FORMULAE

Appendix - A

♦ Derivatives of standard functions

1. $\frac{d}{dx}(c) = 0$

2. (a) $\frac{d}{dx}(x^n) = n x^{n-1}$ (b) $\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}(a)$

3. (a) $\frac{d}{dx}(e^x) = e^x$ (b) $\frac{d}{dx}(e^{ax}) = (e^{ax})(a)$ (c) $\frac{d}{dx}(e^{ax+b}) = (e^{ax+b})(a)$

4. $\frac{d}{dx}(a^x) = a^x \log a$

5. (a) $\frac{d}{dx}(\sin x) = \cos x$ (b) $\frac{d}{dx}(\sin ax) = (\cos ax)(a)$

   (c) $\frac{d}{dx}[\sin(ax+b)] = [\cos(ax+b)](a)$

6. (a) $\frac{d}{dx}(\cos x) = -\sin x$ (b) $\frac{d}{dx}(\cos ax) = (-\sin ax)(a)$

   (c) $\frac{d}{dx}[\cos(ax+b)] = [-\sin(ax+b)](a)$

7. (a) $\frac{d}{dx}(\tan x) = sec^2 x$ (b) $\frac{d}{dx}(\tan ax) = (sec^2 ax)(a)$

   (c) $\frac{d}{dx}[\tan(ax+b)] = [sec^2(ax+b)](a)$

8. (a) $\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$ (b) $\frac{d}{dx}(\cot ax) = [-\mathrm{cosec}^2 ax](a)$

   (c) $\frac{d}{dx}[\cot(ax+b)] = [-\mathrm{cosec}^2(ax+b)](a)$

9. (a) $\frac{d}{dx}(\sec x) = \sec x \tan x$ (b) $\frac{d}{dx}[\sec ax] = [\sec ax \tan ax](a)$

   (c) $\frac{d}{dx}[\sec(ax+b)] = [\sec(ax+b)\tan(ax+b)](a)$

10. (a) $\frac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{cosec}\, x \cot x$ (b) $\frac{d}{dx}(\mathrm{cosec}\, ax) = [-\mathrm{cosec}\, ax \cot ax](a)$

    (c) $\frac{d}{dx}[\mathrm{cosec}(ax+b)] = [-\mathrm{cosec}(ax+b)\cot(ax+b)](a)$

11. $\frac{d}{dx}(\log x) = \frac{1}{x}, x \neq 0$ 12. $\frac{d}{dx}(\log_a x) = \frac{\log_a e}{x}, x \neq 0$

13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ 14. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

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15. $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

16. $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$

17. $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$

18. $\frac{d}{dx}\operatorname{cosec}^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$

19. $\frac{d}{dx}\sinh x = \cosh x$

20. $\frac{d}{dx}\cosh x = \sinh x$

21. $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$

22. $\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x$

23. $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$

24. $\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x$

25. $\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{1+x^2}}$

26. $\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$

27. $\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}$

28. $\frac{d}{dx}\coth^{-1}x = -\frac{1}{x^2-1}$

29. $\frac{d}{dx}\operatorname{sech}^{-1}x = -\frac{1}{x\sqrt{1-x^2}}$

30. $\frac{d}{dx}\mathit{cosech}^{-1}x = -\frac{1}{x\sqrt{x^2+1}}$

Appendix - C

◆ Trigonometry

I. Inter-relations :

1. $\sin\theta = \dfrac{1}{\csc\theta}$ 2. $\csc\theta = \dfrac{1}{\sin\theta}$

3. $\cos\theta = \dfrac{1}{\sec\theta}$ 4. $\sec\theta = \dfrac{1}{\cos\theta}$

5. $\tan\theta = \dfrac{1}{\cot\theta}$ 6. $\cot\theta = \dfrac{1}{\tan\theta}$

7. $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$ 8. $\dfrac{\cos\theta}{\sin\theta} = \cot\theta$

II. Identities

1. $\sin^2\theta + \cos^2\theta = 1$ 2. $1 + \tan^2\theta = \sec^2\theta$ 3. $1 + \cot^2\theta = \csc^2\theta$

III. Radian measure

π radians $= 180^0$

IV. Trigonometric ratios for certain standard angles

θ	0^0	$30^0\ (\pi/6)$	$45^0\ (\pi/4)$	$60^0\ (\pi/3)$	$90^0\ (\pi/2)$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞ (infinity)

$\cot\theta$, $\sec\theta$, $\csc\theta$ are respectively the reciprocals of $\tan\theta$, $\cos\theta$, $\sin\theta$

V. Allied angles

Trigonometrical ratios of $90 \pm \theta$, $180 \pm \theta$, $270 \pm \theta$, $360 \pm \theta$ in terms of those of θ can be found easily by the following rule known as A - S - T - C rule.

(i) When the angle is $90 \pm \theta$ or $270 \pm \theta$ the trigonometrical ratio changes from sine to cosine and vice versa. Also tan $\rightleftarrows$ cot , sec $\rightleftarrows$ cosec

(ii) When the angle is $180 \pm \theta$ or $360 \pm \theta$ the trigonometrical ratio remains the same. i.e., sin → sin , cos → cos, etc.

(iii) In each case the sign + or – is pre multiplied by the A-S-T-C quadrant rule :

S	A
II $(90^0 - 180^0)$	I $(0^0 - 90^0)$
T	C
III $(180^0 - 270^0)$	IV $(270^0 - 360^0)$

A : All ratios are +ve in the I quadrant

S : sin is +ve in the II quadrant

T : tan is +ve in the III quadrant.

C : cos is +ve in the IV quadrant

Note : $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$,
$\sin(n.2\pi + \theta) = \sin\theta$, $\cos(n.2\pi + \theta) = \cos\theta$

Example 1 : $\sin(90^0 - \theta) = \cos\theta$, $\cos(90^0 + \theta) = -\sin\theta$

$\sin(180^0 - \theta) = \sin\theta$, $\tan(180^0 + \theta) = \tan\theta$ etc.

Example 2 : $\sin(135^0) = \sin(90^0 + 45^0) = \cos 45^0 = \dfrac{1}{\sqrt{2}}$

$\tan(315^0) = \tan(270^0 + 45^0) = -\cot 45^0 = -1$

$\cos(225^0) = \cos(180^0 + 45^0) = -\cos 45^0 = -\dfrac{1}{\sqrt{2}}$

$\sin(750^0) = \sin(2 \times 360^0 + 30^0) = \sin 30^0 = \dfrac{1}{2}$ etc.

VI. Compound angle formulae

(i) $\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\sin(A - B) = \sin A\cos B - \cos A\sin B$

(ii) $\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

(iii) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

VII. Formulae to convert a product into sum or difference

(iv) $\left.\begin{array}{r}\sin A\cos B\\ \text{or } \cos B\sin A\end{array}\right\} = \dfrac{1}{2}[\sin(A + B) + \sin(A - B)]$

$\cos A\sin B = \dfrac{1}{2}[\sin(A + B) - \sin(A - B)]$, $\sin(-\theta) = -\sin\theta$

(v) $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

(vi) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$, $\cos(-\theta) = \cos\theta$

VIII. Particular cases of formula (i) to (vi)

$\sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

$= 2\cos^2 A - 1$

$1 + \cos 2A = 2\cos^2 A$

$1 - \cos 2A = 2\sin^2 A$

$\sin A = 2\sin(A/2)\cos(A/2)$

$\cos A = \cos^2(A/2) - \sin^2(A/2)$

$= 1 - 2\sin^2(A/2)$

$= 2\cos^2(A/2) - 1$

$1 + \cos A = 2\cos^2\frac{A}{2}$

$1 - \cos A = 2\sin^2\frac{A}{2}$

$\sin 3A = 3\sin A - 4\sin^3 A$

$\cos 3A = 4\cos^3 A - 3\cos A$

$\sin 2A = \frac{2\tan A}{1+\tan^2 A}$

$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

$\sin A = \frac{2\tan(A/2)}{1+\tan^2(A/2)}$

$\cos A = \frac{1-\tan^2(A/2)}{1+\tan^2(A/2)}$

IX. Formulae to convert a sum or difference into product.

(vii) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(viii) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(ix) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(x) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

X. Hyperbolic functions

We have already said that 'e' whose value is approximately 2.7 is called the exponential constant. Further, if $\log_e y = x$ then $y = e^x$ is called as the exponential function. Hyperbolic functions are defined interm of exponential function as below.

$$\sinh x = \frac{e^x - e^{-x}}{2}; \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

Also, $\tanh x = \frac{\sinh x}{\cosh x}$; $\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$

$\operatorname{sech} x = \frac{1}{\cosh x}$; $\operatorname{cosech} x = \frac{1}{\sinh x}$

XI. Important hyperbolic identities

(i) $\cosh^2 x - \sinh^2 x = 1$ (ii) $1 - \tanh^2 x = \operatorname{sech}^2 x$

(iii) $\coth^2 x - 1 = \operatorname{cosech}^2 x$ (iv) $\cosh^2 x + \sinh^2 x = \cosh 2x$

(v) $2\sinh x\cosh x = \sinh 2x$

XII. Relationship between trigonometric and hyperbolic functions

We have, $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ and $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

Now $\sin(ix) = \frac{e^{-x} - e^{x}}{2i} = (-1)\frac{(e^x - e^{-x})}{2i} = \frac{i^2(e^x - e^{-x})}{2i}$

$\therefore \quad \sin(ix) = i\cdot\frac{e^x - e^{-x}}{2}$ (i.e.,) $\sin(ix) = i\sinh x$

Also, $\cos(ix) = \frac{e^{-x} + e^{x}}{2} = \cosh x$ (i.e.,) $\cos(ix) = \cosh x$

XIII. The following are some of the established **standard limits.**

(i) $\lim\limits_{x\to a}\left(\frac{x^n - a^n}{x - a}\right) = na^{n-1}$, n is any rational number.

(ii) $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$

(iii) $\lim\limits_{x\to 0}\frac{\tan x}{x} = 1$

(iv) $\lim\limits_{x\to 0}(1+x)^{1/x} = e$ or $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

(v) $\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

(vi) $\lim\limits_{x\to\infty} x^{1/x} = 1$

Appendix - D

I. Some standard forms of the Binomial Expansion

For all values of n, when $|x| < 1$, we have

1. $(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \ldots$
2. $(1-x)^n = 1 - \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 - \ldots$
3. $(1-x)^{-n} = 1 + \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 + \ldots$
4. $(1+x)^{-n} = 1 - \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 - \ldots$
5. $(1-x)^{-1} = 1 + x + x^2 + \ldots$
6. $(1+x)^{-1} = 1 - x + x^2 - \ldots$
7. $(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots$
8. $(1+x)^{-2} = 1 - 2x + 3x^2 -$
9. $(1+x)^{-3} = \frac{1}{2}[1.2 - 2.3\,x + 3.4x^2 - \ldots]$

II. Resolution of Rational into partial fraction.

Case (i) : Let $\frac{P(x)}{Q(x)}$ be a proper fraction

[i.e., degree of $P(x)$ is lessthan degree of $Q(x)$]

Type 1: Suppose, $Q(x)$ is factorizable into non-repeated linear factors $a_1x+b_1,\ a_2x+b_2,\ a_3x+b_3$

(i.e.,) $Q(x) = (a_1x+b_1)(a_2x+b_2)(a_3x+b_3)$

$$\frac{P(x)}{Q(x)} = \frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2} + \frac{C}{a_3x+b_3},$$

where A, B, C are constants.

$$\frac{1}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}$$

Type 2 : Suppose, $Q(x)$ contain a repeated linear factor of the form $(ax+b)^r$

$$\frac{P(x)}{Q(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \ldots + \frac{A_r}{(ax+b)^r}$$

Example : $\frac{2x+3}{(x+5)^3} = \frac{A}{x+5} + \frac{B}{(x+5)^2} + \frac{C}{(x+5)^3}$

Type 3 : $Q(x)$ contain a non-repeated quadratic function.

ax^2+bx+c then

$$\frac{P(X)}{Q(x)} = \frac{Ax+B}{ax^2+bx+c}$$

Example : $\frac{x+3}{(x^2+1)(x-3)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-3}$

Type 4 : $Q(x)$ contain a repeated quadratic factor $(ax^2+bx+c)^2$

$$\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2}$$

Case (ii) : Let $\frac{P(x)}{Q(x)}$ be an improper fraction

[i.e., degree of $P(x)$ is greater than or equal to degree of $Q(x)$]

In this case we can write

$\frac{P(x)}{Q(x)} = f(x) + \frac{R(x)}{Q(x)}$ where $f(x)$ is the quotient.

(Integral function) and $\frac{R(x)}{Q(x)}$ is a proper fraction.

III. Seven indeterminants are

1. $\frac{0}{0}$ 2. $\frac{\infty}{\infty}$ 3. $0 \times \infty$ 4. $\infty - \infty$

5. 1^∞ 6. ∞^0 7. 0^0

IV. 1. $\log_a 1 = 0$ 2. $\log_a a = 1$

3. $\frac{a}{0} = \infty$ [where a is any real number except zero]

4. $\frac{0}{a} = 0$ 5. $e = 2.7183$ (approx)

6. $\pi = 3.14$ (approx) 7. $\log_a 0 = -\infty\ (a > 1)$

8. $\log(mn) = \log m + \log n$

9. $\log\left(\frac{m}{n}\right) = \log m - \log n$

10. $\log(m^n) = n \log m$

Differential calculus and Trigonometry

UNIT-I

chapter - I - successive Differentiation.

We have seen that the derivative of a function of x is also a function of x. The new function may be Differen-tiable in which case, the derivative of the first first derivative is called the second derivative of the original function. similarly the derivative of the second derivative is called the third derivative and so on up to the n^{th} derivative.

Example:

Let $y = 4x^5$

$$\frac{dy}{dx} = 20x^4$$

$$\frac{d^2y}{dx^2} = 80x^3$$

$$\frac{d^3y}{dx^3} = 240x^2$$

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The n^{th} derivative:

For certain functions a general Expression involving n may be found for the n^{th} derivative. The usual plan is to find number of successive derivatives, as many as be necessary to discover their law of formation and then by induction write down the n^{th} derivative.

Example: Let $y = e^{ax}$,

$$\frac{dy}{dx} = ae^{ax},$$

$$\frac{d^2y}{dx^2} = a^2e^{ax},$$

$$\frac{d^3y}{dx^3} = a^3e^{ax},$$

.....

.....

....

$$\frac{d^ny}{dx^n} = a^ne^{ax}.$$

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Standard Results

1. Let $y = (ax+b)^m$. Then

$$\frac{dy}{dx} = y_1 = m.a.(ax+b)^{m-1}$$

$$\frac{d^2y}{dx^2} = y_2 = m(m-1)\,a^2(ax+b)^{m-2}$$

$$\frac{d^3y}{dx^3} = y_3 = m(m-1)(m-2)\,a^3(ax+b)^{m-3}$$

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$$\frac{d^ny}{dx^n} = y_n = m(m-1)\cdots\cdots(m-n+1)\,a^n(ax+b)^{m-n}.$$

In particular,

$$D^n(ax-b)^{-1} = (-1)^n n!\, a^n (ax+b)^{n-1}$$

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2. If $y = \log(ax+b)$. then

$y_1 = a(ax+b)^{-1}$

$= \text{-----------}$

$y_n = a \cdot \frac{d^{n-1}}{dx^{n-1}}(ax+b)^{-1}$

$= a(-1)^{n-1}(n-1)!\,a^{n-1}(ax+b)^{-n}$

$= (-1)^{n-1}(n-1)!\,a^{n}(ax+b)^{-n}$.

3. If $y = \sin(ax+b)$, then

$y_1 = a\cos(ax+b) = a\sin\left(\frac{\pi}{2}+ax+b\right)$

$y_2 = a^2\cos\left(\frac{\pi}{2}+ax+b\right) = a^2\sin\left(2\frac{\pi}{2}+ax+b\right)$

$y_3 = a^3\sin\left(3\frac{\pi}{2}+ax+b\right)$

In general

$D^n\sin(ax+b) = a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$

4. Similarly $D(\cos ax+b) = a^n\cos\left(\frac{n\pi}{2}+ax+b\right)$

Put $a=1$ and $b=0$, $D^n(\sin x) = \sin\left(\frac{n\pi}{2}+x\right)$

$D^n(\cos x) = \cos\left(\frac{n\pi}{2}+x\right)$

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Problems:

1. Find y_n, where $y = \dfrac{3}{(x+1)(2x-1)}$

Solution:

Using partial fraction, we get.

$$y = \frac{3}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1} \quad \text{-----} \; (1)$$

$$\Rightarrow \quad 3 = A(2x-1) + B(x+1)$$

Put $x = -1$, $\quad 3 = -3A \Rightarrow A = -1$

Put $x = 1/2 \quad 3 = \frac{3B}{2} \Rightarrow B = 2.$

$\therefore (1) \Rightarrow$

$$y = \frac{2}{2x-1} - \frac{1}{x+1}$$

W.K.T $D^n (ax+b)^{-1} = (-1)^n n!\, a^n (ax+b)^{-(n+1)}$

$$\therefore y = 2\left[\frac{(-1)^n 2^n \cdot n!}{(2x-1)^{n+1}} - \frac{(-1)^n n!}{(x+1)^{n+1}}\right.$$

$$= (-1)^n n! \left\{ \frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}} \right\}$$

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2. Find y_n where $y = \frac{x^2}{(x-1)^2(x+2)}$

Solution:

Using partial fraction, we get

$$y = \frac{x^2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} \quad \text{---} (1)$$

$$\Rightarrow x^2 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Put $x=1$, $1 = 3B \Rightarrow \boxed{B = \frac{1}{3}}$

Put $x=-2$, $4 = 9C \Rightarrow \boxed{C = \frac{4}{9}}$

Put $x=0$, $0 = -2A + 2B + C$

$$0 = -2A + 2\left(\frac{1}{3}\right) + \frac{4}{9}$$

$$\Rightarrow +2A = \frac{10}{9} \Rightarrow \boxed{A = \frac{5}{9}}$$

$\therefore (1) \Rightarrow$

$$y = \frac{5}{9}\frac{1}{(x-1)} + \frac{1}{3}\frac{1}{(x-1)^2} + \frac{4}{9}\frac{1}{(x+2)}$$

w.k.t. $D^n(ax+b)^{-1} = (-1)^n n!\, a^n (ax+b)^{-n-1}$

Hence

$$y_n = \frac{5}{9}\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{1}{3}\frac{(-1)^n (n+1)!}{(x-1)^{n+2}}$$

$$+ \frac{4}{9} \cdot \frac{(-1)^n n!}{(x+2)^{n+1}}.$$

3. Find y_n when $y = \frac{1}{x^2+a^2}$

Solution.

Using partial fractions.

$$y = \frac{1}{x^2+a^2} = \frac{1}{(x+ai)(x-ai)} = \frac{A}{x+ai} + \frac{B}{x-ai} \longrightarrow (1)$$

$\Rightarrow \quad 1 = A(x-ai) + B(x+ai)$

put $x = ai$, $\quad 1 = 2ai\,B \Rightarrow B = \frac{1}{2ai}$

put $x = -ai$, $\quad 1 = -2ai\,A \Rightarrow A = \frac{-1}{2ai}$

$$(1) \Rightarrow y = \frac{1}{2ai}\left[\frac{1}{x-ai} - \frac{1}{x+ai}\right]$$

W.K.T. $D^n(ax+b)^{-1} = (-1)^n n!\, a^n (ax+b)^{-(n+1)}$

$$\therefore y_n = \frac{1}{2ai}\left[\frac{(-1)^n n!}{(x-ai)^{n+1}} - \frac{(-1)^n n!}{(x+ai)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2ai}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right]$$

Home work.

1. Find y_n, where $y = \frac{1}{x^2 - a^2}$

2. Find y_n, where $y = \frac{x^4}{(x-1)(x-2)}$

3. Find y_n, where $y = \frac{x^2}{(x+1)^2(x+2)}$

4. Find y_n, where $y = \frac{x^2}{(x-a)(x-b)(x-c)}$

5. Find y_n, where $y = \frac{x^3}{(x-a)(x-b)(x-c)}$

6. Find y_n, where $y = \frac{1}{4x^2 + 8x + 3}$

Trigonometrical Transformation.

Problems:

1. Find the n^{th} differential coefficient of $\cos x \cdot \cos 2x \cdot \cos 3x$.

Solution:

W.K.T $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$$\therefore \cos x \cdot \cos 2x \cos 3x = \frac{1}{2}\cos 2x\left[\cos 4x + \cos 2x\right]$$

$$= \frac{1}{2}\cos 2x \cos 4x + \frac{1}{2}\cos^2 2x$$

$$= \frac{1}{4}\left[\cos 2x + \cos 6x\right] + \frac{1}{4}(1 + \cos 4x)$$

$$= \frac{1}{4} + \frac{1}{4}(\cos 2x + \cos 4x + \cos 6x)$$

$$\therefore D^n(\cos x \cos 2x \cos 3x)$$

$$= \frac{1}{4}\left\{2^n\cos\left(\frac{n\pi}{2} + 2x\right) + 4^n\cos\left(\frac{n\pi}{2} + 4x\right) + 6^n\cos\left(\frac{n\pi}{2} + 6x\right)\right\}.$$

$$\left[\because D^n\cos(ax+b) = a^n\cos\left(\frac{n\pi}{2} + ax + b\right)\right]$$

2. Find the n^{th} differential coefficient of $\cos^5\theta \sin^7\theta$.

Solution:

Let $x = \cos\theta + i\sin\theta$ then

$$\frac{1}{x} = \cos\theta - i\sin\theta$$

$\therefore x + \frac{1}{x} = 2\cos\theta \ ; \ x - \frac{1}{x} = 2i\sin\theta$

Also by De Moivre's theorem, we have

$x^n = \cos n\theta + i\sin n\theta \ ; \ \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

we have, $2^5\cos^5\theta = \left(x + \frac{1}{x}\right)^5$

and $2^7 i^7 \sin^7\theta = \left(x - \frac{1}{x}\right)^7$

Hence $2^{12} i^7 \cos^5\theta \sin^7\theta = \left(x+\frac{1}{x}\right)^5 \left(x - \frac{1}{x}\right)^7$

$$= \left(x^2 - \frac{1}{x^2}\right)^5 \left(x - \frac{1}{x}\right)^2$$

$$= \left(x^{10} - 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} - \frac{1}{x^{10}}\right)$$

$$\left(x^2 - 2 + \frac{1}{x^2}\right)$$

$$= \left(x^{12} - \frac{1}{x^{12}}\right) - 2\left(x^{10} - \frac{1}{x^{10}}\right) - 4\left(x^8 - \frac{1}{x^8}\right)$$

$$+ 10\left(x^6 - \frac{1}{x^6}\right) + 5\left(x^4 - \frac{1}{x^4}\right) - 20\left(x^2 - \frac{1}{x^2}\right)$$

Hence we have

$$-2^{11}\cos^5\theta \sin^7\theta = \sin 12\theta - 2\sin 10\theta$$

$$-4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta$$

$$-20\sin 2\theta$$

W.K.T $D^n(\sin(ax+b)) = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$

$\therefore D^n(\cos^5\theta \sin^7\theta)$

$$= \frac{-1}{2^{11}}\left\{ 12^n \sin\left(\frac{n\pi}{2} + 12\theta\right) - 10^n 2\sin\left(\frac{n\pi}{2} 10\theta\right)\right.$$

$$-8^n 4\sin\left(\frac{n\pi}{2} + 8\theta\right) + 6^n 10\sin\left(\frac{n\pi}{2} + 6\theta\right)$$

$$\left. + 4^n 5 \sin\left(\frac{n\pi}{2} + 4\theta\right) - 2^n 20 \sin\left(\frac{n\pi}{2} + 2\theta\right)\right\}$$

Home work. Find the n^{th} differential Coefficient of

1. $\sin^3 x \cos^5 x$.
2. $\sin^2 x \cos^3 x$
3. $\sin^3 x$
4. $\cos^4 x$.

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Problems.

1. Prove that if $y^3 - 3ax^2 + x^3 = 0$.

$$\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$$

Proof:

Let $y^3 - 3ax^2 + x^3 = 0$

Diff. with respect to x, we get

$$3y^2 \cdot \frac{dy}{dx} - 3a(2x) + 3x^2 = 0$$

$$3y^2 \cdot y_1 - 6ax + 3x^2 = 0$$

$$\Rightarrow y^2 y_1 - 2ax + x^2 = 0$$

$$\Rightarrow y_1 = \frac{2ax - x^2}{y^2}$$

Again Differentiate w.r.to x, we get

$$y_2 = \frac{y^2(2a-2x) - (2ax-x^2)(2yy_1)}{y^4}$$

$$\left(\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2} \right)$$

$$= \frac{2y\left[y(a-x) - (2ax-x^2)y_1\right]}{y^4}$$

$$= \frac{2}{y^3}\left[y(a-x) - (2ax-x^2)\left(\frac{2ax-x^2}{y^2}\right)\right]$$

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$$y_2 = \frac{2}{y^5}\left[y^3(a-x) - (2ax-x^2)^2\right]$$

$$= \frac{2}{y^5}\left[(3ax^2-x^3)(a-x) - (2ax-x^2)^2\right]$$

$$= \frac{2}{y^5}\left[3a^2x^2 - 3ax^3 - ax^3 + x^4 - 4a^2x^2 - x^4 + 4ax^3\right]$$

$$= \frac{2}{y^5}\left(-a^2x^2\right)$$

$$\therefore y_2 = \frac{-2a^2x^2}{y^5}$$

$$\therefore y_2 + \frac{2a^2x^2}{y^5} = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$$

Successive Differentiation.

Problems:

1. (i) If $x = a(t - \sin t)$, $y = a(1 + \cos t)$ find $\frac{d^2y}{dx^2}$ as a function of t.

(ii) Find $\frac{d^2y}{dx^2}$ if $x = \sqrt{\sin 2t}$ and $y = \sqrt{\cos 2t}$.

Solution:

(i) Given that

$x = a(t - \sin t)$; $y = a(1 + \cos t)$

$\frac{dx}{dt} = a(1 - \cos t)$; $\frac{dy}{dt} = -a \sin t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-a \sin t}{a(1 - \cos t)} = \frac{-\sin t}{1 - \cos t}$$

$$\therefore \frac{dy}{dx} = \frac{-2\sin t/2 \cos t/2}{2\sin^2 t/2} \quad \left[\because \begin{array}{l} \sin A = 2\sin A/2 \cos \frac{A}{2} \\ 1 - \cos A = 2\sin^2 A/2 \end{array}\right.$$

$$= \frac{-\cos t/2}{\sin t/2} = -\cot(t/2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\cot t/2\right)$$

$$\left[\because \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x\right]$$

$$\therefore \frac{d^2y}{dx^2} = -\left[-\operatorname{cosec}^2(t/2)\cdot\frac{1}{2}\cdot\frac{dt}{dx}\right]$$

$$= \frac{1}{2}\operatorname{cosec}^2(t/2)\cdot\frac{1}{a(1-\cos t)}$$

$$= \frac{1}{2}\cdot\frac{1}{\sin^2 t/2}\cdot\frac{1}{2a\sin^2 t/2} \quad \left[\because \frac{1}{\sin\theta} = \operatorname{cosec}\theta\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{4a}\cdot\frac{1}{\sin^4 t/2}.$$

(ii) Given that

$x = \sqrt{\sin 2t}$; $y = \sqrt{\cos 2t}$

$$\frac{dx}{dt} = \frac{1}{2\sqrt{\sin 2t}}\cdot 2\cos 2t \qquad \frac{dy}{dt} = \frac{1}{2\sqrt{\cos 2t}}(-2\sin 2t)$$

$$= \frac{\cos 2t}{\sqrt{\sin 2t}} \qquad = \frac{-\sin 2t}{\sqrt{\cos 2t}}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin 2t/\sqrt{\cos 2t}}{\cos 2t/\sqrt{\sin 2t}}$$

$$= -\frac{\sin 2t}{\cos 2t}\times\frac{\sqrt{\sin 2t}}{\sqrt{\cos 2t}}$$

$$\frac{dy}{dx} = \frac{-(\sin 2t)^{3/2}}{(\cos 2t)^{3/2}}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{-(\sin 2t)^{3/2}}{(\cos 2t)^{3/2}}\right]$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}\right]$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-\left[(\cos 2t)^{3/2} \cdot \frac{3}{2}(\sin 2t)^{1/2} \cdot \cos 2t \cdot 2 \cdot \frac{dt}{dx} - (\sin 2t)^{3/2} \cdot \frac{3}{2}(\cos 2t)^{1/2}(-\sin 2t) \cdot 2 \cdot \frac{dt}{dx}\right]}{\left[(\cos 2t)^{3/2}\right]^2}$$

$$= \frac{-3}{(\cos 2t)^3}\left[(\cos 2t)^{1/2}(\sin 2t)^{1/2}(\cos^2 2t + \sin^2 2t)\right] \cdot \frac{dt}{dx}$$

$$= \frac{-3}{(\cos 2t)^3}\left[(\cos 2t)^{1/2}(\sin 2t)^{1/2}\right] \cdot \left[\frac{\sqrt{\sin 2t}}{\cos 2t}\right]$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-3\sin 2t}{(\cos 2t)^{7/2}}.$$

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2. If $x^3+y^3-3axy=0$. prove that

$$D^2y = \frac{2a^3xy}{(ax-y^2)^3}.$$

Solution:

Given that $x^3+y^3-3axy=0$

Diff. w.r.to x, we get

$$3x^2+3y^2\cdot\frac{dy}{dx} - 3a\left(x\cdot\frac{dy}{dx}+y\cdot 1\right)=0.$$

$$\frac{dy}{dx}(y^2-ax)+x^2-ay=0.$$

$$\therefore \frac{dy}{dx} = \frac{(ay-x^2)}{y^2-ax} = \frac{x^2-ay}{ax-y^2} = y_1$$

$$\frac{d^2y}{dx^2} = \frac{(ax-y^2)(2x-ay_1)-(x^2-ay)(a-2yy_1)}{(ax-y^2)^2}$$

$$=\frac{1}{(ax-y^2)^2}\left[(ax-y^2)\left(2x-a\cdot\left(\frac{x^2-ay}{ax-y^2}\right)\right) - (x^2-ay)\left(a-2y\cdot\left(\frac{x^2-ay}{ax-y^2}\right)\right)\right]$$

$$=\frac{1}{(ax-y^2)^3}\left[(ax-y^2)(2ax^2-2xy^2-ax^2+a^2y) - (x^2-ay)(a^2x-ay^2-2x^2y+2ay^2)\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{(ax-y^2)^3}\Big[2a^2x^3 - 2ax^2y^2 - a^2x^3 + a^3xy - 2axy^2 + 2xy^4 + ax^2y^2 - a^2y^3 - a^2x^3 + ax^2y^2 + 2x^4y - 2ax^2y^2 + a^3xy - a^2y^3 - 2ax^2y^2 + 2a^2y^3\Big]$$

$$= \frac{1}{(ax-y^2)^3}\left[- ax^2y^2 + 2a^3xy + 2xy^4 + 2x^4y\right]$$

$$= \frac{1}{(ax-y^2)^3}\left[2xy(-3axy + y^3 + x^3) + 2a^3xy\right]$$

$$= \frac{1}{(ax-y^2)^3}\left[0 + 2a^3xy\right]$$

$$\therefore \frac{d^2y}{dx^2} = \frac{2a^3xy}{(ax-y^2)^3}.$$

$$\therefore D^2y = \frac{2a^3xy}{(ax-y^2)^3}.$$

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Formation of Equations Involving derivatives.

Problems.

1. If $xy = ae^{x} + be^{-x}$, prove that

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0.$$

Proof:

Let $xy = ae^{x} + be^{-x}$

Now differentiating both sides with respect to x, we have.

$$y + x\cdot\frac{dy}{dx} = ae^{x} - be^{-x}$$

Again Diff. w.r.to x on both sides, we have

$$\frac{dy}{dx} + x\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx} = ae^{x} + be^{-x}.$$

(ie) $x\cdot\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = xy$

(ie) $x\cdot\dfrac{d^2y}{dx^2} + 2\cdot\dfrac{dy}{dx} - xy = 0$

2. Prove that if $y = \sin(m \sin^{-1} x)$,

$(1-x^2) y_2 - x y_1 + m^2 y = 0$.

Proof:

Let $y = \sin(m \sin^{-1} x)$

$$\sin^{-1} y = m \sin^{-1} x.$$

Diff with r. to x on both sides, we have

$$\frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = m \cdot \frac{1}{\sqrt{1-x^2}}$$

Squaring on both sides, we have

$$\frac{1}{(1-y^2)} \cdot \frac{d^2y}{dx^2} = m^2 \cdot \frac{1}{(1-x^2)}$$

$$(1-x^2) \cdot \frac{d^2y}{dx^2} = m^2 (1-y^2)$$

Diff. w.r.to x on both sides, we have

$$(1-x^2) 2 \cdot \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} - 2x \cdot \frac{d^2y}{dx^2} = -2m^2 y \cdot \frac{dy}{dx}.$$

i.e. $(1-x^2) \dfrac{d^2y}{dx^2} - x \cdot \dfrac{dy}{dx} = -m^2 y$

i.e. $(1-x^2) y_2 - x y_1 + m^2 y = 0.$

3. If $x = \sin\theta$, $y = \cos p\theta$, prove that
$(1-x^2)y_2 - xy_1 + p^2 y = 0$

Proof:

Let $x = \sin\theta$, $y = \cos p\theta$.

$$\frac{dx}{d\theta} = \cos\theta \quad ; \quad \frac{dy}{d\theta} = -p\sin p\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -p \cdot \frac{\sin p\theta}{\cos\theta}$$

$$= -p \cdot \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = p^2 \cdot \frac{1-y^2}{1-x^2}$$

i.e. $(1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2(1-y^2)$

i.e. $(1-x^2)\, y_2 = p^2(1-y^2)$

Diff. w.r.to x on both sides, we have

$$(1-x^2) \cdot 2 \cdot \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(-2x)$$
$$= p^2\left(-2y \cdot \frac{dy}{dx}\right)$$

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i.e., $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$

4. If $y = e^{-x}\cos x$, prove that $\frac{d^4y}{dx^4} + 4y = 0$

Proof:

Let $y = e^{-x}\cos x$

Diff. w.r.to x, we have

$$\frac{dy}{dx} = e^{-x}(-\sin x) + \cos x(-e^{-x}) = -e^{-x}(\sin x + \cos x).$$

$$\frac{d^2y}{dx^2} = -e^{-x}(\cos x - \sin x) + (\sin x + \cos x)(e^{-x}).$$

$$= 2e^{-x}\sin x.$$

$$\frac{d^3y}{dx^3} = 2\left[e^{-x}\cdot\cos x + \sin x(-e^{-x})\right]$$

$$= 2e^{-x}(\cos x - \sin x)$$

$$\frac{d^4y}{dx^4} = 2\left\{e^{-x}(-\sin x - \cos x) + (\cos x - \sin x)(-e^{-x})\right\}$$

$$= -4e^{-x}\cos x.$$

$$= -4y$$

i.e., $\frac{d^4y}{dx^4} + 4y = 0.$

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Leibnitz formula for the n^{th} derivative of a product:

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n} v + nC_1 \frac{d^{n-1} u}{dx^{n-1}} \cdot \frac{dv}{dx}$$

$$+ nC_2 \frac{d^{n-2} u}{dx^{n-2}} \cdot \frac{d^2 v}{dx^2} + \cdots\cdots$$

$$+ nC_r \frac{d^{n-r} u}{dx^{n-r}} \cdot \frac{d^r v}{dx^r} + \cdots\cdots$$

$$+ nC_1 \frac{du}{dx} \cdot \frac{d^{n-1} v}{dx^{n-1}} + u \cdot \frac{d^n v}{dx^n}$$

Problems:

1. Find the n^{th} differential coefficient of $x^2 \log x$.

Solution:-

Let $v = x^2$, and $u = \log x$.

$$\frac{d^n}{dx^n}\{x^n \log x\} = \frac{d^n}{dx^n}(\log x) \cdot x^2 +$$

$$nC_1 \frac{d^{n-1}}{dx^{n-1}}(\log x) \cdot \frac{d}{dx}(x^2) + nC_2 \frac{d^{n-2}}{dx^{n-2}}(\log x)$$

$$\frac{d^2}{dx^2}(x^2).$$

w.k.t. $D^n[\log(ax+b)] = (-1)^{n-1} \cdot (n-1)!\, a^n (ax+b)^{-n}$.

$\therefore D^n(x^2\log x) = \dfrac{(-1)^{n-1}(n-1)!}{x^n} \cdot x^2 +$

$n \cdot \dfrac{(-1)^{n-2}(n-2)!}{x^{n-1}} \cdot 2x + \dfrac{n(n-1)}{2} \cdot \dfrac{(-1)^{n-3}(n-3)!}{x^{n-2}} \cdot 2.$

$\therefore D^n(x^2\log x) = \dfrac{(-1)^{n-3} \cdot n(n-1)(n-3)!}{x^{n-2}}.$

---

2. If $y = \sin(m\sin^{-1}x)$, prove that $(1-x^2)y_2 - xy_1 + m^2y = 0$ and $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$.

Proof:

(i) Let $y = \sin(m\sin^{-1}x)$

$\sin^{-1}y = m\sin^{-1}x.$

Diff. w.r. to x, we have

$$\frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = \frac{m}{\sqrt{1-x^2}}$$

$$\therefore (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2(1-y^2)$$

Diff. w.r.to x, we have

$$(1-x^2)\cdot 2\left(\frac{dy}{dx}\right)\cdot\frac{d^2y}{dx^2} - 2x\cdot\left(\frac{dy}{dx}\right)^2 = -2m^2y\cdot\frac{dy}{dx}.$$

i.e. $(1-x^2)\cdot\frac{d^2y}{dx^2} - \frac{dy}{dx} + m^2y = 0$

i.e. $(1-x^2)\,y_2 - y_1 + m^2y = 0$

$\Rightarrow (1-x^2)y_2 = y_1 - m^2y.$

(ii) Taking then n^{th} derivative of each term by Leibnitz's theorem, we have

$$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) = y_{n+1}x + {}^nC_1\, y_n - m^2y_n$$

i.e. $y_{n+2}(1-x^2) - 2nx\,y_{n+1} - n(n-1)y_n$

$$= xy_{n+1} + ny_n - m^2y_n.$$

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iii, $(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} + (m^2-n^2)\,y_n = 0.$

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ie, $(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1}$
$$+ (m^2 - n^2)\,y_n = 0.$$

Problems:

1. Find the n^{th} differential coefficients of $x^2 e^{3x}$.

Solution.

Let $v = x^2$, and $u = e^{3x}$.

$$\frac{d^n}{dx^n}(x^2 e^{3x}) = \frac{d^n}{dx^n}(e^{3x}) \cdot x^2$$

$$+ nC_1 \frac{d^{n-1}}{dx^{n-1}}(e^{3x}) \cdot 2x + nC_2 \frac{d^{n-2}}{dx^{n-2}}(e^{3x}) \cdot 2$$

$$= 3^n e^{3x} \cdot x^2 + n \cdot 3^{n-1} e^{3x} \cdot 2x + \frac{n(n-1)}{2} \cdot 3^{n-2} \cdot e^{3x} \cdot 2$$

$$= e^{3x}\left[3^n x^2 + 2nx\,3^{n-1} + 2n(n-1) \cdot 3^{n-2}\right]$$

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2. Find the n^{th} differential Coefficients of $x^2 \cos x$.

Solution.

Let $v = x^2$ and $u = \cos x$.

$$\frac{d^n}{dx^n}(x^2 \cos x) = \frac{d^n}{dx^n}(\cos x) \cdot x^2$$

$$+ nC_1 \frac{d^{n-1}}{dx^{n-1}}(\cos x) \cdot 2x + nC_2 \frac{d^{n-2}}{dx^{n-2}}(\cos x) \cdot 2$$

$$= \cos\left(\frac{n\pi}{2}+x\right)x^2 + n \cos\left(\frac{(n-1)\pi}{2}+x\right) \cdot 2x$$

$$+ \frac{n(n-1)}{2} \cos\left(\frac{(n-2)\pi}{2}+x\right) \cdot 2$$

$$= x^2 \cos\left(\frac{n\pi}{2}+x\right) + 2nx \cos\left(\frac{(n-1)\pi}{2}+x\right)$$

$$+ n(n-1) \cos\left(\frac{(n-2)\pi}{2}+x\right)$$

Home work.

Find the n^{th} differential coefficient of

(1) xe^x (2) $x \sin x$

(3) $x^3 \sin^3 x$ (4) $x^2 \sin 3x$

3. Find the n^{th} differential coefficients of

(i) $e^x \log x$ (ii) $x^n a^x$

Solution (i) Let $v = \log x$ $u = e^x$

$$D^n(e^x \log x) = D^n(e^x)\log x + nC_1 D^{n-1}(e^x)\cdot D(\log x)$$

$$+ \cdots + nC_1 D^{n-1}(e^x)\, D^n(\log x)$$

$$= e^x \log x + ne^x\left(\frac{1}{x}\right) + \cdots + 1\cdot\frac{e^x(-1)^{n-1}(n-1)!}{x^n}$$

$$= e^x\left[\log x + \frac{n}{x} + \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n}\right].$$

(ii) Let $v = a^x$ $u = x^n$

$$D^n(x^n a^x) = D^n(x^n)\, a^x + nC_1 D^{n-1}(x^n) D^1(a^x)$$

$$+ nC_2 D^{n-2}(x^n) D^2(a^x) + \cdots\cdots,$$

$$nC_n D^{n+n}(x^n)\, D^n(a^x)$$

$$= n!\, a^x + n(n(n-1)\cdots 2)\cdot x\, a^x \log a$$

$$+ \frac{n(n-1)}{2} n(n-1)\cdots 3 x^2 a^x (\log x)^2 + \cdots$$

$$\cdots + x^n \cdots a^x (\log x)^n$$

$$D^n(x^n a^x) = a^x\Big[n! \quad n \cdot \frac{n!}{1!}\, x \log a$$

$$+ \frac{n(n-1)}{2!}\, \frac{n!}{2!}\, x^2 (\log a)^2 + \cdots\cdots$$

$$+ x^n (\log a)^n\Big]$$

$$= a^x n! \Big[1 + nx \log a + \frac{n(n-1)}{2}\, \frac{x^2 (\log a)^2}{2!}$$

$$+ \cdots\cdots + \frac{x^n (\log a)^n}{n!}\Big].$$

Home work.

1. If $y = x^2 e^x$, show that

$$y_n = \frac{1}{2} n(n-1)\, y_2 - n(n-2)\, y_1 + \frac{1}{2}(n-1)(n-2)\, y$$

where y_n stands for $\frac{d^n y}{dx^n}$.

Problems.

1. If $y = \sin^{-1}x$, prove that

(i) $(1-x^2)\, y_2 - x y_1 = 0$ and

(ii) $(1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - n^2 y_n = 0$

Proof :

Let $y = \sin^{-1} x$.

Diff. w.r.to. x, we have

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore y_1 (\sqrt{1-x^2}) = 1$$

$$\Rightarrow y_1^2 (1-x^2) = 1$$

Again Diff. w.r.to x, we have

$$y_1^2(-2x) + (1-x^2)\, 2 y_1 y_2 = 0$$

$$\div 2y_1 \Rightarrow \; -x y_1 + (1-x^2)\, y_2 = 0$$

$$\Rightarrow (1-x^2) y_2 - x y_1 = 0$$

(ii) Let $(1-x^2) y_2 - x y_1 = 0$

Using Lebintz ~~theorem~~ theorem we find n^{th} derivative

$$D^n[(1-x^2) y_2] - D^n (x y_1] = 0$$

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$$D^n(y_2)(1-x^2) + {}^nC_1 D^{n-1}(y_2)(-2x)$$

$$+ {}^nC_2 D^{n-2}(y_2)(-2) - \left[D^n(y_1) \cdot x \right.$$

$$\left. + {}^nC_1 D^{n-1}(y_1)(1) \right] = 0$$

$$y_{n+2}(1-x^2) + n\,y_{n+1}(-2x)$$

$$+ \frac{n(n-1)}{2} y_n (-2) - y_{n+1} x + n y_n = 0$$

$$\therefore y_{n+2}(1-x^2) - x\,y_{n+1}(2n+1)$$

$$- y_n(n(n-1)+n) = 0$$

$$(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - n^2 y_n = 0.$$

---

2. If $y = e^{a\sin^{-1}x}$, prove that

(i) $(1-x^2)y_2 - xy_1 - a^2 y = 0$ and

Hence show that

$$(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - (n^2+a^2)y_n = 0$$

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(32)

Proof:

(i) Let $y = e^{a\sin^{-1}x}$

Diff. w.r.to x, we have

$$y_1 = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}}$$

$$y_1\sqrt{1-x^2} = e^{a\sin^{-1}x} \cdot a.$$

$$\therefore y_1\sqrt{1-x^2} = a \cdot y.$$

$$y_1^2(1-x^2) = a^2y^2$$

Again Diff. w.r.to x, we have.

$$2y_1y_2(1-x^2) + y_1^2(-2x) = a^2 2yy_1$$

$\div 2y_1 \Longrightarrow$

$$(1-x^2)y_2 - xy_1 = a^2y.$$

$$\Rightarrow (1-x^2)y_2 - xy_1 - a^2y = 0.$$

(ii) Let $(1-x^2)y_2 - xy_1 - a^2y = 0$

Use Lebinitz theorem, we have

$$D^n[(1-x^2)y_2] - D^n[y_1 x] - D^n[y a^2] = 0$$

$$D^n(y_2)(1-x^2) + {}^nC_1 D^{n-1}(y_2)(-2x)$$

$$+ {}^nC_2 D^{n-2}(y_2)(-2) - D^n(y_1)(x) + {}^nC_1 D^{n-1}(y_1)\cdot 1$$

$$- D^n(y_1)\cdot a^2 = 0.$$

$$(1-x^2)y_{n+2} - 2nx\cdot y_{n+1} + \frac{n(n-1)(-2)y_n}{2}$$

$$- y_{n+1}\cdot x + ny_n - a^2 y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)\cdot x\, y_{n+1} - (n^2+a^2)y_n = 0$$

UNIT-II- CURVATURE

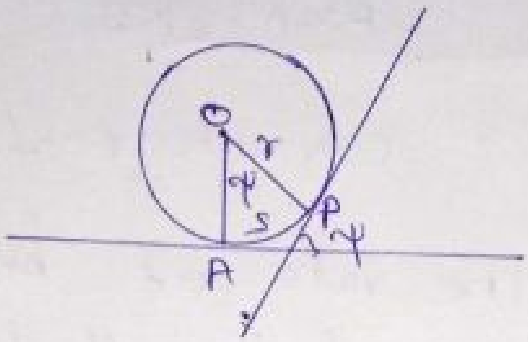
Definition: Curvature of curve.

The rate of bending of a curve in any interval is called the curvature of the curve in that interval.

Definition: Curvature

Consider a curve given by the equation $y = f(x)$. suppose the curve has a definite tangent at each point. Let A be a fixed point on the curve and p be an arbitrary point on the curve. Let s denote the arc length AP. Let ϕ be the angle made by the tangent with the x-axis. Then $\left(\frac{d\phi}{ds}\right)$ is called the curvature of the curve at p.

Thus the curvature is the rate of turning of the tangent w.r.to the arc length.



Note: The curvature of a circle of radius r at any point is $\frac{1}{r}$.

Radius of curvature

The reciprocal of the curvature of the curve at any point is called the radius of curvature at the point and is denoted by ρ.

Hence we have $\rho = \frac{ds}{d\psi}$

Cartesian form:

We know that $\frac{dy}{dx}$ represents the slope of the tangent to the curve $y = f(x)$ at (x, y)

Hence $\frac{dy}{dx} = \tan\psi$.

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Diff. w.r.to. S, we get

$$\left(\frac{d^2y}{dx^2}\right)\left(\frac{dx}{ds}\right) = \sec^2\psi \left(\frac{d\psi}{ds}\right)$$

$$\therefore y_2 \cos\psi = \sec^2\psi \cdot \frac{d\psi}{ds}$$

$$\therefore \frac{ds}{d\psi} = \frac{\sec^3\psi}{y_2}$$

$$= \frac{(1+\tan^2\psi)^{3/2}}{y_2}$$

$$= \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\text{i.e. } \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Note: $y_1 = \frac{dy}{dx}$

$y_2 = \frac{d^2y}{dx^2}$

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Examples:

1. Find the radius of curvature of the curve $y = e^x$ at $(0,1)$

Solution:

Let $y = e^x$ at $(0,1)$

$y = e^x$ at $(0,1)$

$\frac{dy}{dx} = e^x$ $\quad e^0 = 1$

$\frac{d^2y}{dx^2} = e^x$ $\quad e^0 = 1$

$\therefore$ Radius of curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$\therefore \rho = \frac{(1+1)^{3/2}}{1} = 2^{3/2}$

$= 2\sqrt{2}$

$\therefore \rho = 2\sqrt{2}$.

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2 What is the radius of curvature of the curve $x^4+y^4=2$ at $(1,1)$?

Solution:

Let $x^4+y^4=2$ at $(1,1)$

Diff. w.r.to x, we get

$$4x^3+4y^3\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x^3}{y^3}$$

Again Diff. w.r.to x, we get

$$\frac{d^2y}{dx^2}=-\frac{\left[y^3(3x^2)-x^3\,3y^2\cdot\frac{dy}{dx}\right]}{y^6}$$

$$=\frac{-3x^2y+3x^3\frac{dy}{dx}}{y^4}$$

$$=\frac{3(x^3y_1-x^2y)}{y^4}$$

$\frac{dy}{dx}$ at $(1,1)=-1$

$\frac{d^2y}{dx^2}$ at $(1,1)=-6$

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radius of curvature

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

$$\therefore \rho = \frac{(1+1)^{3/2}}{-6} = \frac{2^{3/2}}{-6} = -\frac{\sqrt{2}}{3}.$$

$$\therefore \rho = -\frac{\sqrt{2}}{3}$$

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Unit - II - Radius of curvature.

Problems.

1. Show that the radius of curvature at any point of the catenary $y = c\cosh\frac{x}{c}$ is equal to the length of the portion of the normal intercepted between the curve and the axis of x.

Solution:

Let $y = c.\cosh\frac{x}{c}$

Diff. w.r.to x, we get

$$\frac{dy}{dx} = c\sinh\frac{x}{c}\cdot\left(\frac{1}{c}\right) = \sinh\left(\frac{x}{c}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{c}\cosh\left(\frac{x}{c}\right)$$

Radius of curvature $\rho = \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$\therefore \rho = \frac{\left[1+\sinh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= \frac{\left[\cosh^2 x/c\right]^{3/2}}{\frac{1}{c}\cosh\left(\frac{x}{c}\right)} = \frac{c.\cosh^3 x/c}{\cosh x/c}$$

$\therefore \rho = c\cosh^2 \frac{x}{c} = \frac{y^2}{c}$

$\therefore \rho = \frac{y^2}{c}$.

Again at any point (x, y) the normal

$$= y\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$$= y\cosh\frac{x}{c} = \frac{y^2}{c}$$

$\therefore$ Radius of curvature = length of the normal.

2 If a curve is defined by the parametric equation $x = f(\theta)$ and $y = \phi(\theta)$. Prove that the curvature is

$$\frac{1}{\rho} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}},$$

where dashes denote differentiation with respect to θ.

Solution:

Let $x = f(\theta)$ and $y = \varphi(\theta)$

Let $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \cdot \dfrac{d\theta}{dx}$

$$= \frac{dy}{d\theta} \div \frac{dx}{d\theta}$$

$$= \frac{y'}{x'}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right)$$

$$= \frac{d}{d\theta}\left(\frac{y'}{x'}\right) \cdot \frac{d\theta}{dx}$$

$$= \frac{x' \cdot y'' - y' \cdot x''}{x'^2} \cdot \frac{1}{x'}$$

$$= \frac{y''x' - y'x''}{x'^3}.$$

Radius of curvature $\rho = \dfrac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}.$

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}.$$

$$= \frac{y''x' - y'x''}{x'^3\left[1 + \frac{y'^2}{x'^2}\right]^{3/2}}.$$

$$\frac{1}{\rho} = \frac{\frac{y''x' - y'x''}{x'^3}}{\frac{\left[x'^2 + y'^2\right]^{3/2}}{x'^3}}.$$

$$\frac{1}{\rho} = \frac{x'y'' - y'x''}{\left(x'^2 + y'^2\right)^{3/2}}.$$

Hence proved.

3. Prove that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$ is $4a\cos\frac{\theta}{2}$.

Proof:

Let $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$.

$$\frac{dx}{d\theta} = a(1 + \cos\theta)\,; \quad \frac{dy}{d\theta} = a\sin\theta.$$

$$\frac{d^2x}{d\theta} = -a\sin\theta\,; \quad \frac{d^2y}{d\theta^2} = a\cos\theta$$

Radius of curvature

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}.$$

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$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}.$$

W.K.T. $\frac{1}{\rho} = \frac{x'y'' - y'x''}{\left[x'^2 + y'^2\right]^{3/2}}.$

$$\therefore \frac{1}{\rho} = \frac{a(1+\cos\theta)\,a\cos\theta - a\sin\theta\,(-a\sin\theta)}{\left[a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}.$$

$$= \frac{a^2\cos\theta + a^2\sin^2\theta + a^2\cos^2\theta}{a^3\left[1+\cos^2\theta + 2\cdot1\cdot\cos\theta + \sin^2\theta\right]^{3/2}}$$

$$= \frac{a^2(1+\cos\theta)}{a^3\left[2(1+\cos\theta)\right]^{3/2}}$$

$$= \frac{a^2 \cdot 2\cos^2\theta/2}{a^3\left[4\cos^2\theta/2\right]^{3/2}} = \frac{2\cos^2\theta/2}{2^3\left(\cos^2\theta/2\right)^{3/2}}$$

$$\frac{1}{\rho} = \frac{1}{4a\cos\theta/2}.$$

$$\therefore \rho = 4a\cos\theta/2.$$

Hence proved.

4. Find ρ at the point 't' of the curve
$x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$

Solution

Let $x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$

$$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t)$$
$$= at\cos t$$

$$\frac{dy}{dt} = a(\cos t - \cos t + t\sin t)$$
$$= at\sin t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \tan t$$

$$\therefore \frac{dy}{dx} = \tan t$$

$$\frac{d^2y}{dx^2} = \sec^2 t\,\frac{dt}{dx} = \sec^2 t \cdot \frac{1}{at\cos t} = \frac{1}{at\cos^3 t}$$

Radius of curvature. $\rho = \dfrac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$\therefore \rho = \frac{\left[1 + \tan^2 t\right]^{3/2}}{\frac{1}{at\cos^3 t}} = \frac{\left[1 + \frac{\sin^2 t}{\cos^2 t}\right]^{3/2}}{\frac{1}{at\cos^3 t}}$$

$$\therefore \rho = \frac{\left[\frac{\cos^2 t + \sin^2 t}{\cos^2 t}\right]^{3/2}}{\frac{1}{at\cos^3 t}} = \frac{\frac{1}{\cos^3 t}}{\frac{1}{at\cos^3 t}} = at$$

$$\therefore \rho = at$$

Problems.

1. Find the radius of curvature at $x = 1$ on $y = \frac{\log x}{x}$.

Solution:

Let $y = \left(\frac{1}{x}\right) \log x$. at $x = 1$

$y_1 = \frac{1}{x} \cdot \frac{1}{x} + \log x \left(\frac{-1}{x^2}\right)$ $\quad y_1 = \frac{1}{1^2}(1 - \log 1]$

$= \frac{1}{x^2} + \log x \left(\frac{-1}{x^2}\right)$ $\quad = \frac{1}{1}[1 - 0]$

$= \frac{1}{x^2}\left[1 - \log x\right]$ $\quad = 1$.

$y_2 = \frac{1}{x^2}\left[-\frac{1}{x}\right] + (1 - \log x)\left(\frac{-2}{x^3}\right)$ $\quad y_1 = \frac{-1}{1^3} - \frac{2}{1^3}(1 - \log 1)$

$= \frac{-1}{x^3} - \frac{2}{x^3}(1 - \log x)$ $\quad = -1 - 2(1 - 0)$

$= -3$

Radius of curvature $\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$

$\therefore \rho = \frac{[1+1]^{3/2}}{-3}$

$= \frac{2^{3/2}}{-3} = -\frac{2\sqrt{2}}{3}$.

$|\rho| = \frac{2\sqrt{2}}{3}$ //

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2. Find the radius of curvature at the point (3, 10) on the curve $xy = 30$.

Solution: Let $xy = 30$. at (3,10)

$y = \frac{30}{x}$ at (3,10)

$y_1 = \frac{-30}{x^2}$ $y_1 = \frac{-30}{9} = \frac{-10}{3}$

$y_2 = \frac{60}{x^3}$ $y_2 = \frac{60}{9} = \frac{20}{3}$

Radius of curvature $\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$\therefore \rho = \frac{\left[1 + \left(\frac{-10}{3}\right)^2\right]^{3/2}}{\frac{20}{3}} = \frac{(109)^{3/2}}{3^3} \times \frac{3}{20}$

$= \frac{(109)^{3/2}}{9 \times 20}$

$= \frac{(109)^{3/2}}{180}$ //

3. Find the radius of curvature of the curve $\sqrt{x}+\sqrt{y}=1$ at $\left(\frac{1}{4},\frac{1}{4}\right)$

Solution:

Let $\sqrt{x}+\sqrt{y}=1$

$y=(1-\sqrt{x})^2$

$y_1=2(1-\sqrt{x})\left(0-\frac{1}{2\sqrt{x}}\right)$

$=\frac{\sqrt{x}-1}{\sqrt{x}}$

$=1-\frac{1}{\sqrt{x}}$

$y_1=1-x^{-1/2}$

$y_2=\frac{1}{2}x^{-3/2}$

$=\frac{1/2}{x^{3/2}}$

at $\left(\frac{1}{4},\frac{1}{4}\right)$

$y_1=1-\frac{1}{\sqrt{1/4}}$

$=1-2$

$=-1$

$y_2=\frac{1/2}{(1/4)^{3/2}}$

$=\frac{1}{2}\cdot 8$

$=4.$

Radius of curvature $\rho=\frac{\left[1+(dy/dx)^2\right]^{3/2}}{d^2y/dx^2}$

$\therefore \rho=\frac{\left[1+(-1)^2\right]^{3/2}}{4}=\frac{2^{3/2}}{2^2}=\frac{2\sqrt{2}}{2.\sqrt{2}.\sqrt{2}}$

$\therefore \rho=\frac{1}{\sqrt{2}}$ //

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Home work

Find the radius of curvature for the curves.

a. $xy^3 = a^4$ at (a, a)

b. $\frac{x^2}{9} + \frac{y^2}{16} = 2$ at $(3, 4)$

c. $y^3 = x(x + 2y)$ at $(1, -1)$

d. $4ay^2 = (2a - x)^3$ at $(a, a/2)$

e. $y = 4\sin x - \sin 2x$ at the point $x = \frac{3a}{2}$

f. $y^2 = x^3 + 8$ at $(-2, 0)$

g. $x^3 + y^3 = 3axy$ at the point $x = y = \frac{3a}{2}$.

h. $y = e^x$ at the point where it crosses the y-axis.

Problems.

1. Prove That the radius of curvature at a point $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3}+y^{2/3}=a^{2/3}$ is $3a\sin\theta\cos\theta$.

Proof:

Let $x = a\cos^3\theta$

$$\frac{dx}{d\theta} = 3a\cos^2\theta\,(-\sin\theta)$$

$$= -3a\cos^2\theta\sin\theta$$

Let $y = a\sin^3\theta$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cdot\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}.$$

$$= -\tan\theta.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-\tan\theta)$$

$$= -\sec^2\theta\,\frac{d\theta}{dx}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-\sec^2\theta}{\left(\frac{dx}{d\theta}\right)} = \frac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta} = \frac{1}{3a\cos^4\theta\sin\theta}$$

Radius of curvature

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\therefore \rho = \frac{\left[1+\tan^2\theta\right]^{3/2}}{\left(\frac{1}{3a\cos^4\theta\sin\theta}\right)}$$

$$= (\sec^2\theta)^{3/2} \cdot 3a\cos^4\theta\sin\theta$$

$\rho = 3a\sin\theta\cos\theta$

Hence proved.

Implicit form:

Let $f(x,y)=0$ be the implicit form of the given curve. Then radius of curvature

$$\rho = \frac{\left[f_x^2+f_y^2\right]^{3/2}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}$$

Problems:

1. Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the curve $x^3+y^3=3axy$.

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Solution:

Let $f(x,y) = x^3 + y^3 - 3axy$, At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$f_x = 3x^2 - 3ay$ $\quad f_x = 3\left(\frac{3a}{2}\right)^2 - 3a\left(\frac{3a}{2}\right)$

$= \frac{27}{4}a^2 - \frac{9}{2}a^2$

$= \frac{9}{4}a^2$.

$f_y = 3y^2 - 3ax$ $\quad f_y = 3\left(\frac{3a}{2}\right)^2 - 3a\left(\frac{3a}{2}\right)$

$= \frac{9}{4}a^2$

$f_{xx} = 6x$ $\quad f_{xx} = 6\left(\frac{3a}{2}\right)$

$= 9a$

$f_{yy} = 6y$ $\quad f_{yy} = 6\left(\frac{3a}{2}\right)$

$= 9a$

$f_{xy} = -3a$ $\quad f_{xy} = -3a$.

Radius of curvature

$$\rho = \frac{\left[f_x^2 + f_y^2\right]^{3/2}}{f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2}.$$

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$$\rho = \frac{\left[\left(\frac{9}{4}a^2\right)^2 + \left(\frac{9}{4}a^2\right)^2\right]^{3/2}}{9a\left(\frac{9}{4}a^2\right)^2 - 2(-3a)\left(\frac{9}{4}a^2\right)\left(\frac{9}{4}a^2\right) + 9a\left(\frac{9}{4}a^2\right)^2}$$

$$= \frac{\left[2\left(\frac{9}{4}a^2\right)^2\right]^{3/2}}{(9a + 6a + 9a)\left(\frac{9}{4}a^2\right)^2}$$

$$= \frac{2^{3/2}\left[\frac{9}{4}a^2\right]^3}{(24a)\left(\frac{9}{4}a^2\right)^2}$$

$$= \frac{2\sqrt{2}\left[\frac{9}{4}a^2\right]^3}{(24a)\left(\frac{9}{4}a^2\right)^2}$$

$$= \frac{2\sqrt{2}\left(\frac{9}{4}a^2\right)}{24a}$$

$$= \frac{2\sqrt{2}\cdot 9a}{4\times 24} = \frac{3\sqrt{2}\,a}{16}$$

$$= \frac{3a}{8\sqrt{2}}$$

$$\therefore \rho = \frac{3a}{8\sqrt{2}}.$$

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2. Find the radius of curvature at $(a,0)$ of the curve $xy^2 = a^3 - x^3$

Solution.

Let $f(x,y) = xy^2 - a^3 + x^3$ At $(a,0)$

$f_x = y^2 + 3x^2$ $f_x = 0^2 + 3a^2 = 3a^2$

$f_y = 2xy$ $f_y = 2(a)(0) = 0$

$f_{xx} = 6x$ $f_{xx} = 6(a) = 6a$

$f_{xy} = 2y$ $f_{xy} = 2(0) = 0$

$f_{yy} = 2x$ $f_{yy} = 2(a) = 2a$.

Radius of curvature

$$\rho = \frac{\left[f_x^2 + f_y^2\right]^{3/2}}{f_{xx} f_y^2 - 2f_{xy} f_x f_y + f_{yy} f_x^2}$$

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$$\therefore \rho = \frac{\left[(3a^2)^2 + 0^2\right]^{3/2}}{(6a)(0) - 2(0)(3a^2)(0) + (2a) + (3a^2)^2}$$

$$= \frac{\left[(3a^2)^2\right]^{3/2}}{(2a)(3a^2)^2}$$

$$= \frac{(3a^2)^3}{(2a)(3a^2)^2}$$

$$= \frac{3a^2}{2a}$$

$$= \frac{3}{2} \cdot a$$

$$\therefore \rho = \frac{3}{2} \cdot a.$$

Problems:

1. Find the radius of curvature for the curve $y = 4\sin x - \sin 2x$ at the point where $x = \frac{\pi}{2}$.

Solution:

Let $y = 4\sin x - \sin 2x$

Diff. w. r. to. x, we get

$$\frac{dy}{dx} = 4\cos x - 2\cos 2x$$

$$\left(\frac{dy}{dx}\right)_{x=\pi/2} = 4\cos(\pi/2) - 2\cos 2(\pi/2)$$

$$= 4(0) - 2\cos\pi$$

$$= 2. \quad \because \cos\pi =$$

$$\frac{d^2y}{dx^2} = -4\sin x + 4\sin 2x$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=\pi/2} = -4\sin\pi/2 + 4\sin 2(\pi/2)$$

$$= -4(1) + 0$$

$$= -4.$$

Radius of curvature

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\therefore \rho = \frac{\left[1+2^2\right]^{3/2}}{-4} = \frac{5^{2/3}}{-4} = \frac{-5\sqrt{5}}{4}$$

$$|\rho| = \frac{5\sqrt{2}}{4}.$$

Parametric form:

Let $x = f(t)$ and $y = g(t)$ be the parametric equations of the given curve.

Radius of curvature

$$\rho = \frac{\left(f'^2 + g'^2\right)^{3/2}}{f'g'' - f''g'}.$$

Problems.

1. Find the radius of curvature at any point P of the parabola given by $x = at^2$, $y = 2at$.

solution.

Let $x = at^2 \qquad y = 2at$

$x' = 2at \qquad y' = 2a$

$x'' = 2a \qquad y'' = 0.$

Radius of curvature

$$\rho = \frac{[x'^2 + y'^2]^{3/2}}{x'y'' - y'x''}$$

$$= \frac{(4a^2t^2 + 4a^2)^{3/2}}{0 - (2a)(2a)}$$

$$= \frac{(2a)^3 [1+t^2]^{3/2}}{-(2a)^2}$$

$$= -2a(1+t^2)^{3/2}$$

$$|\rho| = 2a(1+t^2)^{3/2}.$$

---

2. Find the radius of curvature of the curve $x = 3t^2, y = 3t - t^3$ at $t = 1$.

Solution:

Let $x = 3t^2$

$x' = 6t \quad (x')_{t=1} = 6$

$x'' = 6 \quad (x'')_{t=1} = 6$

Let $y = 3t - t^3$

$y' = 3 - 3t^2 \quad (y')_{t=1} = 0$

$y'' = -6t \quad (y'')_{t=1} = -6.$

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Radius of curvature

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

$$= \frac{[6^2 + 0^2]^{3/2}}{-36 - 0}$$

$$= \frac{6^3}{-6^2}$$

$$= -6 \;//$$

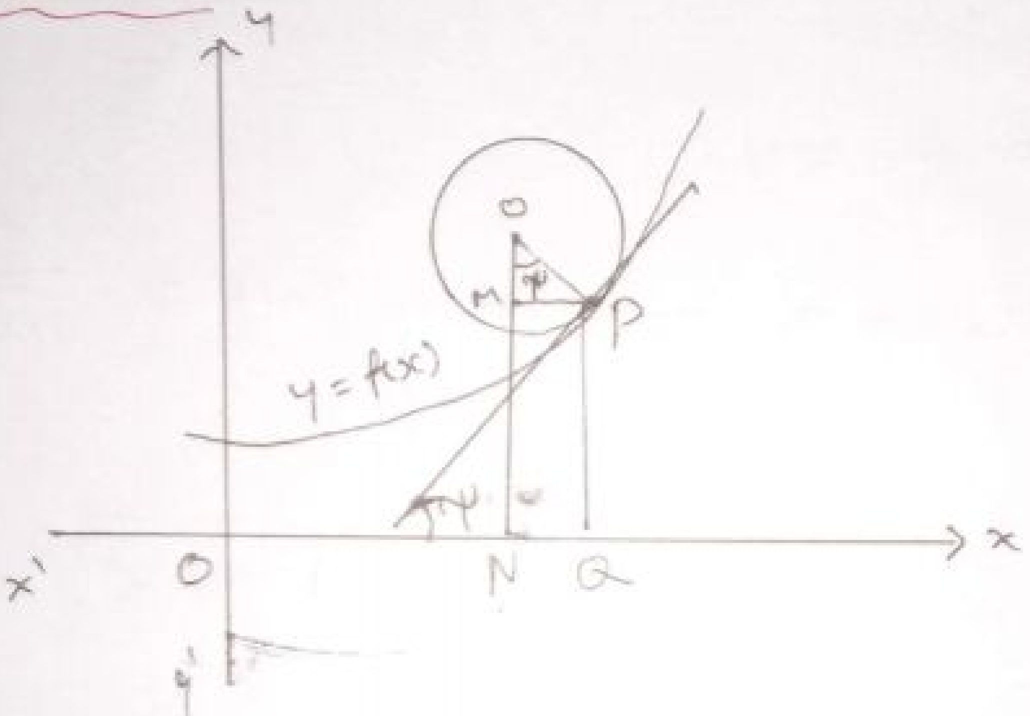
Home work.

1. In the ellipse given by $x = a\cos\theta, y = b\sin\theta$ find the radius of curvature at $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$

2. Find ρ at the point t for the curve $x = a(\cos t - \sin t)$; $y = a(\cos t - \sin t)$

3. For the curve $x = 6t^2 - 3t^4$, $y = 8t^3$. Show that the radius of curvature at the point t is $6t(1+t^2)^2$.

The coordinates of the centre of curvature



Let the centre of curvature of the curve $y = f(x)$ corresponding to the point $P(x, y)$ be X and Y

$$X = ON$$
$$= OQ - NQ = OQ - MP$$
$$= x - PC \sin\psi$$
$$= x - \rho \sin\psi$$
$$Y = NC = NM - MC$$
$$= QP + PC \cos\psi$$
$$= y + \rho \cos\psi$$

If y_1 and y_2 denote $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

We know that $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$ and

$\tan\psi = y_1$

$\therefore \cos\psi = \frac{1}{\sqrt{1+y_1^2}}$ and $\sin\psi = \frac{y_1}{\sqrt{1+y_1^2}}$

$$\therefore X = x - \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{y_1}{(1+y_1^2)^{1/2}}$$

$$= x - \frac{y_1(1+y_1^2)}{y_2}$$

$$Y = y + \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{1}{(1+y_1^2)^{1/2}}$$

$$= y + \frac{1+y_1^2}{y_2}$$

Evolute: The locus of the centre of curvature for a curve is called the evolute of the curve

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Note: Let $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$

The coordinate of the centre of the curvature is

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$Y = y + \frac{(1+y_1^2)}{y_2}$$

Problems:

1. Find the co-ordinates of the centre of curvature of the curve $xy = 2$ at the point $(2,1)$.

Solution:

Let $xy = 2$

$$y = \frac{2}{x}$$

$$\frac{dy}{dx} = -\frac{2}{x^2} \text{ and } \frac{d^2y}{dx^2} = \frac{4}{x^3}.$$

$$\left(\frac{dy}{dx}\right)_{at\,(2,1)} = \frac{-1}{2} \cdot \left(\frac{d^2y}{dx^2}\right)_{(2,1)} = \frac{1}{2}.$$

The coordinates of the centre of curvature is

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$Y = y + \frac{1+y_1^2}{y_2}$$

$$\therefore X = 2 - \frac{\left(-\frac{1}{2}\right)\left(1+\left(\frac{-1}{2}\right)^2\right)}{\frac{1}{2}}$$

$$= 2 + \frac{5}{4} = \frac{13}{4}, = 3\cdot\frac{1}{4}.$$

$$Y = 1 + \frac{1+\left(-\frac{1}{2}\right)^2}{\frac{1}{2}}$$

$$= 1 + 2\left(\frac{5}{4}\right) = \frac{7}{2} = 3\cdot\frac{1}{2}$$

$\therefore$ The centre of curvature is

$$(X, Y) = \left(3\frac{1}{4}, 3\cdot\frac{1}{2}\right).$$

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Problems

1. Show that in the parabola $y^2 = 4ax$ at the point t, $\rho = -2a(1+t^2)^{3/2}$, $X = 2a + 3at^2$, $Y = -2at^3$. Deduce the equation of the evolute

Solution:

Let $x = at^2 \qquad y = 2at$

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{t}\right)$$

$$= \frac{d}{dt}\left(\frac{1}{t}\right) \cdot \frac{dt}{dx}$$

$$= \frac{-1}{t^2} \cdot \frac{1}{2at} = -\frac{1}{2at^3}$$

Radius of curvature

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

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$$\therefore \rho = \frac{\left[1+\left(\frac{1}{t}\right)^2\right]^{3/2}}{-\frac{1}{2at^3}} = \frac{\frac{(t^2+1)^{3/2}}{t^3}}{-1/2at^3}.$$

$$\therefore \rho = -2a(1+t^2)^{3/2}.$$

Co-ordinates of the centre of curvature

$$X = x - \frac{y_1(1+y_1^2)}{y_2} ; Y = y + \frac{1+y_1^2}{y_2}.$$

$$\therefore X = at^2 - \frac{\frac{1}{t}\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}}$$

$$= at^2 + 2a(1+t^2)$$

$$= 2a + 3at^2$$

$$Y = 2at + \frac{1+\frac{1}{t^2}}{-\frac{1}{2at^3}}$$

$$= 2at - 2at(1+t^2)$$

$$= -2at^3$$

$$\therefore X = 2a + 3at^2 \quad \text{——} (1)$$

$$Y = -2at^3 \quad \text{——} (2)$$

To find Evolute:

Eliminating t from x and y

$(1) \Rightarrow \quad 3at^2 = x - 2a$

$$t^2 = \frac{x-2a}{3a}$$

$$\Rightarrow t = \left(\frac{x-2a}{3a}\right)^{1/2}$$

$$(2) \Rightarrow y = -2a\left[\frac{x-2a}{3a}\right]^{3/2}$$

Squaring on both sides, we have

$$y^2 = 4a^2 \frac{(x-2a)^3}{(3a)^3}$$

$$27ay^2 = 4(x-2a)^3$$

$\therefore$ The locus of (x, y) is

$$27ay^2 = 4(x-2a)^3$$

The curve is called a semi-cubical parabola.

2. Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution

Any point on the ellipse is $(a\cos\theta, b\sin\theta)$

Let $x = a\cos\theta$; $y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta$; $\frac{dy}{d\theta} = b\cos\theta$.

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$

$\therefore \frac{dy}{dx} = -\frac{b}{a}\cot\theta$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{b}{a}\cot\theta\right)$

$= -\frac{b}{a}(-\operatorname{cosec}^2\theta) \cdot \frac{d\theta}{dx}$.

$= \frac{b}{a}\operatorname{cosec}^2\theta \cdot \frac{1}{-a\sin\theta}$

$= \frac{-b}{a^2\operatorname{cosec}^3\theta}$.

The co-ordinates of the centre of curvature.

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$Y = y + \frac{1+y_1^2}{y_2}$$

$$X = a\cos\theta - \frac{\left(\frac{-b}{a}\cot\theta\right)\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{\left(\frac{-b}{a^2}\operatorname{cosec}^3\theta\right)}$$

$$= a\cos\theta - a\cot\theta\sin^3\theta\left(1+\frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= a\cos\theta - a\sin^2\theta\cos\theta\left(1+\frac{b^2}{a^2}\cdot\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= a\cos\theta - a\sin^2\theta\cos\theta - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta(1-\sin^2\theta) - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$X = \left(\frac{a^2-b^2}{a}\right)\cos^3\theta$$

$$Y = b\sin\theta + \frac{1+\frac{b^2}{a^2}\cot^2\theta}{\frac{-b}{a^2}\operatorname{cosec}^2\theta}$$

$$= b\sin\theta - \frac{a^2}{b}\sin^2\theta\left(1+\frac{b^2}{a^2}\cot^2\theta\right)$$

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$$y = b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin^3\theta\cot^2\theta$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\cos^2\theta\sin\theta$$

$$= b\sin\theta(1-\cos^2\theta) - \frac{a^2}{b}\sin^3\theta$$

$$= b\sin^3\theta - \frac{a^2}{b}\sin^3\theta$$

$$y = \left(\frac{b^2-a^2}{b}\right)\sin^3\theta$$

$$\therefore x = \left(\frac{a^2-b^2}{a}\right)\cos^3\theta \quad \text{--- (1)}$$

$$y = \left(\frac{b^2-a^2}{b}\right)\sin^3\theta \quad \text{--- (2)}$$

To find the equation of the evolute.

Eliminate θ from x and y, we have.

$$(1) \Rightarrow ax = (a^2-b^2)\cos^3\theta$$

$$(ax)^{2/3} = (a^2-b^2)^{2/3}\cos^2\theta \quad \text{--- (3)}$$

$$(2) \Rightarrow by = (b^2-a^2)\sin^3\theta$$

$$(by)^{2/3} = (b^2-a^2)^{2/3}\sin^2\theta \quad \text{--- (4)}$$

$$\text{i.e., } (by)^{2/3} = (a^2-b^2)^{2/3}\sin^2\theta$$

$(3)+(4) \Rightarrow$

$$(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}[\sin^2\theta+\cos^2\theta]$$

$$\therefore (ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$$

$\therefore$ Locus of (x,y) is

$$(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}.$$

which is the equation of the evolute of the given ellipse.

---

3. Show that the evolute of the cycloid $x=a(\theta-\sin\theta)$ $y=a(1-\cos\theta)$ is another equal cycloid.

Solution:

Let $x=a(\theta-\sin\theta)$; $y=a(1-\cos\theta)$

$$\frac{dx}{d\theta}=a(1-\cos\theta) \; ; \; \frac{dy}{d\theta}=a\sin\theta.$$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\sin\theta}{a(1-\cos\theta)}=\frac{\sin\theta}{1-\cos\theta}$$

$$=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}=\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}.$$

$$\therefore \frac{dy}{dx}=\cot\frac{\theta}{2}.=y_1$$

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$$y_2 = \frac{d^2y}{dx^2} = -\operatorname{cosec}^2\left(\frac{\theta}{2}\right)\left(\frac{1}{2}\right)\left(\frac{d\theta}{dx}\right).$$

$$= \frac{-1}{2}\operatorname{cosec}^2\frac{\theta}{2} \cdot \frac{1}{a(1-\cos\theta)}$$

$$= \frac{-\operatorname{cosec}^2\frac{\theta}{2}}{2a\left(2\sin^2\frac{\theta}{2}\right)}$$

$$\therefore y_2 = \frac{-1}{4a\sin^4\frac{\theta}{2}}.$$

The co-ordinates of the centre of curvature.

$$X = x - \frac{y_1(1+y_1^2)}{y_2}; \quad Y = y + \frac{1+y_1^2}{y_2}.$$

$$X = a(\theta - \sin\theta) - \frac{\frac{\cos\theta/2}{\sin\theta/2}\left(1+\frac{\cos^2\theta/2}{\sin^2\theta/2}\right)}{\left(-\frac{1}{4a\sin^4\theta/2}\right)}$$

$$= a(\theta - \sin\theta) + 4a\sin^4\frac{\theta}{2} \cdot \frac{\cos\theta/2}{\sin\theta/2}\left(\frac{1}{\sin^2\theta/2}\right)$$

$$= a(\theta - \sin\theta) + 4a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= a(\theta - \sin\theta) + 2a\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$= a(\theta - \sin\theta) + 2a\sin\theta$$

$$= a(\theta + \sin\theta)$$

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$$y = a(1-\cos\theta) + \frac{1+\cot^2\frac{\theta}{2}}{\left(\frac{-1}{4a\sin^4\frac{\theta}{2}}\right)}$$

$$= a(1-\cos\theta) - 4a\sin^4\frac{\theta}{2}\left(1+\frac{\cos^2\theta/2}{\sin^2\theta/2}\right)$$

$$= a(1-\cos\theta) - 4a\sin^4\frac{\theta}{2}\left(\frac{1}{\sin^2\theta/2}\right)$$

$$= a(1-\cos\theta) - 4a\sin^2\frac{\theta}{2}.$$

$$= a(1-\cos\theta) - 4a\left(\frac{1-\cos\theta}{2}\right)$$

$$= a(1-\cos\theta) - 2a(1-\cos\theta)$$

$$= a(1-\cos\theta-2+2\cos\theta)$$

$$= a(-1+\cos\theta)$$

$$= -a(1-\cos\theta)$$

$\therefore x = a(\theta+\sin\theta)\ ;\ y = -a(1-\cos\theta)$

$\therefore$ The locus of (x,y) is

$x = a(\theta+\sin\theta)\ ;\ y = -a(1-\cos\theta)$

This is also a cycloid.

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4. Show that the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$.

Solution:

The parametric Equation of the hyperbola are $x = a\sec\theta$; $y = b\tan\theta$.

$\frac{dx}{d\theta} = a\sec\theta \cdot \tan\theta$. $\frac{dy}{d\theta} = b\sec^2\theta$.

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^2\theta}{a\sec\theta\cdot\tan\theta} = \frac{b}{a\sin\theta}$$

$$\therefore y_1 = \frac{b}{a}\operatorname{cosec}\theta.$$

$$y_2 = \frac{d^2y}{dx^2} = +\frac{b}{a}\left(-\operatorname{cosec}\theta\cot\theta\right)\cdot\frac{d\theta}{dx}$$

$$= \frac{-b}{a}\operatorname{cosec}\theta\cdot\cot\theta\cdot\frac{1}{a\sec\theta\cdot\tan\theta}$$

$$= -\frac{b}{a^2}\cdot\frac{\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}}$$

$$= -\frac{b}{a^2}\cdot\cot\theta\cdot\cot\theta\cdot\cot\theta.$$

$$= -\frac{b}{a^2}\cot^3\theta.$$

(41)

Let (X, Y) be the coordinates of the curvature.

then

$$X = x - \frac{y_1(1+y_1^2)}{y_2} \; ; \; Y = y + \frac{1+y_1^2}{y_2}.$$

$$X = a\sec\theta - \frac{\frac{b}{a}\operatorname{cosec}\theta\left(1 + \frac{b^2}{a^2}\operatorname{cosec}^2\theta\right)}{-\frac{b}{a^2}\cot^3\theta}.$$

$$= a\sec\theta + \frac{1}{a}\,\frac{\operatorname{cosec}\theta\,(a^2 + b^2\operatorname{cosec}^2\theta)}{\cot^3\theta}$$

$$= a\sec\theta + \frac{1}{a}\cdot\frac{1}{\sin\theta}\cdot\frac{\sin^3\theta}{\cos^3\theta}\left(a^2 + b^2\frac{1}{\sin^2\theta}\right)$$

$$= a\sec\theta + \frac{1}{a}\,\frac{\sin^2\theta}{\cos^3\theta}\left(a^2 + \frac{b^2}{\sin^2\theta}\right)$$

$$= a\sec\theta + \frac{a\sin^2\theta}{\cos^3\theta} + \frac{b^2}{a\cos^3\theta}$$

$$= a\sec\theta + a\cdot\frac{1-\cos^2\theta}{\cos^3\theta} + \frac{b^2}{a}\cdot\sec^3\theta.$$

$$= a\sec\theta + a\cdot\sec^3\theta - a\sec\theta + \frac{b^2}{a}\sec^3\theta$$

$$X = \frac{(a^2+b^2)}{a}\sec^3\theta$$

$$aX = (a^2+b^2)\sec^3\theta$$

$$(aX)^{2/3} = (a^2+b^2)^{2/3}\sec^2\theta \longrightarrow (1)$$

(42)

$$y = b\tan\theta + \frac{1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta}{-\frac{b}{a^2}\cot^3\theta}$$

$$= b\tan\theta - \frac{a^2+b^2\cdot\frac{1}{\sin^2\theta}}{b\cdot\frac{\cos^3\theta}{\sin^3\theta}}$$

$$= b\tan\theta - \frac{a^2}{b}\cdot\tan^3\theta - b\tan\theta\sec^2\theta.$$

$$by = b^2\tan\theta - a^2\tan^3\theta - b^2\tan\theta(1+\tan^2\theta)$$

$$by = -(a^2+b^2)\tan^3\theta.$$

$$\therefore (by)^{2/3} = (a^2+b^2)^{2/3}\tan^2\theta \quad \text{——} \ (2)$$

$$(1)-(2) \Rightarrow (ax)^{2/3}-(by)^{2/3} = (a^2+b^2)^{2/3}(\sec^2\theta-\tan^2\theta)$$

$$= (a^2+b^2)^{2/3}.$$

$\therefore$ The Locus of (x,y) is

$$(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}.$$

which gives the Equation of the Evolute of the given hyperbola.

5. Show that the Equation of the Evolute of $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$, $y = a \sin t$ is $y = a \cosh \frac{x}{a}$.

Solution:

Let $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan t/2} \cdot \sec^2\left(\frac{t}{2}\right)\left(\frac{1}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{1}{2} \cdot \frac{\cos t/2}{\sin t/2} \cdot \frac{1}{\cos^2 t/2}\right]$$

$$= a\left[-\sin t + \frac{1}{2 \sin t/2 \cos t/2}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$= a\left[\frac{1 - \sin^2 t}{\sin t}\right] = a \cdot \frac{\cos^2 t}{\sin t}$$

Let $y = a \sin t$

$$\frac{dy}{dt} = a \cos t$$

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \cos t}{\left(a \frac{\cos^2 t}{\sin t}\right)} = \frac{\sin t}{\cos t} = \tan t$$

$\therefore y_1 = \tan t$

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$$y_2 = \frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx}.$$

$$y_2 = \sec^2 t \cdot \frac{\sin t}{a\cos^2 t} = \frac{1}{a} \cdot \frac{\sin t}{\cos^4 t}.$$

$$\therefore y_2 = \frac{1}{a} \cdot \frac{\sin t}{\cos^4 t}.$$

Let (X, Y) be the coordinates of the centre of curvature, then,

$$X = x - \frac{y_1(1+y_1^2)}{y_2} \; ; \; Y = y + \frac{1+y_1^2}{y_2}.$$

$$X = a\left(\cos t + \log \tan \tfrac{t}{2}\right) - \frac{\frac{\sin t}{\cos t}\left(1 + \frac{\sin^2 t}{\cos^2 t}\right)}{\left(\frac{\sin t}{a\cos^4 t}\right)}.$$

$$= a\left(\cos t + \log \tan \tfrac{t}{2}\right) - \frac{\sin t}{\cos t} \cdot \frac{1}{\cos^2 t} \cdot \frac{a\cos^4 t}{\sin t}$$

$$= a\left(\cos t + \log \tan \tfrac{t}{2}\right) - a\cos t.$$

$$= a\left[\cos t + \log \tan \tfrac{t}{2} - \cos t\right]$$

$$X = a \log \tan \tfrac{t}{2}.$$

$$Y = a\sin t + \frac{1 + \frac{\sin^2 t}{\cos^2 t}}{\frac{\sin t}{a\cos^4 t}}.$$

$$= a\sin t + \frac{1}{\cos^2 t} \cdot \frac{a\cos^4 t}{\sin t}.$$

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(45)

$$y = a\sin t + \frac{a\cos^2 t}{\sin t}$$

$$= a\left[\frac{\sin^2 t + \cos^2 t}{\sin t}\right] = \frac{a}{\sin t}$$

$$\therefore x = a\log\tan t/2 \;\text{—}\; (1) \;;\; y = \frac{a}{\sin t} \;\text{—}\; (2)$$

$$(1) \Rightarrow \frac{x}{a} = \log\left(\tan\frac{t}{2}\right)$$

(ie) $e^{x/a} = \tan(t/2)$

$|||^{ly}$ $e^{-x/a} = \frac{1}{\tan(t/2)}$

$$\left[\therefore \frac{e^{x/a} + e^{-x/a}}{2} = \cosh\frac{x}{a} \rightarrow \text{formula}\right]$$

$$\therefore \frac{\tan t/2 + \frac{1}{\tan t/2}}{2} = \cosh\frac{x}{a}.$$

$$\frac{1 + \tan^2 t/2}{2\tan t/2} = \cosh\frac{x}{a}.$$

$$\frac{1}{\left[\frac{2\tan t/2}{1+\tan^2 t/2}\right]} = \cosh x/a.$$

i.e., $\frac{1}{\sin t} = \cosh\frac{X}{a}$.

(2) $\Rightarrow \frac{Y}{a} = \frac{1}{\sin t}$

$\therefore \frac{Y}{a} = \cosh\frac{X}{a}$.

$\Rightarrow Y = a\cosh\frac{X}{a}$

The locus of (X, Y) is $Y = a\cosh(\frac{X}{a})$.

$\therefore$ The evolute of the given curve is $y = a\cosh\frac{x}{a}$.

which is the equation of catenary.

---

6. Show that the equation of the evolute of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is

$(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$.

Solution:

Given curve $x^{2/3} + y^{2/3} = a^{2/3}$.

The parametric equations of the given curve are

$x = a\cos^3\theta$, $y = a\sin^3\theta$.

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \; ; \; \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta.$$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta.$$

$$y_2 = \frac{d^2y}{dx^2} = -\sec^2\theta \cdot \frac{d\theta}{dx}$$

$$= -\sec^2\theta \cdot \frac{1}{-3a\cos^2\theta\sin\theta}$$

$$= \frac{1}{3a\sin\theta\cos^4\theta}$$

Let (X, Y) be the coordinates of the centre of curvature. Then

$$X = x - \frac{y_1(1+y_1^2)}{y_2} \; ; \; Y = y + \frac{1+y_1^2}{y_2}.$$

$$X = a\cos^3\theta - \frac{-\tan\theta(1+\tan^2\theta)}{\left[\frac{1}{3a\sin\theta\cos^4\theta}\right]}$$

$$= a\cos^3\theta + 3a\sin\theta\cos^4\theta \cdot \frac{\sin\theta}{\cos\theta}\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right)$$

$$= a\cos^3\theta + 3a\sin^2\theta\cos\theta.$$

$$\therefore X = a\cos^3\theta + 3a\sin^2\theta\cos\theta \longrightarrow (1)$$

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$$y = a\sin^3\theta + \frac{1+\tan^2\theta}{\left(\frac{1}{3a\sin\theta\cos^4\theta}\right)}$$

$$= a\sin^3\theta + 3a\sin\theta\cos^4\theta \cdot \left(1+\frac{\sin^2\theta}{\cos^2\theta}\right)$$

$$= a\sin^3\theta + 3a\sin\theta\cos^2\theta$$

$$\therefore y = a\sin\theta\,(\sin^2\theta + 3\cos^2\theta) \longrightarrow (2)$$

To find the equation of the Evolute,

Eliminate θ from (1) and (2), we have,

$$x+y = a\cos\theta\,(\cos^2\theta + 3\sin^2\theta) + a\sin\theta\,(\sin^2\theta + 3\cos^2\theta)$$

$$= a\cos^3\theta + 3a\cos\theta\sin^2\theta + a\sin^3\theta + 3a\sin\theta\cos^2\theta$$

$$= a\left[\cos^3\theta + \sin^3\theta + 3\cos\theta\sin^2\theta + 3\sin\theta\cos^2\theta\right].$$

$$= a(\cos\theta + \sin\theta)^3.$$

$$(x+y)^{2/3} = a^{2/3}\cdot(\cos\theta + \sin\theta)^2$$

$$= a^{2/3}(\cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta)$$

$$= a^{2/3}(1 + 2\cos\theta\sin\theta)$$

$$\text{|||}^{ly}\ (x-y)^{2/3} = a^{2/3}(1 - 2\cos\theta\sin\theta)$$

$$\therefore (x+y)^{2/3} + (x-y)^{2/3} = a^{2/3}(1+1) = 2a^{2/3}$$

$\therefore$ The locus of (x, y) is

$$(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$$

which is the evolute of the curve.

Note:

Curve	Cartesian Equation	Parametric Equation.
1. Parabola	1. $y^2 = 4ax$ 2. $x^2 = 4ay$	1. $x = at^2,\ y = 2at$ 2. $x = 2at,\ y = at^2$
2. Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos\theta,\ y = b\sin\theta$
3. Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sec\theta$ $y = b\tan\theta$
4. Rectangular hyperbola	$xy = c^2$	$x = ct$ $y = c/t$
5. Astroid	$x^{2/3} + y^{2/3} = a^{2/3}$	$x = a\cos^3\theta$ $y = a\sin^3\theta$

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Home work:

1\. Find the coordinates of the centres of curvature at given points on the curves.

(i) $y = x^2$; $(\frac{1}{2}, \frac{1}{4})$

(ii) $xy = c^2$; (c, c)

2\. Show that in parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $X + Y = 3(x+y)$.

3\. Given the curve $x = 3\cos t + \cos 3t$, $y = 3\sin t - \sin 3t$ find the parametric equation of the Evolute. Find the centre of curvature for $t = 0$.

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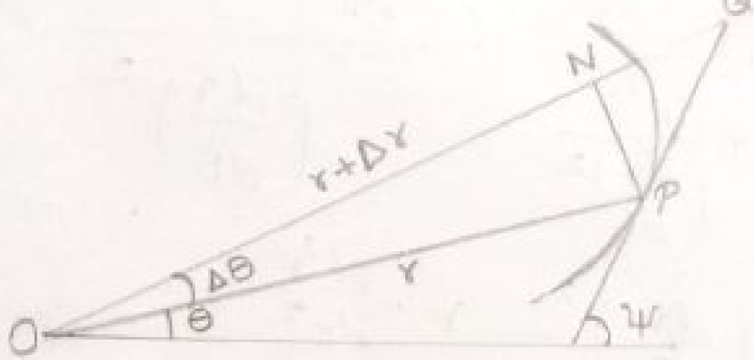
Radius of curvature when the curve is given in polar co-ordinate.



Let us assume that the Equation of the curve in polar coordinates be $r = f(\theta)$.

In the figure,

$$\psi = \theta + \phi$$

$$\frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}$$

we have proved that

$$\tan\phi = r \cdot \frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)}$$

Diff. w.r.to θ, we get

$$\sec^2\phi \frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r \cdot \frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

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$$\frac{d\phi}{d\theta} = \frac{1}{\sec^2\phi} \cdot \frac{\left(\frac{dr}{d\theta}\right)^2 - r \cdot \frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

$$= \frac{1}{1 + \frac{r^2}{\left(\frac{dr}{d\theta}\right)^2}} \cdot \frac{\left(\frac{dr}{d\theta}\right)^2 - r \cdot \frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

$$\frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r \cdot \frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

$$\frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}.$$

$$= 1 + \frac{\left(\frac{dr}{d\theta}\right)^2 - r \cdot \frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}.$$

$$= \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}.$$

We know that $\frac{ds}{d\theta} = \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{1/2}$

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Radius of Curvature

$$\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta} \cdot \frac{d\theta}{d\psi}$$

$$\rho = \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{1/2} \cdot \frac{\left(\frac{dr}{d\theta}\right)^2 + r^2}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \cdot \frac{d^2r}{d\theta^2}}.$$

$$\rho = \frac{\left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \cdot \frac{d^2r}{d\theta^2}}.$$

Problems:

1. Find the radius of curvature of the cardioid $r = a(1 - \cos\theta)$.

Solution:

Let $r = a(1 - \cos\theta)$.

$$\frac{dr}{d\theta} = a \sin\theta$$

$$\frac{d^2r}{d\theta^2} = a \cos\theta$$

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Radius of curvature is

$$\rho = \frac{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{3/2} = \left[a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}$$

$$= \left[a^2(1+\cos^2\theta - 2\cos\theta) + a^2\sin^2\theta\right]^{3/2}$$

$$= \left[2a^2 - 2a^2\cos\theta\right]^{3/2}$$

$$= \left[2a^2(1-\cos\theta)\right]^{3/2}$$

$$= \left[2a^2 \cdot 2\sin^2\theta/2\right]^{3/2}$$

$$= (2a\sin\theta/2)^3$$

$$= 8a^3\sin^3\theta/2.$$

$$r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}$$

$$= a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2\cos\theta(1-\cos\theta)$$

$$= a^2 + a^2\cos^2\theta - 2a^2\cos\theta + 2a^2\sin^2\theta - a^2\cos\theta + a^2\cos^2\theta$$

$$= a^2 + 2a^2 - 3a^2\cos\theta = 3a^2(1-\cos\theta)$$

$$= 3a^2(2\sin^2\theta/2)$$

$$= 6a^2\sin^2\theta/2.$$

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$$\therefore \rho = \frac{8a^3 \sin^3 \frac{\theta}{2}}{6a^2 \sin^2 \theta/2}$$

$$\rho = \frac{4}{3} a . \sin \frac{\theta}{2}.$$

$$\therefore \rho = \frac{2}{3}\left(2a \sin \frac{\theta}{2}\right)$$

$\because r = a(1-\cos\theta)$

$2ar = 2a^2 \cdot 2 \cdot \sin^2 \theta/2$

$= 2^2 a^2 \sin^2 \theta/2$

$$\therefore \rho = \frac{2}{3}\sqrt{2ar}.$$

$\sqrt{2ar} = 2a \sin \theta/2.$

---

2. Show that the radius of curvature of the curve $r^n = a^n \cos n\theta$ is $\dfrac{a^n r^{-n+1}}{n+1}$.

Proof:

Let $r^n = a^n \cos n\theta$.

Taking logarithms on both sides, and differentiating, we get,

$$\log r^n = \log(a^n \cos n\theta)$$

$$n \log r = \log a^n + \log \cos n\theta$$

$$n \log r = n \log a + \log \cos n\theta.$$

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$$n \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}(-\sin n\theta) \cdot n$$

$$n \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = -\tan n\theta \cdot n$$

$$\therefore \frac{dr}{d\theta} = -r\tan n\theta.$$

$$\frac{d^2r}{d\theta^2} = -r \cdot \left[\sec^2 n\theta (n)\right] - \frac{dr}{d\theta} \cdot \tan n\theta$$

$$= -nr\sec^2 n\theta + r\tan^2 n\theta$$

Radius of curvature is

$$\rho = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 - r \cdot \frac{d^2r}{d\theta^2} + 2\left(\frac{dr}{d\theta}\right)^2}.$$

$$\therefore \rho = \frac{\left[r^2 + (-r\tan n\theta)^2\right]^{3/2}}{r^2 - r(-nr\sec^2 n\theta + r\tan^2 n\theta) + 2(-r\tan n\theta)^2}$$

$$= \frac{(r^2 + r^2\tan^2 n\theta)^{3/2}}{r^2 + nr^2\sec^2 n\theta - r^2\tan^2 n\theta + 2r^2\tan^2 n\theta}.$$

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$$\rho = \frac{(r^2)^{3/2}(1+\tan^2 n\theta)^{3/2}}{r^2 + nr^2\sec^2 n\theta - r^2\tan^2 n\theta + 2r^2\tan^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{r^2(1+\tan^2 n\theta) + nr^2\sec^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{r^2\sec^2 n\theta + nr^2\sec^2 n\theta}$$

$$= \frac{r^3\sec^3 n\theta}{r^2\sec^2 n\theta(1+n)}$$

$$= \frac{r\sec n\theta}{1+n}$$

$$= \frac{r}{(n+1)}\left(\cdot\frac{1}{\cos n\theta}\right)$$

$$= \frac{r}{(n+1)} \cdot \frac{1}{\left(\frac{r^n}{a^n}\right)} \qquad \left(\begin{array}{l} r^n = a^n\cos n\theta \\ \therefore \cos n\theta = \frac{r^n}{a^n} \end{array} \right.$$

$$= \frac{a^n}{(n+1)r^{n-1}}$$

$$\rho = \frac{a^n r^{-n+1}}{n+1}.$$

Note:

(i) putting $n = 2$ we get Bernoulli's lemniscate $\therefore \rho = \frac{a^2}{3r}$.

(ii) when $n = -2$, we have a rectangular hyperbola $\therefore \rho = \frac{r^3}{a^2}$.

(iii) when $n = 1/2$, we get cardioid $\therefore \rho = \frac{2}{3}\sqrt{ar}$.

(iv) when $n = -1/2$ we get a parabola $\therefore \rho = \frac{2r^{3/2}}{\sqrt{a}}$.

(v) when $n = 1$, we get a circle $\therefore \rho = a/2$.

Home work..

Find the radius of curvature of the following function.

		Answers
(1)	$r = a(1 + \cos\theta)$ $\longrightarrow$	$4a/3$
(2)	$r = a\sin n\theta$ $\longrightarrow$	$na/2$
(3)	$r = e^{\theta}$ $\longrightarrow$	$\sqrt{2}\, r$
(4)	$r = a\sin^3\left(\frac{\theta}{3}\right)$. $\rightarrow$	$\frac{3a}{4}\sin^2\frac{\theta}{3}$.

UNIT – III

TRIGONOMETRY

Expansions of $\cos^n\theta$ and $\sin^n\theta$

Method: Let $x = \cos\theta + i\sin\theta$.

Then we have,

$x = \cos\theta + i\sin\theta \qquad x^n = \cos n\theta + i\sin n\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta \qquad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$.

By addition and subtraction, we get

$2\cos\theta = x + \frac{1}{x} \qquad x^n + \frac{1}{x^n} = 2\cos n\theta$

$2i\sin\theta = x - \frac{1}{x} \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$.

To express $\cos^n\theta$ in $\cos n\theta$, $\cos(n-2)\theta$, $\cos(n-4)\theta$, etc.

step. 1. Raise $2\cos\theta = x + \frac{1}{x}$ to power n.

step 2. Express RHS in $x^n + \frac{1}{x^n}$,

$x^{n-2} + \frac{1}{x^{n-2}}$, ...

step 3. Use $x^n + \frac{1}{x^n} = 2\cos n\theta$.

To Expand $\sin^n\theta$ in Cosines of multiple angles of θ if n is even, otherwise in sines of multiple angles of θ.

Case (i): n is even.

Step-1. Raise $2i\sin\theta = x - \frac{1}{x}$ to power n.

Step-2. Express RHS in $x^n + \frac{1}{x^n}$, $x^{n-2} + \frac{1}{x^{n-2}}, \dots$

Step-3. Use $x^n + \frac{1}{x^n} = 2\cos n\theta$.

Case (ii): n is odd.

Step 1. Raise $2i\sin\theta = x - \frac{1}{x}$ to power n.

Step 2. Express RHS in $x^n - \frac{1}{x^n}$, $x^{n-2} - \frac{1}{x^{n-2}}, \dots$

Step 3 Use $x^n - \frac{1}{x^n} = 2i\sin n\theta$.

Remark

In the expansion of the product $\cos^m\theta\sin^n\theta$. the working will be simpler if we rewrite RHS as follows

Suppose $m < n$, Then

$$(2\cos\theta)^m(2i\sin\theta)^n = (2\cos\theta)^m(2i\sin\theta)^m(2i\sin\theta)^{n-m}.$$

$$2^{m+n} i^{n} \cos^{m}\theta \sin^{n}\theta = \left(x+\frac{1}{x}\right)^{m}\left(x-\frac{1}{x}\right)^{m}\left(x-\frac{1}{x}\right)^{n-m}$$

$$= \left(x^{2}-\frac{1}{x^{2}}\right)^{m}\left(x-\frac{1}{x}\right)^{n-m}.$$

The method is similar when $n<m$

Pascal's Triangle:

The pascal's triangle, given below, gives the binomial coefficients of $(1+x)^n$, where $n=1,2,\ldots 10$.

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

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Examples:

1. show that $2^5\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$

solution:

Let $x = \cos\theta + i\sin\theta$

Then $2\cos\theta = x + \frac{1}{x}$,

$2\cos n\theta = x^n + \frac{1}{x^n}$.

$(2\cos\theta)^6 = (x + \frac{1}{x})^6$

$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$.

Pascal's Triangle.

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

$\therefore 2^6\cos^6\theta = (x^6 + \frac{1}{x^6}) + 6(x^4 + \frac{1}{x^4}) + 15(x^2 + \frac{1}{x^2}) + 20.$

$2^6\cos^6\theta = 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20.$

$2^5\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10.$

(5)

Home work

2. show that

$$-2^5 \sin^6\theta = \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10$$

3. show that

$$2^6 \cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta.$$

4. show that

$$-2^6 \sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$$

Solution

Let $x = \cos\theta + i\sin\theta$. Then we have

$$2\cos\theta = x + \frac{1}{x} \qquad x^n + \frac{1}{x^n} = 2\cos n\theta$$

$$2i\sin\theta = x - \frac{1}{x} \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta.$$

Pascal's Triangle

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

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(b)

$(2i\sin\theta)^7 = \left(x - \frac{1}{x}\right)^7$

$= x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}$

$= \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right)$

$- 35\left(x - \frac{1}{x}\right)$

$= (2i\sin 7\theta) - 7(2i\sin 5\theta) + 21(2i\sin 3\theta)$

$- 35(2i\sin\theta$

$\div$ by $2i$,

$-2^6\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta$

$- 35\sin\theta$

Home work

1. Prove the results in the following sums.

1. $2^3\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3.$

2. $2^4\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta.$

3. $2^7\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta$
$+ 56\cos 2\theta + 35.$

4. $2^3\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3$

5. $2^4\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta.$

Problems.

1. Show that $2^{10}\cos^5\theta\sin^6\theta = \cos 11\theta - \cos 9\theta$
$- 5\cos 7\theta + 5\cos 5\theta + 10\cos 3\theta - 10\cos\theta$.

Solution:

Let $x = \cos\theta + i\sin\theta$. Then we have

$2\cos\theta = x + \frac{1}{x}$ $\quad x^n + \frac{1}{x^n} = 2\cos n\theta$

$2i\sin\theta = x - \frac{1}{x}$ $\quad x^n - \frac{1}{x^n} = 2i\sin n\theta$.

Pascal's Triangle.

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Now in the expansion of $(1-x)^5$ the coefficients are

$5C_0 \quad -5C_1 \quad 5C_2 \quad -5C_3 \quad 5C_4 \quad -5C_5$

(or) $1 \quad -5 \quad 10 \quad -10 \quad 5 \quad -1$

$$(2\cos\theta)^5(2i\sin\theta)^6 = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^6$$

$$= \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^5\cdot\left(x-\frac{1}{x}\right)$$

$$2^{11} i^6 \cos^5\theta\sin^6\theta = \left(x^2-\frac{1}{x^2}\right)^5\cdot\left(x-\frac{1}{x}\right)$$

(8)

$$-2^{11}\cos^5\theta\,\sin^6\theta$$

$$=\left(x^2-\frac{1}{x^2}\right)^5\left(x-\frac{1}{x}\right)$$

$$=\left\{(x^2)^5-5\cdot(x^2)^4\left(\frac{1}{x^2}\right)+10(x^2)^3\cdot\left(\frac{1}{x^2}\right)^2\right.$$

$$\left.-10\cdot(x^2)^2\cdot\left(\frac{1}{x^2}\right)^3+5(x^2)^1\left(\frac{1}{x^2}\right)^4-(x^2)^0\cdot\left(\frac{1}{x^2}\right)^5\right\}$$

$$\cdot\left\{\left(x-\frac{1}{x}\right)\right\}$$

$$=\left(x^{10}-5x^6+10x^2-\frac{10}{x^2}+\frac{5}{x^6}-\frac{1}{x^{10}}\right)\left(x-\frac{1}{x}\right)$$

$$=\left(x^{11}-5x^7+10x^3-\frac{10}{x}+\frac{5}{x^5}-\frac{1}{x^9}\right)$$

$$+\left(-x^9+5x^5-10x+\frac{10}{x^3}-\frac{5}{x^7}+\frac{1}{x^{11}}\right)$$

$$=\left(x^{11}+\frac{1}{x^{11}}\right)-\left(x^9+\frac{1}{x^9}\right)-5\left(x^7+\frac{1}{x^7}\right)$$

$$+5\left(x^5+\frac{1}{x^5}\right)+10\left(x^3+\frac{1}{x^3}\right)-10\left(x+\frac{1}{x}\right).$$

$$=2\cos 11\theta-2\cos 9\theta-5(2\cos 7\theta)$$

$$+5(2\cos 5\theta)+10(2\cos 3\theta)-10(2\cos\theta)$$

$$\therefore -2^{10}\cos^5\theta\sin^6\theta=\cos 11\theta-\cos 9\theta-5\cos 7\theta$$

$$+5\cos 5\theta+10\cos 3\theta-10\cos\theta.$$

2. Show that $-2^{11}\cos^5\sin^7\theta = \sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta$.

Solution:

Let $x = \cos\theta + i\sin\theta$. Then we have

$2\cos\theta = x + \frac{1}{x}$ $\quad x^n + \frac{1}{x^n} = 2\cos n\theta$

$2i\sin\theta = x - \frac{1}{x}$ $\quad x^n - \frac{1}{x^n} = 2i\sin n\theta$.

$$(2\cos\theta)^5(2i\sin\theta)^7 = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^7$$

$$= \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^2$$

$$= \left(x^2-\frac{1}{x^2}\right)^5\left(x-\frac{1}{x}\right)^2$$

$$2^{12} i^7 \cos^5\theta \sin^7\theta$$

$$= \left\{(x^2)^5 - 5(x^2)^4\left(\frac{1}{x^2}\right) + 10(x^2)^3\left(\frac{1}{x^2}\right)^2 - 10(x^2)^2\left(\frac{1}{x^2}\right)^3 + 5(x^2)^1\left(\frac{1}{x^2}\right)^4 + (x^2)^0\left(\frac{1}{x^2}\right)^5\right\}$$

$$\cdot\left\{x^2 - 2 + \frac{1}{x^2}\right\}$$

[In the expansion of $(1-x)^5$, the coefficients are

1 $\quad -5 \quad 10 \quad -10 \quad 5 \quad -1$

(or) ${}^5C_0 \quad -{}^5C_1 \quad {}^5C_2 \quad -{}^5C_3 \quad {}^5C_4 \quad -{}^5C_5$

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(10)

$2^{12} i^7 \cos^5\theta \sin^7\theta$ -

$= x^{12} - 5x^8 + 10x^4 - 10 + \frac{5}{x^4} - \frac{1}{x^8}$

$- 2x^{10} + 10x^6 - 20x^2 + \frac{20}{x^2} - \frac{10}{x^6} + \frac{2}{x^{10}}.$

$+ x^8 - 5x^4 + 10 + \frac{10}{x^4} + \frac{5}{x^8} - \frac{1}{x^{12}}.$

$= \left(x^{12} - \frac{1}{x^{12}}\right) - 2\left(x^{10} - \frac{1}{x^{10}}\right)$

$- 4\left(x^8 - \frac{1}{x^8}\right) + 10\left(x^6 - \frac{1}{x^6}\right)$

$+ 5\left(x^4 - \frac{1}{x^4}\right) - 20\left(x^2 - \frac{1}{x^2}\right)$

$= 2i\sin 12\theta - 2(2i\sin 10\theta) - 4(2i\sin 8\theta)$

$+ 10(2i\sin 6\theta) + 5(2i\sin 4\theta) - 20(2i\sin 2\theta)$

$\div$ by $2i$, we have.

$-2^{11}\cos^5\theta\sin^7\theta = \sin 12\theta - 2\sin 10\theta$

$- 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta.$

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(11)

3. Find the Expansion of $\cos^9\theta$.

Solution:

Let $x = \cos\theta + i\sin\theta$,

$2\cos\theta = x + \frac{1}{x}$, $x^n + \frac{1}{x^n} = 2\cos n\theta$

$2i\sin\theta = x - \frac{1}{x}$, $x^n - \frac{1}{x^n} = 2i\sin n\theta$

In the expansion of $(1+x)^9$, the coefficients are

$9C_0$ $9C_1$ $9C_2$ $9C_3$ $9C_4$ $9C_5$ $9C_6$

$9C_7$ $9C_8$ $9C_9$.

(or) 1 9 36 84 126 126 84 36 9 1

$$(2\cos\theta)^9 = \left(x + \frac{1}{x}\right)^9$$

$$= x^9 + 9x^7 + 36x^5 + 84x^3 + 126x + \frac{126}{x} + \frac{84}{x^3} + \frac{36}{x^5} + \frac{9}{x^7} + \frac{1}{x^9}.$$

$$= \left(x^9 + \frac{1}{x^9}\right) + 9\left(x^7 + \frac{1}{x^7}\right) + 36\left(x^5 + \frac{1}{x^5}\right) + 84\left(x^3 + \frac{1}{x^3}\right) + 126\left(x + \frac{1}{x}\right)$$

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$$2^9\cos^9\theta = 2\cos 9\theta + 9(2\cos 7\theta) + 36(2\cos 5\theta) + 84(2\cos 3\theta) + 126(2\cos\theta)$$

$\div$ by 2, we have,

$$2^8\cos^9\theta = \cos 9\theta + 9\cos 7\theta + 36\cos 5\theta + 84\cos 3\theta + 126\cos\theta.$$

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4. Find the Expansion of $\sin^{10}\theta$.

Solution:

Let $x = \cos\theta + i\sin\theta$

$2\cos\theta = x + \frac{1}{x}$ $\quad$ $x^n + \frac{1}{x^n} = 2\cos n\theta$

$2i\sin\theta = x - \frac{1}{x}$ $\quad$ $x^n - \frac{1}{x^n} = 2i\sin n\theta.$

In the expansion of $(1-x)^{10}$, the coefficients are,

$10C_0 \; -10C_1 \; 10C_2 \; -10C_3 \; 10C_4 \; -10C_5$

$10C_6 \; -10C_7 \; 10C_8 \; -10C_9 \; 10C_{10}$

(or) $1 \; -10 \; 45 \; -120 \; 210 \; -252$

$210 \; -120 \; 45 \; -10 \; 1.$

(13)

$$(2i\sin\theta)^{10} = x^{10} - 10x^{8} + 45x^{6} - 120x^{4} - 210x^{2} - 252 + \frac{210}{x^2} - \frac{120}{x^4} + \frac{45}{x^6} - \frac{10}{x^8} + \frac{1}{x^{10}}.$$

$$= \left(x^{10} + \frac{1}{x^{10}}\right) - 10\left(x^{8} + \frac{1}{x^{8}}\right) + 45\left(x^{6} + \frac{1}{x^{6}}\right) - 120\left(x^{4} + \frac{1}{x^{4}}\right) + 210\left(x^{2} + \frac{1}{x^{2}}\right) - 252.$$

$$= 2\cos 10\theta - 10(2\cos 8\theta) + 45(2\cos 6\theta) - 120(2\cos 4\theta) + 210(2\cos 2\theta) - 126(2).$$

$\div$ by 2,

$$-2^{9}\sin^{10}\theta = \cos 10\theta - 10\cos 8\theta + 45\cos 6\theta - 120\cos 4\theta + 210\cos 2\theta - 126.$$

(14)

Home work:

1. Find the Expansion of the following.

(i) $\sin^9\theta$ (ii) $\cos^{10}\theta$ (iii) $\sin^8\theta$

Answers

(i) $2^8\sin^9\theta = \sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta$

(ii) $2^9\cos^{10}\theta = \cos 10\theta + 10\cos 8\theta + 45\cos 6\theta + 120\cos 4\theta + 210\cos 2\theta + 126.$

(iii) $2^7\sin^8\theta = \cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35.$

Problems:

1. Show that $-2^6\cos^4\theta\cdot\sin^3\theta = \sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta$.

Solution.

Let $x = \cos\theta + i\sin\theta$ then we have

$2\cos\theta = x + \frac{1}{x}$ $\quad x^n + \frac{1}{x^n} = 2\cos n\theta$

$2i\sin\theta = x - 1/x$ $\quad x^n - \frac{1}{x^n} = 2i\sin n\theta$.

Now in the expansion of $(1-x)^3$, the coefficients are

$3C_0 \quad 3C_1 \quad 3C_2 \quad 3C_3$

(or) $+1 \quad -3 \quad +3 \quad -1$

$$(2\cos\theta)^4(2i\sin\theta)^3 = (x+1/x)^4(x-1/x)^3$$

$$= \left(x+\frac{1}{x}\right)^3\cdot\left(x-1/x\right)^3\cdot\left(x+1/x\right)$$

$$= \left(x^2-\frac{1}{x^2}\right)^3\left(x+1/x\right)$$

$$= \left(x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right)\left(x+1/x\right)$$

$$= \left(x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5}\right)$$

$$+ \left(x^5 - 3x + \frac{3}{x^3} - \frac{1}{x^7}\right)$$

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$$2^7 i^3 \cos^4\theta \sin^3\theta = \left(x^7 - \frac{1}{x^7}\right) + \left(x^5 - \frac{1}{x^5}\right)$$
$$-3\left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right)$$
$$= 2i\sin 7\theta + 2i\sin 5\theta - 3(2i\sin 3\theta)$$
$$-3(2i\sin\theta)$$

$$-2^6 \cos^4\theta \sin^3\theta = \sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta$$

Hence proved.

2. Show that $-2^5 \cos^4\theta \sin^2\theta = \cos 6\theta + 2\cos 4\theta - \cos 2\theta - 2.$

Solution:

Let $x = \cos\theta + i\sin\theta$. Then we have

$$2\cos\theta = x + \frac{1}{x} \qquad x^n + \frac{1}{x^n} = 2\cos n\theta$$
$$2i\sin\theta = x - \frac{1}{x} \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

$$(2\cos\theta)^4 (2i\sin\theta)^2 = (x + 1/x)^4 (x - 1/x)^2$$
$$= (x + 1/x)^2 (x + 1/x)^2 (x - 1/x)^2$$
$$= (x + 1/x)^2 (x^2 - 1/x^2)^2$$

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$$2^6 i^2 \cos^4\theta \sin^2\theta = (x^2 + 2 + 1/x^2)$$
$$\cdot (x^4 - 2 + \frac{1}{x^4})$$

$$= x^6 - 2x^2 + \frac{1}{x^2} + 2x^4 - 4 + \frac{2}{x^4}$$
$$+ x^2 - \frac{2}{x^2} + \frac{1}{x^6}$$

$$= (x^6 + \frac{1}{x^6}) + 2(x^4 + 1/x^4) - 2(x^2 + 1/x^2)$$
$$+ (x^2 + \frac{1}{x^2}) - 4$$
$$= (x^6 + \frac{1}{x^6}) + 2(x^4 + 1/x^4) - (x^2 + 1/x^2) - 4$$
$$= 2\cos 6\theta + 2(2\cos 4\theta) - 2\cos 2\theta - 4$$
$$-2^5 \cos^4\theta \sin^2\theta = \cos 6\theta + 2\cos 4\theta - \cos 2\theta - 2.$$

Home work.

Show that the following

1. $-2^4 \cos^3\theta \sin^2\theta = \cos 5\theta + \cos 3\theta - 2\cos\theta.$
2. $2^6 \cos^3\theta \sin^4\theta = \cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta$
3. $-2^7 \cos^5\theta \sin^3\theta = \sin 8\theta + 2\sin 6\theta - 2\sin 4\theta$
$- 6\sin 2\theta$

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4. $2^8 \cos^4\theta \sin^5\theta = \sin 9\theta - \sin 7\theta - 4\sin 5\theta$
$+ 4\sin 3\theta + 6\sin\theta.$

5. $2^8 \cos^5\theta \sin^4\theta = \cos 9\theta + \cos 7\theta - 4\cos 5\theta$
$- 4\cos 3\theta + 6\cos\theta$

6. $2^5 \cos^2\theta \sin^4\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

7. $2^7 \cos^3\theta \sin^5\theta = \sin 8\theta - 2\sin 6\theta - 2\sin 4\theta$
$+ 6\sin 2\theta.$

8. $2^6 \cos^2\theta \sin^5\theta = \sin 7\theta - 3\sin 5\theta + \sin 3\theta$
$+ 5\sin\theta$

9. $-2^{11} \cos^5\theta \sin^7\theta = \sin 12\theta - 2\sin 10\theta - 4\sin 8\theta$
$+ 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta.$

10. $2^{11} \cos^7\theta \sin^5\theta = \sin 12\theta + 2\sin 10\theta$
$- 4\sin 8\theta - 10\sin 6\theta + 5\sin 4\theta + 20\sin 2\theta.$

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Expansions of $\sin n\theta$ and $\cos n\theta$

$\cos n\theta$ and $\sin n\theta$ can be expressed as polynomials in $\cos\theta$ and $\sin\theta$. The following four trigonometrical functions can be expressed as a polynomial in $\cos\theta$ only and also in $\sin\theta$ only.

$\cos n\theta$ and $\dfrac{\sin n\theta}{\sin\theta\cos\theta}$, where n is even

$\dfrac{\cos n\theta}{\cos\theta}$ and $\dfrac{\sin n\theta}{\sin\theta}$, where n is odd.

To obtain the expansion of $\cos n\theta$

Step-1: use $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

Step-2: Expand RHS as a binomial series.

Step-3: Equate the real parts.

To obtain the Expansion of $\sin n\theta$

Step-4: Equate the imaginary parts.

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From Demovier Theorem,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \longrightarrow (1)$$

From Binomial Theorem,

$$(\cos\theta + i\sin\theta)^n = (c + is)^n$$

$$= c^n + nC_1 c^{n-1}(is)^1 + nC_2 c^{n-2}(is)^2$$

$$+ nC_3 c^{n-3}(is)^3 + nC_4 c^{n-4}(is)^4 + \cdots$$

$$= c^n + i\,nC_1 c^{n-1} s - nC_2 c^{n-2} s^2$$

$$- i\,nC_3 c^{n-3} s^3 + nC_4 c^{n-4} s^4$$

$$+ i\,nC_5 c^{n-5} s^5 + \cdots \longrightarrow (2)$$

$(1) = (2) \Rightarrow$

$$\cos n\theta + i\sin n\theta = c^n + i\,nC_1 c^{n-1} s - nC_2 c^{n-2} s^2$$

$$- i\,nC_3 c^{n-3} s^3 + nC_4 c^{n-4} s^4$$

$$+ i\,nC_5 c^{n-5} s^5 + \cdots$$

Equating real and imaginary part, we have.

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Real part:

$$\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^n\theta \sin^4\theta - nC_6 \cos^{n-6}\theta \sin^6\theta \ldots$$

Imaginary part:

$$\sin n\theta = nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta + nC_5 \cos^{n-5}\theta \sin^5\theta \ldots\ldots$$

Expansions of $\tan n\theta$

To find this expansion, we should use the expansions of $\sin n\theta$ and $\cos n\theta$ as follows.

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$$

$$= \frac{nC_1 c^{n-1} s - nC_3 c^{n-3} s^3 + nC_5 c^{n-5} s^5 \ldots}{c^n - nC_2 c^{n-2} s^2 + nC_4 c^{n-4} s^4 \ldots}$$

Dividing the numerator and denominator by c^n, we get

$$\tan n\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta + nC_5 \tan^5\theta \ldots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta \ldots\ldots}$$

Let $n = 2$, $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Let $n = 3$, $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

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Problems:

1. Expansions of (i) $\sin 9\theta$, (ii) $\cos 10\theta$ (iii) $\tan 12\theta$

Solution:

(i) $\sin n\theta = nC_1 c^{n-1} \cdot s - nC_3 c^{n-3} s^3 + \cdots$

$$\sin 9\theta = 9C_1 c^8 s - 9C_3 c^6 s^3 + 9C_5 c^4 s^5 - 9C_7 c^2 s^7 + 9C_9 c^0 s^9.$$

$$\therefore \sin 9\theta = 9\cos^8\theta \sin\theta - 84\cos^6\theta \sin^3\theta + 126\cos^4\theta \sin^5\theta - 36\cos^2\theta \sin^7\theta + \sin^9\theta.$$

(ii) $\cos n\theta = c^n - nC_2 c^{n-2} s^2 + nC_4 c^{n-4} s^4 \cdots$

$$\cos 10\theta = c^{10} - 10C_2 c^8 s^2 + 10C_4 c^6 s^4 - 10C_6 c^4 s^6 + 10C_8 c^2 s^8 - 10C_{10} c^0 s^{10}.$$

$$\therefore \cos 10\theta = \cos^{10}\theta - 45\cos^8\theta \sin^2\theta + 210\cos^6\theta \sin^4\theta - 210\cos^4\theta \sin^6\theta + 45\cos^2\theta \sin^8\theta - \sin^{10}\theta.$$

(iii) $$\tan n\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta + \cdots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta \cdots}$$

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$$\tan 12\theta = \frac{12C_1\tan\theta - 12C_3\tan^3\theta + 12C_5\tan^5\theta - 12C_7\tan^7\theta + 12C_9\tan^9\theta - 12C_{11}\tan^{11}\theta}{1 - 12C_2\tan^2\theta + 12C_4\tan^4\theta - 12C_6\tan^6\theta + 12C_8\tan^8\theta - 12C_{10}\tan^{10}\theta + 12C_{12}\tan^{12}\theta}.$$

$$\tan 12\theta = \frac{12\tan\theta - 220\tan^3\theta + 792\tan^5\theta - 792\tan^7\theta + 220\tan^9\theta - 12\tan^{11}\theta}{1 - 66\tan^2\theta + 495\tan^4\theta - 924\tan^6\theta + 495\tan^8\theta - 66\tan^{10}\theta + \tan^{10}\theta}$$

---

2. Express $\cos 6\theta$ as a polynomial in (i) $\cos\theta$ (ii) $\sin\theta$.

<u>Solution:</u>

(i) We know that

$$\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6 = (c+is)^6 \text{ say}$$

$$\cos 6\theta = Re\,(c+is)^6$$

$$= Re\left[c^6 + 6c^5 is + 15c^4 i^2 s^2 + 20c^3 i^3 s^3 + 15c^2 i^4 s^4 + 6c i^5 s^5 + i^6 s^6\right]$$

$$= c^6 - 15c^4 s^2 + 15c^2 s^4 - s^6$$

$$= c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3$$

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$$\cos 6\theta = c^6 - 15c^4 + 15c^6 + 15c^2(1-2c^2+c^4) - (1-3c^2+3c^4-c^6).$$

$$= c^6(1+15+15+1) + c^4(-15-30-3) + c^2(15+3) - 1$$

$$= 32c^6 - 48c^4 + 18c^2 - 1;$$

$$\therefore \cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$

(ii) $\cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6.$

$$= (1-s^2)^3 - 15(1-s^2)^2 \cdot s^2 + 15(1-s^2)s^4 - s^6.$$

$$= (1-3s^2+3s^4-s^6) - 15(1-2s^2+s^4)s^2 + 15(1-s^2)s^4 - s^6.$$

$$= 1 + s^2(-3-15) + s^4(3+30+15) + s^6(-1,-15-15-1)$$

$$= 1 - 18s^2 + 48s^4 - 32s^6.$$

$$\therefore \cos 6\theta = 1 - 18\sin^2\theta + 48\sin^4\theta - 32\sin^6\theta.$$

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3. Show that $\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1$.

Solution:

$$\cos 8\theta + i\sin 8\theta = (\cos\theta + i\sin\theta)^8 = (c+is)^8 \text{ say.}$$

$$\cos 8\theta = Re[c+is]^8$$

$$= c^8 + 8C_2\, c^6(i^2 s^2) + 8C_4\, c^4(i^4 s^4) + 8C_6\, c^2(i^6 s^6) + i^8 s^8$$

$$= c^8 - 28c^6 s^2 + 70c^4 s^4 - 28c^2 s^6 + s^8$$

$$= c^8 - 28c^6(1-c^2) + 70c^4(1-c^2)^2 - 28c^2(1-c^2)^3) + (1-c^2)^4$$

$$= c^8 - 28c^6(1-c^2) + 70c^4(1-2c^2+c^4) - 28c^2(1-3c^2+3c^4-c^6) + (1-4c^2+6c^4-4c^6+c^8)$$

$$= c^8(1+28+70+28+1) + c^6(-28-140-84-4) + c^4(70+84+6) + c^2(-28-4) + 1$$

$$= 128c^8 - 256c^6 + 160c^4 - 32c^2 + 1$$

$$\therefore \cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1.$$

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Home work

1. Expand the following in powers of $\cos\theta$ and $\sin\theta$

(a) $\sin 4\theta$ (c) $\cos 6\theta$ (e) $\cos 7\theta$

b) $\sin 6\theta$ d) $\sin 7\theta$. f) $\sin 5\theta$.

2. Prove the following.

a) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

b) $\cos 4\theta = 1 - 8\sin^2\theta + 8\sin^4\theta$

c) $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$.

d) $\cos 6\theta = 1 - 18\sin^2\theta + 48\sin^4\theta - 32\sin^6\theta$.

e) $\cos 8\theta = 1 - 32\sin^2\theta + 160\sin^4\theta - 256\sin^6\theta + 128\sin^8\theta$.

Problems.

1\. (a) Express $\frac{\sin 6\theta}{\sin\theta\cos\theta}$ as a polynomial in (i) $\cos\theta$ (ii) $\sin\theta$.

(iii) Show that $\frac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$.

Solution:

(i) We know that

$$\cos 6\theta + i\sin 6\theta = [\cos\theta + i\sin\theta]^6 = (c+is)^6 \text{ say.}$$

$$\sin 6\theta = \text{Im}[c+is]^6$$

$$= \text{Im}[c^6 + 6c^5 is + 15c^4 i^2 s^2 + 20c^3 i^3 s^3 + 15c^2 i^4 s^4 + 6c i^5 s^5 + i^6 s^6]$$

$$= 6c^5 s - 20c^3 s^3 + 6cs^5$$

$$\frac{\sin 6\theta}{\sin\theta\cos\theta} = 6c^4 - 20c^2 s^2 + 6s^4$$

$$= 6c^4 - 20c^2(1-c^2) + 6(1-c^2)^2$$

$$= 6c^4 - 20c^2 + 20c^4 + 6(1-2c^2+c^4)$$

$$= c^4(6+20+6) - c^2(20+12) + 6$$

$$= 32c^4 - 32c^2 + 6.$$

$$\frac{\sin 6\theta}{\sin\theta\cos\theta} = 32\cos^4\theta - 32\cos^2\theta + 6$$

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(ii) $\frac{\sin 6\theta}{\sin\theta\cos\theta} = 6c^4 - 20c^2s^2 + 6s^4$

$= 6(1-s^2)^2 - 20(1-s^2)s^2 + 6s^4$

$= 6(1-2s^2+s^4) - 20s^2 + 20s^4 + 6s^4$

$= s^4(6+20+6) - s^2(12+20) + 6$

$\frac{\sin 6\theta}{\sin\theta\cos\theta} = 32s^4 - 32s^2 + 6.$

(iii) $\sin 6\theta = 6c^5s - 20c^3s^3 + 6cs^5$

$\frac{\sin 6\theta}{\sin\theta} = 6c^5 - 20c^3s^2 + 6cs^4$

$= 6c^5 - 20c^3(1-c^2) + 6c(1-s^2)^2$

$= 6c^5 - 20c^3 + 20c^5 + 6c(1-2s^2+s^4)$

$= c^5(6+20+6) - c^3(20+12) + 6c.$

$= 32c^5 - 32c^3 + 6c.$

$\therefore \frac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta.$

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2. Express $\frac{\cos 5\theta}{\cos\theta}$ as a polynomial in

(i) $\cos\theta$ (ii) $\sin\theta$.

Solution: (i)

We know that

$(\cos 5\theta + i\sin 5\theta) = (\cos\theta + i\sin\theta)^5 = (c+is)^5$ say.

$\cos 5\theta = Re\,(c+is)^5$

$= Re\,[c^5 + 5c^4 is + 10c^3 i^2 s^2 + 10c^2 i^3 s^3 + 5c\,i^4 s^4 + i^5 s^5]$

$= c^5 - 10c^3 s^2 + 5cs^4.$

$\frac{\cos 5\theta}{\cos\theta} = c^4 - 10c^2 s^2 + 5s^4$

$= c^4 - 10c^2(1-c^2) + 5(1-c^2)^2$

$= c^4 - 10c^2 + 10c^4 + 5(1 - 2c^2 + c^4)$

$= c^4(1+10+5) - c^2(10+10) + 5$

$= 16c^4 - 20c^2 + 5$

$\therefore \frac{\cos 5\theta}{\cos\theta} = 16\cos^4\theta - 20\cos^2\theta + 5.$

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(ii) $\frac{\cos 5\theta}{\cos\theta} = (1-s^2)^2 - 10(1-s^2)s^2 + 5s^4$

$= (1-2s^2+s^4) - 10s^2 + 10s^4 + 5s^4$

$= s^4(1+10+5) - s^2(2+10) + 1$

$= 16s^4 - 12s^2 + 1$

$\therefore \frac{\cos 5\theta}{\cos\theta} = 16\sin^4\theta - 12\sin^2\theta + 1.$

---

3. Express $\frac{\cos 7\theta}{\cos\theta}$ as a polynomial in

(i) $\cos\theta$ (ii) $\sin\theta$.

Solution:

(i) We know that

$(\cos 7\theta + i\sin 7\theta) = (\cos\theta + i\sin\theta)^7 = (c+is)^7$ say.

$\cos 7\theta = \text{Re}\,(c+is)^7$

$= \text{Re}(c^7 + 7c^6 is + 21c^5 i^2 s^2 + 35c^4 i^3 s^3$

$+ 35c^3 i^4 s^4 + 21c^2 i^5 s^5 + 7c i^6 s^6 + i^7 s^7)$

$= c^7 - 21c^5 s^2 + 35c^3 s^4 - 7cs^6.$

$$\frac{\cos 7\theta}{\cos\theta} = c^6 - 21c^4s^2 + 35c^2s^4 - 7s^6$$

$$= c^6 - 21c^4(1-c^2) + 35c^2(1-c^2)^2 - 7(1-c^2)^3$$

$$= c^6 - 21c^4 + 21c^6 + 35c^2(1-2c^2+c^4)$$

$$-7(1-3c^2+3c^4-c^6)$$

$$= c^6(1+21+35+7) + c^4(-21-70-21)$$

$$+ c^2(35+21) - 7$$

$$= 64c^6 - 112c^4 + 56c^2 - 7$$

$$\therefore \frac{\cos 7\theta}{\cos\theta} = 64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7.$$

(ii) $$\frac{\cos 7\theta}{\cos\theta} = (1-s^2)^3 - 21(1-s^2)^2 \cdot s^2 + 35(1-s^2)s^4 - 7s^6$$

$$=(1-3s^2+3s^4 s^6) - 21(s^2-2s^4+s^6)$$

$$+ 35(s^4-s^6) - 7s^6.$$

$$= s^6(-1-21-35-7) + s^4(3+42+35)$$

$$+ s^2(-3-21)+1$$

$$= 1 - 24s^2 + 80s^4 - 64s^6.$$

$$\frac{\cos 7\theta}{\cos\theta} = 1 - 24\sin^2\theta - 80\sin^4\theta - 64\sin^6\theta.$$

4. Express $\frac{\sin 9\theta}{\sin\theta}$ in terms of $\sin\theta$.

Solution:

We know that

$(\cos 9\theta + i\sin 9\theta) = (\cos\theta + i\sin\theta)^9 = (c+is)^9$ say

$\sin 9\theta = Im(c+is)^9$

$= Im[c^9 + 9c^8 is + 36c^7 i^2 s^2$
$+ 84c^6 i^3 s^3 + 126c^5 i^4 s^4 + 126c^4 i^5 s^5$
$+ 84c^3 i^6 s^6 + 36c^2 i^7 s^7 + 9c i^8 s^8 + i^9 s^9]$

$= 9c^8 s - 84c^6 s^3 + 126c^4 s^5 - 36c^2 s^7 + s^9$

$\frac{\sin 9\theta}{\sin\theta} = 9c^8 - 84c^6 s^2 + 126c^4 s^4 - 36c^2 s^6 + s^8$

$= 9(1-s^2)^4 - 84(1-s^2)^3 s^2$
$+ 126(1-s^2)^2 s^4 - 36(1-s^2)s^6 + s^8$.

$\cancel{= 9 - 120s^2 + 432s^4 - 576s^6 + 256s^8}$

$= 9[1 - 4s^2 + 6s^4 - 4s^6 + s^8] - 84(1 - 3s^2 + 3s^4 + s^6)$
$\cdot s^2 + 126(1 - 2s^2 + s^4)s^4 - 36(1-s^2)s^6 + s^8$

$= 9 - 120s^2 + 432s^4 - 576s^6 + 256s^8$.

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$$\therefore \frac{\sin 9\theta}{\sin\theta} = 9 - 120\sin^2\theta + 432\sin^4\theta - 576\sin^6\theta + 256\sin^8\theta.$$

Home work.

1. Express $\frac{\cos 9\theta}{\cos\theta}$ in terms of $\cos\theta$.

Ans: $$\frac{\cos 9\theta}{\cos\theta} = 256\cos^8\theta - 576\cos^6\theta + 432\cos^4\theta - 120\cos^2\theta + 9.$$

2. Express $\frac{\sin 7\theta}{\sin\theta}$ as a polynomial in (i) $\cos\theta$ (ii) $\sin\theta$.

Ans (i) $$\frac{\sin 7\theta}{\sin\theta} = 64\cos^6\theta - 80\cos^4\theta + 24\cos^2\theta - 1$$

(ii) $$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta.$$

3. Express $\frac{\sin 5\theta}{\sin\theta}$ as a polynomial in (i) $\cos\theta$ (ii) $\sin\theta$.

Ans: (i) $$\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$$

(ii) $$\frac{\sin 5\theta}{\sin\theta} = 5 - 20\sin^2\theta + 16\sin^4\theta.$$

4. Expansion of $\frac{\sin 11\theta}{\sin\theta}$ and find the terms of $\sin\theta$.

Ans:
$$\frac{\sin 11\theta}{\sin\theta} = 11 - 220\sin^2\theta + 1232\sin^4\theta - 2816\sin^6\theta + 2816\sin^8\theta - 1024\sin^9\theta.$$

Expansions of $\sin\theta$, $\cos\theta$, $\tan\theta$ in θ

Maclaurin's Series for $f(x)$ is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

(i) If $f(x) = \sin x$. $f(0) = 0$

Then $f'(x) = \cos x$ $\quad f'(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$

$f'''(0) = -\cos x$ $\quad f'''(0) = -1$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

IIIly (ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

(iii) $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

1. Series for $\cos\theta$.

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!} \cos^{n-2}\theta \sin^2\theta + \cdots$$

Setting $n\theta = x$ and replacing θ by x/n.

$$\cos x = \cos^n x/n - \frac{n(n-1)}{2!} \cos^{n-2} x/n \sin^2 x/n + \cdots$$

$$= \cos^n x/n - \frac{n(n-1)}{2!} \cos^{n-2} x/n \frac{\sin^2 x/n}{(x/n)^2} \left(\frac{x}{n}\right)^2 + \cdots$$

$$= \cos^n x/n - \frac{1(1-1/n)}{2!} \cos^{n-2}\left(\frac{x}{n}\right) \left(\frac{\sin x/n}{x/n}\right)^2 \cdot x^2 + \cdots$$

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Taking limit as $n \to \infty$ and using

$\lim_{n \to \infty} \frac{\sin x/n}{x/n} = 1$, we have

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\cdots$$

2. Series for $\sin\theta$.

$$\sin n\theta = n\cos^{n-1}\theta \sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta \sin^3\theta + \cdots\cdots$$

setting $n\theta = x$ and replacing θ by x/n.

$$\sin x = n\cos^{n-1} x/n \sin x/n - \frac{n(n-1)(n-2)}{3!}\cos^{n-3} x/n \sin^3 x/n + \cdots$$

$$= n\cos^{n-1} x/n \frac{\sin x/n}{(x/n)}\cdot(x/n)$$

$$- \frac{n(n-1)(n-2)}{3!}\cos^{n-3} x/n \frac{\sin^3 x/n}{(x/n)^3}\cdot(x/n)^3 + \cdots\cdots$$

$$= \cos^{n-1} x/n \left(\frac{\sin x/n}{x/n}\right)\cdot x$$

$$- \frac{1(1-1/n)(1-2/n)}{3!}\cos^{n-3} x/n \left(\frac{\sin x/n}{x/n}\right)^3 \cdot x^3 + \cdots\cdots$$

Taking limit as $n \to \infty$ and using $\lim_{n\to\infty} \frac{\sin x/n}{x/n} = 1$,

we have.

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\cdots$$

3. Series for $\tan x$ as far as the term x^5.

$$\tan x = \frac{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots}{1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)\left[1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)\right]^{-1}$$

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} \cdots\right)\left[1 + \left(\frac{x^2}{2} - \frac{x^4}{24}\right) + \left(\frac{x^2}{2}\right)^2 + \cdots\right]$$

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} \cdots\right)\left[1 + \frac{x^2}{2} + x^4\left(-\frac{1}{24} + \frac{1}{4}\right) + \cdots\right]$$

$$= \left(x - \frac{x^3}{6} + \frac{x^5}{120} \cdots\right)\left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \cdots\right)$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$$

$$\therefore \tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$$

Problems:

1. Find $\displaystyle\lim_{\theta \to 0} \frac{n\sin\theta - \sin n\theta}{\theta(\cos\theta - \sin n\theta)}$.

Solution: we have $\dfrac{n\sin\theta - \sin n\theta}{\theta(\cos\theta - \sin n\theta}$

$$= \frac{n\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) - \left(n\theta - \frac{n^3\theta^3}{3!} + \frac{n^5\theta^5}{5!} \cdots\right)}{\theta\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right) - \theta\left(n\theta - \frac{n^3\theta^3}{3!} + \frac{n^5\theta^5}{5!} \cdots\right)}$$

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$$= \frac{\frac{\theta^3}{3!}(n^3-n) + \frac{\theta^5}{5!}(n-n^5) \ldots}{\theta - n\theta^2 - \frac{\theta^3}{2!} + \frac{n^3\theta^4}{3!} + \frac{\theta^5}{4!} \ldots}$$

$$= \frac{\frac{\theta^2}{3!}(n^3-n) + \frac{\theta^4}{5!}(n-n^5) \ldots}{1 - n\theta - \frac{\theta^2}{2!} + \frac{n^3\theta^3}{3!} + \frac{\theta^4}{4!} \ldots}$$

Taking $\theta \to 0$, we have

$$\lim_{\theta \to 0} \frac{n\sin\theta - \sin n\theta}{\theta(\cos\theta - \sin n\theta)} = 0.$$

---

2. Find $\lim_{\theta \to 0} \frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$

Solution:

$$\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta} + 1}$$

$$= \frac{\sin\theta + 1 - \cos\theta}{\sin\theta - 1 + \cos\theta}$$

$$= \frac{\left(\theta - \frac{\theta^3}{3!} + \ldots\right) + 1 - \left(1 - \frac{\theta^2}{2!} + \ldots\right)}{\left(\theta - \frac{\theta^3}{3!} + \ldots\right) - 1 + \left(1 - \frac{\theta^2}{2!} + \ldots\right)}$$

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$$\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \frac{\theta+\frac{\theta^2}{2!}-\frac{\theta^3}{3!}\cdots}{\theta-\frac{\theta^2}{2!}-\frac{\theta^3}{3!}\cdots}$$

$$= \frac{1+\frac{\theta}{2!}-\frac{\theta^2}{3!}\cdots}{1-\frac{\theta}{2!}-\frac{\theta^2}{3!}\cdots}$$

Taking $\theta \to 0$, we have

$$\lim_{\theta\to 0} \frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = 1.$$

---

3. Evaluate $\lim\limits_{x\to \pi/2} \dfrac{\sin x + \cos 2x}{\cos^2 x}$.

Solution

Put $x = \theta + \frac{\pi}{2}$. As $x \to \frac{\pi}{2}$, $\theta \to 0$.

$$\therefore \lim_{x\to \pi/2} \frac{\sin x+\cos 2x}{\cos^2 x} = \lim_{\theta\to 0} \frac{\sin(\theta+\frac{\pi}{2})+\cos(2\theta+\pi)}{\cos^2(\theta+\frac{\pi}{2})}$$

$$= \lim_{\theta\to 0} \frac{\cos\theta-\cos 2\theta}{\sin^2\theta}$$

$$= \lim_{\theta\to 0} \frac{\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}\cdots\right)-\left(1-\frac{4\theta^2}{2!}+\frac{16\theta^4}{4!}\cdots\right)}{\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}\cdots\right)^2}$$

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$$\lim_{x \to \pi/2} \frac{\sin x + \cos 2x}{\cos^2 x} = \lim_{\theta \to 0} \frac{\frac{3}{2}\theta^2 - \frac{5}{8}\theta^4 \ldots}{\theta^2\left(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} \ldots\right)^2}$$

$$= \lim_{\theta \to 0} \frac{\frac{3}{2} - \frac{5}{8}\theta^2 \ldots}{\left(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} \ldots\right)^2}$$

$$= \frac{3}{2}.$$

$$\therefore \lim_{x \to \pi/2} \frac{\sin x + \cos 2x}{\cos^2 x} = \frac{3}{2}$$

---

4. Evaluate $\lim_{x \to 0} \frac{\sin 2x - 2\sin x}{x^3}$

Solution:

$$\lim_{x \to 0} \frac{\sin 2x - 2\sin x}{x^3}$$

$$= \lim_{x \to 0} \frac{\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} \ldots\right) - 2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\right)}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{x^3}{6}\left[(-8+2) + \text{terms of higher powers of } x\right]}{x^3}$$

$$= \lim_{x \to 0} \left[-1 + \text{terms of higher powers of } x\right]$$

$$= -1.$$

$$\therefore \lim_{x \to 0} \frac{\sin 2x - 2\sin x}{x^3} = -1$$

Home work:

Evaluate the followings.

(a). $\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}$; (b) $\lim_{\theta \to 0} \frac{\cot \theta - \cos 2\theta}{\cot 3\theta}$.

(c) $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$ (d) $\lim_{\theta \to \pi/2} \frac{1 - \sin\theta}{\cos^2\theta - 2\cos^2 3\theta}$.

(e) $\lim_{\theta \to \pi/2} \frac{\cos\theta - \sin 2\theta}{\cos 3\theta}$ (f) $\lim_{x \to 0} \frac{\sin 2x - 2\sin x}{x^3}$

Answers:

(a). $\frac{1}{2}$ (b) $\frac{9}{2}$ (c) $\frac{1}{2}$ (d) $\frac{-1}{34}$ (e) $\frac{1}{3}$. (f) -1.

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Problems.

1. If $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$. show that θ is Equal to $3^\circ 1'$ nearly.

Solution:

$$\frac{2165}{2166} = 1 - \frac{1}{2166}$$

and hence is nearly 1. So θ is small. Hence neglecting powers higher than 3.

$$\frac{\sin\theta}{\theta} = \frac{\theta - \frac{\theta^3}{3!}}{\theta} = 1 - \frac{1}{6}\theta^2$$

$$\therefore 1 - \frac{1}{6}\theta^2 = 1 - \frac{1}{2166}$$

$$\therefore \theta^2 = \frac{6}{2166} = \frac{1}{361}$$

$$\therefore \theta = \frac{1}{19} \text{ radian.}$$

$$= \frac{1}{19} \times \frac{180}{\pi} \qquad 1\text{-radian} = \frac{180^\circ}{\pi}.$$

$$= 3^\circ 0' 56''$$

2. If $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$, show that $\theta = 1^\circ 58'$ approximately.

Solution.

$$\frac{5045}{5046} = 1 - \frac{1}{5046}$$

and hence, is nearly 1. So θ is small

Hence neglecting powers higher than 3.

$$\frac{\sin\theta}{\theta} = \frac{\theta - \frac{\theta^3}{3!}}{\theta} = 1 - \frac{\theta^2}{6}.$$

$$\therefore 1 - \frac{\theta^2}{6} = 1 - \frac{1}{5046}$$

$$\theta^2 = \frac{6}{5046} = \frac{1}{841}$$

$$\therefore \theta = \frac{1}{29} \text{ radian}$$

$$= \frac{1}{29} \times \frac{180^\circ}{\pi}$$

$$= 1^\circ 58' 32''.$$

3. Find the approximate value of θ, if

$$\frac{\tan\theta}{\theta} = \frac{2524}{2523}.$$

Solution :

$$\frac{2524}{2523} = 1 + \frac{1}{2523}$$

and hence is nearly 1. So θ is small.

Hence neglecting powers higher than 3.

$$\frac{\tan\theta}{\theta} = \frac{\theta + \frac{\theta^3}{3}}{\theta} = 1 + \frac{\theta^2}{3}.$$

$$\therefore 1 + \frac{\theta^2}{3} = 1 + \frac{1}{2523}$$

$$\theta^2 = \frac{3}{2523} = \frac{1}{841}$$

$$\theta = \frac{1}{29} \text{ radian}$$

$$= \frac{1}{29} \times \frac{180^\circ}{\pi}$$

$$= 1^\circ\, 58'\, 32''.$$

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Home work.

Find the approximate values of θ in radians if

(i) $\frac{\sin\theta}{\theta} = \frac{863}{864}$ (ii) $\frac{\sin\theta}{\theta} = \frac{1013}{1014}$

(iii) $\frac{\sin\theta}{\theta} = \frac{599}{600}$

Answers:

(i) $1 - \frac{\theta^2}{6} = 1 - \frac{1}{864}$; $\theta = \frac{1}{12}$ radian

(ii) $1 - \frac{\theta^2}{6} = 1 - \frac{1}{1014}$; $\theta = \frac{1}{13}$ radian

(iii) $1 - \frac{\theta^2}{6} = 1 - \frac{1}{600}$; $\theta = \frac{1}{10}$ radian.

Problems:

1. Show that if $\cos\left(\frac{\pi}{3}+\theta\right) = 0.49$, then θ is $40'$ nearly.

Solution.

$\cos\frac{\pi}{3} = \frac{1}{2}$, So θ is small

$$\cos\left(\frac{\pi}{3}+\theta\right) = \cos\frac{\pi}{3}\cos\theta - \sin\frac{\pi}{3}\sin\theta$$

$$= \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$$

$[\cos(A+B) = \cos A\cos B - \sin A \sin B]$

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$\therefore \cos\left(\frac{\pi}{3}+\theta\right) = \frac{1}{2}\left(1-\frac{\theta^2}{2!}+\dots\right) - \frac{\sqrt{3}}{2}\left(\theta-\frac{\theta^3}{3!}+\dots\right)$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}\theta$, neglecting $\theta^2, \theta^3 \dots$

$\cos\frac{\pi}{3} = 0.50$ and 0.49 is nearly 0.50.

$\therefore \quad 0.49 = \frac{1}{2} - \frac{\sqrt{3}}{2}\theta$

$\therefore \theta = 0.0115$ radian

$= 0.0115 \times \frac{180}{\pi}$

$= 0^\circ 39' 32''$

$= 40'$ nearly.

---

2. Solve approximately in radians

$\sin\left(\frac{\pi}{3}+x\right) = 0.87$

Solution:

W.K.T. $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\left(\frac{\pi}{3}+x\right) = 0.87$

$\sin\frac{\pi}{3}\cos x + \cos\frac{\pi}{3}\sin x = 0.87$

$\left[\because \sin(A+B) = \sin A\cos B + \cos A\sin B\right]$

$$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = 0.87$$

$$\frac{\sqrt{3}}{2}\left(1 - \frac{x^2}{2!} + \cdots\right) + \frac{1}{2}\left(x - \frac{x^3}{3!} + \cdots\right) = 0.87$$

$$\frac{\sqrt{3}}{2} + \frac{1}{2}x = 0.87$$

$$\Rightarrow x = \left(0.87 - \frac{\sqrt{3}}{2}\right) \times \frac{2}{1}$$

$$= 0.008 \text{ radian}$$

$$x = 0.008 \times \frac{180}{\pi}$$

$$= 0^\circ\, 27'\, 30''$$

$$= 28' \text{ nearly.}$$

Home work.

Establish the following results

1. If $\sin\left(\frac{\pi}{6} + \theta\right) = 0.51$, $\theta = 0.012$ radian nearly

2. If $\tan\left(\frac{\pi}{4} - \theta\right) = 1.001$; $\theta = -\frac{1}{2001}$ radian

Hint:

1. $\frac{1}{2} + \frac{\sqrt{3}}{2}\theta = 0.51$ 2. $\frac{1-\theta}{1+\theta} = 1.001$.

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Appendix - C

◆ Trigonometry

I. Inter-relations :

1. $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$ 2. $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

3. $\cos\theta = \dfrac{1}{\sec\theta}$ 4. $\sec\theta = \dfrac{1}{\cos\theta}$

5. $\tan\theta = \dfrac{1}{\cot\theta}$ 6. $\cot\theta = \dfrac{1}{\tan\theta}$

7. $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$ 8. $\dfrac{\cos\theta}{\sin\theta} = \cot\theta$

II. Identities

1. $\sin^2\theta + \cos^2\theta = 1$ 2. $1 + \tan^2\theta = \sec^2\theta$ 3. $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

III. Radian measure

π radians $= 180^\circ$

IV. Trigonometric ratios for certain standard angles

θ	0°	$30^\circ\ (\pi/6)$	$45^\circ\ (\pi/4)$	$60^\circ\ (\pi/3)$	$90^\circ\ (\pi/2)$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞ (infinity)

$\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$ are respectively the reciprocals of $\tan\theta$, $\cos\theta$, $\sin\theta$

V. Allied angles

Trigonometrical ratios of $90 \pm \theta$, $180 \pm \theta$, $270 \pm \theta$, $360 \pm \theta$ in terms of those of θ can be found easily by the following rule known as A - S - T - C rule.

(i) When the angle is $90 \pm \theta$ or $270 \pm \theta$ the trigonometrical ratio changes from sine to cosine and vice versa. Also tan $\rightleftarrows$ cot , sec $\rightleftarrows$ cosec

(ii) When the angle is $180 \pm \theta$ or $360 \pm \theta$ the trigonometrical ratio remains the same. i.e., sin $\rightarrow$ sin , cos $\rightarrow$ cos, etc.

(iii) In each case the sign + or − is pre multiplied by the A-S-T-C quadrant rule :

S	A
II $(90^\circ - 180^\circ)$	I $(0^\circ - 90^\circ)$
T	C
III $(180^\circ - 270^\circ)$	IV $(270^\circ - 360^\circ)$

A : All ratios are +ve in the I qudarant

S : sin is +ve in the II quadrant

T : tan is +ve in the III quadrant.

C : cos is +ve in the IV quadrant

Note : $\sin(-\theta) = -\sin\theta,\ \cos(-\theta) = \cos\theta,$
$\sin(n\,.\,2\pi + \theta) = \sin\theta,\ \cos(n\,.\,2\pi + \theta) = \cos\theta$

Example 1 : $\sin(90^\circ - \theta) = \cos\theta,\ \cos(90^\circ + \theta) = -\sin\theta$

$\sin(180^\circ - \theta) = \sin\theta,\ \tan(180^\circ + \theta) = \tan\theta$ etc.

Example 2 : $\sin(135^\circ) = \sin(90^\circ + 45^\circ) = \cos 45^\circ = \dfrac{1}{\sqrt{2}}$

$\tan(315^\circ) = \tan(270^\circ + 45^\circ) = -\cot 45^\circ = -1$

$\cos(225^\circ) = \cos(180^\circ + 45^\circ) = -\cos 45^\circ = -\dfrac{1}{\sqrt{2}}$

$\sin(750^\circ) = \sin(2 \times 360^\circ + 30^\circ) = \sin 30^\circ = \dfrac{1}{2}$ etc.

VI. Compound angle formulae

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

(ii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

(iii) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

VII. Formulae to convert a product into sum or difference

(iv) $\left.\begin{array}{r}\sin A \cos B \\ \text{or } \cos B \sin A\end{array}\right\} = \dfrac{1}{2}[\sin(A + B) + \sin(A - B)]$

$\cos A \sin B = \dfrac{1}{2}[\sin(A + B) - \sin(A - B)],\quad \sin(-\theta) = -\sin\theta$

(v) $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

(vi) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$, $\cos(-\theta) = \cos\theta$

VIII. Particular cases of formula (i) to (vi)

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$= 1 - 2\sin^2 A$$

$$= 2\cos^2 A - 1$$

$$1 + \cos 2A = 2\cos^2 A$$

$$1 - \cos 2A = 2\sin^2 A$$

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\sin A = 2\sin(A/2)\cos(A/2)$$

$$\cos A = \cos^2(A/2) - \sin^2(A/2)$$

$$= 1 - 2\sin^2(A/2)$$

$$= 2\cos^2(A/2) - 1$$

$$1 + \cos A = 2\cos^2\frac{A}{2}$$

$$1 - \cos A = 2\sin^2\frac{A}{2}$$

$$\sin A = \frac{2\tan(A/2)}{1+\tan^2(A/2)}$$

$$\cos A = \frac{1-\tan^2(A/2)}{1+\tan^2(A/2)}$$

IX. Formulae to convert a sum or difference into product.

(vii) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(viii) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(ix) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(x) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

X. Hyperbolic functions

We have already said that 'e' whose value is approximately 2.7 is called the exponential constant. Further, if $\log_e y = x$ then $y = e^x$ is called as the exponential function. Hyperbolic functions are defined interm of exponential function as below.

$$\sinh x = \frac{e^x - e^{-x}}{2}; \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

Also, $\tanh x = \dfrac{\sinh x}{\cosh x}$; $\coth x = \dfrac{1}{\tanh x} = \dfrac{\cosh x}{\sinh x}$

$\operatorname{sech} x = \dfrac{1}{\cosh x}$; $\operatorname{cosech} x = \dfrac{1}{\sinh x}$

XI. Important hyperbolic identities

(i) $\cosh^2 x - \sinh^2 x = 1$; (ii) $1 - \tanh^2 x = \operatorname{sech}^2 x$

(iii) $\coth^2 x - 1 = \operatorname{cosech}^2 x$ (iv) $\cosh^2 x + \sinh^2 x = \cosh 2x$

(v) $2 \sinh x \cosh x = \sinh 2x$

XII. Relationship between trigonometric and hyperbolic functions

We have, $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ and $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

Now $\sin(ix) = \dfrac{e^{-x} - e^{x}}{2i} = (-1)\dfrac{(e^{x} - e^{-x})}{2i} = \dfrac{i^2(e^{x} - e^{-x})}{2i}$

$\therefore \sin(ix) = i \cdot \dfrac{e^{x} - e^{-x}}{2}$ (i.e.,) $\sin(ix) = i \sinh x$

Also, $\cos(ix) = \dfrac{e^{-x} + e^{x}}{2} = \cosh x$ (i.e.,) $\cos(ix) = \cosh x$

6.2. Hyperbolic functions.

Circular functions. Using Euler's relations,

$$e^{ix} = \cos x + i \sin x, \; e^{-ix} = \cos x - i \sin x,$$

the six circular functions are defined algebraically as

$\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$	$\sec x = \dfrac{2}{e^{ix} + e^{-ix}}$
$\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$	$\operatorname{cosec} x = \dfrac{2i}{e^{ix} - e^{-ix}}$
$\tan x = \dfrac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$	$\cot x = \dfrac{i(e^{ix} + e^{-ix})}{e^{ix} - e^{-ix}}$

Definitions of hyperbolic functions. The six functions,

$$\left.\begin{matrix}\cosh x \\ \operatorname{sech} x\end{matrix}\right\} \quad \left.\begin{matrix}\sinh x \\ \operatorname{cosech} x\end{matrix}\right\} \quad \left.\begin{matrix}\tanh x \\ \coth x\end{matrix}\right\}$$

which are defined below are called hyperbolic functions.

Hyperbolic cosine of x	$\cosh x = \dfrac{e^{x} + e^{-x}}{2}$
Hyperbolic sine of x	$\sinh x = \dfrac{e^{x} - e^{-x}}{2}$
Hyperbolic tangent of x	$\tanh x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

Hyperbolic secant, hyperbolic cosecant, hyperbolic cotangent of x are the reciprocals of $\cosh x$, $\sinh x$, $\tanh x$.

6.2.1. Relations between circular and hyperbolic functions.

Considering $\cos ix$, $\sin ix$, $\tan ix$, we have

1. $\cos ix = \dfrac{e^{i(ix)} + e^{-i(ix)}}{2} = \dfrac{e^{-x} + e^{x}}{2} = \cosh x.$

2. $\sin ix = \dfrac{e^{i(ix)} - e^{-i(ix)}}{2i} = \dfrac{e^{-x} - e^{x}}{2i} = -\dfrac{e^{x} - e^{-x}}{2i}$

$$= i^2 \frac{e^{x} - e^{-x}}{2i} = i\,\frac{e^{x} - e^{-x}}{2} = i \sinh x.$$

3. $\tan ix = \dfrac{\sin ix}{\cos ix} = \dfrac{i \sinh x}{\cosh x} = i \tanh x.$

These three results are very important and so we shall tabulate them.

$\cos ix = \cosh x$
$\sin ix = i \sinh x$
$\tan ix = i \tanh x$

4. $\sec ix = \dfrac{1}{\cos ix} = \dfrac{1}{\cosh x} = \operatorname{sech} x.$

5. $\operatorname{cosec} ix = \dfrac{1}{\sin ix} = \dfrac{1}{i \sinh x} = \dfrac{i}{i^2} \cdot \dfrac{1}{\sinh x} = -i \operatorname{cosech} x.$

6. $\cot ix = \dfrac{1}{\tan ix} = \dfrac{1}{i \tanh x} = -i \coth x.$

6.2.2. Formulae in hyperbolic functions.

Corresponding to the formulae in circular functions, we get similar formulae in hyperbolic functions either by using definitions or by using the relations between circular and hyperbolic functions.

Method 1. Using definitions, we shall prove

$$\cosh^2 x - \sinh^2 x = 1.$$

$$\text{LHS} = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{4 \cdot e^x e^{-x}}{4} = 1.$$

Method 2. In $\cos^2 x + \sin^2 x = 1$, changing x into ix and using $\cos ix = \cosh x$, $\sin ix = i \sinh x$,

$$\cos^2 ix + \sin^2 ix = 1 \text{ or } (\cosh x)^2 + (i \sinh x)^2 = 1$$

i.e., $$\cosh^2 x - \sinh^2 x = 1.$$

The important formulae in circular functions and the corresponding formulae in hyperbolic functions are tabulated below.

In circular functions	In hyperbolic functions
$\cos^2 x + \sin^2 x = 1$ $\sec^2 x = 1 + \tan^2 x$ $\operatorname{cosec}^2 x = 1 + \cot^2 x$	$\cosh^2 x - \sinh^2 x = 1$ $\operatorname{sech}^2 x = 1 - \tanh^2 x$ $-\operatorname{cosech}^2 x = 1 - \coth^2 x$
$\cos 2x = 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$ $= \cos^2 x - \sin^2 x$ $\sin 2x = 2 \sin x \cos x$ $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$ $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$ $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$	$\cosh 2x = 2\cosh^2 x - 1$ $= 1 + 2\sinh^2 x$ $= \cosh^2 x + \sinh^2 x$ $\sinh 2x = 2\sinh x \cosh x$ $\sinh(\theta + \phi) = \sinh\theta\cosh\phi + \cosh\theta\sinh\phi$ $\sinh(\theta - \phi) = \sinh\theta\cosh\phi - \cosh\theta\sinh\phi$ $\cosh(\theta + \phi) = \cosh\theta\cosh\phi + \sinh\theta\sinh\phi$
$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$ $\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$ $\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$ $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$	$\cosh(\theta - \phi) = \cosh\theta\cosh\phi - \sinh\theta\sinh\phi$ $\sinh 2\theta = \frac{2\tanh\theta}{1 - \tanh^2\theta}$ $\cosh 2\theta = \frac{1 + \tanh^2\theta}{1 - \tanh^2\theta}$ $\tanh 2\theta = \frac{2\tanh\theta}{1 + \tanh^2\theta}$

Note. To get the hyperbolic formula from its corresponding circular formula

(i) replace the circular names by hyperbolic names

(ii) and also change the signs of products of two sines, products of two tangents, etc.

6.2.3. Real and imaginary parts.

We shall find the real and imaginary parts of

$$\sin(\theta + i\phi), \quad \cos(\theta + i\phi), \quad \tan(\theta + i\phi)$$
$$\sinh(\theta + i\phi), \quad \cosh(\theta + i\phi), \quad \tanh(\theta + i\phi)$$

which are tabulated below and of certain others.

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Circular Functions

Function	Real part	Imaginary part
$\sin(\theta + i\phi)$	$\sin\theta \cosh\phi$	$\cos\theta \sinh\phi$
$\cos(\theta + i\phi)$	$\cos\theta \cosh\phi$	$-\sin\theta \sinh\phi$
$\tan(\theta + i\phi)$	$\dfrac{\sin 2\theta}{\cos 2\theta + \cosh 2\phi}$	$\dfrac{\sinh 2\phi}{\cos 2\theta + \cosh 2\phi}$

Hyperbolic Functions

Function	Real part	Imaginary part
$\sinh(\theta + i\phi)$	$\sinh\theta \cos\phi$	$\cosh\theta \sin\phi$
$\cosh(\theta + i\phi)$	$\cosh\theta \cos\phi$	$\sinh\theta \sin\phi$
$\tanh(\theta + i\phi)$	$\dfrac{\sinh 2\theta}{\cosh 2\theta + \cos 2\phi}$	$\dfrac{\sin 2\phi}{\cosh 2\theta + \cos 2\phi}$

Results. In finding the real and imaginary parts, use

$$\cos ix = \cosh x, \quad \sin ix = i \sinh x.$$

1. $\sin(\theta + i\phi)$, $\sin(\theta - i\phi)$

$$\begin{aligned}\sin(\theta + i\phi) &= \sin\theta \cos i\phi + \cos\theta \sin i\phi \\ &= \sin\theta \cosh\phi + \cos\theta\,(i \sinh\phi) \\ &= \sin\theta \cosh\phi + i \cos\theta \sinh\phi.\end{aligned}$$

Replacing i by $-i$, we have

$$\sin(\theta - i\phi) = \sin\theta \cosh\phi - i \cos\theta \sinh\phi.$$

2. $\cos(\theta + i\phi)$, $\cos(\theta - i\phi)$

$$\begin{aligned}\cos(\theta + i\phi) &= \cos\theta \cos i\phi - \sin\theta \sin i\phi \\ &= \cos\theta \cosh\phi - \sin\theta\,(i \sinh\phi) \\ &= \cos\theta \cosh\phi - i \sin\theta \sinh\phi.\end{aligned}$$

Replacing i by $-i$, we have

$$\cos(\theta - i\phi) = \cos\theta \cosh\phi + i \sin\theta \sinh\phi.$$

3. $\tan(\theta + i\phi)$, $\tan(\theta - i\phi)$

Results. $\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$,

$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$.

$$\tan(\theta + i\phi) = \frac{\sin(\theta + i\phi)}{\cos(\theta + i\phi)} = \frac{\sin(\theta + i\phi)\cos(\theta - i\phi)}{\cos(\theta + i\phi)\cos(\theta - i\phi)}$$

$$= \frac{\tfrac{1}{2}[\sin(\theta + i\phi + \theta - i\phi) + \sin(\theta + i\phi - \theta + i\phi)]}{\tfrac{1}{2}[\cos(\theta + i\phi + \theta - i\phi) + \cos(\theta + i\phi - \theta + i\phi)]}$$

$$= \frac{\tfrac{1}{2}[\sin 2\theta + \sin 2i\phi]}{\tfrac{1}{2}[\cos 2\theta + \cos 2i\phi]} = \frac{\sin 2\theta + (i \sinh 2\phi)}{\cos 2\theta + (\cosh 2\phi)}$$

$$= \frac{\sin 2\theta}{\cos 2\theta + \cosh 2\phi} + i\frac{\sinh 2\phi}{\cos 2\theta + \cosh 2\phi}.$$

Replacing i by $-i$, we have

$$\tan(\theta - i\phi) = \frac{\sin 2\theta}{\cos 2\theta + \cosh 2\phi} - i\frac{\sinh 2\phi}{\cos 2\theta + \cosh 2\phi}.$$

4. $\cot(\theta + i\phi)$, $\cot(\theta - i\phi)$ **(2003)**

Results. $\cos A \sin B = \tfrac{1}{2}[\sin(A+B) - \sin(A-B)]$,
$\sin A \sin B = -\tfrac{1}{2}[\cos(A+B) - \cos(A-B)]$.

$$\cot(\theta + i\phi) = \frac{\cos(\theta + i\phi)}{\sin(\theta + i\phi)} = \frac{\cos(\theta + i\phi)\sin(\theta - i\phi)}{\sin(\theta + i\phi)\sin(\theta - i\phi)}$$

$$= \frac{\tfrac{1}{2}[\sin(\theta + i\phi + \theta - i\phi) - \sin(\theta + i\phi - \theta + i\phi)]}{-\tfrac{1}{2}[\cos(\theta + i\phi + \theta - i\phi) - \cos(\theta + i\phi - \theta + i\phi)]}$$

$$= -\frac{\sin 2\theta - \sin 2i\phi}{\cos 2\theta - \cos 2i\phi} = -\frac{\sin 2\theta - i\sinh 2\phi}{\cos 2\theta - \cosh 2\phi}.$$

$$\therefore \quad \cot(\theta + i\phi) = -\frac{\sin 2\theta}{\cos 2\theta - \cosh 2\phi} + i\frac{\sinh 2\phi}{\cos 2\theta - \cosh 2\phi}.$$

Replacing i into $-i$, we have

$$\cot(\theta - i\phi) = -\frac{\sin 2\theta}{\cos 2\theta - \cosh 2\phi} - i\frac{\sinh 2\phi}{\cos 2\theta - \cosh 2\phi}.$$

5. $\sinh(\theta + i\phi)$

Result. $\sin ix = i \sinh x$ or $\sinh x = \dfrac{\sin ix}{i}$.

$$\sinh(\theta + i\phi) = \frac{\sin i(\theta + i\phi)}{i} = \frac{\sin(i\theta - \phi)}{i}$$

$$= \frac{i\sin(i\theta - \phi)}{i^2} = -i\sin(i\theta - \phi)$$

$$= -i[\sin i\theta \cos\phi - \cos i\theta \sin\phi]$$

$$= -i\,[(i \sinh\theta)\cos\phi - (\cosh\theta)\sin\phi]$$

$$= \sinh\theta\cos\phi + i\cosh\theta\sin\phi.$$

6. cosh $(\theta + i\phi)$

Result. $\cos ix = \cosh x$ or $\cosh x = \cos ix$.

$$\cosh(\theta + i\phi) = \cos i(\theta + i\phi) = \cos(i\theta - \phi)$$

$$= \cos i\theta\cos\phi + \sin i\theta\sin\phi$$

$$= [\cosh\theta]\cos\phi + [i\sinh\theta]\sin\phi.$$

7. tanh $(\theta + i\phi)$.

Results. $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$,

$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$.

$$\tanh(\theta + i\phi) = \frac{\tan i(\theta + i\phi)}{i} = \frac{i\tan(i\theta - \phi)}{i^2}$$

$$= -i\tan(i\theta - \phi) = -i\,\frac{\sin(i\theta - \phi)}{\cos(i\theta - \phi)}$$

$$= -i\,\frac{\sin(i\theta - \phi)\cos(i\theta + \phi)}{\cos(i\theta - \phi)\cos(i\theta + \phi)}$$

$$= -i\,\frac{\frac{1}{2}[\sin(2i\theta) + \sin(-2\phi)]}{\frac{1}{2}[\cos(2i\theta) + \cos(-2\phi)]}$$

$$= -i\,\frac{[i\sinh 2\theta] - \sin 2\phi}{[\cosh 2\theta] + \cos 2\phi}$$

$$= \frac{\sinh 2\theta}{\cosh 2\theta + \cos 2\phi} + i\,\frac{\sin 2\phi}{\cosh 2\theta + \cos 2\phi}.$$

Lastly we shall express $\operatorname{cosec}(\theta + i\phi)$ and $\sec(\theta + i\phi)$ in $a + ib$ form.

8. cosec $(\theta + i\phi)$, **sec** $(\theta + i\phi)$

$$\operatorname{cosec}(\theta + i\phi) = \frac{1}{\sin(\theta + i\phi)} = \frac{\sin(\theta - i\phi)}{\sin(\theta + i\phi)\sin(\theta - i\phi)}$$

$$= \frac{\sin(\theta - i\phi)}{-\frac{1}{2}[\cos 2\theta - \cos 2i\phi]}$$

$$= \frac{2[\sin\theta\cos i\phi - \cos\theta\sin i\phi]}{\cos 2i\phi - \cos 2\theta}$$

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$$= \frac{2 \sin \theta \cosh \phi}{\cosh 2\phi - \cos 2\theta} - i \frac{2 \cos \theta \sinh \phi}{\cosh 2\phi - \cos 2\theta}.$$

$$\sec (\theta + i\phi) = \frac{1}{\cos (\theta + i\phi)}$$

$$= \frac{\cos (\theta - i\phi)}{\cos (\theta + i\phi) \cos (\theta - i\phi)}$$

$$= \frac{\cos (\theta - i\phi)}{\frac{1}{2} [\cos 2\theta + \cos 2i\phi]}$$

$$= \frac{2 [\cos \theta \cos i\phi + \sin \theta \sin i\phi]}{\cos 2\theta + \cos 2i\phi}$$

$$= \frac{2 \cos \theta \cosh \phi}{\cos 2\theta + \cosh 2\phi} + i \frac{2 \sin \theta \sinh \phi}{\cos 2\theta + \cosh 2\phi}.$$

UNIT- IV

HYPERBOLIC FUNCTIONS

Problems

1. Find the expansions of $\cosh^4\theta$ and $\sinh^4\theta$.

Solution:

W.K.T $\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$ and $\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$.

$$\therefore \cosh^4\theta = \left[\frac{e^{\theta}+e^{-\theta}}{2}\right]^4$$

$$= \frac{e^{4\theta}+4e^{2\theta}+6+4e^{-2\theta}+e^{-4\theta}}{2^4}$$

$$= \frac{1}{2^3}\left[\frac{(e^{4\theta}+e^{-4\theta})+4(e^{2\theta}+e^{-2\theta})+6}{2}\right]$$

$$= \frac{1}{2^3}\left[\frac{e^{4\theta}+e^{-4\theta}}{2}+4\cdot\frac{e^{2\theta}+e^{-2\theta}}{2}+\frac{6}{2}\right]$$

$$= \frac{1}{2^3}\left[\cosh 4\theta + 4\cosh 2\theta + 3\right]$$

$$\therefore \cosh^4\theta = \frac{1}{2^3}\left[\cosh 4\theta + 4\cosh 2\theta + 3\right].$$

$$\sinh^4\theta = \left[\frac{e^{\theta}-e^{-\theta}}{2}\right]^4$$

$$= \frac{e^{4\theta}-4e^{2\theta}+6-4e^{-2\theta}+e^{-4\theta}}{2^4}$$

$$= \frac{1}{2^3}\left[\frac{(e^{4\theta}+e^{-4\theta})-4(e^{2\theta}+e^{-2\theta})+6}{2}\right]$$

$$= \frac{1}{2^3}\left[\frac{e^{4\theta}+e^{4\theta}}{2}-4\,\frac{e^{2\theta}+e^{-2\theta}}{2}+\frac{6}{2}\right]$$

$$= \frac{1}{2^3}\left[\cosh 4\theta - 4\cosh 2\theta + 3\right].$$

$$\therefore \sinh^4\theta = \frac{1}{2^3}\left(\cosh 4\theta - 4\cosh 2\theta + 3\right).$$

2. Prove the following.

(i) $\cosh^2 x + \sinh^2 x = \cosh 2x$

(ii) $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$

(iii) $\tanh 2x = \dfrac{2\tan x}{1+\tanh^2 x}$

Proof:

(i) We know that $\cos^2 x - \sin^2 x = \cos 2x$

Replacing x by ix, we get.

③

$$(\cos ix)^2 - (\sin ix)^2 = \cos 2ix.$$

$$(\cosh x)^2 - (i \sinh x)^2 = \cosh 2x$$

$$\therefore \cosh^2 x + \sinh^2 x = \cosh 2x \qquad \left(\because \begin{array}{l} \cos ix = \cosh x \\ \sin ix = i \sinh x \end{array}\right.$$

(ii) We know that $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

$$\therefore (\cos ix)^2 = \frac{1}{2}(\cos 2ix + 1)$$

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

(iii) w.k.T $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

$$\therefore \tan 2ix = \frac{2\tan ix}{1-\tan^2 ix} \qquad (\tan ix = i\tanh x)$$

$$\therefore i\tanh 2x = \frac{2i\tanh x}{1-(i\tanh x)^2}$$

$$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}.$$

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3. Prove that $\frac{1+\tanh x}{1-\tanh x} = \cosh 2x + \sinh 2x$

Proof:

$$\frac{1+\tanh x}{1-\tanh x} = \frac{1+\frac{\sinh x}{\cosh x}}{1-\frac{\sinh x}{\cosh x}}$$

$$= \frac{\cosh x + \sinh x}{\cosh x - \sinh x}$$

$$= \frac{\cosh x + \sinh x}{\cosh x - \sinh x} \times \frac{\cosh x + \sinh x}{\cosh x + \sinh x}$$

$$= \frac{\cosh^2 x + \sinh^2 x + 2\sinh x \cosh x}{\cosh^2 x - \sinh^2 x}$$

$$= (\cosh^2 x + \sinh^2 x) + 2\sinh x \cosh x$$

$$= \cosh 2x + \sinh 2x.$$

$$\left(\begin{aligned} \cosh 2x &= \cosh^2 x + \sinh^2 x \\ \sinh 2x &= 2\sinh x \cosh x. \end{aligned} \right.$$

$$\therefore \frac{1+\tanh x}{1-\tanh x} = \cosh 2x + \sinh 2x.$$

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Problems:

1. Prove that $\sinh(A+B) = \sinh A \cdot \cosh B + \cosh A \sinh B$.

Proof:

We know that $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

Replace A by iA, B by iB, we have

$$\sin(iA+iB) = \sin iA \cos iB + \cos iA \sin iB$$

$$i\sinh(A+B) = i\sinh A \cosh B + \cosh A \cdot i\sinh B.$$

$$\therefore \sinh(A+B) = \sinh A \cosh B + \cosh A \cdot \sinh B.$$

$$\left[\because \begin{array}{l} \cos ix = \cosh x \\ \sin ix = i\sinh x \end{array}\right.$$

---

2. If $\tan\frac{x}{2} = \tanh x/2$, show that $\cos x \cosh x = 1$.

Proof:

We know that $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

$$\therefore \cos\theta = \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2} \;;\; \therefore \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2} \quad \to (1)$$

Changing x into ix, we have

$$\cos ix = \frac{1-\tan^2 ix/2}{1+\tan^2 ix/2}$$

$$\left[\because \begin{array}{l} \cos ix = \cosh x \\ \tan ix = i\tanh x. \end{array}\right.$$

(6)

$$\therefore \cosh x = \frac{1-(i\tanh x/2)^2}{1+(i\tanh x/2)^2}$$

$$= \frac{1+\tanh^2 x/2}{1-\tanh^2 x/2}$$

$$\cosh x = \frac{1+\tan^2 x/2}{1-\tan^2 x/2} \quad [\because \tan x/2 = \tanh x/2]$$

$$\therefore \cosh x = \frac{1}{\cos x}.$$

$$\therefore \cos x \cosh x = 1.$$

---

3. If $\sin(A+iB) = x+iy$, then.

(i) Show that $x = \sin A \cosh B$.

(ii) Show that $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$

(iii) Show that $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$

Proof:

Given that $x+iy = \sin(A+iB)$

$$x+iy = \sin A \cos iB + \cos A \sin iB$$

$$= \sin A \cosh B + i \cos A \sinh B.$$

Equating real and imaginary parts, we get.

$$\left.\begin{aligned} x &= \sin A \cosh B \\ y &= \cos A \sinh B \end{aligned}\right\} \longrightarrow (1)$$

$$\therefore \cosh B = \frac{x}{\sin A}\ ; \sinh B = \frac{y}{\cos A}.$$

Using $\cosh^2 B - \sinh^2 B = 1$, we get

$$\therefore \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1 \longrightarrow (2)$$

From (1), $\sin A = \frac{x}{\cosh B}$; $\cos A = \frac{y}{\sinh B}$.

using $\sin^2 A + \cos^2 A = 1$

$$\therefore \frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1. \longrightarrow (3)$$

---

4. If $\log[\sin(\theta + i\phi)] = A + iB$, prove that

$$2e^{2A} = \cosh 2\phi - \cos 2\theta$$

Proof: Given that

$$\log[\sin(\theta + i\phi)] = A + iB$$

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(8)

$$e^{\log(\sin(\theta+i\phi))} = e^{A+iB}$$

$$\therefore \sin(\theta+i\phi) = e^{A}\cdot e^{iB}$$

$$\sin\theta\cosh\phi + i\cos\theta\cdot\sinh\phi = e^{A}(\cos B + i\sin B)$$

$$\therefore e^{A}\cos B = \sin\theta\cosh\phi$$

$$e^{A}\sin B = \cos\theta\sinh\phi$$

Squaring and adding, we get

$$e^{2A} = \sin^2\theta\cosh^2\phi + \cos^2\theta\sinh^2\phi$$

$$= (1-\cos^2\theta)\cosh^2\phi + \cos^2\theta(\cosh^2\phi - 1)$$

$$(\because \cos^2\theta + \sin^2\theta = 1$$

$$\cosh^2\phi - \sinh^2\phi = 1$$

$$= \cosh^2\phi - \cos^2\theta$$

$$= \frac{1}{2}(1+\cosh 2\phi) - \frac{1}{2}(1+\cos 2\theta)$$

$$e^{2A} = \frac{1}{2}[\cosh 2\phi - \cos 2\theta].$$

$$[\because \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

5. If $x+iy = c\cos(A-iB)$, find the value of

$$\left(\frac{x}{\cosh B}\right)^2 + \left(\frac{y}{\sinh B}\right)^2$$

Solution

Given that $x+iy = c\cdot\cos(A-iB)$

$\therefore x+iy = c[\cos A\cosh B + i\sin A\sinh B]$.

Equating real and imaginary parts, we get.

$x = c\cos A\cosh B.$

$y = c\sin A\sinh B.$

$$\therefore \left(\frac{x}{\cosh B}\right)^2 + \left(\frac{y}{\sinh B}\right)^2 = (c\cos A)^2 + (c\sin A)^2$$

$$= c^2$$

//.

6. If $\cos(A+iB) = x+iy$, then show that

(i) $\frac{x^2}{\cos^2 A} - \frac{y^2}{\sin^2 A} = 1$, $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$.

(ii) $x^2\operatorname{sech}^2 B + y^2\operatorname{cosech}^2 B = 1$.

Solution:

Given that. $x+iy = \cos(A+iB)$.

$\therefore x+iy = \cos A \cos iB - \sin A \sin iB$

$= \cos A \cosh B - i \sin A \sinh B$

Equating real and imaginary parts, we get.

$$\left.\begin{aligned} x &= \cos A \cosh B \\ y &= -\sin A \sinh B \end{aligned}\right\} \longrightarrow (1)$$

$\therefore \cosh B = \dfrac{x}{\cos A}$; $\sinh B = \dfrac{-y}{\sin A}$

$\therefore$ Using $\cosh^2 B - \sinh^2 B = 1$, we get

$$\frac{x^2}{\cos^2 A} - \frac{y^2}{\sin^2 A} = 1$$

Again from (1), we have,

$\cos A = \dfrac{x}{\cosh B}$; $\sin A = \dfrac{-y}{\sinh B}$.

$\therefore$ Using $\cos^2 A + \sin^2 A = 1$, we get.

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1.$$

$\Rightarrow x^2 \operatorname{sech}^2 B + y^2 \operatorname{cosech}^2 B = 1.$

Problems.

1. If $\cos(A+iB) = \cos\theta + i\sin\theta$, show that $\cos 2A + \cosh 2B = 2$.

Solution: Given that $\cos(A+iB) = \cos\theta + i\sin\theta$.

We write $\cos(A+iB) = e^{i\theta}$

and $\cos(A-iB) = e^{-i\theta}$

$\therefore \cos(A+iB)\cdot\cos(A-iB) = e^{i\theta}\cdot e^{-i\theta} = 1$

$\therefore \frac{1}{2}\left\{\cos[A+iB+A-iB] + \cos(A+iB-A+iB)\right\} = 1$

$\therefore \frac{1}{2}[\cos 2A + \cos 2iB] = 1$

$\therefore \cos 2A + \cosh 2B = 2$.

2. If $\cos(x+iy) = r(\cos\theta + i\sin\theta)$, Then prove that

(i) $r^2 = \frac{1}{2}(\cos 2x + \cosh 2y)$

(ii) $\tan\theta = -\tan x \tanh y$

(iii) $y = \frac{1}{2}\log\frac{\sin(x-\theta)}{\sin(x+\theta)}$.

(iv) $e^{2x} = \frac{\sin(x-\theta)}{\sin(x+\theta)}$.

Solution: Given that $\cos(x+iy) = r(\cos\theta + i\sin\theta)$

(i) We write $\cos(x+iy) = re^{i\theta}$

and the conjugate $\cos(x-iy) = re^{-i\theta}$.

$\therefore \cos(x+iy)\cdot\cos(x-iy) = re^{i\theta}\cdot r\cdot e^{-i\theta} = r^2$

$\frac{1}{2}\{\cos(x+iy+x-iy) + \cos(x+iy-x+iy)\} = r^2$

$\frac{1}{2}\{\cos 2x + \cos 2iy\} = r^2$

$\therefore r^2 = \frac{1}{2}(\cos 2x + \cosh 2y)$.

(ii) Given $\cos(x+iy) = r(\cos\theta + i\sin\theta)$

$\cos x \cos iy - \sin x \sin iy = r(\cos\theta + i\sin\theta)$

$\cos x \cosh y - i\sin x \sinh y = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts, we get

$\cos x \cosh y = r\cos\theta \longrightarrow (1)$

$-\sin x \sinh y = r\sin\theta \longrightarrow (2)$

$\frac{(2)}{(1)} \Rightarrow \tan\theta = -\tan x \tanh y. \longrightarrow (3)$

(iii) $(3) \Rightarrow \tan\theta = -\tan x \tanh y.$

$\therefore \tanh y = -\frac{\tan\theta}{\tan x}.$

(13)

$\therefore y = \tanh^{-1}\left(-\frac{\tan\theta}{\tan x}\right)$.

We know that $\tanh^{-1}x = \frac{1}{2}\log\frac{1+x}{1-x}$.

$$\therefore y = \frac{1}{2}\log\frac{1-\frac{\tan\theta}{\tan x}}{1+\frac{\tan\theta}{\tan x}}$$

$$= \frac{1}{2}\log\frac{\tan x - \tan\theta}{\tan x + \tan\theta}$$

Multiplying both the nr. and dr. by $\cos x\cos\theta$

$$\therefore y = \frac{1}{2}\log\frac{\sin x\cos\theta - \cos x\sin\theta}{\sin x\cos\theta + \cos x\sin\theta}.$$

$$y = \frac{1}{2}\log\frac{\sin(x-\theta)}{\sin(x+\theta)}. \quad \text{—— (4)}$$

(iv) (4) $\Rightarrow$, $2y = \log\frac{\sin(x-\theta)}{\sin(x+\theta)}$

$$\therefore e^{2y} = e^{\log\frac{\sin(x-\theta)}{\sin(x+\theta)}}$$

$$\therefore e^{2y} = \frac{\sin(x-\theta)}{\sin(x+\theta)} \quad (\because e^{\log x} = x.$$

Inverse Hyperbolic Functions.

Inverse hyperbolic functions are defined in the same way as inverse circular functions are defined.

i.e., If $\sinh x = y$ then $x = \sinh^{-1} y$.

If $\cosh x = y$ then $x = \cosh^{-1} y$.

Now we shall obtain the values of $\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$.

(i) $\sinh^{-1} x = \log(x + \sqrt{x^2+1})$

(ii) $\cosh^{-1} x = \pm \log(x + \sqrt{x^2-1})$

(iii) $\tanh^{-1} x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$.

Proof:

(i) Denote $\sinh^{-1} x$ by y.

Then $x = \sinh y$

$$= \frac{e^y - e^{-y}}{2}$$

$$\therefore 2x = e^y - \frac{1}{e^y}.$$

$$2x(e^y) = (e^y)^2 - 1$$

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$\therefore (e^y)^2 - 2x(e^y) - 1 = 0$

$\therefore e^y = \dfrac{2x \pm \sqrt{4x^2+4}}{2} = \dfrac{2x \pm 2\sqrt{x^2+1}}{2}$

$\therefore e^y = x \pm \sqrt{x^2+1}$

But e^y lies between 0 and ∞.

$\therefore$ it does not become negative.

Hence $\log e^y = \log(x \pm \sqrt{x^2+1})$

$\therefore \quad y = \log(x \pm \sqrt{x^2+1})$

(ii) Denote $\cosh^{-1}x$ by y.

Then $x = \cosh y = \dfrac{e^y + e^{-y}}{2}$.

$\therefore 2x = e^y + \dfrac{1}{e^y}$

$(2x)e^y = (e^y)^2 + 1$

$\therefore (e^y)^2 - 2x(e^y) + 1 = 0$

$\therefore e^y = \dfrac{2x \pm \sqrt{4x^2-4}}{2} = \dfrac{2x \pm 2\sqrt{x^2-1}}{2}$

$\therefore e^y = x \pm \sqrt{x^2-1}$

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Now $x-\sqrt{x^2-1}$ can be rewritten as

$$x-\sqrt{x^2-1} = \frac{(x-\sqrt{x^2-1})(x+\sqrt{x^2-1})}{(x+\sqrt{x^2-1})}$$

$$= \frac{x^2-(x^2-1)}{x+\sqrt{x^2-1}} = \frac{1}{x+\sqrt{x^2-1}}$$

$$= (x+\sqrt{x^2-1})^{-1}$$

$$\therefore e^y = (x+\sqrt{x^2-1})^{\pm 1}$$

$$\therefore \log e^y = \log (x+\sqrt{x^2-1})^{\pm 1}$$

$$y = \pm \log (x+\sqrt{x^2-1})$$ //.

(iii) Denote $\tanh^{-1} x$ by y.

Then $x = \tanh y$

$$= \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$= \frac{e^y - \left(\frac{1}{e^y}\right)}{e^y + \left(\frac{1}{e^y}\right)} = \frac{(e^y)^2 - 1}{(e^y)^2 + 1}$$

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$$x\left[(e^y)^2+1\right] = (e^y)^2-1$$

i.e. $(e^y)^2(x-1) = -(x+1)$

$$\therefore (e^y)^2 = \frac{-(x+1)}{x-1} = \frac{1+x}{1-x}.$$

$$\therefore e^y = \left(\frac{1+x}{1-x}\right)^{1/2}.$$

$$\log e^y = \log\left(\frac{1+x}{1-x}\right)^{1/2}$$

$$\therefore y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) //$$

Problems:

1. Find the real and Imaginary part of $\tan^{-1}(x+iy)$.

Solution: Let $\tan^{-1}(x+iy) = a+ib$ Then

$\tan(a+ib) = x+iy$

$\tan(a-ib) = x-iy$.

(13)

(i) To find the real part, use

$$2a = (a+ib) + (a-ib)$$

We know that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

$$\therefore \tan 2a = \tan[(a+ib) + (a-ib)]$$

$$= \frac{\tan(a+ib) + \tan(a-ib)}{1 - \tan(a+ib)\cdot\tan(a-ib)}$$

$$= \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}$$

$$= \frac{2x}{1-(x^2+y^2)}$$

$$\therefore 2a = \tan^{-1}\left(\frac{2x}{1-(x^2+y^2)}\right)$$

$$\therefore a = \frac{1}{2}\tan^{-1}\left[\frac{2x}{1-(x^2+y^2)}\right]$$

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(19)

(ii) To find the Imaginary part, use

$$2ib = (a+ib) - (a-ib)$$

$$\tan(2ib) = \frac{\tan(a+ib) - \tan(a-ib)}{1+\tan(a+ib)\tan(a-ib)}$$

$$\left(\because \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}\right.$$

$$\therefore i\tanh 2b = \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}$$

$$= \frac{2yi}{1+(x^2+y^2)}$$

$$\therefore \tanh 2b = \frac{2y}{1+x^2+y^2}$$

$$2b = \tanh^{-1}\left[\frac{2y}{1+x^2+y^2}\right]$$

$$\therefore b = \frac{1}{2}\tanh^{-1}\left[\frac{2y}{1+x^2+y^2}\right].$$

But $\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

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Thus now.

$$b = \frac{1}{2}\left[\frac{1}{2}\log\frac{\left[1 + 2y/(1+x^2+y^2)\right]}{\left[1 - 2y/(1+x^2+y^2)\right]}\right]$$

$$= \frac{1}{4}\log\frac{(1+x^2+y^2)+2y}{(1+x^2+y^2)-2y}.$$

$$= \frac{1}{4}\log\frac{x^2+(1+y^2+2y)}{x^2+(1+y^2-2y)}$$

$$b = \frac{1}{4}\log\frac{x^2+(1+y)^2}{x^2+(1-y)^2} \;//$$

---

2. If $\tan(\theta+i\phi) = \cos\alpha + i\sin\alpha$, then show that

(i) $\theta = \frac{n\pi}{2} + \frac{\pi}{4}$.

(ii) $\phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$

<u>Solution:</u>

Now $\tan(\theta+i\phi) = \cos\alpha + i\sin\alpha$

$$\theta + i\phi = \tan^{-1}(\cos\alpha + i\sin\alpha).$$

(21)

But, we know that,

$$\tan^{-1}(x+iy) = \frac{1}{2}\tan^{-1}\frac{2x}{1-(x^2+y^2)} + \frac{i}{4}\log\frac{x^2+(1+y)^2}{x^2+(1-y)^2}$$

i) $\theta = \frac{1}{2}\tan^{-1}\frac{2\cos\alpha}{1-(\cos^2\alpha+\sin^2\alpha)}$

$$= \frac{1}{2}\tan^{-1}\frac{2\cos\alpha}{1-1}$$

$$= \frac{1}{2}\tan^{-1}(\infty)$$

$$= \frac{1}{2}\left(n\pi+\frac{\pi}{2}\right)$$

$$= \frac{n\pi}{2}+\frac{\pi}{4}.$$

(ii) $\phi = \frac{1}{4}\log\frac{\cos^2\alpha+(1+\sin\alpha)^2}{\cos^2\alpha+(1-\sin\alpha)^2}$

$$= \frac{1}{4}\log\frac{(\cos^2\alpha+\sin^2\alpha)+1+2\sin\alpha}{(\cos^2\alpha+\sin^2\alpha)+1-2\sin\alpha}$$

$$= \frac{1}{4}\log\frac{2(1+\sin\alpha)}{2(1-\sin\alpha)}.$$

(22)

$$\therefore \phi = \frac{1}{4}\log\frac{1+\sin\alpha}{1-\sin\alpha}$$

$$= \frac{1}{4}\log\frac{1-\cos\left(\frac{\pi}{2}+\alpha\right)}{1+\cos\left(\frac{\pi}{2}+\alpha\right)}$$

$(\because \cos(90+\theta) = \sin\theta.$

$$= \quad \frac{1-\cos A}{2} = \sin^2\frac{A}{2}, \quad \frac{1+\cos A}{2} = \cos^2\frac{A}{2}$$

$$\therefore \phi = \frac{1}{4}\log\frac{2\sin^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}$$

$$= \frac{1}{4}\log\tan^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$$= \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$(\because \log a^b = b\log a.$

$$\therefore \phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

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3. If $\tan^{-1}(2-i) = x+iy$, show that $4y = -\log 2$.

Solution: We know that

$$Im[\tan^{-1}(a+ib)] = \frac{1}{4}\log\frac{a^2+(1+b)^2}{a^2+(1-b)^2}$$

Now $a=2$, and $b=1$, we have.

$$Im[\tan^{-1}(2-i)] = \frac{1}{4}\log\frac{2^2+(1-1)^2}{2^2+(1+1)^2}$$

$$\therefore y = \frac{1}{4}\log\frac{4}{8} = \frac{1}{4}\log\frac{1}{2}$$

$$\therefore 4y = -\log 2 \text{ //}$$

---

4. Separate into real and imaginary parts $\tanh(1+i)$

Solution:

We know that

$$\tan(ix) = i\tanh x.$$

Put $x = 1+i$, $i\tanh(1+i) = \tan i(1+i)$
$$= \tan(i-1).$$

(24)

$$\therefore i \tanh(1+i) = \frac{\sin(i-1)}{\cos(i-1)}$$

$$= \frac{2\cos(i+1)\sin(i-1)}{2\cos(i+1)\cos(i-1)}$$

We know that

$$\cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$$

$$\therefore i \tanh(1+i) = \frac{\sin 2i - \sin 2}{\cos 2i + \cos 2}.$$

$$= \frac{i\sinh 2 - \sin 2}{\cosh 2 + \cos 2}$$

$$\therefore \tanh(1+i) = \frac{\sinh 2 + i\sin 2}{\cosh 2 + \cos 2}.$$

$$\therefore \text{Real part} = \frac{\sinh 2}{\cosh 2 + \cos 2}.$$

$$\text{Imaginary part} = \frac{\sin 2}{\cosh 2 + \cos 2}.$$

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5. If $\tan(x+iy) = u+iv$, prove that

$$\frac{u}{v} = \frac{\sin 2x}{\sinh 2y}.$$

Proof:

$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)}$$

$$= \frac{2\cos(x-iy)\sin(x+iy)}{2\cos(x-iy)\cos(x+iy)}$$

$$= \frac{\sin(2x) + \sin(2iy)}{\cos(2x) + \cos(2iy)}$$

$$= \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}.$$

This expression is given as $u+iv$

as $\tan(x+iy) = u+iv$.

$$\therefore u = \frac{\sin 2x}{\cos 2x + \cosh 2y}, \quad v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}.$$

$$\therefore \frac{u}{v} = \frac{\sin 2x}{\sinh 2y}.$$

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(26)

6. If $\cosh u = \sec\theta$, show that

$$u = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

Solution: Let $\cosh u = \sec\theta$

$$\therefore u = \cosh^{-1}(\sec\theta)$$

$$= \log_e\left(\sec\theta + \sqrt{\sec^2\theta - 1}\right)$$

$$= \log_e(\sec\theta + \tan\theta)$$

$$\left(\because \cosh^{-1}x = \log\left(x+\sqrt{x^2-1}\right)\right)$$

$$u = \log_e\left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)$$

$$= \log_e\left(\frac{1+\sin\theta}{\cos\theta}\right) \quad \left(\because \sin A = \frac{2\tan A/2}{1+\tan^2 A/2}\right.$$

$$\cos A = \frac{1-\tan^2 A/2}{1+\tan^2 A/2}$$

$$= \log_e\left\{\left(1+\frac{2\tan\theta/2}{1+\tan^2\theta/2}\right) \div \left(\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}\right)\right\}$$

$$= \log_e\frac{1+\tan\theta/2}{1-\tan\theta/2} = \log_e\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

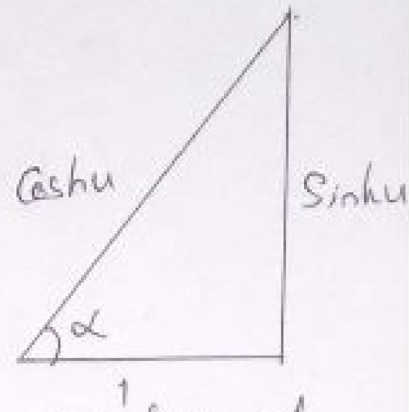
$$\therefore u = \log_e\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) //.$$

Implications of the relations.

(i) $\sinh u = \tan\alpha$

(ii) $\cosh u = \sec\alpha$

(iii) $\tanh u = \sin\alpha$.



When u is real the minimum value of $\cosh u = 1$. Now $\cosh^2 u = 1 + \sinh^2 u$.

So, given an u, a right angled triangle can be drawn with hypotenuse as $\cosh u$ and the other two sides as $\sinh u, 1$.

If α is the angle opposite to the side $\sinh u$, then,

(i) $\sinh u = \dfrac{\sinh u}{1} = \dfrac{opp}{adj.} = \tan\alpha$

(ii) $\cosh u = \dfrac{\cosh u}{1} = \dfrac{hyp}{adj} = \sec\alpha$

(iii) $\tanh u = \dfrac{\sinh u}{\cosh u} = \dfrac{opp}{hyp} = \sin\alpha$.

Now we shall show that, if one of (A) is given, then we can obtain the other two as follows.

(i) If $\sinh u = \tan\alpha$, then

$$\cosh u = \sqrt{1+\sinh^2 u} = \sqrt{1+\tan^2\alpha}$$
$$= \sec\alpha.$$

$\therefore \cosh u = \sec\alpha$

$$\tanh u = \frac{\sinh u}{\cosh u} = \frac{\tan\alpha}{\sec\alpha}$$
$$= \frac{\sin\alpha}{\cos\alpha\sec\alpha}$$
$$= \sin\alpha.$$

$\therefore \tanh u = \sin\alpha.$

(ii) If $\cosh u = \sec\alpha$, then

$$\sinh u = \sqrt{\cosh^2 u - 1} = \sqrt{\sec^2\alpha - 1} = \tan\alpha.$$

$\therefore \sinh u = \tan\alpha.$

$$\tanh u = \frac{\sinh u}{\cosh u} = \frac{\tan\alpha}{\sec\alpha} = \frac{\sin\alpha}{\cos\alpha\sec\alpha} = \sin\alpha.$$

$\therefore \tanh u = \sin\alpha.$

(iii) If $\tanh u = \sin\alpha$, then

$$\sinh u = \frac{1}{\operatorname{cosech} u} = \frac{1}{\sqrt{\coth^2 u - 1}}$$
$$= \frac{1}{\sqrt{\operatorname{cosec}^2\alpha - 1}} = \frac{1}{\cot\alpha} = \tan\alpha.$$

$\therefore \sinh u = \tan\alpha.$

$$\cosh u = \frac{1}{\operatorname{sech} u} = \frac{1}{\sqrt{1-\tanh^2 u}}$$

$$= \frac{1}{\sqrt{1-\sin^2\alpha}} = \frac{1}{\cos\alpha} = \sec\alpha.$$

$\therefore \cosh u = \sec\alpha.$

Result:

1. If (i) $\sinh u = \tan\alpha$ (or)

(ii) $\cosh u = \sec\alpha$ (or)

(iii) $\tanh u = \sin\alpha$. Then prove that

$u = \log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right).$

Proof: (i) $\sinh u = \tan\alpha.$

$$u = \sinh^{-1}(\tan\alpha)$$

$$= \log\left(\tan\alpha + \sqrt{1+\tan^2\alpha}\right)$$

$$= \log(\tan\alpha + \sec\alpha) \quad \text{--- (1)}$$

$$= \log\left(\frac{\sin\alpha}{\cos\alpha} + \frac{1}{\cos\alpha}\right)$$

$$= \log\left(\frac{1+\sin\alpha}{\cos\alpha}\right)$$

$$= \log\left(\frac{1-\cos(\pi/2+\alpha)}{\sin(\pi/2+\alpha)}\right)$$

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$$u = \log \frac{2\sin^2(\pi/4 + \alpha/2)}{2\sin(\frac{\pi}{4}+\alpha/2)\cos(\pi/4+\alpha/2)}$$

$$= \log\tan(\pi/4+\alpha/2).$$

(ii) $\cosh u = \sec\alpha.$

$$u = \cosh^{-1}(\sec\alpha)$$

$$= \log(\sec\alpha + \sqrt{\sec^2\alpha - 1})$$

$$= \log(\sec\alpha + \tan\alpha)$$

Rest is same as in (i) after (1).

$$\Rightarrow u = \log\tan(\pi/4 + \alpha/2)$$

(iii) $\tanh u = \sin\alpha.$

$$u = \tanh^{-1}(\sin\alpha).$$

$$= \frac{1}{2}\log\frac{1+\sin\alpha}{1-\sin\alpha}$$

$$= \frac{1}{2}\log\frac{(1+\sin\alpha)(1+\sin\alpha)}{(1-\sin\alpha)(1+\sin\alpha)}$$

$$= \frac{1}{2}\log\frac{(1+\sin\alpha)^2}{1-\sin^2\alpha}$$

$$= \frac{1}{2}\log\frac{(1+\sin\alpha)^2}{\cos^2\alpha}.$$

(31)

$$u = \frac{1}{2}\log\left(\frac{1+\sin\alpha}{\cos\alpha}\right)^2$$

$$= \log\frac{1+\sin\alpha}{\cos\alpha}.$$

$$\therefore u = \log(\sec\alpha + \tan\alpha)$$

Rest is same as in (i) after (1).

$$\Rightarrow u = \log\tan(\pi/4 + \alpha/2).$$

2. If $u = \log\tan(\pi/4 + \alpha/2)$ then prove that.

(i) $\sinh u = \tan\alpha$

(ii) $\cosh u = \sec\alpha$

(iii) $\tanh u = \sin\alpha$.

Proof: Given that $u = \log\tan(\pi/4 + \alpha/2)$

Take exponential on bothsides, we get.

$$\therefore e^u = \tan(\pi/4 + \alpha/2)$$

$$= \frac{\tan\pi/4 + \tan\alpha/2}{1 - \tan\pi/4\tan\alpha/2}$$

$$= \frac{1+\tan\alpha/2}{1-\tan\alpha/2}$$

$$= \frac{\cos\alpha/2 + \sin\alpha/2}{\cos\alpha/2 - \sin\alpha/2} \quad \text{--- (1)}$$

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$$e^{u} = \frac{(\cos\alpha/2 + \sin\alpha/2)(\cos\alpha/2 + \sin\alpha/2)}{(\cos\alpha/2 - \sin\alpha/2)(\cos\alpha/2 + \sin\alpha/2)}$$

$$= \frac{\cos^2\alpha/2 + \sin^2\alpha/2 + 2\sin\alpha/2\cos\alpha/2}{\cos^2\alpha/2 - \sin^2\alpha/2}$$

$$= \frac{1+\sin\alpha}{\cos\alpha}$$

$$= \sec\alpha + \tan\alpha. \quad \text{———} (2)$$

$$e^{-u} = \frac{1}{\sec\alpha + \tan\alpha}$$

$$= \frac{\sec\alpha - \tan\alpha}{(\sec\alpha + \tan\alpha)(\sec\alpha - \tan\alpha)}$$

$$= \frac{\sec\alpha - \tan\alpha}{\sec^2\alpha - \tan^2\alpha}$$

$$= \sec\alpha - \tan\alpha. \quad \text{———} (3)$$

using (2), and (3), we get

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(i) $\frac{e^u - e^{-u}}{2} = \frac{2\tan\alpha}{2}$

$\sinh u = \tan\alpha.$

(ii) $\frac{e^u + e^{-u}}{2} = \frac{2\sec\alpha}{2}$

$\therefore \cosh u = \sec\alpha$

(iii) $\frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{2\tan\alpha}{2\sec\alpha}$

$\therefore \tanh u = \sin\alpha.$

HOME WORK

(3). If $\tanh u/2 = \tan \alpha/2$, then show that

(i) $\sinh u = \tan\alpha$

(ii) $\cosh u = \sec\alpha$

(iii) $\tanh u = \sin\alpha.$

Hint $\tanh u/2 = \tan \alpha/2$

$\frac{u}{2} = \tanh^{-1}(\tan \alpha/2) = \frac{1}{2}\log\left(\frac{1+\tan\alpha/2}{1-\tan\alpha/2}\right)$

$u = \log \frac{1+\tan\alpha/2}{1+\tan\alpha/2}$ (or) $e^u = \frac{1+\tan\alpha/2}{1-\tan\alpha/2}$

Rest is as in the working after ① in Result 2. (page no. 31).

$$u = \log\left\{\frac{1+\frac{2\tan\theta/2}{1+\tan^2\theta/2}}{\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}}\right\}$$

$$= \log\left\{\frac{\frac{1+\tan^2\theta/2+2\tan\theta/2}{1+\tan^2\theta/2}}{\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}}\right\}$$

$$= \log\left\{\frac{(1+\tan\theta/2)^2}{(1+\tan\theta/2)(1-\tan\theta/2)}\right\}$$

$$= \log\left\{\frac{1+\tan\theta/2}{1-\tan\theta/2}\right\}$$

$$= \log\left\{\frac{\tan\pi/4+\tan\theta/2}{1-\tan\pi/4\tan\theta/2}\right\} \quad \left(\because \tan\frac{\pi}{4}=1\right)$$

$$= \log\tan\left(\pi/4+\theta/2\right)$$

$$\left(\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}\right.$$

$$\therefore u = \log\tan\left(\pi/4+\theta/2\right)$$

Assignment - I

1. Find the n^{th} differential coefficient of $\cos^5\theta \sin^7\theta$.

2. (i) If $x = a(t - \sin t)$, $y = a(1 + \cos t)$ find $\frac{d^2y}{dx^2}$ as a function of t.

   (ii) Find $\frac{d^2y}{dx^2}$ if $x = \sqrt{\sin 2t}$ and $y = \sqrt{\cos 2t}$.

3. If $y = \sin(m \sin^{-1}x)$, prove that $(1-x^2)y_2 - xy_1 + m^2y = 0$ and $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

4. Find the n^{th} differential coefficient of (i) $e^x \log x$ (ii) $x^n a^x$.

5. If $y = e^{a\sin^{-1}x}$, prove that (i) $(1-x^2)y_2 - xy_1 - a^2y = 0$ and Hence show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

6. Prove that the radius of curvature of any point of the cycloid $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$ is $4a\cos\frac{\theta}{2}$.

7. Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = 1$ at $(1/4, 1/4)$

8. Prove that the radius of curvature of a point $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $3a\sin\theta\cos\theta$.

(2)

9. Show that in the parabola $y^2 = 4ax$ at the point t, $\rho = -2a(1+t^2)^{3/2}$, $x = 2a + 3at^2$, $y = -2at^3$. Deduce the equation of the evolute.

10. Show that Evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ is another equal cycloid

11. Show that the radius of curvature of the curve $r^n = a^n \cos n\theta$ is $\frac{a^n r^{-n+1}}{n+1}$

12. Find the radius of curvature of the Cardioid $r = a(1-\cos\theta)$

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Assignment - II

1. Show that $-2^{10}\cos^5\theta\sin^6\theta = \cos 11\theta - \cos 9\theta - 5\cos 7\theta + 5\cos 5\theta + 10\cos 3\theta - 10\cos\theta$.

2. Find the expansion of (i) $\cos^9\theta$, (ii) $\sin^{10}\theta$

3. Find the expansion of.
(i) $\cos 9\theta$ (ii) $\cos 10\theta$ (iii) $\tan 12\theta$.

4. Show that $\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1$.

5. Express $\frac{\cos 5\theta}{\cos\theta}$ as a polynomial in
(i) $\cos\theta$ (ii) $\sin\theta$.

6. Express $\frac{\sin 9\theta}{\sin\theta}$ in terms of $\sin\theta$

7. Find $\lim\limits_{\theta \to 0} \frac{n\sin\theta - \sin n\theta}{\theta(\cos\theta - \sin n\theta)}$

8. If $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$, show that θ is Equal to $3^\circ 1'$ nearly.

9. If $\log[\sin(\theta + i\phi)] = A + iB$, prove that $2e^{2A} = \cosh 2\phi - \cos 2\theta$.

10. Find the real and Imaginary parts of $\tan^{-1}(x+iy)$.

11. If $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$, then show that

(i) $\theta = \frac{n\pi}{2} + \frac{\pi}{4}$

(ii) $\phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$

12. If $\cosh u = \sec\theta$, show that $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$.

=== * ===

①

UNIT-5

Logarithms of Complex quantities

Definition:

If u and z be any two Complex quantities such that $z = e^{u}$, Then u is called the logarithm of z and we write $u = \log_e z$ or simply $u = \log z$.

Note: We know, by Euler's relation that $\cos\theta + i\sin\theta = e^{i\theta}$.

$\therefore$ for any integral value of n,

$$e^{i2n\pi} = \cos 2n\pi + i\sin 2n\pi,$$

$$= 1 + 0$$

$$= 1$$

$\therefore$ for any Complex number z whose polar form is $r(\cos\theta + i\sin\theta)$

$$z = re^{i\theta} \times 1 = re^{i\theta} \cdot e^{i2n\pi}$$

$$= re^{i\theta + i2n\pi}$$

$$\operatorname{Log} z = \log r e^{i\theta + i2n\pi}$$

$$= \log r + \log e^{i\theta + 2n\pi i}$$

$$= \log r + i\theta + 2n\pi i$$

$$\therefore \operatorname{Log} z = \log r + i(\theta + 2n\pi),$$

$n = 0, \pm 1, \pm 2, \ldots$

In this $\log r + i\theta$ is called the principal value of $\log z$ and is denoted by

$$\log z = \log r + i\theta.$$

So that $\operatorname{Log} z = \log z + i2n\pi$.

Result. $(a+ib)^{x+iy} = e^{(x+iy)\log(a+ib)}$

To find the logarithm of $x+iy$

Let $\log_e(x+iy) = \alpha + i\beta$

$$\therefore x+iy = e^{\alpha+i\beta}$$

$$= e^{\alpha} \cdot e^{i\beta}$$

$$= e^{\alpha}(\cos\beta + i\sin\beta)$$

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(3)

$\therefore x = e^{\alpha}\cos\beta \,;\; y = e^{\alpha}\sin\beta$

Hence $x^2+y^2 = e^{2\alpha}(\cos^2\beta+\sin^2\beta)$

$\therefore e^{2\alpha} = x^2+y^2$

$\therefore \log e^{2\alpha} = \log(x^2+y^2)$

$2\alpha = \log(x^2+y^2)$

$\therefore \alpha = \frac{1}{2}\log(x^2+y^2)$

and $\qquad \frac{y}{x} = \frac{e^{\alpha}\sin\beta}{e^{\alpha}\cos\beta}.$

$\therefore \tan\beta = \frac{y}{x} \qquad \therefore \beta = \tan^{-1}\frac{y}{x}.$

$\therefore \log_e(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$

$= \log r + i\beta$

where $r = \sqrt{x^2+y^2}$ and $\tan\theta = y/x$.

$\therefore \log_e(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right).$

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(4)

General value of logarithm of $x+iy$

Let $\log_e(x+iy) = \alpha + i\beta$.

Then $x+iy = e^{\alpha+i\beta}$

$= e^{\alpha} \cdot e^{i\beta}$

$= e^{\alpha}(\cos\beta + i\sin\beta)$

$= e^{\alpha}\{\cos(2n\pi+\beta) + i\sin(2n\pi+\beta)\}$

$= e^{\alpha} \cdot e^{i(2n\pi+\beta)}$

$= e^{\alpha + 2n\pi i + i\beta}$.

$\therefore$ $\alpha + i\beta + 2n\pi i$ is the value of $\log_e(x+iy)$.

This is called the general value and is written with a capital letter

$\mathrm{Log}_e(x+iy) = \alpha + i\beta + 2n\pi i$

where n is any integer.

If $n=0$,

$$\log_e(x+iy) = \alpha + i\beta$$

$$\therefore \operatorname{Log}_e(x+iy) = \log_e(x+iy) + 2n\pi i$$

It is clear that logarithm of a complex quantity has more than one value.

$$\therefore \operatorname{Log}_e(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi i$$

Cor. 1: put $y=0$.

Then $\operatorname{Log} x = \frac{1}{2}\log(x^2) + 2n\pi i$

$$= \log x + 2n\pi i$$

$$\therefore \operatorname{Log} x = \log x + 2n\pi i$$

Hence the logarithm of a real positive quantity is many valued and that the principal value of the logarithm in its ordinary logarithm which is real.

Cor. 2. Let $y=0$ and x be negative.
(say $-x_1$)

If $\log(x+iy) = \alpha+i\beta$, then

$$x = e^{\alpha}\cos\beta, \quad y = e^{\alpha}\sin\beta.$$

In this case $e^{\alpha}\cos\beta = -x_1$

and $e^{\alpha}\sin\beta = 0$.

i.e. $x_1^2 = e^{2\alpha}$ $\therefore e^{\alpha} = x_1$

$\therefore e^{\alpha} = x_1$

$\cos\beta = -1$ and $\sin\beta = 0$

$\therefore \beta = \pi$.

$$\log(-x_1) = \log x_1 + i\pi$$

$$\therefore \operatorname{Log}(-x_1) = \log x_1 + i(2n+1)\pi$$

Hence the principal value of the logarithm of a negative quantity is imaginary.

Cor. 3. Put $x = 0$.

$$\text{Log}_e(iy) = \frac{1}{2}\log(y^2) + i\tan^{-1}(\infty) + 2n\pi i$$

$$= \log y + i\frac{\pi}{2} + 2n\pi i$$

$$= \log y + i\left(2n + \frac{1}{2}\right)\pi.$$

Hence the logarithm of a purely imaginary quantity consists of two parts, one real and other imaginary.

Problems.

1. Find the logarithm of i

Solution

$$i = \cos \pi/2 + i \sin \pi/2$$

$$= e^{i(\pi/2)}$$

$$= e^{i(2n\pi + \pi/2)}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\text{Log}\, i = \log e^{i(2n\pi + \pi/2)}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\therefore \text{Log}\, i = i(\pi/2 + 2n\pi), \quad n = 0, \pm 1, \pm 2 \ldots$$

---

2. Find the logarithm of $\sqrt{i}$

Solution

$$\sqrt{i} = \left(\cos \pi/2 + i \sin \pi/2\right)^{1/2}$$

$$= \cos \pi/4 + i \sin \pi/4$$

$$= e^{i(\pi/4)}$$

$$= e^{i(\pi/4)} \cdot e^{i 2n\pi}$$

$$= e^{i(\pi/4 + 2n\pi)}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\operatorname{Log} \sqrt{i} = i(\pi/4 + 2n\pi), \quad n = 0, \pm 1, \pm 2, \ldots$$

---

3. Find the logarithm of $-i$.

Solution

$$-i = \cos(-\pi/2) + i \sin(-\pi/2)$$

$$= e^{-i(\pi/2)}$$

$$= e^{-i(\pi/2)} \cdot e^{i 2n\pi}$$

$$= e^{-i(\pi/2) + 2n\pi i}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\operatorname{Log}(-1) = -i(\pi/2) + i 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

---

4. Find the real and imaginary parts of $Log(a+ib)$.

Solution

$a+ib = r(\cos\theta + i\sin\theta)$

where $r = \sqrt{a^2+b^2} = \tan^{-1} b/a$.

$\therefore r = re^{i\theta} \cdot e^{i2n\pi} = re^{i\theta + i2n\pi}$

$Log(a+ib) = \log r + i\theta + i2n\pi$

$= \log\sqrt{a^2+b^2} + i\tan^{-1} b/a + i2n\pi$

$\therefore$ Real part $= \log(\sqrt{a^2+b^2})$

Imaginary part $= \tan^{-1} b/a + 2n\pi$.

where $n = 0, \pm1, \pm2, \ldots$

---

5. Find the values of i^i

Solution

$i^i = e^{\log(i^i)}$

$= e^{i[\log i + 2n\pi i]}$

But $i = \cos\pi/2 + i\sin\pi/2 = e^{i\pi/2}$.

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$\therefore \log i = i\pi/2.$

So $i^i = e^{i[i\pi/2 + 2n\pi i]}$

$= e^{-\pi/2 - 2n\pi}$

$= e^{-(4n+1)\pi/2}, \quad n = 0, \pm 1, \pm 2, \ldots$

$\therefore i^i = e^{-(4n+1)\pi/2}, \quad n = 0, \pm 1, \pm 2, \ldots$

6. Find $\text{Log}(1-i)$.

Solution:

W.k.T. $\text{Log}\, x = \log x + 2n\pi i$

and $\text{Log}_e(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x) + 2n\pi i$

$\text{Log}(1-i) = \log(1-i) + 2n\pi i$

$= \frac{1}{2}\log(1^2+(-1)^2) + i\tan^{-1}\left(\frac{-1}{1}\right) + 2n\pi i$

$= \frac{1}{2}\log 2 + i\tan^{-1}(-1) + 2n\pi i$

$= \frac{1}{2}\log 2 + i\frac{3\pi}{4} + 2n\pi i$

$\text{Log}(1-i) = \frac{1}{2}\log 2 + i\left(\frac{3\pi}{4} + 2n\pi\right).$

Problems.

1. If $\log \sin(\theta + i\phi) = L + iB$. prove that

$$2e^{2L} = \cosh 2\phi - \cos 2\theta.$$

Proof: Given that

$$L + i\beta = \log \sin(\theta + i\phi)$$

$$= \log(\sin\theta \cos i\phi + \cos\theta \sin i\phi)$$

$$= \log(\sin\theta \cosh\phi + i\cos\theta \sinh\phi)$$

$$= \frac{1}{2}\log\left\{(\sin\theta\cosh\phi)^2 + (\cos\theta\sinh\phi)^2\right\}$$

$$+ i\tan^{-1}\left(\frac{\cos\theta\sinh\phi}{\sin\theta\cosh\phi}\right).$$

$$\therefore L = \frac{1}{2}\log\left\{(\sin\theta\cosh\phi)^2 + (\cos\theta\sinh\phi)^2\right\}$$

$$2L = \log\left\{(\sin\theta\cosh\phi)^2 + (\cos\theta\sinh\phi)^2\right\}$$

$$e^{2L} = (\sin\theta\cosh\phi)^2 + (\cos\theta\sinh\phi)^2$$

$$= \sin^2\theta\cosh^2\phi + \cos^2\theta\sinh^2\phi$$

$$= \frac{1-\cos 2\theta}{2}\cdot\cosh^2\phi + \frac{1+\cos 2\theta}{2}\sinh^2\phi$$

$$= \frac{1}{2}\left\{(\cosh^2\phi + \sinh^2\phi) - \cos 2\theta(\cosh^2\phi - \sinh^2\phi)\right\}$$

$$2e^{2L} = \cosh 2\phi - \cos 2\theta.$$

$$\left[\because \begin{array}{l}\cosh^2\phi + \sinh^2\phi = \cosh 2\phi \\ \cosh^2\phi - \sinh^2\phi = 1\end{array}\right.$$

Hence proved

2. Deduce the expansion of $\tan^{-1}x$ in powers of x from the expansion of $\log(a+ib)$.

Solution: We know that

$$\log(a+ib) = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right)$$

put $a=1$, $b=x$.

$$\therefore \log(1+ix) = \frac{1}{2}\log(1+x^2) + i\tan^{-1}(x).$$

Equating imaginary part on both sides, we get.

$$\tan^{-1}x = \text{imaginary part of } \log(1+ix).$$

$$= \text{imaginary part}\left(ix - \frac{(ix)^2}{2} + \frac{(ix)^3}{3} \cdots\cdots\right)$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots\cdots$$

$$\left(\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots\cdots\right)$$

$$\therefore \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots\cdots$$

---

3. Reduce $(\alpha+i\beta)^{x+iy}$ to the form $A+iB$.

Solution:

$$(\alpha+i\beta)^{(x+iy)} = e^{\log(\alpha+i\beta)^{(x+iy)}}$$

$$= e^{(x+iy)\log(\alpha+i\beta)}$$

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$$(\alpha+i\beta)^{(x+iy)} = e^{(x+iy)\{\log(\alpha+i\beta)+2n\pi i\}}$$

$$= e^{x+iy\{\log r+i\theta+2n\pi i\}}$$

where $r=\sqrt{\alpha^2+\beta^2}$ and $\theta=\tan^{-1}\frac{\beta}{\alpha}$

$$(\alpha+i\beta)^{x+iy} = \left(e^{x\log r-y(\theta+2n\pi)}\right)\cdot\left(e^{i\{y\log r+x(\theta+2n\pi)\}}\right)$$

$$= e^{x\log r-y(\theta+2n\pi)}\left[\cos\{y\log r+x(\theta+2n\pi)\} + i\sin\{y\log r+x(\theta+2n\pi)\}\right].$$

$$\therefore A = e^{x\log r-y(\theta+2n\pi)}\left[\cos\{y\log r+x(\theta+2n\pi)\}\right]$$

$$B = e^{x\log r-y(\theta+2n\pi)}\left[\sin\{y\log r+x(\theta+2n\pi)\}\right].$$

---

4. Show that $\log_i i = \frac{4n+1}{4m+1}$, where m and n are integers.

Solution

Let $\log_i i = x+iy$. Then

$i = i^{x+iy}$.

Taking the general value of the logarithm on both sides, we have,

$\text{Log}\, i = \text{Log}\, i^{x+iy}$.

$\therefore x+iy \,\text{Log}\, i = \text{Log}\, i$

$$x+iy = \frac{\text{Log}_e i}{\text{Log}_e i}$$

$$= \frac{(2n+\frac{1}{2})\pi i}{(2m+\frac{1}{2})\pi i}$$

$$= \frac{(4n+1)}{(4m+1)}$$

$\therefore x+iy = \frac{4n+1}{4m+1}$, where n, m are integers.

---

5. Find the general value of $\text{Log}_{(-3)}(-2)$.

Solution

Let $\text{Log}_{-3}(-2) = x+iy$

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$$\therefore (x+iy) = \frac{\log_e(-2)}{\log_e(-3)}$$

$$(x+iy)\,\mathrm{Log}_e(3) = \mathrm{Log}_e(-2)$$

i.e., $(x+iy)\{\log 3 + i(2m+1)\pi\}$

$$= \log 2 + i(2n+1)\pi.$$

Equating real and imaginary parts, we get

$$x\log 3 - y(2m+1)\pi = \log 2 \longrightarrow (1)$$

$$y\log 3 + x(2m+1)\pi = (2n+1)\pi \longrightarrow (2)$$

Solving the Equations (1) & (2), we get.

$$x = \frac{(2m+1)(2n+1)\pi^2 + (\log 2)(\log 3)}{(\log 3)^2 + (2m+1)^2\pi^2}$$

and

$$y = \frac{\log 3\,(2n+1)\pi - (2m+1)\pi\log 2}{(\log 3)^2 + (2m+1)^2\pi^2}$$

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6. Find the value of (i) $\log(1+i)$ (ii) $\log(-e)$

Solution:

(i) $\log(1+i)$

Let $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= \sqrt{2} \cdot e^{i\pi/4}$

$= \sqrt{2}\, e^{i(\pi/4 + 2n\pi)}$

$\log(1+i) = \log\sqrt{2} + \log e^{i(\pi/4 + 2n\pi)}$

$= \log 2^{1/2} + i(\pi/4 + 2n\pi)$

$= \frac{1}{2}\log 2 + i\left(\frac{\pi}{4} + 2n\pi\right)$

$\therefore \log(1+i) = \frac{1}{2}\log 2 + i\left(\frac{\pi}{4} + 2n\pi\right)$

(ii) Let $-e = e(\cos\pi + i\sin\pi)$

$= e \cdot e^{i\pi} = e \cdot e^{i(\pi + 2n\pi)}$

$\therefore \log(-e) = \log e + \log e^{i(\pi + 2n\pi)}$

$= 1 + i\pi(2n+1).$

$\therefore \log(-e) = 1 + i\pi(2n+1).$

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7. Prove that

$$\log\left(\frac{1}{1-e^{i\theta}}\right) = \log\left(\frac{\operatorname{cosec}\theta/2}{2}\right) + i\left(2n\pi + \pi/2 - \theta/2\right).$$

Proof:

$$\frac{1}{1-e^{i\theta}} = \left(1-e^{i\theta}\right)^{-1}$$

$$= \left[1-(\cos\theta + i\sin\theta)\right]^{-1}$$

$$= \left[(1-\cos\theta) - i\sin\theta\right]^{-1}$$

$$= \left[2\sin^2\theta/2 - i\,2\sin\theta/2\cos\theta/2\right]^{-1}$$

$$= (2\sin\theta/2)^{-1}\left[\sin\theta/2 - i\cos\theta/2\right]^{-1}$$

$$= \frac{\operatorname{cosec}\theta/2}{2}\left[\cos(\pi/2-\theta/2) + i\sin(\pi/2-\theta/2)\right]^{-1}$$

$(\because \cos(90-\theta) = \sin\theta,\ \sin(90-\theta) = \cos\theta$.

$x = \cos\theta + i\sin\theta$

$\frac{1}{x} = \cos\theta - i\sin\theta$

$\left(\frac{1}{x}\right)^{-1} = (\cos\theta - i\sin\theta)^{-1}$

$\therefore x = (\cos\theta - i\sin\theta)^{-1}$

$$\therefore \frac{1}{1-e^{i\theta}} = \frac{\operatorname{cosec}\theta/2}{2}\left[\cos(\pi/2-\theta/2) - i\sin(\pi/2-\theta/2)\right].$$

$$= \frac{\operatorname{cosec}\theta/2}{2}\, e^{i(\pi/2-\theta/2)+2n\pi i}$$

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Taking logarithm on both sides, we have

$$\log\left(\frac{1}{1-e^{i\theta}}\right) = \log\left(\frac{\operatorname{cosec}\theta/2}{2}\right) \cdot \log e^{i(\pi/2-\theta/2+2n\pi)}$$

$$\therefore \log\left(\frac{1}{1-e^{i\theta}}\right) = \log\left(\frac{\operatorname{cosec}\theta/2}{2}\right) + i\left(\pi/2-\theta/2+2n\pi\right)$$

8. If $i^{a+ib} = a+ib$, prove that $a^2+b^2 = e^{-(4n+1)\pi b}$.

Proof: Let $i^{a+ib} = a+ib \longrightarrow (1)$

$$i^{a+ib} = e^{\log i^{a+ib}}$$

$$= e^{a+ib.\log i}$$

$$= e^{a+ib\, i(4n+1)\pi/2}$$

$$(\log i = i(4n+1)\pi/2$$

$$= e^{(ai-b)(4n+1)\pi/2}$$

$$i^{a+ib} = e^{ai(4n+1)\pi/2} \cdot e^{-b(4n+1)\pi/2}$$

$$= e^{-b(4n+1)\pi/2} \cdot \left[e^{ia(4n+1)\pi/2} \right]$$

$$= e^{-b(4n+1)\pi/2} (\cos\theta + i\sin\theta)$$

Here $\theta = a(4n+1)\pi/2$,

$$\therefore a+ib = e^{-b(4n+1)\pi/2} (\cos\theta + i\sin\theta)$$

Equating real and imaginary part, we get

$a = e^{-b(4n+1)\pi/2} \cos\theta$ and

$b = e^{-b(4n+1)\pi/2} \cdot \sin\theta$.

$$a^2+b^2 = \left(e^{-b(4n+1)\pi/2} \cos\theta\right)^2 + \left(e^{-b(4n+1)\pi/2} \cdot \sin\theta\right)^2$$

$$= e^{-b(4n+1)\pi} \cdot \cos^2\theta + e^{-b(4n+1)\pi} \cdot \sin^2\theta$$

$$= e^{-b(4n+1)\pi} (\cos^2\theta + \sin^2\theta)$$

$$\therefore a^2+b^2 = e^{-b(4n+1)\pi}$$

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