^{m+n}} \int_0^\infty \frac{x^{m-1}}{(1+bx/a)^{m+n}}\,dx$$

Put,

$$\frac{bx}{a} = t \Rightarrow x = \frac{at}{b}$$

$$\frac{b}{a}\,dx = dt$$

$$dx = \frac{a}{b}\,dt.$$

$$= \frac{1}{a^{m+n}} \int_0^\infty \frac{(at/b)^{m-1}}{(1+t)^{m+n}} \cdot \frac{a\,dt}{b}$$

$$= \frac{1}{a^{m+n}} \left(\frac{a}{b}\right)^{m-1} \frac{a}{b} \cdot \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

$$\left[\because \beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx\right]$$

$$= \frac{a^m}{b^m} \cdot \frac{1}{a^m a^n} \beta(m,n)$$

$$= \frac{1}{a^n b^m} \beta(m,n) \Rightarrow \text{R.H.S}$$

Hence $\therefore$ L.H.S = R.H.S // Proved.

Prove that,

$$\frac{\beta(m+1,n)}{\beta(m,n)} = \frac{m}{m+n}$$

Soln :

L.H.S =>

$$\frac{\beta(m+1,n)}{\beta(m,n)}$$

$$= \frac{\Gamma m+1\,\Gamma n \,/\, \Gamma m+n+1}{\Gamma m\,\Gamma n \,/\, \Gamma m+n}$$

$$= \frac{m\cancel{\Gamma m}\,\cancel{\Gamma n} \,/\, (m+n)\,\cancel{\Gamma m+n}}{\cancel{\Gamma m}\,\cancel{\Gamma n} \,/\, \cancel{\Gamma m+n}}$$

$$= \frac{m}{m+n} \Rightarrow \text{R.H.S}$$

Hence

L.H.S = R.H.S // proved.

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