BHARATHIDASAN UNIVERSITY, TIRUCHIRAPPALL – 620 024.

B.Sc. Computer Science, B.Sc. Information Technology, BCA & B.Sc. Software Development Programmes

[16SACMA1]

ALLIED MATHEMATICS

ALLIED COURSE I

ALGEBRA AND CALCULUS

Objects :

1. To lean the basic concepts in the integration
2. To train the students to solve the problems in Theory of Equations

UNIT I

Theory of Equations: Relation between roots & coefficients – Transformations of Equations – Diminishing ,Increasing & multiplying the roots by a constant- Forming equations with the given roots –Rolle's Theorem, Descarte's rule of Signs(statement only) –simple problems.

UNIT II

Matrices : Singular matrices – Inverse of a non-singular matrix using adjoint method - Rank of a Matrix – Consistency - Characteristic equation, Eigen values, Eigen vectors – Cayley Hamilton's Theorem (proof not needed) –Simple applications only

UNIT III

Differentiation: Maxima & Minima – Concavity , Convexity – Points of inflexion - Partial differentiation – Euler's Theorem - Total differential coefficients (proof not needed) –Simple problems only.

UNIT IV

Integration : Evaluation of integrals of types

1] $\int \frac{px+q}{ax^2+bx+c}dx$ 2] $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ 3] $\int \frac{dx}{a+b\sin x}$

4] $\int \frac{dx}{a+b\cos x}$

Evaluation using Integration by parts – Properties of definite integrals – Fourier Series in the range (0 , 2 π) – Odd & Even Functions – Fourier Half range Sine & Cosine Series

UNIT V

Differential Equations: Variables Separables – Linear equations – Second order of types $(a D^2 + b D + c) y = F (x)$ where a,b,c are constants and $F (x)$ is one of the following types (i) e^{Kx} (ii) sin (kx) or cos (kx) (iii) x^n , n being an integer (iv) e^{Kx} f (x)

TEXT BOOK(S)

1. T.K. Manickavasagam Pillai & others, Algebra, Volume I, S.V Publications , 1985 Revised Edition (Units I, II)
2. S. Narayanan, T.K. Manicavachagam Pillai, Calculus, Vol. II, S. Viswanathan Pvt Limited, 2003. (Units III, IV and V)

REFERENCE(S)

1. M.L. Khanna, Differential Calculus, Jaiprakashnath and Co., Meerut-2004.

*****

Government Arts College Grade-I, Ariyalur.

Class : I st B.Sc, Computer Science,

Subject : "Algebra and Calculus" "Allied paper - Mathematics - M1"

Unit-I

(1)

CHAPTER 6

THEORY OF EQUATIONS

§ 1. If n is a positive integer and $a_0, a_1, a_2, \ldots a_n$ are constants and $a_0 \neq 0$, an expression of the form

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n \quad \ldots (1)$$

is called a polynomial in x of the n^{th} degree. A constant may be regarded as a polynomial of degree zero.

The equation obtained by putting the polynomial (1) equal to zero,

i.e., $f(x) = a_0 x^n + a_1^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0$ is called an algebraic equation of the n^{th} degree. An equation is not altered if all its terms be divided by any quantity. Dividing the equation by a_0, we can make the coefficient of x^n in the above equation equal to unity. Then the equation can be written in the form

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0.$$

Equations of the first, second, third, fourth ... degrees are known as linear, quadratic, cubic, biquadratic ... equations respectively. The term independent of x is called the absolute term.

Any value of x for which the polynomial $f(x)$ vanishes is called a root of the equation $f(x) = 0$.

The main object of the theory of equations is to find the roots of the equation $f(x) = 0$, i.e., to solve the equation.

In this chapter unless otherwise stated $f(x)$ represents always a polynomial in x. We can easily see that $f(x)$ is a continuous function of x for all values of x.

§ 2. Remainder Theorem. If $f(x)$ is a polynomial, then $f(a)$ is the remainder when $f(x)$ is divided by $x - a$.

Divide the polynomial $f(x)$ by $x - a$ until a remainder is obtained which does not involve x.

Let the quotient be Q (x) and remainder R.

Then $f(x) \equiv (x - a)\,Q(x) + R$.

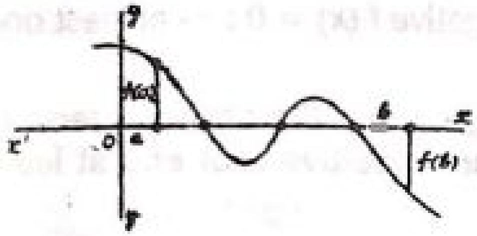
Substituting $x = a$ in the above equation, we get $f(a) = R$.

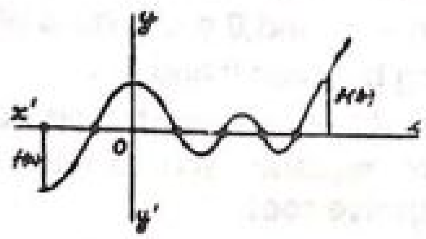
Cor. If $f(a) = 0$, the polynomial $f(x)$ has the factor $x - a$, i.e., if a be the root of the equation $f(x) = 0$, then $x - a$ is a factor of the polynomial $f(x)$.

§ 3. If $f(a)$ and $f(b)$ are of different signs, then at least one root of the equation $f(x) = 0$ must lie between a and b.

As x changes gradually from a to b, the function $f(x)$ changes gradually from $f(a)$ and $f(b)$ and therefore must pass through all intermediate values, but since $f(a)$ and $f(b)$ have different signs the value zero must be between them, i.e., $f(x)$ assumes the value zero for atleast for one value of x between a and b.

This theorem can be proved by means of drawing the graph of the function $y = f(x)$. Since $f(a)$ and $f(b)$ have different signs, the graph $y = f(x)$ must cross the x-axis at least once between a and b.

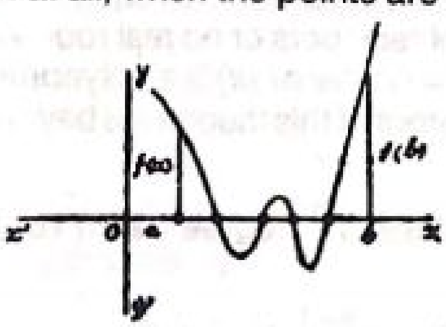


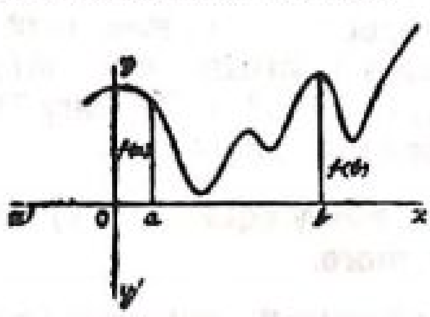


At the point where the graph crosses the x-axis there is a real root of $f(x) = 0$.

∴ There is atleast one real root between a and b.

§ 4. If $f(a)$ and $f(b)$ have the same sign, it does not follow that $f(x) = 0$ has no root between a and b. It is evident that when two points are connected by a curve, the portions of the curve between these points must cut the axis at an odd number of times when the points are on opposite sides of the axis and an even number of times or not at all, when the points are on the same side of the axis.





Hence we get the following results :-

(1) If $f(a)$ and $f(b)$ have like signs, an even number of roots of $f(x) = 0$ lie between a and b or else there is no root between a and b.

(2) If $f(a)$ and $f(b)$ have unlike signs, an odd number of roots of $f(x) = 0$ lie between a and b.

§ 5. If $f(x) = 0$ is an equation of odd degree, it has atleast one real root whose sign is opposite to that of the last term.

Let $f(x)$ be $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n$.

Substituting $-\infty, 0, +\infty$, for x in $f(x)$, we get

$$f(-\infty) = -\infty \text{ since } n \text{ is odd.}$$
$$f(0) = p_n$$
$$f(+\infty) = +\infty.$$

Hence if p_n is positive $f(x) = 0$ has at least one root lying between $-\infty$ and 0 and if p_n is negative $f(x) = 0$ has at least one root lying between 0 and $+\infty$.

§ 6. If $f(x) = 0$ is of even degree and the absolute term is negative, equation has at least one positive root and at least one negative root.

Let $f(x)$ be $x^n + p_1 x^{n-1} + \ldots p_n$

Here n is even and p_n is negative.

$$f(-\infty) = +\infty \text{ since n is even}$$
$$f(0) = p_n = \text{a negative quantity}$$
$$f(+\infty) = +\infty.$$

Hence $f(x) = 0$ has at least one root lying between $-\infty$ and 0, and at least another lying between 0 and $+\infty$.

§ 7. We have proved that every equation except one of an even degree with a positive last term has a real root. Such an equation of even degree may have even number of real roots or no real root. We shall assume that every equation $f(x) = 0$ where $f(x)$ is a polynomial in x has a root real or imaginary. The proof of this theorem is beyond the scope of this book.

§ 8. Every equation $f(x) = 0$ of the n^{th} degree has n roots and no more.

Let $f(x)$ be the polynomial $a_0 x^n + a_1 x^{n-1} + \ldots + a_n$.

We assume that every equation $f(x) = 0$ has at least one root real or imaginary.

Let α_1 be a root of $f(x) = 0$.

Then $f(x)$ is exactly divisible by $x - \alpha_1$, so that

$$f(x) = (x - \alpha_1)\,\phi_1(x)$$

where $\phi_1(x)$ is a rational integral function of degree $n - 1$.

Again $\phi_1(x) = 0$ has a root real or imaginary and let that root be α_2.

Then $\phi_1(x)$ is exactly divisible by $x - \alpha_2$, so that

$$\phi_1(x) = (x - \alpha_2)\,\phi_2(x)$$

where $\phi_2(x)$ is a rational integral function of degree $n - 2$.

$$\therefore\; f(x) = (x - \alpha_1)(x - \alpha_2)\,\phi_2(x).$$

By continuing in this way, we obtain

$$f(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)\,\phi_n(x)$$

where $\phi_n(x)$ is of degree $n - n$, i.e., zero.

$\therefore\;\; \phi_n(x)$ is a constant.

Equating the coefficients of x^n on both sides we get

$$\phi_n(x) = \text{coefficients of } x^n$$
$$= a_0$$

$$\therefore\;\; f(x) = a_0 (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n).$$

Hence the equation $f(x) = 0$ has n roots, since $f(x)$ vanishes when x has any one of the values $\alpha_1, \alpha_2, \ldots \alpha_n$. If x is given any value different from any one of these n roots, then no factor of $f(x)$ can vanish and the equation is not satisfied. Hence $f(x) = 0$ cannot have more than n roots.

Example 1. If α be a real root of the cubic equation $x^3 + px^2 + qx + r = 0$, of which the coefficients are real, show that the other two roots of the equation are real, if

$$p^2 \geq 4q + 2p\alpha + 3\alpha^2.$$

Proof

Since α is a root of the equation, $x^3 + px^2 + qx + r$ is exactly divisible by $x - \alpha$.

$\therefore$ Let $x^3 + px^2 + qx + r \equiv (x - \alpha)(x^2 + ax + b)$.

Equating the coefficients of powers of x on both sides, we get

$$p = -\alpha + a$$
$$q = -a\alpha + b$$
$$r = -b\alpha.$$

$\therefore\ a = p + \alpha$ and $b = q + a\alpha = q + \alpha(p + \alpha)$

$$= q + p\alpha + \alpha^2.$$

The other two roots of the equation are the roots of

$x^2 + (p + \alpha)x + q + p\alpha + \alpha^2 = 0$

which are real if $(p + \alpha)^2 - 4(q + p\alpha + \alpha^2) \geq 0$

$$\text{i.e., } p^2 - 2p\alpha - 4q - 3\alpha^2 \geq 0$$

$$\text{i.e., } p^2 \geq 4q + 2p\alpha + 3\alpha^2.$$

Example 2. If $x_1, x_2, x_3 \ldots, x_n$ are the roots of the equation $(a_1 - x)(a_2 - x) \ldots (a_n - x) + k = 0$, then show that $a_1, a_2, \ldots, a_n$ are the roots of the equation.

$$(x_1 - x)(x_2 - x) \ldots (x_n - x) - k = 0.$$

Since $x_1, x_2, x_3, \ldots. x_n$ are the roots of the equation

$$(a_1 - x)(a_2 - x) \ldots (a_n - x) + k = 0,$$

we have

$(a_1 - x)(a_2 - x) \ldots (a_n - x) + k \equiv (x_1-x)(x_2-x) \ldots (x_n-x)$

$\therefore\ (x_1-x)(x_2-x) \ldots (x_n-x) - k \equiv (a_1-x)(a_2-x) \ldots (a_n-x).$

$\therefore\ a_1, a_2, a_3 \ldots, a_n$ are the roots of

$$(x_1 - x)(x_2 - x) \ldots (x_n - x) - k = 0.$$

Example 3. Show that if a, b, c are real, the roots of

$$\frac{1}{x+a} + \frac{1}{x+b} + \frac{1}{x+c} = \frac{3}{x} \text{ are real.}$$

Simplifying we get

$$x(x+b)(x+c) + x(x+c)(x+a) + x(x+a)(x+b)$$
$$- 3(x+a)(x+b)(x+c) = 0.$$

Let $f(x)$ be the expression on the left-hand side. It can easily be seen that $f(x)$ is a quadratic function of x.

$\therefore\ f(-a) = -a(b-a)(c-a)$

$f(-b) = -b(c-b)(a-b)$

$f(-c) = -c(a-c)(b-c).$

Without loss of generality let us assume that $a > b > c$ and a, b, c are all positive.

Then $a-b, b-c, a-c$ are positive.

$\therefore\ f(-a) = -$ ve.

$f(-b) = +$ve. (−ve)

$f(-c) = -$ ve.

$\therefore$ The equation has at least one real root between $-a$ and $-b$, and another between $-b$ and $-c$.

The equation can have only two roots since $f(x) = 0$ is a quadratic equation.

$\therefore$ The roots of the equations are real.

Exercises 41

1. If $f(x)$ is a rational integral function of x and a, b are unequal show that the remainder in the division of $f(x)$ by $(x-a)(x-b)$ is

$$\frac{(x-a)f(b)-(x-b)f(a)}{b-a}$$

2. If $x^3 + 3px + q$ has a factor of the form $x^2 - 2ax + a^2$, show that $q^2 + 4p^3 = 0$.

3. If $px^3 + qx + r$ has a factor of the form $x^2 + ax + 1$, prove that $p^2 = pq + r^2$.

4. If $px^5 + qx^2 + r$ has factor of the form $x^2 + ax + 1$, prove that $(p^2 - r^2)(p^2 - r^2 + qr) = p^2q^2$.

5. Prove that $x^4 + px^3 + rx + s^2$ is a perfect square, if

$ps = \pm r$ and $p^2 + 8s = 0$.

Hence solve $x^4 - 4x^3 + 8x + 4 = 0$.

6. Find the relation between p and q in order that the equation $x^3 - px + q = 0$ may be put in the form $(x^2 + mx + n)^2 = x^4$. Hence or otherwise solve the equation $8x^3 - 36x + 27 = 0$.

7. If L, B, Y are the roots of the equation $(x+a)(x+b)(x+c) = d$, prove that a, b, c are the roots of the equation $(x + L)(x + B)(x + Y) = d$.

8. If a, b, c are all positive, show that all the roots of

$$\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} = \frac{1}{x} \text{ are real.}$$ (B.Sc. 1988)

9. If $a > b > c > d$ and E, A, B, C, D are positive, show that the equation

$$E + \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \frac{D}{x-d} = 0$$

has one root between a and b, one root between b and c and one between c and d and if $E > 0$, there is a root $> d$ and if $E < 0$, there is a root $< a$.

10. If $a < b < c < d$, show that the roots of

$$(x - a)(x - c) = k(x - b)(x - d)$$

are real for all values of k.

11. If a, b, c, f, g, h be all real, show that all the roots of the equation

$$(x-a)(x-b)(x-c) - f^2(x-a) - g^2(x-b) - h^2(x-c) - 2fgh = 0$$

are all real. (Assume that $a > b > c$).

§ 9. In an equation with real coefficients, imaginary roots occur in pairs.

Let the equation be $f(x) = 0$ and let $\alpha + i\beta$ be an imaginary root of the equation. We shall show that $\alpha - i\beta$ is also a root.

We have $(x - \alpha - i\beta)(x - \alpha + i\beta) = (x - \alpha)^2 + \beta^2$ ... (1)

If $f(x)$ is divided by $(x - \alpha)^2 + \beta^2$, let the quotient be $Q(x)$ and the remainder be $Rx + R'$.

Here $Q(x)$ is of degree $(n - 2)$.

$\therefore \; f(x) = \{(x - \alpha)^2 + \beta^2\} Q(x) + Rx + R'$ ... (2)

Substituting $(\alpha + i\beta)$ for x in the equation (2), we get

$$f(\alpha+i\beta) = \{(\alpha+i\beta-\alpha)^2 + \beta^2\} Q(\alpha+i\beta) + R(\alpha+i\beta) + R'$$
$$= R(\alpha + i\beta) + R'$$

but $f(\alpha+i\beta) = 0$ since $\alpha + i\beta$ is a root of $f(x) = 0$.

$\therefore \; R(\alpha+i\beta) + R' = 0.$

Equating to zero the real and imaginary parts

$R\alpha + R' = 0$ and $R\beta = 0$.

Since $\beta \neq 0$, $R = 0$ and so $R' = 0$.

$\therefore\ f(x) = \{(x-\alpha)^2 + \beta^2\}\, Q(x)$.

$\therefore\ \alpha - i\beta$ is also a root of $f(x) = 0$.

Example 1. Form a rational cubic equation which shall have for roots $1, 3 - \sqrt{-2}$.

Since $3 - \sqrt{-2}$ is a root of the equation, $3 + \sqrt{-2}$ is also a root. So we have to form an equation whose roots are $1, 3 + \sqrt{-2}, 3 - \sqrt{-2}$.

Hence the required equation is

$(x-1)(x-3-\sqrt{-2})(x-3+\sqrt{-2}) = 0$

i.e., $(x-1)\{(x-3)^2 + 2\} = 0$

i.e., $(x-1)(x^2 - 6x + 11) = 0$

i.e., $x^3 - 7x^2 + 17x - 11 = 0$.

Example 2. Solve the equation $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$ of which one root is $-1 + \sqrt{-1}$. (B.Sc.1991)

Imaginary roots occur in pairs. Hence $-1 - \sqrt{-1}$ is also a root of the equation.

$\therefore$ The expression on the left side of equation has the factors

$(x + 1 - \sqrt{-1})(x + 1 + \sqrt{-1})$.

$\therefore$ The expression on the left side is exactly divisible by

$(x+1)^2 + 1$, *i.e.*, $x^2 + 2x + 2$.

Dividing $x^4 + 4x^3 + 5x^2 + 2x - 2$ by $x^2 + 2x + 2$, we get the quotient $x^2 + 2x - 1$.

$\therefore\ x^4 + 4x^3 + 5x^2 + 2x - 2 = (x^2+2x+2)(x^2+2x-1)$.

Hence the other roots are obtained from $x^2 + 2x - 1 = 0$.

Thus the other roots are $-1 \pm \sqrt{2}$.

Example 3. Show that $\dfrac{a^2}{x-\alpha} + \dfrac{b^2}{x-\beta} + \dfrac{c^2}{x-\gamma} - x + \delta = 0$.

has only real roots if $a, b, c, \alpha, \beta, \gamma, \delta$ are all real.

If possible let $p + iq$ be a root. Then $p - iq$ is also a root.

Substituting these values for x, we have

$$\frac{a^2}{p+iq-\alpha}+\frac{b^2}{p+iq-\beta}+\frac{c^2}{p+iq-\gamma}-p-iq+\delta=0 \quad \dots (1)$$

$$\frac{a^2}{p-iq-\alpha}+\frac{b^2}{p-iq-\beta}+\frac{c^2}{p-iq-\gamma}-p+iq+\delta=0 \quad \dots (2)$$

Substracting (2) from (1), we get

$$-\frac{2a^2iq}{(p-\alpha)^2+q^2}-\frac{2b^2iq}{(p-\beta)^2+q^2}-\frac{2c^2iq}{(p-\gamma)^2+q^2}-2iq=0$$

i.e., $$-2iq\left\{\frac{a^2}{(p-\alpha)^2+q^2}+\frac{b^2}{(p-\beta)^2+q^2}+\frac{c^2}{(p-\gamma)^2+q^2}+1\right\}=0$$

This is only possible when $q=0$ since the other factor cannot be zero. In that case the roots are real.

Article 3 can also be used to prove this result.

§ 10. In an equation with rational coefficients irrational roots occur in pairs.

Let $f(x)=0$ denote the equation and suppose that $a+\sqrt{b}$ is a root of the equation where a and b are rational and $\sqrt{b}$ is irrational. We now show that $a-\sqrt{b}$ is also a root of the equation

$$(x-a-\sqrt{b})(x-a+\sqrt{b})=(x-a)^2-b \quad \dots (1)$$

If $f(x)$ is divided by $(x-a)^2-b$, let the quotient be $Q(x)$ and the remainder be $Rx+R'$.

Here $Q(x)$ is a polynomial of degree $n-2$.

$$\therefore \ f(x)=\{(x-a)^2-b\}\,Q(x)+Rx+R' \quad \dots (2)$$

Substituting $a+\sqrt{b}$ for x in (2), we get

$$f(a+\sqrt{b})=\{(a+\sqrt{b}-a)^2-b\}\,Q(a+\sqrt{b})+R(a+\sqrt{b})+R$$
$$=R(a+\sqrt{b})+R'$$

but $f(a+\sqrt{b})=0$, since $a+\sqrt{b}$ is a root of $f(x)=0$.

$$\therefore \ Ra+R'+R\sqrt{b}=0.$$

10

Equating the rational and irrational parts, we have

$Ra + R' = 0$ and $R = 0$.

$\therefore\ R' = 0$.

Hence $f(x) = \{(x-a)^2 - b\}\, Q(x)$

$= (x - a - \sqrt{b})(x - a + \sqrt{b})\, Q(x)$.

$\therefore\ a - \sqrt{b}$ is a root of $f(x) = 0$.

Example 1. Frame an equation with rational coefficients, one of whose roots is $\sqrt{5} + \sqrt{2}$. (B.Sc.1990)

Then the other roots are $\sqrt{5} - \sqrt{2}$, $-\sqrt{5} + \sqrt{2}$, $-\sqrt{5} - \sqrt{2}$.

Hence the required equation is

$$(x - \sqrt{5} - \sqrt{2})(x - \sqrt{5} + \sqrt{2})(x + \sqrt{5} - \sqrt{2})(x + \sqrt{5} + \sqrt{2}) = 0$$

i.e., $\{(x - \sqrt{5})^2 - 2\}\{(x + \sqrt{5})^2 - 2\} = 0$

i.e., $(x^2 - 2x\sqrt{5} + 3)(x^2 + 2x\sqrt{5} + 3) = 0$

i.e., $(x^2 + 3)^2 - 4x^2 . 5 = 0$

i.e., $x^4 - 14x^2 + 9 = 0$.

Example 2. Solve the equation $x^4 - 5x^3 + 4x^2 + 8x - 8 = 0$ given that one of the roots is $1 - \sqrt{5}$. (B.Sc.1994)

Since the irrational roots occur in pairs, $1 + \sqrt{5}$ is also a root.

The factors corresponding to these roots are

$(x - 1 + \sqrt{5})(x - 1 - \sqrt{5})$, i.e., $(x-1)^2 - 5$

i.e., $x^2 - 2x - 4$.

Dividing $x^4 - 5x^3 + 4x^2 + 8x - 8$ by $x^2 - 2x - 4$, we get the quotient $x^2 - 3x + 2$.

$\therefore\ x^4 - 5x^3 + 4x^2 + 8x - 8 = (x^2-2x-4)(x^2-3x+2)$

$= (x^2-2x-4)(x-1)(x-2)$.

The roots of the equation are $1 \pm \sqrt{5}$, 1, 2.

Exeroises 42

1. Find the equation with rational coefficients whose roots are

(1) $4\sqrt{3}$, $5 + 2\sqrt{-1}$.　　(2) $1 + 5\sqrt{-1}$, $5 - \sqrt{-1}$.

(3) $\sqrt{-1} - \sqrt{5}$.　　(4) $-\sqrt{3} + \sqrt{-2}$.

2. Solve the equation $x^4 + 2x^3 - 5x^2 + 6x + 2 = 0$ given that $1 + \sqrt{-1}$ is a root of it. (B.Sc.1990)

3. Solve the equation $3x^3 - 4x^2 + x + 88 = 0$ which has a root $2 - \sqrt{-7}$. (B.Sc.1988 & 1990)

4. Solve $x^4 - 4x^2 + 8x + 35 = 0$ given that $2 + i\sqrt{3}$ is a root of it. (B.Sc.1993)

5. Solve the equation $x^4 - 6x^3 + 11x^2 - 10x + 2 = 0$ given that $2 + \sqrt{3}$ is a root of the equation. (B.Sc.1993)

6. Given that $-2 + \sqrt{-7}$ is a root of the equation

$$x^4 + 2x^2 - 16x + 77 = 0,$$

solve it completely.

7. Solve the equation $x^5 - x^4 + 8x^2 - 9x - 15 = 0$, one root being $-\sqrt{3}$ and another $1 + 2\sqrt{-1}$.

8. Show that the equation

$$\frac{a^2}{x-a'} + \frac{b^2}{x-b'} + \frac{c^2}{x-c'} + \dots + \frac{k^2}{x-k'} = x - m,$$

where $a, b, c \dots k$ are all different cannot have an imaginary root.

9. One root of the equation $3x^5 - 4x^4 - 42x^3 + 56x^2 + 27x - 36 = 0$ is $\sqrt{2} + \sqrt{5}$. Find the remaining roots.

10. Solve the equation $x^6 - 4x^5 - 11x^4 + 40x^3 + 11x^2 - 4x - 1 = 0$ given that one root is $\sqrt{2} - \sqrt{3}$. (B.Sc.1990)

11. Solve the equation $2x^6 - 3x^5 + 5x^4 + 6x^3 - 27x + 81 = 0$ given that one root is $\sqrt{2} - \sqrt{-1}$.

§11. Relations between the roots and coefficients of equations.

Let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0.$$

If this equation has the roots $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n$, then we have

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n$$

$= (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$

$= x^n - \sum \alpha_1 . x^{n-1} + \sum \alpha_1 \alpha_2 x^{n-2} - \dots + (-1)^n \alpha_1 \alpha_2 \dots \alpha_n$

$= x^n - S_1 x^{n-1} + S_2 x^{n-2} - \dots + (-1)^n S_n$

where S_r is the sum of the products of the quantities $\alpha_1, \alpha_2, \dots \alpha_n$ taken r at a time.

Equating the coefficients of like powers on both sides, we have

$- p_1 = S_1 =$ sum of the roots.

$(-1)^2 p_2 = S_2 =$ sum of the products of the roots taken two at a time.

$(-1)^3 p_3 = S_3 =$ sum of the products of the roots taken three at a time.

$(-1)^n p_n = S_n =$ product of the roots.

If the equation is

$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0,$

divide each term of the equation by a_0.

The equation becomes

$$x^n + \frac{a_1}{a_0} x^{n-1} + \frac{a_2}{a_0} x^{n-2} + \dots + \frac{a_{n-1}}{a_0} x + \frac{a_n}{a_0} = 0$$

and so we have

$$\sum \alpha_1 = -\frac{a_1}{a_0}$$

$$\sum \alpha_1 \alpha_2 = \frac{a_2}{a_0}$$

$$\sum \alpha_1 \alpha_2 \alpha_3 = -\frac{a_3}{a_0}$$

$$\alpha_1 \alpha_2 \alpha_3 \dots \alpha_n = (-1)^n \frac{a_n}{a_0}.$$

These n equations are of no help in the general solution of an equation but they are often helpful in the solution of numerical equations when some special relation is known to exist among the roots. The method is illustrated in the examples given below.

Example 1. Show that the roots of the equation $x^3 + px^2 + qx + r = 0$ are in Arithmetical progression if

$$2p^3 - 9pq + 27r - 0.$$

Show that the above condition is satisfied by the equation

$$x^3 - 6x^2 + 13x - 10 = 0.$$

Hence or otherwise solve the equation. (B.Sc.1988)

Let the roots of the equation $x^3 + px^2 + qx + r = 0$ be

$$\alpha - \delta, \alpha, \alpha + \delta.$$

We have from the relation of the roots and coefficients

$$\alpha - \delta + \alpha + \alpha - \delta = -p$$
$$(\alpha - \delta)\alpha + (\alpha - \delta)(\alpha + \delta) + \alpha(\alpha + \delta) = q$$
$$(\alpha - \delta)\alpha(\alpha + \delta) = -r.$$

Simplifying these equations, we get

$$3\alpha = -p \quad \text{... (1)}$$
$$3\alpha^2 - \delta^2 = q \quad \text{... (2)}$$
$$\alpha^3 - \alpha\delta^2 = -r \quad \text{... (3)}$$

From (1), $\alpha = -\frac{p}{3}$.

From (2), $\delta^2 = 3\left(-\frac{p}{3}\right)^2 - q = \frac{p^2}{3} - q.$

Substituting these values in (3), we get

$$\left(-\frac{p}{3}\right)^3 - \left(-\frac{p}{3}\right)\left(\frac{p^2}{3} - q\right) = -r$$

i.e., $2p^3 - 9pq + 27r = 0.$

In the equation $x^3 - 6x^2 + 13x - 10 = 0.$

$$p = -6, \; q = 13, \; r = -10.$$

$$\therefore \; 2p^3 - 9pq + 27r = 2(-6)^3 - 9(-6)13 + 27(-10)$$
$$= 0.$$

$\therefore$ The condition is satisfied and so the roots of the equation are in arithmetical progression. In this case the equations (1), (2), (3) become

$$3\alpha = 6$$

$$3\alpha^2 - \delta^2 = 13$$

$$\alpha^3 - \alpha\delta^2 = 10.$$

$\therefore\ \alpha = 2,\ 12 - \delta^2 = 13.\ \therefore\ \delta^2 = -1$

i.e., $\delta = \pm i$.

The roots are $2 - i, 2, 2 + i$.

Example 2. Find the condition that the roots of the equation $ax^3 + 3bx^2 + 3cx + d = 0$ may be in geometric progression.

Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$ whose roots are in geometric progression. (B.Sc.1989)

Let the roots of the equation be $\frac{k}{r}$, k and kr.

$$\therefore\ \frac{k}{r} + k + kr = -\frac{3b}{a} \quad \text{... (1)}$$

$$\frac{k^2}{r} + k^2 + k^2r = \frac{3c}{a} \quad \text{... (2)}$$

$$k^3 = -\frac{d}{a} \quad \text{... (3)}$$

From (1), $k\left(\frac{1}{r} + 1 + r\right) = -\frac{3b}{a}$.

From (2), $k^2\left(\frac{1}{r} + 1 + r\right) = \frac{3c}{a}$.

Dividing one by the other, we get $k = -\frac{c}{b}$.

Substituting this value of k in (3), we get $\left(-\frac{c}{b}\right)^3 = -\frac{d}{a}$.

$$\therefore\ ac^3 = b^3 d.$$

In the equation $27x^3 + 42x^2 - 28x - 8 = 0$.

$$\frac{k}{r} + k + kr = -\frac{42}{27} \quad \text{... (4)}$$

$$\frac{k^2}{r} + k^2 + k^2r = -\frac{28}{27} \quad \text{... (5)}$$

$$k^3 = \frac{8}{27} \qquad \text{... (6)}$$

$$\therefore \quad k = \frac{2}{3}.$$

Substituting the value of k in (4), we get

$$\frac{2}{3}\left(\frac{1}{r} + 1 + r\right) = -\frac{42}{27}$$

i.e., $3r^2 + 10r + 3 = 0.$

$(3r + 1)(r + 3) = 0.$

$$\therefore \quad r = -\frac{1}{3} \text{ or } -3.$$

For both the values of r, the roots are $-2, \frac{2}{3}, -\frac{2}{9}$.

Example 3. Solve the equation $81x^3 - 18x^2 - 36x + 8 = 0$ whose roots are in harmonic progression. (B.Sc. 1994)

Let the roots be α, β, γ.

Then $\frac{2}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma}$

i.e., $2\gamma\alpha = \beta\gamma + \alpha\beta$ ... (1)

From the relation between the coefficients and the roots we have

$$\alpha + \beta + \gamma = \frac{18}{81} \qquad \text{... (2)}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{36}{81} \qquad \text{... (3)}$$

$$\alpha\beta\gamma = -\frac{8}{81} \qquad \text{... (4)}$$

From (1) and (3), we get

$$2\gamma\alpha + \gamma\alpha = -\frac{36}{81}$$

$$\text{i.e., } 3\gamma\alpha = -\frac{36}{81}.$$

$$\therefore \quad \gamma\alpha = -\frac{4}{27} \qquad \text{... (5)}$$

Substituting this value of $\gamma\alpha$ in (4), we get

$$\beta\left(\frac{4}{27}\right) = -\frac{8}{81}.$$

$$\therefore\ \beta = \frac{2}{3}.$$

From (2), we have

$$\alpha + \gamma = \frac{18}{81} - \frac{2}{3} = -\frac{4}{9} \qquad \dots (6)$$

From (5) and (6), we get

$$\alpha = \frac{2}{9} \text{ and } \gamma = -\frac{2}{3}.$$

$\therefore$ The roots are $\frac{2}{9}, \frac{2}{3}$ and $-\frac{2}{3}$.

Example 4. If the sum of two roots of the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

equals the sum of the other two, prove that $p^3 + 8r = 4pq$.

(B.A. 1988)

Let the roots of the equation be α, β, γ and δ

Then $\alpha + \beta = \gamma + \delta$ ... (1)

From the relation of the coefficients and the roots, we have

$$\alpha + \beta + \gamma + \delta = -p \qquad \dots (2)$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q \qquad \dots (3)$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r \qquad \dots (4)$$

$$\alpha\beta\gamma\delta = s \qquad \dots (5)$$

From (1) and (2), we get

$$2(\alpha + \beta) = -p \qquad \dots (6)$$

(3) can be written as

$$\alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta) = q$$

$$\text{i.e., } (\alpha\beta + \gamma\delta) + (\alpha + \beta)^2 = q \qquad \dots (7)$$

(4) can be written as

$$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -r$$

$$\text{i.e., } (\alpha + \beta)(\alpha\beta + \gamma\delta) = -r \qquad \dots (8)$$

From (6) and (7), we get

$$\alpha\beta + \gamma\delta + \frac{p^2}{4} = q.$$

$$\therefore\ \alpha\beta + \gamma\delta = q - \frac{p^2}{4} \qquad \ldots (9)$$

From (8), we get

$$-\frac{p}{2}(\alpha\beta + \gamma\delta) = -r$$

$$\text{i.e., } \alpha\beta + \gamma\delta = \frac{2r}{p} \qquad \ldots (10)$$

Equating (9) and (10), we get

$$q - \frac{p^2}{4} = \frac{2r}{p}$$

i.e., $4pq - p^3 = 8r$

i.e., $p^3 + 8r = 4pq.$

Example 5. Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ given that two of its roots are equal in magnitude and opposite in sign.

Let the roots of the equation be $\alpha, \beta, \gamma, \delta$.

Here $\gamma = -\delta$

i.e., $\gamma + \delta = 0$ ... (1)

From the relations of the roots and coefficients

$\alpha + \beta + \gamma + \delta = 2$ ... (2)

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 4$ ... (3)

$\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta = -6$ ... (4)

$\alpha\beta\gamma\delta = -21$ ... (5)

From (1) and (2), we get $\alpha + \beta = 2$ ... (6)

(3) can be written as $\alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta) = 4.$

$\therefore\ \alpha\beta + \gamma\delta = 4$ ... (7)

(4) can be written as $\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -6$

i.e., $\gamma\delta(\alpha + \beta) = -6$ ... (8)

From (6) and (8), we get $\gamma\delta = -3$ ... (9)

but $\gamma + \delta = 0 \quad \therefore \quad \gamma = \sqrt{3}\,,\ \delta = -\sqrt{3}$.

From (7) and (9), we get $\alpha\beta = 7$

$\therefore \quad \alpha$ and β are the roots of $x^2 - 2x + 7 = 0$.

$\therefore \quad \alpha = 1 + \sqrt{-6}\,,\ \beta = 1 - \sqrt{-6}$

$\therefore$ The roots of the equation are $\pm\sqrt{3}\,,\ 1 \pm \sqrt{-6}$

Example 6. Find the condition that the general biquadratic equation $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$ may have two pairs of equal roots.

Let the roots be $\alpha, \alpha, \beta, \beta$.

From the relations of coefficients and roots

$$2\alpha + 2\beta = -\frac{4b}{a} \qquad \text{... (1)}$$

$$\alpha^2 + \beta^2 + 4\alpha\beta = \frac{6c}{a} \qquad \text{... (2)}$$

$$2\alpha\beta^2 + 2\alpha^2\beta = -\frac{4d}{a} \qquad \text{... (3)}$$

$$\alpha^2\beta\delta = \frac{e}{a} \qquad \text{... (4)}$$

From (1), we get $\alpha + \beta = -\dfrac{2b}{a}$ ... (5)

From (3), we get $2\alpha\beta(\alpha + \beta) = -\dfrac{4d}{a}$.

$$\therefore \quad \alpha\beta = \frac{d}{b} \qquad \text{... (6)}$$

From (5) and (6), we get that α, β are the roots of the equation

$$x^2 + \frac{2b}{a}x + \frac{d}{b} = 0.$$

$$\therefore \quad ax^4 + 4bx^3 + 6cx^2 + 4dx + e \equiv a\left(x^2 + \frac{2b}{a}x + \frac{d}{b}\right)^2.$$

Comparing coefficients

$$6c = a\left(\frac{4b^2}{a^2} + \frac{2d}{b}\right) \text{ and } e = \frac{ad^2}{b^2}.$$

$$\therefore \quad 3abc = a^2d + 2b^3 \text{ and } eb^2 = ad^2.$$

§ 15. Transformations of equations.

If an equation is given, it is possible to transform this equation into another whose roots bear with the roots of the original equation a given relation. Such a transformation often helps us to solve equations easily or to discuss the nature of the roots of the equations. We shall explain here the most important elementary transformations of equations.

§ 15.1. Roots with signs changed.

To transform an equation into another whose roots are numerically the same as those of the given equation but opposite in sign.

Let $\alpha_1, \alpha_2, \dots \alpha_n$ be the root s of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = 0.$$

Then we have

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n \equiv (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n).$$

Changing x into $-x$, we have

$$(-x)^n + p_1(-x)^{n-1} + p_2(-x)^{n-2} + \dots + p_n \equiv (-x - \alpha_1)(-x - \alpha_2) \dots (-x - \alpha_n)$$

$\therefore$ The roots of the equation

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \dots \pm p_n = 0,$$

are $-\alpha_1, -\alpha_2, \dots, -\alpha_n$.

Therefore to effect the required transformation, we have to substitute $-x$ for x in the given equation ; that is to change the sign of every alternate term of the given equation beginning with the second.

§ 15.2. Roots multiplied by a given number.

To transform an equation into another whose roots are m times that of the given equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n \equiv (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$$

Instead of x substitute $\dfrac{y}{m}$.

We get

$$\left(\frac{y}{m}\right)^n + p_1\left(\frac{y}{m}\right)^{n-1} + p_2\left(\frac{y}{m}\right)^{n-2} + \ldots + p_n$$
$$\equiv \left(\frac{y}{m} - \alpha_1\right)\left(\frac{y}{m} - \alpha_2\right)\ldots\left(\frac{y}{m} - \alpha_n\right)$$

Multiplying both sides by m^n,

$$y^n + mp_1 y^{n-1} + m^2p_2y^{n-2} + \ldots + m^np_n$$
$$\equiv (y - m\alpha_1)(y - m\alpha_2)\ldots(y - m\alpha_n).$$

$\therefore$ The equation $y^n + mp_1y^{n-1} + m^2p_2y^{n-2} + m^np_n = 0$ has the roots $m\alpha_1, m\alpha_2, \ldots m\alpha_n$.

Hence to effect this transformation we have to multiply the successive terms beginning with the second by

$$m, m^2, m^3, \ldots, m^n.$$

This transformation is useful for the purpose of removing the coefficient of the first term of an equation when it is other than unity and generally for removing the fractional coefficients from an equation.

Example 1. Change the equation $2x^4-3x^3+3x^2-x+2=0$ into another the coefficient of whose highest term will be unity.

Multiply the roots by 2. Then the transformed equation becomes

$$2x^4 - 3 . 2x^3 + 3 . 2^2x^2 - 2^3x + 2 . 2^4 = 0$$
$$\text{i.e., } 2x^4 - 6x^3 + 12x^2 - 8x + 32 = 0.$$

Dividing by 2, we get

$$x^4 - 3x^3 + 6x^2 - 4x + 16 = 0.$$

Example 2. Remove the fractional coefficients from the equation $x^3 - \frac{1}{4}x^2 + \frac{1}{3}x - 1 = 0.$

Multiply the roots by 12.

We get the transformed equation as

$$x^3 - \frac{1}{4} . 12x^2 + \frac{1}{3} . 12^2x - 1 . 12^3 = 0$$
$$\text{i.e., } x^3 - 3x^2 + 48x - 1728 = 0.$$

Example 3. Remove the fractional coefficients from the equation $x^3 + \frac{1}{4}x^2 - \frac{1}{16}x + \frac{1}{72} = 0$.

To transform the equation into another whose roots are multiplied by m, we get

$$x^3 + \frac{m}{4}x^2 - \frac{m^2}{16}x + \frac{m^3}{72} = 0$$

$$\text{i.e., } x^3 + \frac{m}{2^2}x^2 - \frac{m^2}{2^4}x + \frac{m^3}{2^3 \cdot 3^2} = 0.$$

If $m = 12$, the fractions $\frac{m}{2}, \frac{m^2}{2^4}, \frac{m^3}{2^3 \cdot 3^2}$ will be integers.

Hence we have to multiply the roots by 12. The equation becomes

$$x^3 + \frac{12}{2^2}x^2 - \frac{12^2}{2^4}x + \frac{12^3}{2^3 \cdot 3^2} = 0$$

$$\text{i.e., } x^3 + 3x^2 - 9x + 24 = 0.$$

Exercises 46

1. Find the equation whose roots are the roots of

$$x^5 + 6x^4 + 6x^3 - 7x^2 + 2x - 1 = 0$$

with the signs changed.

2. Change the sign of the roots of the equation

$$x^7 + 4x^5 + x^3 - 2x^2 + 7x + 3 = 0.$$

3. Transform the equation $3x^3 + 4x^2 + 5x - 6 = 0$ into one in which the coefficient of x^3 is unity and all coefficients are integral. (B.Sc.1990)

4. Remove the fractional coefficients from the equation

(1) $x^3 + \frac{3}{2}x^2 + \frac{5}{18}x + \frac{1}{108} = 0.$

(2) $2x^3 + \frac{3}{2}x^2 - \frac{1}{8}x - \frac{3}{16} = 0.$ (B.Sc.1990)

5. Transform the equation $3x^4 - \frac{5}{2}x^3 + \frac{7}{6}x^2 - x + \frac{7}{18} = 0$ into another with integral coefficients and for the coefficient of the first term unity.

§ 15.3. Reciprocal roots.

To transform an equation into another whose roots are the reciprocals of the roots of the given equation.

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0.$$

We have

$$x^n+p_1x^{n-1}+p_2x^{n-2}+..+p_n \equiv (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n).$$

Put $x = \frac{1}{y}$, we have

$$\left(\frac{1}{y}\right)^n + p_1\left(\frac{1}{y}\right)^{n-1} + p_2\left(\frac{1}{y}\right)^{n-2} + \dots + p_n$$
$$= \left(\frac{1}{y} - \alpha_1\right)\left(\frac{1}{y} - \alpha_2\right)\dots\left(\frac{1}{y} - \alpha_n\right)$$

Multiplying throughout by y^n, we have

$$p_ny^n + p_{n-1}y^{n-1} + p_{n-2}y^{n-2} + \dots + p_1y + 1$$
$$= (\alpha_1\alpha_2\dots\alpha_n)\left(\frac{1}{\alpha_1}-y\right)\left(\frac{1}{\alpha_2}-y\right)\dots\left(\frac{1}{\alpha_n}-y\right).$$

Hence the equation

$$p_ny^n + p_{n-1}y^{n-1} + p_{n-2}y^{n-2} + \dots + p_1y + 1 = 0$$

has roots $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \dots, \frac{1}{\alpha_n}$.

§ 16. Reciprocal equation.

If an equation remains unaltered when x is changed into its reciprocal, it is called a reciprocal equation.

Let $x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0$ ... (1)

be a reciprocal equation. When x is changed into its reciprocal $\frac{1}{x}$, we get the transformed equation

$$p_nx^n + p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \dots + p_1x + 1 = 0$$

i.e., $x^n + \frac{p_{n-1}}{p_n}x^{n-1} + \frac{p_{n-2}}{p_n}x^{n-2} + \ldots + \frac{p_1}{p_n}x + \frac{1}{p_n} = 0$ (2)

Since (1) is a reciprocal equation, it must be the same as (2).

$\therefore \quad \frac{p_{n-1}}{p_n} = p_1, \frac{p_{n-2}}{p_n} = p_2 \cdots \frac{p_1}{p_n} = p_{n-1}$ and $\frac{1}{p_n} = p_n$.

$\therefore \quad p_n^2 = 1.$

$\therefore \quad p_n = \pm 1.$

Case i. $p_n = 1$.

Then $p_{n-1} = p_1, p_{n-2} = p_2, p_{n-3} = p_3, \ldots$

In this case the coefficients of the terms equidistant from the beginning and the end are equal in magnitude and have the same sign.

Case ii. $p_n = -1$, we have

$$p_{n-1} = -p_1, p_{n-2} = -p_2, \ldots p_1 = -p_{n-1}.$$

In this case the terms equidistant from the beginning and the end are equal in magnitude but different in sign.

§ 16.1. Standard form of reciprocal equations.

If α be a root of a reciprocal equation, $\frac{1}{\alpha}$ must also be a root, for it is a root of the transformed equation and the transformed equation is identical with the first equation. Hence the roots of a reciprocal equation occur in pairs

$$\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}, \ldots$$

When the degree is odd one of its roots must be its own reciprocal.

$$\gamma = \frac{1}{\gamma}$$

i.e., $\gamma^2 = 1$.

i.e., $\gamma = \pm 1$.

If the coefficients have all like signs, then -1 is a root ; if the coefficients of the terms equidistant from the first and last have opposite signs, then $+1$ is a root. In either case the degree of an equation can be depressed by unity if we divide the equation by

$x + 1$ or by $x - 1$. The depressed equation is always a reciprocal equation of even degree with like signs for its coefficeints.

If the degree of a given reciprocal equation is even, say $n = 2m$ and if terms equidistant from the first and last have opposite signs, then

$$p_m = -p_m,$$

i.e., $p_m = 0$, so that in this type of reciprocal equations, the middle term is absent. Such an equation may be written as

$$x^{2m} - 1 + p_1 x (x^{2m-2} - 1) + \ldots = 0.$$

Dividing by $x^2 - 1$, this reduces to a reciprocal equation of like signs of even degree. Hence all reciprocal equations may be reduced to an even degree reciprocal equation with like sign, and so an even degree reciprocal equation with like signs is considered as the standard form of reciprocal equations.

§ 16.2. A reciprocal equation of the standard form can always be depressed to another of half the dimensions.

It has been shown in the previous article that all reciprocal equations can be reduced to a standard form, in which the degree is even and the coefficients of terms equidistant from the beginning and the end are equal and have the same sign.

Let the standard reciprooal equation be

$$a_0 x^{2m} + a_1 x^{2m-1} + a_2 x^{2m-2} + \ldots + a_m x^m + \ldots + a_1 x + a_0 = 0.$$

Dividing by x^m and grouping the terms equally distant from the ends, we have

$$a_0 \left(x^m + \frac{1}{x^m}\right) + a_1 \left(x^{m-1} + \frac{1}{x^{m-1}}\right) + \ldots$$

$$+ a_{m-1} \left(x + \frac{1}{x}\right) + a_m = 0$$

Let $x + \frac{1}{x} = z$ and $x^r + \frac{1}{x^r} = X_r$.

We have the relation $X_{r+1} = z \, . \, X_r - X_{r-1}$.

Giving r in succession the values 1, 2, 3, ...

we have $X_2 = z X_1 - X_0 = z^2 - 2$

$$X_3 = z X_2 - X_1 = z^3 - 3z$$

$$X_4 = zX_3 - X_2 = z^4 - 4z^2 + 2$$
$$X_5 = zX_4 - X_3 = z^5 - 5z^3 + 5z$$

and so on. Substituting these values in the above equation, we get an equation of the m^{th} degree in z. To every root of the reduced equation in z, correspond two roots of the reciprocal equation. Thus if k be a root of the reduced equation, the quadratic $x + \frac{1}{x} = k$, *i.e.*, $x^2 - kx + 1 = 0$ gives the two corresponding roots $\frac{k \pm \sqrt{k^2 - 4}}{2}$ of the given reciprocal equation.

Example 1. Find the roots of the equation

$$x^5 + 4x^4 + 3x^3 + 3x^2 + 4x + 1 = 0 \qquad \text{(B.Sc.1991)}$$

This is a reciprocal equation of odd degree with like signs.

$\therefore$ $(x + 1)$ is a factor of $x^5 + 4x^4 + 3x^3 + 3x^2 + 4x + 1$.

The equation can be written as

$$x^5 + x^4 + 3x^4 + 3x^3 + 3x^2 + 3x + x + 1 = 0$$

i.e., $x^4(x + 1) + 3x^3(x + 1) + 3x(x + 1) + 1(x + 1) = 0$

i.e., $(x + 1)(x^4 + 3x^3 + 3x + 1) = 0$.

$\therefore$ $x + 1 = 0$ or $x^4 + 3x^3 + 3x + 1 = 0$.

Dividing by x^2, we get $\left(x^2 + \frac{1}{x^2}\right) + 3\left(x + \frac{1}{x}\right) = 0$.

Put $x + \frac{1}{x} = z$. $\therefore$ $x^2 + \frac{1}{x^2} = z^2 - 2$.

$$\therefore \; z^2 - 2 + 3z = 0$$
$$\therefore \; z = \frac{-3 \pm \sqrt{17}}{2}.$$

Hence $x + \frac{1}{x} = \frac{-3 \pm \sqrt{17}}{2}$

i.e., $2x^2 + (3 + \sqrt{17})x + 2 = 0$

or $2x^2 + (3 - \sqrt{17})x + 2 = 0$.

From these equations x can be found.

Example 2. Solve the equation

$$6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0. \qquad \text{(B.Sc.1994)}$$

This is a reciprocal equation of odd degree with unlike signs. Hence $x - 1$ is a factor of the left-hand side.

The equation can be written as follows :

$$6x^5 - 6x^4 + 5x^4 - 5x^3 - 38x^3 + 38x^2 + 5x^2 - 5x + 6x - 6 = 0$$

i.e., $6x^4(x-1) + 5x^3(x-1) - 38x^2(x-1) + 5x(x-1) + 6(x-1) = 0$

i.e., $(x-1)(6x^4 + 5x^3 - 38x^2 + 5x + 6) = 0$

$\therefore\ x = 1$ or $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0.$

We have to solve the equation $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0.$

Dividing by x^2, $6x^2 + 5x - 38 + \frac{5}{x} + \frac{6}{x^2} = 0$

i.e., $6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0.$

Put $x + \frac{1}{x} = z$. $\therefore\ x^2 + \frac{1}{x^2} = z^2 - 2.$

The equation becomes

$6(z^2 - 2) + 5z - 38 = 0$

i.e., $6z^2 + 5z - 50 = 0$

i.e., $(2z - 5)(3z + 10) = 0.$

$\therefore\ z = \frac{5}{2}$ or $-\frac{10}{3}$

i.e., $x + \frac{1}{x} = \frac{5}{2}$ or $x + \frac{1}{x} = -\frac{10}{3}$

i.e., $2x^2 - 5x + 2 = 0$ or $3x^2 + 10x + 3 = 0$

i.e., $(2x - 1)(x - 2) = 0$ or $(3x + 1)(x + 3) = 0$

i.e., $x = \frac{1}{2}$ or 2 or $-\frac{1}{3}$ or $-3.$

$\therefore$ The roots of the equation are $1, \frac{1}{2}, 2, -\frac{1}{3}$ and $-3.$

Example 3. Solve the equation

$$6x^6 - 35x^5 + 56x^4 - 56x^2 + 35x - 6 = 0.$$

There is no mid-term and this is a reciprocal equation of even degree with unlike signs. We can easily see that $x^2 - 1$ is a factor of the expression on left-hand side of the equation.

The equation can be written as

$$6(x^6 - 1) - 35x(x^4 - 1) + 56x^2(x^2 - 1) = 0$$

i.e., $6(x^2 - 1)(x^4 + x^2 + 1) - 35x(x^2 - 1)(x^2 + 1) + 56x^2(x^2 - 1) = 0$

i.e., $(x^2 - 1)(6x^4 - 35x^3 + 62x^2 - 35x + 6) = 0$

i.e., $x = 1$ or -1 or $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0.$

Dividing by x^2, we get $6\left(x^2 + \frac{1}{x^2}\right) - 35\left(x + \frac{1}{x}\right) + 62 = 0.$

Put $x + \frac{1}{x} = z$, then $x^2 + \frac{1}{x^2} = z^2 - 2.$

$\therefore\ 6(z^2 - 2) - 35z + 62 = 0$

i.e., $6z^2 - 35z + 50 = 0$

i.e., $(3z - 10)(2z - 5) = 0$

i.e., $z = \frac{10}{3}$ or $\frac{5}{2}.$

$\therefore\ x + \frac{1}{x} = \frac{10}{3}$ or $x + \frac{1}{x} = \frac{5}{2}$

i.e., $3x^2 - 10x + 3 = 0$ or $2x^2 - 5x + 2 = 0$

i.e., $(x - 3)(3x - 1) = 0$ or $(x - 2)(2x - 1) = 0$

i.e., $x = 3$ or $\frac{1}{3}$ or 2 or $\frac{1}{2}.$

$\therefore$ The roots of the equation are $1, -1, 3, \frac{1}{3}, 2$ and $\frac{1}{2}.$

Exercises 47

Solve the following equations :-

1. $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0.$ (B.Sc.1987)

2. $x^4 + 3x^3 - 3x - 1 = 0.$

3. $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0.$ (B.Sc.1990)

4. $60x^4 - 736x^3 + 1433x^3 - 736x + 60 = 0.$ (B.Sc.1991)

5. $2x^6 - 9x^5 + 10x^4 - 3x^3 + 10x^2 - 9x + 2 = 0.$

6. $2x^6 - x^5 - 2x^3 - x + 2 = 0.$

7. $x^5 + x^4 + x^3 + x^2 + x + 1 = 0.$ (B.Sc.1989)

8. $2x^5 + x^4 + x + 2 = 12x^2 (x + 1).$

9. $6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0.$ (B.Sc.1988)

10. $x^5 + 4x^4 + x^3 + x^2 + 4x + 1 = 0.$

11. $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0.$

12. $x^5 - 5x^3 + 5x^2 - 1 = 0.$

13. $x^{10} - 3x^8 + 5x^6 - 5x^4 + 3x^2 - 1 = 0.$ (B.Sc.1988)

14. $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0.$ (B.Sc.1990)

15. $x^6 + 2x^5 + 2x^4 - 2x^2 - 2x - 1 = 0.$

16. $x^4 - x^3 - 8x^2 + x + 1 = 0.$

§ 17. To increase or decrease the roots of a given equation by a given quantity.

Let the roots of the given equation

$$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0$$

be $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ and suppose we require the equation whose roots are $\alpha_1 - h, \alpha_2 - h, \alpha_3 - h, \dots, \alpha_n - h$.

We have $f(x) \equiv a_0 (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$.

In this if we change x into $y + h$, we have

$$f(y+h) \equiv a_0 (y+h - \alpha_1)(y+h - \alpha_2) \dots (y+h - \alpha_n).$$

The right-hand side vanishes when

$$y = \alpha_r - h, \ (r = 1, 2, \dots, n).$$

Hence if an equation is to be transformed into another whose roots are those of the first diminished by h, substitute $y + h$ for x in the given equation. Then we obtain the transformed equation as

$$a_0 (y+h)^n + a_1 (y+h)^{n-1} + a_2 (y+h)^{n-2} + \dots + a_n = 0 \quad \dots (1)$$

6. Determine the value of a such that the equation

$x^3 - 12x + a = 0$ has only one real root.

7. Find the range of values of k for which the following equations have real roots :-

(1) $x^3 + 4x^2 + 5x + 2 + k = 0.$

(2) $2x^3 - 9x^2 + 12x - k = 0.$

(3) $3x^4 - 4x^3 - 12x^2 + k = 0.$

(4) $x^4 - 14x^2 + 24x - k = 0.$

(5) $x^4 + 4x^3 - 8x^2 + k = 0.$

(6) $3x^4 + 8x^3 - 6x^2 - 24x + k = 0.$

(7) $2x^3 - 15x^2 + 36x + k = 0.$

8. Discuss the nature of the roots of the equation

$3x^4 + 8x^3 - 30x^2 - 72x + k = 0$

for different values of k.

9. Show that the equation $3x^4 + 8x^3 - 6x^2 - 24x + r = 0$ has four real roots if $-13 < r < -8$, two real roots if $-8 < r < 19$ or if $k < -13$ and no real root if $r > 19$.

28

§ 26. Multiple roots.

If $f(x)$ is a polynomial in x and the equation $f(x) = 0$ has m roots equal to α, then $f(x)$ must be of the form $(x - \alpha)^m \phi(x)$ where $\phi(\alpha) \neq 0$.

$$\therefore \quad f'(x) = (x - \alpha)^m \phi'(x) + m(x - \alpha)^{m-1} \phi(x)$$
$$= (x - \alpha)^{m-1} \{(x - \alpha)\phi'(x) + m\phi(x)\}.$$

Hence $(x - \alpha)^{m-1}$ is a common factor of $f(x)$ and $f'(x)$ and it is easily seen that $(x - \alpha)^{m-1}$ will not be a common factor unless $f(x)$ is divisible by $(x - \alpha)^m$. Hence the multiple roots of $f(x)$, if any, are to be deducted by finding the greatest common factors of $f(x)$ and $f'(x)$ by the usual algebraic process. We may then state a rule for finding the multiple roots of an equation $f(x) = 0$ as follows :

(1) Find $f'(x)$.

(2) Find the H.C.F. of $f(x)$ and $f'(x)$.

(3) Find the roots of the H.C.F.

Each different root of the H.C.F. will occur once more in $f(x)$ than it does in the H.C.F.

Example 1. Find the multiple roots of the equation

$$x^4 - 9x^2 + 4x + 12 = 0. \quad \text{(B.Sc.1988)}$$

$$f(x) = x^4 - 9x^2 + 4x + 12$$

$$f'(x) = 4x^3 - 18x + 4.$$

The H.C.F. of $f(x)$ and $f'(x)$ is easily found to be $x - 2$.

Hence $(x-2)^2$ is a factor of $f(x)$.

The remaining factors are easily ascertained.

Thus we find $f(x) = (x-2)^2(x+1)(x+3)$.

$\therefore$ The roots of $f(x) = 0$ are 2, 2, -1 and -3.

Example 2. Find the values of a for which

$$ax^3 - 9x^2 + 12x - 5 = 0$$

has equal roots and solve the equation in one case.

$$f(x) = ax^3 - 9x^2 + 12x - 5.$$

$$f'(x) = 3ax^2 - 18x + 12.$$

Find the H.C.F. of

$$ax^3 - 9x^2 + 12x - 5$$

and $3ax^2 - 18x + 12$.

The process of finding the H.C.F. is exhibited below :

$\frac{1}{3}x$	$ax^3 - 9x^2 + 12x - 5$	$3ax^2 - 18x + 12$	$-a$
	$ax^3 - 6x^2 + 4x$	$3ax^2 - 8ax + 5a$	
	$-3x^2 + 8x - 5$	$(8a-18)x + 12 - 5a$	
	$-3(8a-18)x^2 + 8(8a-18)x - 5(8a-18)$		
	$3(8a-18)x^2 + 3(12-5a)x$		
	$(49a-108)x - 5(8a-18)$		

If $f(x)$ and $f'(x)$ have a common linear factor

$$\frac{49a - 108}{8a - 18} = \frac{-5(8a-18)}{12 - 5a}.$$

i.e., $25a^2 - 104a + 108 = 0$

i.e., $(25a - 54)(a - 2) = 0$.

$\therefore \quad a = 2$ or $\frac{54}{25}$.

If $a = 2$, the linear factor on both sides becomes $-10x + 10$ and $-2x + 2$.

$\therefore$ The H.C.F. is $(x - 1)$.

Hence $(x - 1)^2$ is a factor of $2x^3 - 9x^2 + 12x - 5$.

$\therefore$ The remaining factor is $2x - 5$.

$\therefore$ The roots of the equation are $1, 1, \frac{5}{2}$.

Example 3. Find the condition that the cubic equation $ax^3 + 3bx^2 + 3cx + d = 0$ has two equal roots and when the condition is satisfied, find the equal roots.

Let α be the equal root.

Then α is also a root of $f'(x) = 0$, where

$$f(x) = ax^3 + 3bx^2 + 3cx + d.$$

$$\therefore \quad a\alpha^3 + 3b\alpha^2 + 3c\alpha + d = 0 \qquad \ldots (1)$$

and $$3a\alpha^2 + 6b\alpha + 3c = 0$$

i.e., $$a\alpha^2 + 2b\alpha + c = 0 \qquad \ldots (2)$$

Subtracting the product of (2) and α from (1), we get

$$b\alpha^2 + 2c\alpha + d = 0 \qquad \ldots (3)$$

From (2) and (3), we have

$$\frac{\alpha^2}{2(bd - c^2)} = \frac{\alpha}{bc - ad} = \frac{1}{2(ac - b^2)}.$$

$$\therefore \quad (bc - ad)^2 = 4(bd - c^2)(ac - b^2)$$

and $$\alpha = \frac{1}{2}\left(\frac{bc - ad}{ac - b^2}\right).$$

Example 4. Find the condition that the equations

$$ax^3 + 3bx + c = 0,\ a'x^3 + 3b'x + c' = 0$$

should have a common root. When this condition is satisfied, show that the common root is a double root of the equation

$$2(ab' - a'b)x^3 + (ac' - a'c)x^2 + (bc' - b'c) = 0.$$

Let α be the common root of the equation.

Then $a\alpha^3 + 3b\alpha + c = 0$ ... (1)

$a'\alpha^3 + 3b'\alpha + c' = 3$ ... (2)

Multiplying (1) by a' and (2) by a and subtracting, we get

$3(a'b - ab')\alpha + ca' - ac' = 0$

$$\therefore \alpha = -\frac{ac' - a'c}{3(ab' - a'b)} \quad \text{... (3)}$$

Substituting in (1), we get

$$-\frac{a(ac' - a'c)^3}{27(ab' - a'b)^3} - \frac{3b(ac' - a'c)}{3(ab' - a'b)} + c = 0$$

i.e., $a(a'c - a'c)^3 + 27b(ac' - a'c)(ab' - a'b)^2$

$- 27c(ab' - a'b)^2 = 0.$

If α is a double root of the equation

$2(ab' - a'b)x^3 + (ac' - a'c)x^2 + (bc' - b'c) = 0$ ... (4)

α is also a root of

$$6(ab' - a'b)x^2 + 2x(ac' - a'c) = 0$$

i.e., $6(ab' - a'b)\alpha^2 + 2\alpha(ac' - a'c) = 0$

i.e., $2\alpha\{3(ab' - a'b)\alpha + (ac' - a'c)\} = 0.$

α cannot be equal to zero.

$\therefore \alpha = -\dfrac{(ac' - a'c)}{3(ab' - a'b)}$ which is the same as (3) .

$\therefore$ α is a double root of the equation (4).

Exercises 54

1. Find the multiple roots of the equations :

(1) $4x^4 + 24x^3 + 49x^2 + 45x + 25 = 0.$

(2) $4x^3 - 12x^2 - 15x - 4 = 0.$

(3) $x^4 - 6x^3 + 13x^2 - 24x + 36 = 0.$

12. Show that $x^9 + x^8 + x^4 + x^2 + 1 = 0$ has one real root which is negative and eight imaginary roots. (B.Sc.1991)

§ 25. Rolles' Theorem.

Between two consecutive real roots a and b of the equation $f(x) = 0$ where $f(x)$ is a polynomial, there lies at least one real root of the equation $f'(x) = 0$.

Let $f(x)$ be $(x-a)^m (x-b)^n \phi(x)$ where m and n are positive integers and $\phi(x)$ is not divisible by $(x-a)$ or by $(x-b)$. Since a and b are consecutive real roots of $f(x)$, the sign of $\phi(x)$ in the interval $a \le x \le b$ is either positive throughout or negative throughout, for if it changes its sign between a and b, then there is a root of $\phi(x) = 0$ that is of $f(x) = 0$ lying between a and b, which is contrary to the hypothesis that a and b are consecutive roots.

$$\therefore \; f'(x) = (x-a)^m n (x-b)^{n-1} \phi(x) + m(x-a)^{m-1}(x-b)^n \phi(x) + (x-a)^m (x-b)^n \phi'(x)$$
$$= (x-a)^{m-1}(x-b)^{n-1} \chi(x),$$

where $\chi(x) = \{m(x-b) + n(x-a)\}\phi(x) + (x-a)(x-b)\phi'(x)$.

$$\therefore \; \chi(a) = m(a-b)\phi(a)$$
$$\chi(b) = n(b-a)\phi(b).$$

$\chi(a)$ and $\chi(b)$ have differnt signs since $\phi(a)$ and $\phi(b)$ have the same sign.

$\therefore \; \chi(x) = 0$ has atleast one root between a and b.

Hence $f'(x) = 0$ has at least one root between a and b.

Cor. 1. If all the roots of $f(x) = 0$ are real, then all the roots of $f'(x) = 0$ are also real.

If $f(x) = 0$ is a polynomial of degree n, $f'(x) = 0$ is a polynomial of degree $n-1$ and each root of $f'(x) = 0$ lies in each of the $(n-1)$ intervals between the n roots of $f(x) = 0$.

Cor. 2. If all the roots of $f(x) = 0$ are real, then the roots of

$f'(x) = 0$, $f''(x) = 0$, $f''' = 0$ are real.

Cor. 3. At the most only one real root of $f(x) = 0$ can lie between two consecutive roots of $f'(x) = 0$, that is the real roots of $f'(x) = 0$ separate those of $f(x) = 0$.

Cor. 4. If $f'(x) = 0$ has r real roots, then $f(x) = 0$ cannot have more than $(r+1)$ real roots.

Cor. 5. $f(x)=0$ has at least as many imginary roots as $f'(x)=0$.

Example 1. Find the nature of the roots of the equation

$$4x^3 - 21x^2 + 18x + 20 = 0.$$

Let us consider the function $f(x) = 4x^3 - 21x^2 + 18x + 20$.

We have $f'(x) = 12x^2 - 42x + 18$
$= 6(2x-1)(x-3)$.

Hence the real roots of $f'(x) = 0$ are $\frac{1}{2}$ and 3. So, the roots of $f(x) = 0$, if any will be in the intervals between $-\infty$ and $\frac{1}{2}$, $\frac{1}{2}$ and 3, 3 and $+\infty$ respectively.

x :	$-\infty$	$\frac{1}{2}$	3	∞
$f(x)$:	$-$	$+$	$-$	$+$

$\therefore$ $f(x)$ must vanish, once in each of the above intervals.

Hence $f(x) = 0$ has three real roots.

Example 2. Show that the equation $3x^4-8x^3-6x^2+24x-7=0$ has one positive, one negative and two imginary roots.

Let $f(x)$ be $3x^4 - 8x^3 - 6x^2 + 24x - 7$.

We have $f'(x) = 12x^3 - 24x^2 - 12x + 24$
$= 12(x+1)(x-1)(x-2)$.

The roots of $f'(x) = 0$ are -1, $+1$, $+2$.

x :	$-\infty$	-1	$+1$	$+2$	$+\infty$
$f(x)$:	$+$	$-$	$+$	$+$	$+$

$\therefore$ $f(x) = 0$ has a real root lying between -1 and $-\infty$, one between -1 and $+1$ and two imaginary roots.

We know that $f(+1) = +$, $f(0) = -$.

$\therefore$ The real root lying between -1 and $+1$ lies between 0 and $+1$. Hence it is a positive root. The other real root lies between -1 and $-\infty$ and so it is a negative root.

Example 3. Discuss the reality of the roots

$$x^4 + 4x^3 - 2x^2 - 12x + a = 0$$

for all real values of a.

Let $f(x)$ be $x^4 + 4x^3 - 2x^2 - 12x + a$.

$\therefore \ f'(x) = 4x^3 + 12x^2 - 4x - 12$

$= 4(x + 1)(x - 1)(x + 3).$

$\therefore$ The roots of $f'(x) = 0$ are -3, -1 and 1.

x :	$-\infty$	-3	-1	1	$+\infty$
$f(x)$:	$+$	$a - 9$	$7 + a$	$a - 9$	$+$

If $a - 9$ is negative and $7 + a$ is positive, the four roots of $f(x)$ are real.

$\therefore$ If $-7 < a < 9$, $f(x) = 0$ has four real roots.

If $a > 9$, then $f(x)$ is positive throughout and hence all the roots of $f(x) = 0$ are imaginary.

If $a < -7$, the signs of $f(x)$ at $-\infty, -3, -1, 1, \infty$ are respectively $+, -, -, -, +$.

Hence $f(x) = 0$ has two real roots and two imaginary roots.

Exercises 53

1. Prove that all the roots of the equation $x^3 - 18x + 25 = 0$ are real. (B.Sc.1990)

2. Find the nature of the roots of the equation

(1) $4x^3 - 21x^2 + 18x + 30 = 0$.

(2) $2x^3 - 9x^2 + 12x + 3 = 0$.

(3) $x^4 + 4x^3 - 20x^2 + 10 = 0$.

3. Prove that if $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} \dots + \frac{a_{n-1}}{2} + \frac{a_n}{1} = 0$, then the equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0$ has at least one root between 0 and 1.

4. Show that the equation $f(x) = (x-a)^3 + (x-b)^3 + (x-c)^3 = 0$ has one real and two imaginary roots.

5. Find the limits to the value of c in order that the equation $x^3 - 3x + c = 0$ may have all its roots real.

22. If the roots of the equation $x^4 - ax^3 + bx^2 - abx + 1 = 0$ are $\alpha, \beta, \gamma, \delta$, show that

$$(\alpha+\beta+\gamma)(\alpha+\beta+\delta)(\alpha+\gamma+\delta)(\beta+\gamma+\delta) = 1.$$

§ 23. Without actually solving an equation, it is possible to determine the nature of the roots of that equation. The following articles will give the various methods of determining the nature of the roots of the equation.

§ 24. Descartes' Rule of signs.

35

An equation $f(x) = 0$ cannot have more positive roots than there are changes of sign in $f(x)$.

Let $f(x)$ be a polynomial whose signs of the terms are

$$+ + - - - + - + + + - + -.$$

In this there are seven changes of sign including changes from + to − and from − to +. We shall show that if this polynomial be multiplied by a binomial (corresponding to a positive root) whose signs of the terms are + −, the resulting polynomial will have atleast one more change of sign than the original. Writing down only the signs of the terms in the multiplication, we have

$$\begin{array}{l} + + - - - + - + + + - + - \\ \underline{\qquad\qquad\qquad\qquad\qquad\quad + -} \\ \quad - - + + + - + - - - + - + \\ \underline{+ + - - - + - + + + - + -\quad} \\ + \pm - \pm \pm + - + \pm \pm - + - + \end{array}$$

Here in the last line tha ambiguous sign ± is placed wherever there are two different signs to be added. Here we see in the product

(1) an ambiguity replaces each continuation of sign in the original polynomial ;

(2) the sign before and after an ambiguity or a set of ambiguities are unlike ; and

(3) a change of sign is introduced in the end.

Let us take the most unfavourable case and suppose that all the ambiguities are replaced by continuations, then the sign of the terms become

$$+ + - - - + - + + + - + - +$$

The number of changes of sign is 8. Thus even in the most unfavourable case there is one more change of sign than the number of changes of sign in the original polynomial. Therefore we may conclude in general that the effect of multiplication of a binomial factor $x - \alpha$ is to introduce at least one change of sign.

Suppose the product of all the factors corresponding to negative and imginary roots of $f(x) = 0$ be a polynomial F (x). The effect of multipluing F (x) by each of the factors $x - \alpha$, $x - \beta$, $x - \gamma$,.... corresponding to the positive roots, α, β, γ is to introduce at least one change of sign for each, so that when the complete product is formed containing all the roots, we have the resulting polynomial which has at least as many changes of signs as it has positive roots. This is **Descartes' rule** of signs.

§ 24.1. Descartes' rule of signs for negative roots.

Let $f(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$.

By substituting $-x$ instead of x in the equations, we get

$$f(-x) = (-x - \alpha_1)(-x - \alpha_2) \cdots (-x - \alpha_n).$$

The roots of $f(-x) = 0$ are $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$.

$\therefore$ The negative roots of $f(x) = 0$ become the positive roots of

$$f(-x) = 0.$$

Hence to find the maximum number of negative roots of $f(x) = 0$, it is enough to find the maximum number of positive roots of $f(-x) = 0$.

So we can enunciate Descartes' rule for negative roots as follows.

No equation can have a greater number of negative roots then there are changes of sign in the terms of the polynomial $f(-x)$.

24.2. Using Descartes' rule of signs we can ascertain whether an equation $f(x) = 0$ has imginary roots or not.

We can find the maximum number for positive roots and also for negative roots. The degree of the equation will give the total numbers of roots of the equation. So if the sum of the maximum numbers of positive roots and negative roots is less than the degree of the equation, we are sure of the existence of imginary roots. Take for example the equation $x^7 + 8x^5 - x + 9 = 0$. The series of signs of the terms are as follows :

$$+ + - +.$$

The number of changes of signs is 2 and the equation cannot have more than two positive roots.

Now change x into $-x$, we get

$$-x^7 + 8x^5 + x + 9 = 0$$

i.e., $$x^7 + 8x^5 - x - 9 = 0.$$

The series of signs of the terms are

$$+ + - -$$

and the number of changes of sign is only one and so the equation cannot have more than one negative root.

Hence in the equation there cannot exist more than three real roots. Since it is a seventh degree equation, it has seven roots real or imaginary.

Therefore the given equation has at least four imginary roots.

§ 24.3. An equation $f(x) = 0$ is called complete when all powers of x from n^{th} to the constant term are present. In an complete equation, we can easily see that the sum of the number of changes of sign in $f(x)$ and $f(-x)$ is exactly equal to the degree of the equation. Hence this rule can be used to detect the imginary roots only in incomplete equations.

Example. Determine completely the nature of the roots of the equation $x^5 - 6x^2 - 4x + 5 = 0$. (B.Sc.1994)

The series of signs of the terms are $+ - - +$.

Here there are two changes of sign.

Hence there cannot be more than two positive roots.

Changing x into $-x$, the equation becomes

$$-x^5 - 6x^2 + 4x + 5 = 0$$

i.e., $$x^5 + 6x^2 - 4x - 5 = 0.$$

The series of the signs of the terms are

$$+ + - -.$$

Here there is only one change of sign.

∴ There cannot be more than one negative root.

So the equation has got at the most three real roots. The total number of roots of the equation is 5. Hence there are at least two imginary roots for the equation. We can also determine the limits between which the real roots lie.

$x =$	$-\infty$	-2	-1	0	1	2	∞
$x^5 - 6x^2 - 4x + 5 =$	$-$	$-$	$+$	$+$	$-$	$+$	$+$

The positive roots lie between 0 and 1, and 1 and 2, the negative root between -2 and -1.

Unit-I is Completed

Exercises 52

1. Show that the equation $x^7 - 3x^4 + 2x^3 - 1 = 0$ has at least four imginary roots. (B.Sc.1992)

2. Show that $x^6 + 3x^2 - 5x + 1 = 0$ has at least four imaginary roots. (B.Sc.1993)

3. Find the number of real roots of the equation

$$x^3 + 18x - 6 = 0.$$

4. Prove that the equation $x^4 + 3x - 1 = 0$ has two real and two imginary roots.

5. Find the number of imaginary roots of the equation

$$x^5 + 5x - 7 = 0.$$

6. Find the nature of the roots of the equation

$$x^4 + 15x^2 + 7x - 11 = 0.$$

7. Show that (1) the equation $x^3 + qx + r = 0$ where q and r are essentially positive has one negative and two imaginary roots, and (2) equation $x^3 - qx + r = 0$, has one negative root and the other two roots are either imaginary or positive.

8. Find the number of real roots of $x^7 - x^5 - x^4 - 6x^2 + 7 = 0$. (B.Sc. 1988)

9. Show that $12x^7 - x^4 + 10x^3 - 28 = 0$ has at least four imaginary roots.

10. Show that the equation $x^n - 1 = 0$ has, when n is even, two real roots 1 and -1 and no other real root and when n is odd, the real root is 1 and no other real root.

11. Show that the equation $x^n + 1 = 0$ has, when n is even, no real root and when n is odd, the real root is -1 and no other real root.

Government Arts College Grade-1, Ariyalur-621713

Class : Ist B.Sc, Computer science

subject : "Algebra and calculus"

Unit-II Allied - Mathematics 1

CHAPTER 2

MATRICES

I. §1. A **matrix** is defined to be a rectangular array of numbers arranged into rows and columns. It is written as follows :-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

The above array is called an m by n matrix (written $m \times n$) since it has m rows and n columns. It should be noted that a matrix has no numerical value. It is simply a convenient way of representing arrays of numbers. Two arrays containing the same numbers but differing in which the numbers appear constitute different matrices.

The individual numbers in the array are called the elements. The elements may be positive or negative integers or they may be fractions (i.e., rational numbers) or irrational numbers or complex numbers. The elements may be algebraic expressions in one or more variables, or functions.

Ex: §1.1. **Notation.** Brackets [ ], parenteses ( ) or the form ‖ ‖ is used to enclose the rectangular array of numbers. We shall adopt the notation [ ] to represent a matrix. An alternative notation is $[a_{rs}]$ in which the general element a_{rs} of the array is enclosed within square brackets. For a complete description $r = 1, 2, 3, \dots m; s = 1, 2, 3, \dots n$ should be added.

§ 1.2. Special types of matrices.

(i) A **row matrix** is a matrix with only one row.

E.g. [2, 1, 3].

(ii) A **column vector** is a matrix with only one column.

$$\text{E.g.} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

To save space column vectors are sometimes written as rows of numbers enclosed with curly brackets ; the example already given may be written as { – 1, 2, 3 }.

(iii) A **square matrix** is one in which the number of rows is equal to the number of columns.

If A is the square matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

then the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$$

is called the determinant of the matrix A and it is denoted by $|A|$ or det A. Only when a matrix is a square one, it will have a determinant.

The sum of the diagonal elements of a square matrix is called **the trace** of the matrix.

If $|A| = 0$, then the matrix is called a **singular matrix**. If $|A| \neq 0$, then the matrix is called a **non-singular matrix**.

(iv) A square matrix A whose elements above the principal diagonal (or below the principal diagonal) are all zero is called a **triangular matrix**.

$$\text{E.g.} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ -4 & 3 & -2 & 0 \\ 2 & 1 & 6 & 0 \end{bmatrix} ; \begin{bmatrix} 3 & -1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

(v) **Diagonal matrix** is a square matrix of any order with zero elements everywhere except on the main diagonal.

$$\text{E.g.,} \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix}$$

(vi) **Scalar matrix** is a diagonal matrix in which all the elements along the main diagonal are equal.

$$\text{E.g.,} \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_1 \end{bmatrix}$$

(vii) **Unit matrix** is a scalar matrix in which all the elements along the main diagonal are unity.

If it is a square matrix of order $n \times n$ it is a unit matrix of order n. It is usually denoted by I_n.

$$I_2 \text{ is } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ I_3 \text{ is } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A unit matrix of order n is usually denoted by I_n

$$\text{Hence } I_n = [\delta_{rs}] = \begin{bmatrix} 1 & 0 & 0 & 0 & .. & .. & 0 \\ 0 & 1 & 0 & 0 & .. & .. & 0 \\ 0 & 0 & 1 & 0 & .. & .. & 0 \\ 0 & 0 & 0 & 1 & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & .. & .. & 1 \end{bmatrix}$$

where δ_{rs} is called the knonecker delta and is defined by

$$\delta_{rs} = 1 \text{ if } r = s$$
$$= 0 \text{ if } r \neq s \text{ for } r, s = 1, 2, \ldots n.$$

(vii) Null or zero matrix. If all the elements in a matrix are zeros, it is called a null or zero matrix and is denoted by 0.

(viii)

$$\text{E.g., (i)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(ix) **Transpose matrix.** If rows and columns are interchanged in a matrix A, we obtain a second matrix that is called the transpose of the original matrix and is denoted by A^t.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then } A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\text{If } A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \text{ then } A^t = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

In general if $A = [a_{rs}]$, then $A^t = [a_{sr}]$.

§ 2. Scalar multiplication of a matrix. If all the elements of a matrix A are multiplied by a constant k, then the resulting matrix is kA.

$$\text{E.g., If } A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & -2 & 2 \end{bmatrix}, \text{ then } 2A = \begin{bmatrix} 4 & -6 & 8 \\ 2 & -4 & 4 \end{bmatrix}$$

$$\text{If } A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \text{ then } -3A = \begin{bmatrix} -6 & -3 & 3 \\ -9 & 0 & 6 \\ 0 & -3 & -3 \end{bmatrix}$$

Multiplication by a scalar obeys

(i) Commutative law i.e., $kA = Ak$

(ii) Associative law i.e., $k_1(k_2A) = (k_1k_2)A = k_1k_2A$.

§ 3. Equality of matrices. Two matrices A and B are said to be equal if they are identical, i.e., the corresponding elements are equal. Matrix A cannot be equal to matrix B unless the number of rows and columns in A is the same as the number of rows and columns in B.

Using the above definition of equality we can express certain relationships more compactly.

For example the equation.

$$\begin{bmatrix} 2x+3y & 2a-3b \\ 2x-3y & 2a+3b \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix}$$

can be written in the place of the four equations

$2x + 3y = 4$; $2x - 3y = 2$

$2a - 3b = -2$ and $2a + 3b = 3$.

§ 4. Addition of matrices. Matrices are added, by adding together corresponding elements of the matrices. Hence only matrices of the same order may be added together. The result of addition of two matrices is a matrix of the same order whose elements are the sum of the same elements of the corresponding positions in the original matrices.

$$\text{E.g.,} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \\ a_5+b_5 & a_6+b_6 \end{bmatrix}$$

In general if $A = [a_{rs}]$, $B = [b_{rs}]$

Then $A + B = C$, where $C = [a_{rs} + b_{rs}]$.

Two matrices that have the same order are conformable with respect to addition ; two matrices with different orders are not conformable with respect to addition and the process of addition will be considered to be meaningless for them.

§4.1. From the definition of addition of two matrices, the following properties can easily be proved :-

(i) Matrix addition is commutative.

If two matrices A and B are conformable for addition, A+B=B+A.

(ii) Matrix addition is associative.

If A,B,C are matrices of the same order. (A+B)+C=A+(B+C).

(iii) If A is a $m \times n$ matrix and 0 be the $m \times n$ zero matrix, then

$$A + 0 = 0 + A = A.$$

§5. Substraction. Two matrices of the same order are subtracted by subtracting their corresponding elements.

If $A = [a_{rs}]$ and $B = [b_{rs}]$, then $A - B = [a_{rs} - b_{rs}]$.

We can easily see that $A - B = A + (-1) B$.

If two matrices are equal, then their difference is a zero matrix.

$$A - A = 0.$$

Example 1. Given $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & 0 & 6 \end{bmatrix}$; $B = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

compute 3A − 4B

$$3A - 4B = 3\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & 0 & 6 \end{bmatrix} - 4\begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 6 \\ 9 & 3 & 12 \\ 15 & 0 & 18 \end{bmatrix} - \begin{bmatrix} 8 & 4 & -4 \\ 12 & 0 & -8 \\ 0 & 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3-8 & 0-4 & 6+4 \\ 9-12 & 3-0 & 12+8 \\ 15-0 & 0-4 & 18-4 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -4 & 10 \\ -3 & 3 & 20 \\ 15 & -4 & 14 \end{bmatrix}$$

Example 2. Find values of x, y, z and w that satisfy the matrix relationship

$$\begin{bmatrix} x+3 & 2y+5 \\ z+4 & 4x+5 \\ w-2 & 3w+1 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ -4 & 2x+1 \\ 2w+5 & -20 \end{bmatrix}$$

From the equality of these two matrices we get the equations.

$$x+3 = 1, \qquad 4x+5 = 2x+1$$

$$2y+5 = -5, \qquad w-2 = 2w+5$$

$$z+4 = -4, \qquad 3w+1 = -20$$

Solving these equations we get

$$x = -2,\ y = -5,\ z = -8,\ w = -7.$$

Example 3. Solve the equation for the matrix A

$$3A + \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 4 \end{bmatrix}$$

$$3A = \begin{bmatrix} -2 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2-4 & 2-(-1) \\ 1-(-2) & 4-1 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 3 & 3 \end{bmatrix}$$

$$\therefore A = \frac{1}{3}\begin{bmatrix} -6 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

Exercises 8

1. If $A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 2 & -2 \\ 2 & 1 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 4 & 4 \\ 4 & 1 & 6 \\ 7 & 10 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & 5 & 5 \\ 6 & 0 & 1 \\ 8 & 9 & 0 \end{bmatrix}$

evaluate the following:

(i) $3A - 2B + C$ (ii) $4A - 3B - 2C$ (iii) $5A - 3(B - C)$.

2. Solve the equation for the matrix X:

$$\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix} + 2X = \begin{bmatrix} -3 & 4 \\ 5 & -1 \end{bmatrix}$$

3. If A, B, C be the matrices of example 1, solve the equation

$$2(X + B) = 3\left(\frac{3}{2}X + A\right) + C.$$

4. Determine the matrices X and Y from the equations

$$X + Y = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}; \; X - Y = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$$

5. X and Y are 3 x 3 matrices. Determine these from the equations

$$3X + 2Y = I, \quad 2X - Y = 0.$$

6. Find values for a, b, c that satisfy the matrix relationship:

(i) $\begin{bmatrix} a+3 & 3a-2b \\ -3a-c & a+b+c \end{bmatrix} = \begin{bmatrix} 2 & -7+2b \\ b+4 & 2a \end{bmatrix}$

(ii) $3\begin{bmatrix} 2 & a \\ b & c \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ -1 & 2c \end{bmatrix} + \begin{bmatrix} 4 & a+2 \\ b+c & 3 \end{bmatrix}$

7. Show that (i) $(A + B)^t = A^t + B^t$ where A and B are matrices of the same order. (ii) $(kA)^t = kA^t$ where k is some scalar.

§ 6.1. Symmetric matrix. A matrix which is unchanged by transposition is called a symmetric matrix. Such a matrix is necessarily square

$$\text{E.g.,} \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 1 \\ 3 & 1 & 5 \end{bmatrix}$$

Hence if $A = [a_{rs}]$ is symmetric then $A = A^t$ i.e., $[a_{rs}] = [a_{sr}]$

§ 6.2. Skew symmetric matrix.

If the element a_{rs} of a matrix is the same as the element a_{sr} with a negative sign i.e., $a_{rs} = -a_{sr}$ then the matrix is called a skew symmetric. The diagonal elements in a skew symmetric matrix are zeros since $a_{rr} = -a_{rr}$ i.e., $a_{rr} = 0$.

Hence a skew symmetric matrix is a square matrix with all its elements along the main diagonal zeros and $a_{rs} = -a_{sr}$.

$$\text{E.g., } \begin{bmatrix} 0 & 2 & 3 & -1 \\ -2 & 0 & 4 & -3 \\ -3 & -4 & 0 & 1 \\ 1 & 3 & -1 & 0 \end{bmatrix}$$

Hence if A is skew symmetric then $A = -A^t$.

§ 6.3. Hermitian and skew Hermitian matrices.

If the elements a_{rs} of the matrix A are complex numbers, the matrix formed by the conjugates of a_{rs} which are denoted by $\overline{a_{rs}}$ is called the conjugate of the matrix and is denoted by $\overline{A}$.

Hence $\overline{A} = [\overline{a_{rs}}]$

$$\text{If } A = \begin{bmatrix} 2+i & 3-2i \\ 2+3i & 4+3i \end{bmatrix} \text{ then } \overline{A} = \begin{bmatrix} 2-i & 3+2i \\ 2-3i & 4-3i \end{bmatrix}$$

If A is a square matrix and A is the transpose of its conjugate, such a matrix is called a **Hermitian matrix.**

If A is Hermitian matrix, then $A = \left(\overline{A}\right)^t$ or $A^t = \overline{A}$

The elements of Hermitian matrix satisfy the relationship

$$a_{rs} = \overline{a_{sr}}.$$

Hence for the elements along the diagonal $a_{rr} = \overline{a_{rr}}$

Such is the case only when a_{rr} is real. Therefore the diagonal elements of Hermitian matrix are real.

Example of a Hermitian matrix :

$$\begin{bmatrix} a & b+ic & d-ie \\ b-ic & f & -g+ih \\ d+ie & -g-ih & h \end{bmatrix}$$

A Hermitain matrix with real elements is a symmetric matrix.

If A is a square matrix and if A is the negative of the transpose of its conjugate such a matrix is called a skew Hermitian matrix.

Hence if A is a skew Hermitian matrix.

$$A = -(\overline{A})^t \text{ or } A^t = -\overline{A}.$$

The elements of a skew Hermitian matrix satisfy the relationships $a_{rs} = -\overline{a}_{sr}$.

For elements along the diagonal $a_{rr} = -\overline{a}_{rr}$

∴ They are purely imaginary or zero.

Examples of skew Hermitian matrices are

$$\begin{bmatrix} ia & -b+ic & d+ie \\ b+ic & if & g-ih \\ -d+ie & -g-ih & -ik \end{bmatrix} ; \begin{bmatrix} 0 & a+ib & c-id \\ -a+ib & 0 & e+if \\ -c-id & -e+if & 0 \end{bmatrix}$$

Every real skew hermitian matrix is a skew symmetric matrix.

Example. Show that any real square matrix A may be written as the sum of a symmetric matrix R and a skew symmetric matrix S, where $R = \frac{1}{2}(A + A^t)$ and $S = \frac{1}{2}(A - A^t)$

Hence represent the matrix $\begin{bmatrix} 2 & 1 & 4 \\ 8 & -1 & 3 \\ 3 & -5 & 0 \end{bmatrix}$

as the sum of a symmetric and a skew symmetric matrix. (B.Sc. 1994)

Let A be $[a_{rs}]$. Then $A^t = [a_{sr}]$

$$\therefore \quad R = \frac{1}{2}(A + A^t) = \frac{1}{2}[a_{rs} + a_{sr}]$$

R is a symmetric matrix since in R if r and s are interchanged, we get the same result i.e., $R = R^t$

$$S = \frac{1}{2}(A - A^t) = \frac{1}{2}[a_{rs} - a_{sr}]$$

$$S^t = \frac{1}{2}[a_{sr} - a_{rs}] = -\frac{1}{2}[a_{rs} - a_{rs}]$$

$\therefore \quad S = -S^t.$

Hence S is skew symmetric.

$$R + S = \frac{1}{2}[a_{rs} + a_{sr}] + \frac{1}{2}[a_{rs} - a_{sr}] = [a_{rs}] = A.$$

Hence any matrix A can be represented as the sum of a symmetric matrix and a skew symmetrix matrix.

Let $A = \begin{bmatrix} 2 & 1 & 4 \\ 8 & -1 & 3 \\ 3 & -5 & 0 \end{bmatrix}$. Then $A^t = \begin{bmatrix} 2 & 8 & 3 \\ 1 & -1 & -5 \\ 4 & 3 & 0 \end{bmatrix}$

$$\therefore \quad A + A^t = \begin{bmatrix} 4 & 9 & 7 \\ 9 & -2 & -2 \\ 7 & -2 & 0 \end{bmatrix}$$

$$\text{Hence } R = \frac{1}{2}(A + A^t) = \begin{bmatrix} 2 & \frac{9}{2} & \frac{7}{2} \\ \frac{9}{2} & -1 & -1 \\ \frac{7}{2} & -1 & 0 \end{bmatrix}$$

$$S = \frac{1}{2}(A - A^t) = \frac{1}{2}\begin{bmatrix} 0 & -7 & 1 \\ 7 & 0 & 8 \\ -1 & -8 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{7}{2} & \frac{1}{2} \\ \frac{7}{2} & 0 & 4 \\ -\frac{1}{2} & -4 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & \frac{9}{2} & \frac{7}{2} \\ \frac{9}{2} & -1 & -1 \\ \frac{7}{2} & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{7}{2} & \frac{1}{2} \\ \frac{7}{2} & 0 & 4 \\ -\frac{1}{2} & -4 & 0 \end{bmatrix}$$

Exercises 9

1. If A and B are symmetric matrices of the same order, show that A + B is symmetric.

2. If A and B are skew symmetric matrices of the same order, show that A + B is skew symmetric.

3. If A is skew symmetric and has an odd order n, show that $|A| = 0$.

[Hint : $A^t = -A$, $|A| = |A^t| = |-A| = (-1)^n |A|$]

4. Represent the matrix $\begin{bmatrix} 3 & -1 & 0 & 8 \\ 4 & 2 & -3 & -1 \\ 1 & 3 & -6 & 5 \\ -5 & 0 & -7 & -2 \end{bmatrix}$

as the sum of a symmetric and a skew symmetric matrix.

5. Prove that (i) if A is Hermitian, then iA is skew Hermitian

(ii) if A is skew Hermitian iA is Hermitian.

6. If A is a square matrix prove that $(\overline{A})^t + A$ is Hermitian and $(\overline{A})^t - A$ is skew Hermitian. Hence deduce that any square matrix A can be expressed as the sum of a Hermitian and a skew Hermitian matrix. Hence represent the matrix

$$\begin{bmatrix} 2+3i & 1+i & 2-4i \\ -1+2i & 3 & 3+2i \\ 6 & 0 & 5-i \end{bmatrix}$$

as the sum of a Hermitian and a skew Hermitian matrix.

7. If A is Hermitian show that it can be written as R + iS, where R is a real symmetric and S is a real skew symmetrical matrix.

8. Show that every square matrix can be expressed uniquely as P + iQ where P and Q are Hermitian.

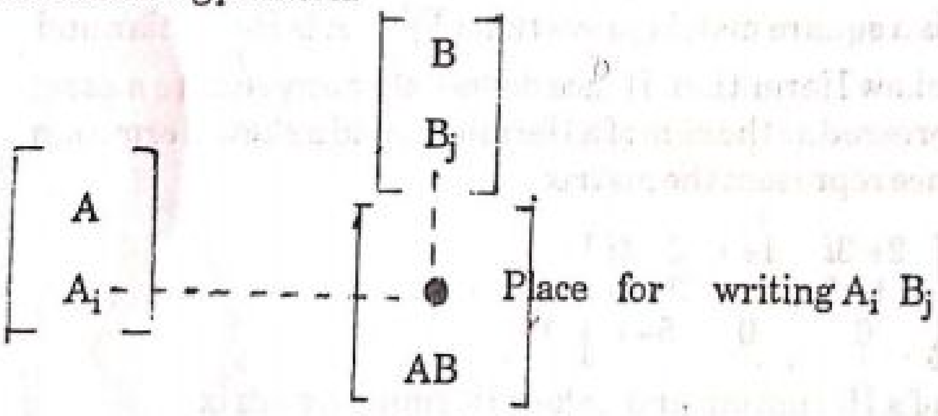
§7. Multiplication of Matrices.

Definition. If A be the row matrix $(a_1, a_2, a_3 \ldots a_n)$ and B the column vector $(b_1, b_2, b_3 \ldots b_n)$, the **inner product** of A and B is given by the formula $a_1b_1 + a_2b_2 + a_3b_3 + \ldots a_nb_n$ and is denoted by the notation A.B.

§7.1. If A is a $m \times n$ matrix with rows $A_1, A_2 \ldots A_m$ and B is a $n \times p$ matrix with columns $B_1, B_2, \ldots B_p$, then the product AB is a $m \times p$ matrix C whose elements are given by the formula $C_{ij} = A_i \cdot B_j$.

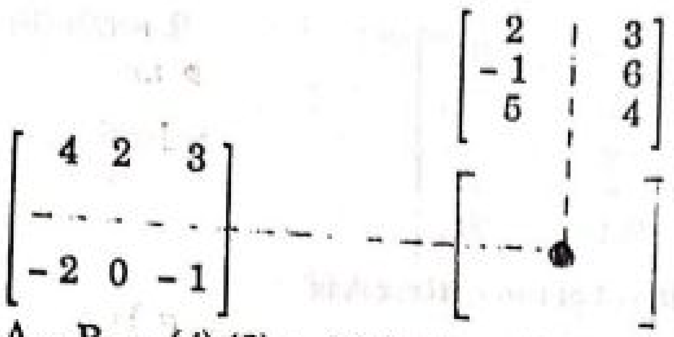
$$\text{Hence } C = AB = \begin{bmatrix} A_1 \cdot B_1 & A_1 \cdot B_2 & \ldots & A_1 \cdot B_p \\ A_2 \cdot B_1 & A_2 \cdot B_2 & \ldots & A_2 \cdot B_p \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ A_m \cdot B_1 & A_m \cdot B_2 & \ldots & A_m \cdot B_p \end{bmatrix}$$

The simplest way to carry this out in practice is by the following geometric scheme. Given two matrices A and B to be multiplied, first write A and then write B immediately above and to the right of A in the following position:



$$\text{Thus to multiply} \begin{bmatrix} 4 & 2 & 3 \\ -2 & 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 3 \\ -1 & 6 \\ 5 & 4 \end{bmatrix}$$

we write the matrices as below :



$A_1 . B_1 = (4)(2) + (2)(-1) + (3)(5) = 21$

$A_1 . B_2 = (4)(3) + (2)(6) + (3)(4) = 36$

$A_2 . B_1 = (-2)(2) + (0)(-1) + (-1)(5) = -9$

$A_2 . B_2 = (-2)(3) + (0)(6) + (-1)(4) = -10$

Hence $AB = \begin{bmatrix} 21 & 36 \\ -9 & -10 \end{bmatrix}$

Only if the number of columns in A is equal to the number of rows in B, then AB exists. It is important for the student to become familiar with the various combinations of matrix orders for which multiplicationispossible.

Order of A.	Order of B.	Order of AB
2 x 3	3 x 3	2 x 3
3 x 4	4 x 3	3 x 3
3 x 3	3 x 3	3 x 3
m x n	n x p	m x p
3 x 4	2 x 4	Does not exist
1 x n	n x 1	1 x 1
m x n	n x 1	m x 1

Cor. If A is $m \times n$ matrix and B is a $n \times p$ matrix, then

$$AB = A\,[B_1\ B_2\ B_3 \ldots B_p]$$
$$= [AB_1\ AB_2\ AB_3 \ldots AB_p]$$

where B_1, B_2 ... B_p are column vectors.

Cor. Let A be the square matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Then the determinant of the matrix A is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

If $|A| = 0$, A is called is a **singularmatrix.**

§ 7.2. With the help of matrix multiplication simultaneous equations can be expressed in a compact form.

Consider the set of equations

$$a_1x + a_2y + a_3z = d_1$$

$$b_1x + b_2y + b_3z = d_2$$

$$c_1x + c_2y + c_3z = d_2$$

This is can be written as AX = D

$$\text{where } A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} ; \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; \; D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

§7.3. Matrix multiplication does not obey all the rules for the multiplication of ordinary numbers.

(a) Matrix multiplication is not commutative

i.e., $AB \neq BA$.

(i) If AB exists, BA need not exist.

$$\text{E.g., if } A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 0 & 4 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ -2 & 1 \end{bmatrix},$$

$$\text{then } AB = \begin{bmatrix} -5 & 10 \\ -5 & 5 \\ 0 & 10 \end{bmatrix} \text{ but BA does not exist.}$$

(ii) Even if AB and BA exist they may be of different orders.

E.g., if $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \end{bmatrix}$

AB is of order 3 x 3 and is equal to $\begin{bmatrix} 1 & 8 & 9 \\ -6 & 7 & 1 \\ 4 & 17 & 21 \end{bmatrix}$

but BA is of order 2 x 2 and is equal to $\begin{bmatrix} 18 & 28 \\ 1 & 11 \end{bmatrix}$

(iii) Even if AB and BA are of the same order they need not be equal.

E.g., if $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & -2 \\ 4 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & -3 \\ 3 & -1 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 13 & 3 & 5 \\ -1 & 10 & 9 \\ 1 & 9 & 16 \end{bmatrix}$ but $BA = \begin{bmatrix} 24 & 3 & -5 \\ -2 & 4 & 5 \\ 3 & 2 & 11 \end{bmatrix}$

(b) If AB = 0, neither A nor B is zero.

E.g.,1. if $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ but neither A nor B is zero.

Eg.,2. $A = \begin{bmatrix} -6 & -4 & -2 \\ -9 & -6 & -3 \\ 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

(c) If AB = AC, then B need not be equal to C when $A \neq 0$.

$$AB = AC$$

$$\therefore \quad AB - AC = 0$$

$$\text{i.e., } A(B - C) = 0$$

Hence A or (B − C) need not be equal to zero.

E.g., Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$

and $C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

We can easily see that $AB = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 2 \\ 3 & 2 & -7 \end{bmatrix} = AC$ but $B \neq C$.

§7.4. The associative and distributive laws hold good in matrix multiplication.

If A, B, C are such that multiplication operations are defined.

(i) (AB) C = A (BC).

(ii) A (B + C) = AB + AC.

We shall illustrate these two results by taking numerical examples.

(i) Let A, B, C be respectively the matrices

$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 1 \\ -1 & 3 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 3 \\ 2 & 12 & 9 \end{bmatrix}.$$

$$\therefore \ (AB)\,C = \begin{bmatrix} 4 & 6 & 3 \\ 2 & 12 & 9 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 1 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 23 & 37 & 24 \\ 31 & 77 & 36 \end{bmatrix}$$

$$BC = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 1 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 19 & 10 \\ 5 & -1 & 4 \end{bmatrix}$$

$$\therefore \ A\,(BC) = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 9 & 19 & 10 \\ 5 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 37 & 24 \\ 31 & 77 & 36 \end{bmatrix}.$$

$\therefore$ (AB) C = A (BC).

(ii) Let A, B, C be respectively the matrices

$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 3 \\ 2 & 12 & 9 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 8 & 5 \\ -5 & 4 & -5 \end{bmatrix}$$

$$\therefore\ AB + AC = \begin{bmatrix} 4 & 6 & 3 \\ 2 & 12 & 9 \end{bmatrix} + \begin{bmatrix} -1 & 8 & 5 \\ -5 & 4 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 14 & 8 \\ -3 & 16 & 4 \end{bmatrix}$$

$$\therefore\ B + C = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 2 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\therefore\ A(B + C) = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 0 & 5 & 2 \\ 3 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 14 & 8 \\ -3 & 16 & 4 \end{bmatrix}$$

$\therefore A(B + C) = AB + BC.$

§ 7.5. The following results are easily seen :-

(i) If 0 is a zero matrix, then $A0 = 0A = 0$.

(ii) If A is a square matrix and I is the unit matrix of the same order, then $AI = IA = A$.

(iii) The product of two diagonal matrices A and B of order *n* is another diagonal matrix of order *n* and $AB = BA$.

(iv) The product of two scalar matrices of the same order is a scalarmatrix.

(v) If A and B are two square matrices, then $|A|\ |B| = |AB|$

If $AB = 0$, then either $|A|$ or $|B|$ is zero.

(vi) If A is square matrix, AA can easily be found and is denoted by A^2.

Similarly $AAA = A^2 A = A(A^2) = A^3$.

In general $A^m A^n = A^{m+n} = A^n A^m$

§7.6. A matrix is said to be *idempotent* if $A^2 = A$.

E.g., $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example 1. Show that the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

satisfies the equation $A(A - 1)(A + 2I) = 0$.

$$A - I = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 3 & 0 & 3 \\ -5 & 2 & 5 \end{bmatrix}$$

$$A + 2I = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 1 \\ 3 & 3 & 3 \\ -5 & 2 & -2 \end{bmatrix}$$

Hence $(A - I)(A + 2I)$

$$= \begin{bmatrix} 1 & -3 & 1 \\ 3 & 0 & 3 \\ -5 & 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & -3 & 1 \\ 3 & 3 & 3 \\ -5 & 2 & -4 \end{bmatrix} = \begin{bmatrix} -10 & -10 & -10 \\ -3 & -3 & -3 \\ 11 & 11 & 11 \end{bmatrix}$$

$\therefore \; A(A - I)(A + 2I)$

$$= \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} \begin{bmatrix} -10 & -10 & -10 \\ -3 & -3 & -3 \\ 11 & 11 & 11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \; A(A - I)(A + 2I) = 0.$

Example 2. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, show that A satisfies the equation $A^2 - 5A - 2I = 0$. Using this result determine A^5.

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$5A = 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$$

$$2I = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\therefore \; A^2 - 5A - 2I = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 - 5 - 2 & 10 - 10 - 0 \\ 15 - 15 - 0 & 22 - 20 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \; A^2 - 5A - 2I = 0.$

Hence $A^2 = 5A + 2I$

$\therefore\ A^4 = (5A + 2I)(5A + 2I)$

$= 25A^2 + 20AI + 4I^2$

$= 25A^2 + 20A + 4I$

$= 25(5A + 2I) + 20A + 4I$

$= 125A + 50I + 20A + 4I$

$= 145A + 54I.$

Hence $A^5 = 145A^2 + 54IA$

$= 145(5A + 2I) + 54A$

$= 725A + 290I + 54A$

$= 779A + 290I$

$$= 779\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 290\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 779 & 1558 \\ 2337 & 3116 \end{bmatrix} + \begin{bmatrix} 290 & 0 \\ 0 & 290 \end{bmatrix}$$

$$= \begin{bmatrix} 1069 & 1558 \\ 2337 & 3406 \end{bmatrix}$$

Example 3. Show that the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ satisfies the equation $A^2 = -I$. Use this result to calculate the 16th power of the matrix $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I.$$

$$B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$= A + I.$

$\therefore\ B^2 = (A + I)(A + I)$

$= A^2 + 2AI + I^2$

$= A^2 + 2A + I$

$= 2A$, since $A^2 = -I$.

$\therefore \quad B^{16} = (B^2)^8 = (2A)^8$

$= 2^8 (A^2)^4$

$= 2^8 (-I)^4$ since $A^2 = -I$.

$= 2^8 I$

$= \begin{bmatrix} 256 & 0 \\ 0 & 256 \end{bmatrix}$

Exercises 10

1. If $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$

show that AB = BA, AC = CA and BC = CB.

2. Evaluate AB – BA, when

(i) $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -3 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & -1 \end{bmatrix}$

3. Verify the result (AB) C = A (BC), when

(i) $A = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 2 & 5 \\ 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 2 & 3 & 4 \\ 0 & -1 & 2 & 3 \end{bmatrix}$

4. If $A = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

show by computation that

30. If A and B be square skew symmetrical matrices of the same order, show that AB is symmetrical if and only if AB = BA.

§8. Inverse matrix. Let A be any matrix. If a matrix B exists such that AB = BA = I, then B is called the inverse matrix of A.

Since AB and BA exist and equal to a square matrix, A and B must be square matrices of the same order.

If an inverse matrix to A exists, then it is unique.

Let B and C be the inverse matrices to A.

Then AB = BA = I and AC = CA = I.

Pre-multiplying AB by C we get CAB = CI

ie., IB = CI i.e., B = C.

The inverse of A is denoted by A^{-1}. Hence $AA^{-1} = A^{-1}A = I$

§8.1 Adjoint matrix.

Let A be the square matix $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$

Then |A| is the determinated of the matrix A

Then $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$

Let the co-factors of the elements $a_{11}, a_{12}, \dots\dots\dots$ in the determinant be $A_{11}, A_{12} \dots\dots$

Then the transpose of the matrix

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & A_{n3} & \dots & A_{nn} \end{bmatrix}$$

i.e., the matrix $\begin{bmatrix} A_{11} & A_{21} & A_{31} & \dots & A_{n1} \\ A_{12} & A_{22} & A_{32} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & A_{3n} & \dots & A_{nn} \end{bmatrix}$.

is called the adjoint of the matrix A and is denoted by adj A.

§ 8.2. Relationship between adjoint and inverse matrices.

We get

$$A\,(\text{adj }A) = \begin{bmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & |A| \end{bmatrix}$$

since $a_{11}A_{11} + a_{12}A_{12} + \dots a_{1n}A_{1n} = |A|$

$a_{21}A_{21} + a_{22}A_{22} + \dots a_{2n}A_{2n} = |A|$

....

....

$a_{n1}A_{n1} + a_{n2}A_{n2} + \dots a_{nn}A_{nn} = |A|$

$a_{11}A_{21} + a_{12}A_{22} + \dots a_{1n}A_{2n} = 0$

....

$a_{11}A_{n1} + a_{12}A_{n2} + \dots a_{1n}A_{nn} = 0$

$a_{21}A_{11} + a_{22} + \dots a_{2n}A_{nn} = 0$

$$\therefore A\,(\text{adj }A) = |A| \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$= |A|\, I.$

Hence $A\left(\frac{\text{adj } A}{|A|}\right) = I.$

Similarly we can show that $\left(\frac{\text{adj } A}{|A|}\right) A = I$

$\therefore \frac{\text{adj } A}{|A|}$ is the inverse matrix of A.

$\frac{\text{adj } A}{|A|}$ is also called the reciprocal of the matrix and is denoted by

A^{-1}.

The inverse of A exists only when $|A| \neq 0$, i.e., when A is non-singular.

§8.3 The necessary and sufficient condition for a square matrix A to possess the inverse is that $|A|$ is not zero. i.e., A is non-singular.

Let A^{-1} be the inverse of A.

$\therefore \; AA^{-1} = I$

Hence $|A|\,|A^{-1}| = |I| = 1$

$\therefore \; |A| \neq 0$ and $|A^{-1}| \neq 0$

$\therefore$ The condition $|A| \neq 0$ is necessary.

Let $|A| \neq 0$

$$AA^{-1} = A\left\{\frac{1}{|A|}\text{adj } A\right\}$$

$$= \frac{1}{|A|} A\,(\text{adj } A)$$

$$= \frac{1}{|A|}\begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = I.$$

Similarly $A^{-1}A = I$.

Hence the condition is sufficient.

§8.4. $(A^t)^{-1} = (A^{-1})^t$

Let A be the matrix $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$

Then $A^t = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}$

$$A^{-1} = \frac{(\text{adj})\,A}{|A|} = \frac{I}{|A|}\begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$$

$$\therefore \; \left(A^{-1}\right)^t = \frac{I}{|A|}\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}$$

$$\left(A^t\right)^{-1} = \frac{(\text{adj}\,A^t)}{|A^t|} = \frac{I}{|A|}\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} \text{ since } |A^t| = |A|.$$

Hence $(A^t)^{-1} = (A^{-1})^t$.

§8.5. Inverse of AB is $B^{-1}A^{-1}$.

Let A and B be non-singular square matrices and their inverses be respectively A^{-1} and B^{-1}.

Since A and B are non-singular

$|A| \neq 0, |B| \neq 0 \therefore |AB| \neq 0.$

We have $AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

(By associative law)

$= AIA^{-1}$

$= AA^{-1}$

$= I.$

Similarly $(B^{-1}A^{-1})AB = I$

$\therefore AB(B^{-1}A^{-1}) = (B^{-1}A^{-1})AB = I.$

$\therefore (AB)^{-1} = B^{-1}A^{-1}.$

Cor. (i) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(ii) $(A^2)^{-1} = (A^{-1})^2.$

(iii) $(A^n)^{-1} = (A^{-1})^n$

Example 1. Find the inverse of $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix}$

Let the matrix be A $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and

the co-factors of the determinant be $A_{11}, A_{12}, \dots A_{33}$.

$A_{11} = \begin{vmatrix} 8 & 2 \\ 9 & -1 \end{vmatrix} = -26;\ A_{12} = -\begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix} = 11;$

$A_{13} = \begin{vmatrix} 3 & 8 \\ 4 & 9 \end{vmatrix} = -5;\quad A_{21} = -\begin{vmatrix} 2 & -1 \\ 9 & -1 \end{vmatrix} = -7$

$A_{22} = \begin{vmatrix} 1 & -1 \\ 4 & -1 \end{vmatrix} = 3;\quad A_{23} = -\begin{vmatrix} 1 & 2 \\ 4 & 9 \end{vmatrix} = -1$

$A_{31} = \begin{vmatrix} 2 & -1 \\ 8 & 2 \end{vmatrix} = 12;\quad A_{32} = -\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = -5;$

$A_{33} = \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 2; \ |A| = 1.$

$$\therefore \ A^{-1} = \begin{bmatrix} -26 & -7 & 12 \\ 11 & 3 & -5 \\ -5 & -1 & 2 \end{bmatrix}$$

Example 2. Find A satisfying the matrix equation

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$$

Let B and C be respectively the matrices

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$$

Then $BAC = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$

$$B^{-1} BAC\, C^{-1} = B^{-1} \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} C^{-1}$$

$$\text{i.e., } IAI = B^{-1} \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} C^{-1}$$

$$\text{i.e., } A = B^{-1} \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} C^{-1}$$

$B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \qquad \therefore \ B^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$C = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \qquad \therefore \ C^{-1} = -\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$

$$\therefore \ A = -\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$= -\begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix} = \begin{bmatrix} -24 & -13 \\ 34 & 18 \end{bmatrix}$$

Example 3. Show that $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the

equation $A^2 - 4A - 5I = 0$.

Hence determine its inverse

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$4A = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}; \ 5I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \ A^2 - 4A - 5I = 0.$

Multiplying by A^{-1}, we have

$A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I = 0$

i.e., $A - 4I - 5A^{-1} = 0.$

$\therefore \ 5A^{-1} = A - 4I$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$\therefore \ A^{-1} = \begin{bmatrix} -\frac{3}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \end{bmatrix}$$

§9. Inner Product.

If $X = (x_1 x_2 x_3 \dots x_n)$ and $Y = (y_1 y_2 y_3 \dots y_n)$ are the column vectors of a matrix A, then $x_1y_1 + x_2y_2 + \dots + x_ny_n$ is called the inner product of the vectors X and Y. The inner product of X with itself is $x_1^2 + x_2^2 + \dots + x_n^2$.

$x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$ is called the length of the vector X.

If the length of the vector is unity, it is called a normal or unit vector.

If $x_1y_1 + x_2y_2 + \dots + x_ny_n = 0$, the vectors X and Y are said to be orthogonal.

The same definitions of inner product and orthogonal are extended to rows of matrices also.

§9.1. Orthogonal matrices.

A square matrix P is said to be orthogonal if $PP^t = I$

i.e., $P^t = P^{-1}$

§9.2. Properties of orthogonal matrices.

1. Every orthogonal matrix commutes with its transpose

We have $P^t = P^{-1}$

$\therefore \; PP^t = PP^{-1} = I$

Also $P^tP = P^{-1}P = I$

$\therefore \; P^tP = PP^t.$

2. Product of two orthogonal matrices is orthogonal. (B.Sc.1994)

Let P and Q be two orthogonal matrices.

$\therefore \; P^t = P^{-1}$ and $Q^t = Q^{-1}$

$\therefore \; P^t Q^t = P^{-1}Q^{-1}$

i.e., $(QP)^t = (QP)^{-1}$

$\therefore$ QP is orthogonal.

Similarly we can show that PQ is orthogonal.

3. The inverse of an orthogonal matrix is orthogonal.

(B.Sc.1994)

Let P be an orthogonal matrix $\therefore \; P^{-1} = P^t$

Hence $(P^{-1})^{-1} = (P^t)^{-1} = (P^{-1})^t$

$\therefore$ P^{-1} is orthogonal.

4. A matrix is orthogonal if and only if its rows and columns are mutually orthogonal normal vectors.

$$\text{Let P be } \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \dots \\ \dots \\ X_n \end{bmatrix}$$

$$\text{Then } P^t = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix} = (X_1, X_2, \dots X_n)$$

$$\therefore \; PP^t = \begin{bmatrix} X_1(X_1^t) & X_1(X_2^t) & \dots & X_1(X_n^t) \\ X_2(X_1^t) & X_2(X_2^t) & \dots & X_2(X_n^t) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ X_n(X_1^t) & X_n(X_2^t) & \dots & X_n(X_n^t) \end{bmatrix} \quad (1)$$

If P is orthogonal, $PP^t = I$

$\therefore$ $X_1(X_1^t) = 1, X_2(X_2^t) = 1 \dots X_n(X_n^t) = 1$ and

$X_1(X_2^t) = 0, \dots (X_1)(X_n^t) = 0,$

$X_2(X_1^t) = 0, \dots X_2(X_n^t) = 0,$

. . . .

$X_n(X_1^t) = 0, \dots X_n(X_{n-1}^t) = 0.$

i.e., $X_i(X_j^t) = 1$ if $i = j$

$= 0$ if $i \neq j$

i.e., $X_i(X_j^t) = \delta_{ij}$ (knonecker delta)

Hence the rows are normal and are mutually orthogonal. Similarly we can show that the column vectors of the matrix are normal and mutually orthogonal.

Hence the condition is necessary.

We can easily show that the condition is sufficient.

If in (1) $X_i (X_j^t) = \delta_{ij}$

i.e., $X_i (X_j^t) = 1$ if $i = j$

$= 0$ if $i \neq j$.

then $PP^t = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = I$

$\therefore$ P is orthogonal.

5. If P is orthogonal, $|P| = \pm 1$

Example. Show that $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$ is orthogonal

The given matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 3 & 6 & -6 \\ 6 & 3 & 6 \\ 6 & -6 & -3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} = A^t$$

$\therefore$ A is orthogonal.

§ 10. Solution of simultaneous equations.

Consider the equations

$$a_1x + a_2y + a_3z = d_1$$

$$b_1x + b_2y + b_3z = d_2$$

$$c_1x + c_2y + c_3z = d_3$$

If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$, then

the system of equations can be expressed as the matrix equation

$$AX = D.$$

Pre-multiplying with A^{-1}, we get

$$A^{-1}AX = A^{-1}D, \text{ i.e., } IX = A^{-1}D, \text{ i.e., } X = A^{-1}D$$

$$\text{i.e., } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

The product on the left is a 3 x 1 matrix.

Hence equating the corresponding elements in X and $A^{-1}D$, the values of x, y, z are determined.

Example. Solve the equations

$$6x + 2y - 2z = 6$$

$$-2x + 2y + 2z = 2$$

$$2x + 2y + 2z = 6$$

This system of equations can be put in the form

$$\begin{bmatrix} 6 & 2 & -2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 6 \end{bmatrix}$$

If A is the matrix $\begin{bmatrix} 6 & 2 & -2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ the equation

is written as $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 6 \end{bmatrix}$ $\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 6 \\ 2 \\ 6 \end{bmatrix}$

A^{-1} can be seen as $\begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore\ x = 1,\ y = 1,\ z = 1.$

Exercises 11

1. Find the inverses of the matrices :

(i) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$, (iii) $\begin{bmatrix} 4 & 2 & 1 \\ -3 & 0 & 5 \\ -1 & 1 & 6 \end{bmatrix}$

2. If $A = \begin{bmatrix} 7 & 4 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$, prove by numerical computation $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$.

3. Verify numerically that the inverse of the symmetric matrix $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 6 & 2 \\ 5 & 2 & 1 \end{bmatrix}$ is also symmetric.

4. If $A = \begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}$, find $A + A^{-1}$.

5. Determine the inverse matrix of A where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$.

Hence find the matrix X satisfying the matrix equation

$$AX = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{bmatrix}$$

6. Determine A from the following matrix equations :-

(i) $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} A = \begin{bmatrix} 4 & -6 \\ 2 & 1 \end{bmatrix}$; (ii) $A \begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

(33)

III. Rank of a Matrix.

III **11. Rank of a matrix.** A sub-matrix of a given matrix A is defined to be either A itself for any array remaining after certain rows and columns are deleted from A.

For example the matrix $\begin{bmatrix} 3 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix}$

has as sub-matrices first itself, next the matrices

$\begin{bmatrix} 3 & 2 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$[3 \ \ 2 \ \ -1], \ [0 \ \ -2 \ \ 1], \ [3 \ \ 2], [2 \ \ -1],$

$[3 \ \ -1], [0 \ \ -2], \ [-2 \ \ 1], [0 \ \ 1]$

and finally the matrices containing its individual elements. The determinants of the square sub-matrices are called the minors of A.

The rank of an $m \times n$ matrix A is r if and only if every minor in A of order $r + 1$ vanishes while there is at least one minor of order r which does not vanish.

Clearly $r \leq$ minimum of m and n. A null matrix is of rank 0.

If A is non-singular matrix of order $n \times n$, then A is of rank n.

The rank of the transpose of a matrix is the same as that of the original matrix.

Example. Find the rank of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{bmatrix}$

Minor of third order = $\begin{vmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{vmatrix} = 0.$

The minors of order 2 are obtained by deleting any one row and any one column.

One of the minors of orders 2 is $\begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix}$. Its value is 8.

Hence the rank of the given matrix is 2.

12. The determination of the rank of a matrix involves sometimes the evaluation of many determinants and if the order of the matrix is higher than 3 x 3, then, the work becomes tedious. A comparatively easier method of finding the rank of a matrix is evolved in the following articles.

13. The following operations on a matrix are called **elementary transformations** of a matrix :-

(i) Interchange of two rows (or columns).

(ii) Multiplication of all elements of a row (or column) by the same non-zero number.

(iii) Addition of the elements of a row (or column) multiplied by the same number, to the corresponding elements of another row (or column).

We can easily show that these elementary transformations do not change the rank of a matrix.

Two matrices are said to be equivalent if it is possible to pass one to the other by a chain of elementary transformations.

If A and B are equaivalent matrices, we write A ~ B.

We can easily see that equivalent matrices have the same rank.

§13.1. This result gives us a method to determine the rank of a matrix easily. Replace it, say matrix, A, by an equivalent matrix B, all whose elements above the leading diagonal are zeros. Then we can determine the rank by inspection. The method is illustrated in the following worked out examples.

Example 1. Find the rank of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & 8 & -1 \\ 3 & 16 & -2 \end{bmatrix} \; c_1, c_1 + c_2, c_3 - 2c_1.$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 4 & 16 & -2 \\ 3 & 16 & -2 \end{bmatrix} \; r_1, 2r_2, r_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 16 & -2 \end{bmatrix} \; r_1, r_2 - r_3, r_3$$

We can easily see that the rank is 2.

Example 2. Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & 2 \end{bmatrix} \qquad \text{(B.Sc.1994)}$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & 2 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -2 & 1 \\ -1 & 0 & -2 & 5 \end{bmatrix} \quad c_1, c_2 - 2c_1, c_3 + c_1, c_4 - 3c_1$$

$$\cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ 10 & 0 & -10 & 5 \\ -1 & 0 & -2 & 5 \end{bmatrix} \quad r_1, 5r_2, r_3$$

$$\cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ 11 & 0 & -8 & 0 \\ -1 & 0 & -2 & 5 \end{bmatrix} \quad r_1, r_2 - r_3, r_3$$

$$\cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ 11 & -8 & 0 & 0 \\ -1 & -2 & 5 & 0 \end{bmatrix} \quad c_1, c_3, c_4, c_2.$$

The third order determinant $\begin{vmatrix} 1 & 0 & 0 \\ 11 & -8 & 0 \\ -1 & -2 & 5 \end{vmatrix}$ is not zero.

Hence the rank of the given matrix is 3.

Exercises 12

Find the ranks of the following matrices :-

1. $\begin{bmatrix} 3 & 4 & -6 \\ 2 & -1 & 7 \\ 1 & -2 & 8 \end{bmatrix}$ 2. $\begin{bmatrix} 3 & 2 & 1 \\ 4 & -1 & -2 \\ -6 & 7 & 8 \end{bmatrix}$ (B.Sc.1993)

3. $\begin{bmatrix} 3 & 11 & 1 & 5 \\ 5 & 13 & -1 & 11 \\ -2 & 2 & 4 & -8 \end{bmatrix}$ 4. $\begin{bmatrix} 1 & 7 & 3 & -3 \\ 7 & -20 & -2 & 25 \\ 5 & -2 & 4 & 7 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 4 & 6 & 2 & 2 \\ 2 & 10 & 9 & 5 & 7 \\ 3 & 10 & 21 & 5 & 4 \end{bmatrix}$ 6. $\begin{bmatrix} 2 & -3 & -1 & 1 \\ 3 & 4 & -4 & -3 \\ 0 & 17 & -5 & -9 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 4 & 1 & -1 \\ -3 & 2 & 3 & 1 \\ 0 & 7 & 6 & 3 \end{bmatrix}$ 8. $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & -2 & -1 & 4 \\ 3 & 3 & 1 & 2 \\ 6 & 0 & 3 & 7 \end{bmatrix}$

9. For what values of k are the following matrices are of rank 3?

(i) $\begin{bmatrix} 6 & 3 & 5 & 9 \\ 5 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 2 & 1 & 1 & k \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$

§ 14. A system of m homogeneous linear equations in n unknowns.

Let the equations be

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

....

....

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

The equations can be put in the form

$$AX = 0 \text{ where A is the } m \times n \text{ matix } \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and X is the column matrix of order n $(x_1, x_2, \dots x_n)$ and 0 is the null column matrix of order m (0 .... 0)

$x_1 = x_2 = \dots = x_n = 0$ clearly satisfies the equations whatever the matrix A may be. This solution is called trivial. We shall try to find the non- trivial solutions of the equations if they exist and the conditions for the existence of the non-trivial solutions.

§14.1. If A is a matrix of order $m \times n$ and rank r, then AX = 0 have a non-trivial solution if $r < n$.

Suppose that a non-trivial solution exists.

Let us suppose that $r = n$.

Since $r \leq m$, $m \geq n$.

The rank of A is n. Hence there is at least one non-zero minor of A of order n.

Arrange the equations in such a way that the sub-matrix A_n of A, consisting of its first n rows is non-singular. The first n equations may be written.

$A_n X = 0$, where $| A_n | \neq 0$.

A_n^{-1} exists and we have

$$(A_n^{-1}) A_n X = A_n^{-1} (0)$$

$$\text{i.e., } (A_n^{-1} A_n) X = 0$$

$$\text{i.e., } IX = 0$$

$$\text{i.e., } X = 0$$

i.e., $x_1 = x_2 = \ldots = x_n = 0$.

This contradiction proves that $r \neq n$. Hence, since $r \leq n$, it follows that $r < n$. Conversely it can be shown that if $r < n$, $AX = 0$ has non-trivial solutions.

Note. If the number of equations m is equal to the number of unknown n, the condition $r < n$ reduces to the familiar form $|A| = 0$.

If there are fewer equations than unknowns, i.e, if $m < n$, then r is certainly less than n and the equations always have a non-trivial solution.

If $m > n$, the condition is satisfied only if every matrix formed from n rows of the matrix A is singular.

§14.2. If A and B are n-rowed squared matrices, not null matices such that $AB = 0$, then both A and B are singular.

Let $B_1 \ldots B_n$ be the column vectors of B

Then $AB = [\, AB_1 \;\; AB_2 \; \ldots . \; AB_n \,]$

$= [\; 0 \quad 0 \qquad\qquad 0 \;]$

Since B is not a null matrix, $B_k \neq 0$ for at least one k.

Expanding $AB_k = 0$ we get

$$a_{11}b_{1k} + a_{12}b_{2k} + \dots + a_{1n}b_{nk} = 0$$

$$a_{21}b_{1k} + a_{22}b_{2k} + \dots + a_{2n}b_{nk} = 0$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{n1}b_{1k} + a_{n2}b_{2k} + \dots + a_{nn}b_{nk} = 0$$

Considering these n equations as n linear equations containing $b_{1k}, b_{2k}, \dots b_{nk}$ we get that these n equations will contain non-trivial solutions if $|A| = 0$ i.e., the matrix is singular.

Taking the transpose we get $B^t A^t = 0$. Applying the same arguments as before we have $|B^t| = 0$ i.e., $|B| = 0$ i.e., B is singular.

§14.3. If AB = 0 and A is non-singular, then B = 0.

We get $AB = A\,[\,B_1\ B_2 \dots\ B_n\,]$

$= [\,AB_1\ AB_2 \dots\ AB_n\,]$

$= [\ 0\quad 0\ \dots\ 0\ ]$

$\therefore\ AB_k = 0$ where $k = 1, 2, \dots n$.

Taking $k = 1$ and expanding $AB_1 = 0$, we get

$$a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1n}b_{n1} = 0$$

$$a_{21}b_{11} + a_{22}b_{21} + \dots + a_{2n}b_{n1} = 0$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{n1}b_{11} + a_{n2}b_{21} + \dots + a_{nn}b_{n1} = 0$$

Since $|A| \neq 0$, these equations have trivial solutions i.e., $b_{11} = 0$, $b_{21} = 0$, ... $b_{n1} = 0$, considering these equations as homogeneous equations in the variables $b_{11}, b_{21}, \dots b_{n1}$

$\therefore\ B_1 = 0$

Similarly we can show that $B_2 = 0, B_3, \ldots B_n = 0$

Hence $B = 0$

Cor. If $AB = AC$, then $B = C$ if A is non-singular.

$$AB - AC = 0$$

i.e., $A(B - C) = 0$

Since A is non-singular $B - C = 0$

i.e., $B = C$.

Note. If A and B are both singular and if $AB = 0$, it does not follow that $BA = 0$.

for example

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

but $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

§ 15. Linear dependence and independence of vectors.

A system of n vectors of the same order

$$\xi_1, \xi_2, \ldots \xi_n$$

is said to be linearly dependent if there exist n numbers $k_1, k_2, \ldots k_n$ not all zero such that

$$k_1 \xi_1 + k_2 \xi_2 + \ldots k_n \xi_n = 0$$

where 0 is a null vector of the same order.

A system of n vectors $\xi_1, \xi_2, \ldots \xi_n$ of the same order is said to be linearly independent if they satisfy the realtion.

$$k_1 \xi_1 + k_2 \xi_2 + \ldots k_n \xi_n = 0 \text{ only if } k_1 = k_2 = \ldots = k_n = 0.$$

$X = k_1 \xi_1 + k_2 \xi_2 + \ldots + k_n \xi_n$ where $k_1, k_2, \ldots k_n$ are scalar is called a linear combination of $\xi_1, \xi_2, \ldots \xi_n$.

Let $A_1, A_2, A_3, \ldots A_n$ be column vectors of the $m \times n$ matrix A.

These columns are linearly dependent if there exist numbers $k_1, k_2, \ldots k_n$ not all zero, such that

(10)

$$k_1A_1 + k_2A_2 + ... + k_nA_n = 0.$$

Since each vector is of order m, this is equivalent to m equations in the n unknowns. These equations can be put in the form $Ak = 0$.

Where k is the column vector $(k_1, k_2 ... k_n)$. Hence the columns of A are linearly dependent if and only if the equations have non-trivial solutions.

In the particular case when A is a square matrix, its columns are linearly dependent if and only if $|A| = 0$.

If $|A| = 0$, then $|A^t| = 0$, so that the columns of A^t, i.e., the rows of A are also linearly dependent.

If $|A| \neq 0$, then the columns and rows of the matrix A are linearly independent.

If A is any $m \times n$ matrix of rank r, then A contains at least one non-vanishing minor $|A_r|$ of order r and the columns of A_r are linearly independent. Since the rank is of order r, every minor of order $r + 1$ is zero. Hence no set of $(r + 1)$ columns of A can be linearly independent. Similarly the r rows of A which contain A_r are linearly independent but no set of $(r+1)$ rows is linearly independent. Hence we can define the rank of a matrix A as the maximum number of linearly independent rows (or columns) of A.

§15.1. If the rank of a matrix A of order $m \times n$ is r, then there is atleast one set of r linearly independent columns (rows) of A and every other column (row) can be written as a linear combination of any such set.

Let $(C_1, C_2, ... C_n)$ denote a $m \times n$ matrix A. If A has rank r, then sub- matrix of order r is non-singular. Let $C_{k-1}, C_{k-2}, ... C_{k.r}$ be the columns of A passing through this sub-matrix. Since the matrix $[C_{k.1}\ C_{k.2} ... C_{k.r}]$ has a rank r, the vectors $C_{k.1}\ C_{k.2} ... C_{k.r}$ are linearly independent. If now C_j denotes any column of A, the matrix $[C_{k.1}\ C_{k.2} ... C_{k.r}\ C_r]$ has still rank r and hence the columns, $C_{k.1}\ C_{k.2} ... C_{k.r}\ C_j$ are linearly dependent i.e., $\lambda_1, \lambda_2, ... \lambda_{r+1}$ exist such that

$$\lambda_1 C_{k.1} + \lambda_2 C_{k.2} + ... \lambda_r C_{k.r} + \lambda_{r+1} C_j = 0$$

Hence C_j can be written as a linear combination of columns $C_{k.1}, C_{k.2} \dots C_{k.r}$.

Cor. If a square matrix is non-singular if and only if its columns (rows) are linearly independent.

§ 16. System of non-homogeneous linear equations.

We have seen under Crammer's rule that a system of n equations in n unknown has a unique solution if the determinant formed by the coefficients of the equations is not equal to zero, i.e., when the rank of matrix formed by the coefficients in n.

Let us consider in this article a system of m equations with n unkonwn with no restrictions between m and n.

Let the system of equations be

$$a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n = b_m$$

These equations can be put in the form AX = B,

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{mn} & a_{mn} \end{bmatrix}; \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix}; \; B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \dots \\ b_m \end{bmatrix}$$

$$\text{The matrix } \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix} \text{ is called the}$$

augmented matrix and is denoted by [A,B].

Since every determinant in [A] occurs in [A,B], the rank of A cannot exceed [A,B] and also the rank of [A,B] cannot exceed the rank of [A] + 1.

Hence (i) rank of A < rank of [A,B]

or (ii) rank of A = rank of [A,B]

Let us consider the case (i).

Because of this relation, the greatest non-vanishing determinant of [A,B] must contain the column B.

Hence B is linearly independent of the columns of A and there are no x_i such that

$$\sum_{i=1}^{n} x_i [A_i] = B_i$$

where the $[A_i]$ represent the columns of A. Hence there do not exist any $x_1, x_2 \ldots x_n$ satisfying these equations. Hence there is no solution for the given set of equations and hence they are inconsistent. Let us consider the case rank of [A] = rank of [A,B].

Let each be equal to r. Hence every column of [A,B] can be expressed as a linear combination of r linearly independent columns of A.

Since [B] is a column of [A,B] there must exist numbers x_i such that $\sum_{i=1}^{n} x_i [A_i] = B_i$

Hence there is at least one solution to the system of equations.

Thus if rank of A < rank of [A,B], the system of equation is inconsistent and there is no solution and if rank of A = rank of [A,B], there is always one solution to the system of equations and hence the system is consistent.

Example 1. Discuss the consistency of the following sets of equations:-

(i) $4x + 3y = 12$ (ii) $4x + 3y = 12$

$3x + 2.25y = 9$ $3x + 2.25y = 8.$

(iii) $3x + 2y + z = 7$

$2x + y - 2z = 8$

$4x + 3y + 4z = 20.$

(i) In this case $A = \begin{bmatrix} 4 & 3 \\ 3 & 2.25 \end{bmatrix}$

and $[A,B] = \begin{bmatrix} 4 & 3 & 12 \\ 3 & 2.25 & 9 \end{bmatrix}$

We can easily see that the rank of A is 1 and the rank of [A,B] is also I.

Thus the set of equations are consistent. Hence there is at least one solution. We can easily see that the system of equations has an infinite number of solutions.

The second equation is $\frac{3}{4}$ of the first equation and the solutions are given by $x = 3 - \frac{3}{4}y$ for any y.

(ii) In this case $A = \begin{bmatrix} 4 & 3 \\ 3 & 2.25 \end{bmatrix}$ and its rank 1.

$[A,B] = \begin{bmatrix} 4 & 3 & 12 \\ 3 & 2.25 & 8 \end{bmatrix}$ and its rank is 2.

No solution exists and the equations are inconsistent.

The first equation is $4x + 3y = 12$.

The second equation multiplied by $\frac{4}{3}$ gives $4x + 3y = \frac{32}{3}$.

Here the two equations are inconsistent.

(iii) In this case $[A] = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -2 \\ 4 & 3 & 4 \end{bmatrix}$

and $[A,B] = \begin{bmatrix} 3 & 2 & 1 & 7 \\ 2 & 1 & -2 & 8 \\ 4 & 3 & 4 & 20 \end{bmatrix}$

$|A| = 0$ and rank of $[A] = 2$.

The rank of $[A,B] = 3$.

Hence the system of equations are inconsistent. If the second and third equations are added and divided by 2, we get the equation $3x + 2y + z = 14$.

This is inconsistent with the first equation.

Example 2. Investigate for what values of a, b the simultaneous equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + az = b$$

have (i) no solution, (ii) a unique solution and (iii) an infinite number of solutions.

The above system of equations have a unique solution if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & a \end{vmatrix} \neq 0$$

i.e., $a - 3 \neq 0$ i.e., $a \neq 3$.

If $a = 3$, the system of equations become

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + 3z = b$$

In this case $[A] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ and $[A,B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & 3 & b \end{bmatrix}$

$$[A] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} [r_1, r_2, r_3 - r_2]$$

Hence its rank is 2.

$$[A,B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & 0 & b-10 \end{bmatrix} [r_1, r_2, r_3 - r_2]$$

If $b \neq 10$, the rank of [A,B] is 3.

Hence in this case, the system of equations has no solution.

If $b = 10$, the rank of [A,B] is 2.

Hence the equations are consistent. In this case the number of solutions is infinite.

V. [Characteristic equation, Eigen values, Eigen vectors]

do not have a solution unless $a + c = 2b$.

V. §16. Eigenvalues and Eigenvectors.

In many important applications of matrices the following problem arises : Given a matrix A of order n, determine the scalar λ and the non-zero vectors X which simultaneously satisfy the equation

$$AX = \lambda X$$

Let us prove these results by taking a matrix of third order, but the proof is general.

Let A be $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and X be $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Hence the equation $AX = \lambda X$ becomes

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

i.e., $$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} - \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix} = 0$$

$$\therefore \quad (a_{11} - \lambda) x_1 + a_{12} x_2 + a_{13} x_3 = 0$$

$$a_{21}x_1 + (a_{22} - \lambda) x_2 + a_{23} x_3 = 0$$

$$a_{31}x_1 + a_{32} x_2 + (a_{33} - \lambda) x_3 = 0.$$

These equations have non-trivial solutions only when

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

The expansion of this determinant gives a polynomial of degree 3 in λ which is denoted by $\phi(\lambda)$.

The equation $\phi(\lambda) = 0$ is called the **characteristic equation** of the matrix A.

The roots of this equation are called the **characteristic values or latent values or eigenvalues** of the matrix A.

Let the roots of the characteristic equation be λ_1, λ_2, λ_3. These roots may be real or imaginary but for the present we shall assume that these roots are real and distinct.

If we substitute λ_1 in equations (1) we can get values of x_1, x_2, x_3 satisfying the equations of (1). Corresponding to each root we get a set of values x_1, x_2, x_3 satisfying these equations.

Let them be $X_1 = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$; $X_2 = \begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix}$; $X_3 = \begin{bmatrix} x_1''' \\ x_2''' \\ x_3''' \end{bmatrix}$

The vectors X_1, X_2, X_3 are called the characteristic or eigen-vectors of the matrix.

Let P be the matrix formed by the eigenvectors X_1, X_2, X_3

i.e., $P = [X_1, X_2, X_3]$

$$AP = A[X_1, X_2, X_3]$$

$$= [AX_1, AX_2, AX_3]$$

Since X_1, X_2, X_3 satisfy the equations

$AX = \lambda X$ when $\lambda = \lambda_1, \lambda_2, \lambda_3$, we have

$AX_1 = \lambda_1 X_1$, $AX_2 = \lambda_2 X_2$, $AX_3 = \lambda_3 X_3$

$$\therefore \; AP = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$$

$$= [X_1 \; X_2 \; X_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$= PD$$

i.e., $P^{-1}AP = D$

Here D is a diagonal matrix.

This process of finding P such that $P^{-1}AP = D$ is called **diagonalising the matrix** A.

Note 1. The characteristic equation of the matrix A is $|A - \lambda I| = 0$

2. If the roots of the characteristic equation are not distinct, it may not be possible to diagonalise the matrix A.

Cor.i. $P^{-1}AP = D \quad \therefore A = PDP^{-1}$

Cor.ii. $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Its characteristic equation is $\begin{vmatrix} \lambda_1 - \lambda & 0 & 0 \\ 0 & \lambda_2 - \lambda & 0 \\ 0 & 0 & \lambda_3 - \lambda \end{vmatrix} = 0$

i.e., $(\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) = 0$

Hence the eigenvalues of D are $\lambda_1, \lambda_2, \lambda_3$

Hence A and D have the same characteristic equation and the same eigenvalues.

Cor.iii. The eigenvectors of a matrix are linearly independent.

We have to show that if $C_1X_1 + C_2X_2 + C_3X_3 = 0$ then

$$C_1 = C_2 = C_3 = 0.$$

Let us assume that C_1, C_2, C_3 exist such that

$$C_1X_1 + C_2X_2 + C_3X_3 = 0 \qquad \dots (1)$$

Multiplying this equation (1) by A we get

$$C_1AX_1 + C_2AX_2 + C_3AX_3 = 0$$

$$\text{i.e., } C_1\lambda_1X_1 + C_2\lambda_2X_2 + C_3\lambda_3X_3 = 0 \qquad \dots (2)$$

Multiplying this equation (2) by A we get

$$C_1\lambda_1AX_1 + C_2\lambda_2AX_2 + C_3\lambda_3AX_3 = 0$$

$$\text{i.e., } C_1\lambda_1^2X_1 + C_2\lambda_2^2X_2 + C_3\lambda_3^2X_3 = 0 \qquad \dots (3)$$

These three equations (1), (2), (3) may be written in the form

$$[C_1X_1 \quad C_2X_2 \quad C_3X_3] \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{bmatrix} = 0 \qquad \dots (4)$$

If $\lambda_1, \lambda_2, \lambda_3$ are all unequal, then

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{vmatrix} \neq 0 \text{ and hence the matrix}$$

$$B = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{bmatrix} \text{ is non-singular and hence an}$$

inverse of this matrix exists.

If we multiply equation (4) on the right by the inverse of the matrix B, we have

$[C_1X_1 \quad C_2X_2 \quad C_3X_3] = 0$

Since no X is zero, it implies that

$$C_1 = 0, \ C_2 = 0, \ C_3 = 0$$

Hence X_1, X_2, X_3 are linearly independent.

Cor.iv. The determinant of the matrix A is equal to the product of its eigenvalues and is numerically equal to the absolute term of the characteristic equation.

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of the matrix. Then

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = -(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$$

Putting $\lambda = 0$ on both sides we get $|A| = \lambda_1 \lambda_2 \lambda_3$

Cor.v. The sum of the elements on the diagonal A is the sum of the eigenvalues of the matrix.

The characteristic equation is

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

Sum of the eigenvalues = Sum of the roots of the characteristic equation.

$$= -\frac{\text{Coeff.of } \lambda^2}{\text{Coeff.of } \lambda^3}$$

λ^2 and λ^3 occur only in the term $(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)$ when the determinant is expanded.

Coeff. of $\lambda^3 = -1$

Coeff. of $\lambda^2 = a_{11} + a_{22} + a_{33}$

Hence $a_{11} + a_{22} + a_{33}$ = sum of the eigenvalues of the matrix A.

Example 1. Diagonalise the matrix $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

The characteristic equation is

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

i.e., $(\lambda - 1)(\lambda + 2)(\lambda - 3) = 0$

$\therefore \lambda = 1, -2$ or 3.

When $\lambda = 1$, the equations become

$$x_1 - 2x_2 + 3x_3 = 0$$

$$x_1 + x_3 = 0$$

$$x_1 + 3x_2 - 2x_3 = 0$$

Hence $\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{5} \therefore x_1 = -1, x_2 = 1, x_3 = 1$

$\therefore X_1 = (-1, 1, 1)$

Similarly for the value of $\lambda = -2$, the eigenvector is

$X_2 = (11, 1, -14)$

and for $\lambda = 3$, the eigenvector $X_3 = (1, 1, 1)$

Hence $P = \begin{bmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{bmatrix}$

We can easily see that $P^{-1} = \frac{1}{30}\begin{bmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 3 & 12 \end{bmatrix}$

Hence

$$\begin{bmatrix} -2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 3 & 12 \end{bmatrix}$$

Example 2. Show that if λ is an eigenvalue of the matrix A, then λ^n is an eigenvalue of A^n, where n is a positive interger.

Let P be the matrix which is such that $P^{-1}AP = D$ where

$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A.

Hence $(P^{-1}AP)(P^{-1}AP) = D.D$

i.e., $P^{-1}A(PP^{-1})AP = D^2$

i.e., $P^{-1}AIAP = D^2$

i.e., $P^{-1}A^2P = D^2$

Multiplying this equation by $P^{-1}AP$ on both sides we get

$(P^{-1}A^2P)(P^{-1}AP) = D^2(P^{-1}AP)$

i.e., $P^{-1}A^2(PP^{-1})AP = D^2 D$

i.e., $P^{-1}A^2IAP = D^3$

i.e., $P^{-1}A^3P = D^3$

Continuing these process, we get $P^{-1}A^nP = D^n$.

Hence A^n and D^n have the same eigenvalues.

$$D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

$\therefore$ The eigenvalues of D^n are $\lambda_1^n, \lambda_2^n, \lambda_3^n$.

Hence the eigenvalues of A^n are $\lambda_1^n, \lambda_2^n, \lambda_3^n$.

§ 16.1. Similar matrices.

Two matrices A and B are said to be similar if there exists a non-singular matrix P such that $P^{-1}AP = B$.

If D is the diagonal matrix whose diagonal elements are the eigenvalues of the matrix A, then A and D are similar matrices.

§ 16.2. If A and B are similar matrices, they have the same characteristic equation.

Since A and B are similar, a matrix P exists such that

$$B = P^{-1}AP$$

$$\therefore \; B - \lambda I = P^{-1}AP - \lambda I$$

$$= P^{-1}AP - P^{-1}\lambda IP$$

$$= P^{-1}(A - \lambda I)P$$

Hence $|B - \lambda I| = |P^{-1}(A - \lambda I)P|$

$$= |P^{-1}|\,|A - \lambda I|\,|P|$$

$$= |P^{-1}|\,|P|\,|A - \lambda I|$$

$$= |P^{-1}P|\,|A - \lambda I|$$

$$= |I|\,|A - \lambda I|$$

$$= |A - \lambda I|$$

The characteristic equations of A and B are respectively $|A - \lambda I| = 0$ and $|B - \lambda I| = 0$. Hence they are equal.

Cor. Two similar matrices have the same eigenvalues.

VI. Cayley Hamilton's Theorem (52)

§163. Cayley-Hamilton Theorem.

Every matrix satisfies its characteristic equation.

Let A be a matrix of order n.

The matrix $[A - \lambda I]$ is $\begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{22} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{bmatrix}$

Let $|A - \lambda I|$ be $a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_n\lambda^n$

We have $\{ \text{adj}[A - \lambda I]\} [A - \lambda I] = |A - \lambda I| I$ since

$[(\text{adj } A)(A) = |A| I]$

Hence adj $[A - \lambda I]$ is of the form

$B_0 + B_1\lambda + B_2\lambda^2 + \dots + B_{n-1}\lambda^{n-1}$

where $B_0, B_1, B_2, \dots B_{n-1}$ are matrices. of order n.

$\therefore \ (B_0 + B_1\lambda + B_2\lambda^2 + \dots + B_{n-1}\lambda^{n-1}) [A - \lambda I]$

$= (a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_n\lambda^n) I.$

Equating the different powers of λ on both sides we get

$$B_0A = a_0 I$$
$$B_1A - B_0 = a_1 I$$
$$B_2A - B_1 = a_2 I$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$
$$B_{n-1}A - B_{n-2} = a_{n-1} I$$
$$-B_{n-1} = a_n I$$

Multiplying these equations successively by $I, A, A^2, \dots A^{n-1}, A^n$ and adding we get $a_n A^n + a_{n-1}A^{n-1} + \dots + a_1 A + a_0 I = 0.$

Hence A satisfies its characteristic equation.

§ 16.4. An important application of the Cayley-Hamilton Theorem is to express the inverse of a matrix in terms of powers of A. We have shown that

$$a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n = 0$$

where $a_0 \neq 0$ and $|A| \neq 0$.

$$\therefore\ a_0 I = -a_1 A - a_2 A^2 - \dots - a_n A^n$$

Premultiplying by A^{-1}, we get

$$a_0 A^{-1} I = a_1 A^{-1} A - a_2 A^{-1} A^2 - \dots a_n A^{-1} A^n$$

i.e., $a_0 A^{-1} = -a_1 I - a_2 A - \dots - a_n A^{n-1}$

$$\therefore\ A^{-1} = -\frac{a_1}{a_0} I - \frac{a_2}{a_0} A - \dots - \frac{a_n}{a_0} A^{n-1}$$

§16.5. Another important use is to calculate the higher powers of the matrices.

This is illustrated in examples 2 and 3 given below.

Example 1. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$ and hence determine its inverse.

The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{vmatrix} = 0.$$

Simplifying we get $\lambda^3 - 13\lambda + 12 = 0$.

Hence the matrix A satisfies the equation

$$A^3 - 13A + 12I = 0.$$

Premultiplying by A^{-1}, we have

$$A^2 - 13I + 12A^{-1} = 0.$$

$$\therefore\ 12A^{-1} = 13I - A^2$$

$$A^2 = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 8 & 6 & 2 \\ -1 & 7 & -2 \\ 31 & -18 & 11 \end{bmatrix}$$

$$\therefore \; 12A^{-1} = 13 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 6 & 2 \\ -1 & 7 & -2 \\ 31 & -18 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{bmatrix} - \begin{bmatrix} 8 & 6 & 2 \\ -1 & 7 & -2 \\ 31 & -18 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -6 & -2 \\ 1 & 6 & 2 \\ -31 & 18 & 2 \end{bmatrix}$$

$$\therefore \; A^{-1} = \frac{1}{12} \begin{bmatrix} 5 & -6 & -2 \\ 1 & 6 & 2 \\ -31 & 18 & 2 \end{bmatrix}$$

Example 2. If $A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$ determine A^n in terms of A.

The characteristic equation is given by

$$\begin{vmatrix} 4-\lambda & -2 \\ 3 & 3-\lambda \end{vmatrix} = 0.$$

i.e., $\lambda^2 - 7\lambda + 6 = 0.$

Hence A satisfies the equation

$A^2 - 7A + 6I = 0.$

Let $\lambda^n = f(\lambda)(\lambda^2 - 7\lambda + 6) + p\lambda + q$

when $\lambda = 1$ or 6, $\lambda^2 - 7\lambda + 6 = 0.$

$$\therefore \; 1^n = p + q, \qquad 6^n = 6p + q$$

$$\therefore \; p = \frac{6^n - 1}{5}, \qquad q = \frac{6 - 6^n}{5}$$

$$\therefore \; \lambda^n = f(\lambda)(\lambda^2 - 7\lambda + 6) + \frac{(6^n - 1)\lambda + (6 - 6^n) I}{5}$$

Hence $A^n = f(A)(A^2 - 7A + 6I) + \dfrac{(6^n - 1)A + (6 - 6^n) I}{5}$

$$= \frac{1}{5}[(6^n - 1)A + (6 - 6^n)I] \text{ since } A^2 - 7A + 6I = 0$$

$$= \frac{6^n - 1}{5}\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} + \frac{6 - 6^n}{5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 3. Calculate A^4 when $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

The characteristic equation of the matrix A is

$$\begin{vmatrix} 1-\lambda & 3 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^2 - 5\lambda - 2 = 0$

$\therefore\ A^2 - 5A - 2I = 0$

Hence $A^2 = 5A + 2I$

$$\therefore\ A^4 = (5A + 2I)(5A + 2I)$$

$$= 25A^2 + 20A + 4I$$

$$= 25(5A + 2I) + 20A + 4I$$

$$= 145A + 54I$$

$$= 145\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + 54\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 145 & 435 \\ 290 & 580 \end{bmatrix} + \begin{bmatrix} 54 & 0 \\ 0 & 54 \end{bmatrix}$$

$$= \begin{bmatrix} 199 & 435 \\ 290 & 634 \end{bmatrix}$$

Unit – II is Completed.

Exercises 14

1. Find the eigenvalues of the following matrices :-

(i) $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 0 \\ 5 & 6 & 7 \end{bmatrix}$

(1)

Unit - III

CHAPTER III

SUCCESSIVE DIFFERENTIATION

§ 1.1. (We have seen that the derivative of a function of x is (in general) also a function of x. The new function may be differentiable, in which case, the derivative of the first derivative is called the second derivative of the original function. Similarly the derivative of the second derivative is called the third derivative ; and so on up to the n^{th} derivative.)

Thus if
$$y = 4x^5,$$
$$\frac{dy}{dx} = 20x^4,$$
$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 80x^3,$$
$$\frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\} = 240\,x^2, \text{ etc.}$$

The symbols of the successive derivates are usually abbreviated as follows :-

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2y.$$
$$\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = D^3y.$$
$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = D^ny.$$

If $y = f(x)$, the successive derivatives are also denoted by

$f'(x), f''(x), \ldots. f^n(x),$

$y', y'', \ldots\ldots y^{(n)},$

$y_1, y_2, \ldots\ldots y_n.$

§ 1.2. The n^{th} derivative.

For certain functions a general expression involving n may be found for the n^{th} derivative. The usual plan is to find number of successive derivatives, as many as be necessary to discover their law of formation and then by induction write down the n^{th} derivative.

For example, if $y = e^{ax}$,

$$\frac{dy}{dx} = ae^{ax},$$

$$\frac{d^2 y}{dx^2} = a^2 e^{ax},$$

Then we can write $\dfrac{d^n y}{dx^n} = a^n e^{ax}$.

§ 1.3. Standard results.

1. If $y = (ax + b)^m$, then

$$y_1 = m \, . \, a \, (ax + b)^{m-1}$$

$$y_2 = m \, (m - 1) \, a^2 \, (ax + b)^{m-2}$$

$$y_3 = m \, (m - 1) \, (m - 2) \, a^3 \, (ax + b)^{m-3}$$

Hence $y_n = m \, (m - 1) \ldots (m - n + 1) \, a^n \, (ax + b)^{m-n}$

In particular, $D^n \, (ax - b)^{-1} = (-1)^n \, n \, ! \, a^n \, (ax + b)^{-n-1}$

(B.Sc. 1977)

2. If $y = \log (ax + b)$ then

$$y_1 = a \, (ax + b)^{-1}$$

$$y_n = a \frac{d^{n-1}}{dx^{n-1}} (ax + b)^{-1}$$

$$= a \, (-1)^{n-1} \, (n - 1) \, ! \, a^{n-1} \, (ax + b)^{-n}$$

$$= (-1)^{n-1} \, (n - 1) \, ! \, a^n \, (ax + b)^{-n}$$

3. If $y = \sin(ax + b)$, then

$$y_1 = a\cos(ax+b) = a\sin\left(\frac{\pi}{2} + ax + b\right)$$

Thus the effect of a differentiation is to multiply by a and increase the angle by $\frac{\pi}{2}$.

$$y_2 = a^2\cos\left(\frac{\pi}{2} + ax + b\right) = a^2\sin\left(2\frac{\pi}{2} + ax + b\right)$$

$$y_3 = a^3\sin\left(3\frac{\pi}{2} + ax + b\right)$$

In general, $D^n\sin(ax+b) = a^n\sin\left(\frac{n\pi}{2} + ax + b\right)$

4. Similarly $D\cos(ax+b) = a^n\cos\left(\frac{n\pi}{2} + ax + b\right)$

Corollaries : Putting $a = 1$ and $b = 0$.

$$D^n(\sin x) = \sin\left(\frac{n\pi}{2} + x\right),$$

$$D^n(\cos x) = \cos\left(\frac{n\pi}{2} + x\right).$$

5. If $y = e^{ax}\sin(bx+c)$, then

$$y_1 = e^{ax}b\cos(bx+c) + ae^{ax}\sin(bx+c).$$

Putting $a = r\cos\phi$ and $b = r\sin\phi$, we have

$$y_1 = re^{ax}\sin(bx+c+\phi).$$

Thus the effect of a differentiation is to multiply by r and increase the angle by ϕ.

Similarly $y_2 = r^2e^{ax}\sin(bx+c+2\phi)$ ....

In general,

$$D^n\left\{e^{ax}\sin(bx+c)\right\} = r^ne^{ax}\sin(bx+c+n\phi),$$

where $r = (a^2+b^2)^{1/2}$ and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

6. Similarly

$$D^n\left\{e^{ax}\cos(bx+c)\right\} = r^n e^{ax}\cos(bx+c+n\phi),$$

where r and ϕ have the same meanings as before.

§1.4. Fractional expressions of the form $\frac{f(x)}{\phi(x)}$, both functions being algebraic and rational, can be differentiated n times by splitting them into partial fractions.

Examples.

Ex. 1. Find y_n where $y = \frac{3}{(x+1)(2x-1)}$ (B.Sc. 1986)

Resolving into partial fractions, we obtain

$$y = \frac{2}{2x-1} - \frac{1}{(x+1)}$$

$$\therefore \quad y_n = \frac{2(-1)^n 2^n . n!}{(2x-1)^{n+1}} - \frac{(-1)^n n!}{(x+1)^{n+1}}.$$

$$= (-1)^n n!\left\{\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}}\right\}$$

Ex. 2. Find y_n when $y = \frac{x^2}{(x-1)^2(x+2)}$

Let $\frac{x^2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$

Then we easily find that $A = \frac{5}{9}$, $B = \frac{1}{3}$ and $C = \frac{4}{9}$

$$\therefore y = \frac{5}{9}\frac{1}{x-1} + \frac{1}{3}\frac{1}{(x-1)^2} + \frac{4}{9}\frac{1}{(x+2)}$$

Hence

$$Y_n = \frac{5}{9}\frac{n!(-1)^n}{(x-1)^{n+1}} + \frac{(n+1)!(-1)^n}{3(x-1)^{n+2}} + \frac{4}{9}\frac{(-1)^n n!}{(x+2)^{n+1}}$$

Scanned with CamScanner

$$=(-1)^n n!\left\{\frac{5}{9(x-1)^{n+1}}+\frac{n+1}{3(x-1)^{n+2}}+\frac{4}{9(x+2)^{n+1}}\right\}$$

Ex. 3. Find y_n when $y = \dfrac{1}{x^2+a^2}$ (B.Sc. 1990)

$$y = \frac{1}{x^2+a^2} = \frac{1}{2ai}\left[\frac{1}{x-ai}-\frac{1}{x+ai}\right]$$

$$\therefore\ y_n = \frac{(-1)^n n!}{2ai}\left[\frac{1}{(x-ai)^{n+1}}-\frac{1}{(x+ai)^{n+1}}\right].$$

§ 1.5. Trigonometrical transformation.

It is possible to break up products of powers of sines and consines into a sum by Trigomometrical methods.

Examples.

Ex. 1. Find the n^{th} differential coefficient of

$\cos x \cdot \cos 2x \cdot \cos 3x$. (B.Sc. 1985)

$$\cos x \cos 2x \cos 3x = \frac{1}{2}\cos 2x(\cos 4x+\cos 2x)$$

$$= \frac{1}{2}\cos 2x\cos 4x+\frac{1}{2}\cos^2 2x.$$

$$= \frac{1}{4}(\cos 2x+\cos 6x)+\frac{1}{4}(1+\cos 4x)$$

$$= \frac{1}{4}+\frac{1}{4}(\cos 2x+\cos 4x+\cos 6x)$$

$$\therefore\ D^n(\cos x\cos 2x\cos 3x)$$

$$=\frac{1}{4}\left\{2^n\cos\left(\frac{n\pi}{2}+2x\right)+4^n\cos\left(\frac{n\pi}{2}+4x\right)+6^n\cos\left(\frac{n\pi}{2}+6x\right)\right\}$$

Ex. 2. Find the n^{th} differential coefficient of $\cos^5\theta\sin^7\theta$.

Let $x = \cos\theta + i\sin\theta$; then $\dfrac{1}{x} = \cos\theta - i\sin\theta$.

$$\therefore \quad x + \frac{1}{x} = 2\cos\theta \,;\; x - \frac{1}{x} = 2i\sin\theta.$$

Also by De Moivre's Theorem, we have

$$x^n = \cos n\theta + i\sin n\theta \,;\; \frac{1}{x^n} = \cos n\theta - i\sin n\theta$$

so that $x^n + \dfrac{1}{x^n} = 2\cos n\theta$ and $x^n - \dfrac{1}{x^n} = 2i\sin n\theta$

We have $2^5\cos^5\theta = \left(x + \dfrac{1}{x}\right)^5$ and

$$2^7 i^7 \sin^7\theta = \left(x - \frac{1}{x}\right)^7$$

Hence $2^{12} i^7 \cos^5\theta \sin^7\theta = \left(x + \dfrac{1}{x}\right)^5 \left(x - \dfrac{1}{x}\right)^7$

$$= \left(x^2 - \frac{1}{x^2}\right)^5 \left(x - \frac{1}{x}\right)^2$$

$$= \left(x^{10} - 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} - \frac{1}{x^{10}}\right)\left(x^2 - 2 + \frac{1}{x^2}\right)$$

$$= \left(x^{12} - \frac{1}{x^{12}}\right) - 2\left(x^{10} - \frac{1}{x^{10}}\right) - 4\left(x^8 - \frac{1}{x^8}\right)$$

$$+ 10\left(x^6 - \frac{1}{x^6}\right) + 5\left(x^4 - \frac{1}{x^4}\right) - 20\left(x^2 - \frac{1}{x^2}\right)$$

Hence we have

$$-2^{11}\cos^5\theta\sin^7\theta = \sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta.$$

Scanned with CamScanner

$$D^n (\cos^5 \theta \sin^7 \theta) = -\tfrac{1}{2^{11}} \left\{ 12^n \sin\left(\frac{n\pi}{2} + 12\theta\right) \right.$$

$$- 10^n\, 2 \sin\left(\frac{n\pi}{2} + 10\theta\right) - 8^n\, 4 \sin\left(\frac{n\pi}{2} + 8\theta\right)$$

$$+ 6^n\, 10 \sin\left(\frac{n\pi}{2} + 6\theta\right) + 4^n\, 5 \sin\left(\frac{n\pi}{2} + 4\theta\right)$$

$$\left. - 2^n\, 20 \sin\left(\frac{n\pi}{2} + 2\theta\right) \right\}$$

Exercises 13.

1. Find the n^{th} differential coefficient of :-

(1) $\sin^3 x$.

(2) $\cos^4 x$.

(3) $\sin^3 x \cos^5 x$.

(4) $\sin^2 x \cos^3 x$.

(5) $\sin x \,.\, \sin 2x \sin 3x$.

(6) $e^x \sin x$.

(7) $e^{4x} \sin^2 x$.

(8) $e^x \sin x \sin 2x$.

(9) $\dfrac{ax + b}{cx + d}$.

(10) $e^{5x} \sin^3 ax$.

(11) $\dfrac{1}{4x^2 + 8x + 3}$

(12) $\dfrac{x^2}{(x + 1)^2 (x + 2)}$.

(13) $\dfrac{x^4}{(x - 1)(x - 2)}$.

(14) $\log (4 - x^2)$.(B.Sc.1988)

(15) $\dfrac{1}{x^2 - a^2}$.(B.Sc. 1990)

(16) $\dfrac{1}{4x^2 - 1}$.

(17) $\dfrac{x^2}{(x - a)(x - b)(x - c)}$.

(18) $\dfrac{x^3}{(x - a)(x - b)(x - c)}$.

(B.Sc. 1986)

2. Prove that if $y^3 - 3ax^2 + x^3 = 0$, $\dfrac{d^2 y}{dx^2} + \dfrac{2a^2 x^2}{y^5} = 0$.

(B.Sc. 1989)

(ii) If $x = t^3 + 1$ and $y = t^2 - 2$, show that

$$\left(\frac{dy}{dx}\right)^4 \Big/ \frac{d^2y}{dx^2} \text{ is constant.}$$

(B.Tech. 1984)

9. Find y_n when

(i) $y = \tan^{-1}\dfrac{x}{a}$.

(ii) $y = \dfrac{1}{(x+a)^2 + b^2}$

(iii) $y = \dfrac{1}{(x^2+a^2)(x^2+b^2)}$

(iv) $y = \dfrac{x}{(x-1)^2(x+2)}$

10. If $y = \sin 2ax + \cos ax$, prove that

$$y_n = a^n \left\{1 + (-1)^n \sin 2ax\right\}^{1/2} \qquad \text{(B.Sc. 1990)}$$

§ 1.6. Formation of equations involving derivatives.

When a relation between x and y is given, we can in many cases deduce from it a relation between the variables x,y and the derivatives of y with respect to x as the following examples will show.

Examples.

Ex. 1. If $xy = ae^x + be^{-x}$, prove that

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0. \qquad \text{(B.Sc. 1988)}$$

Here $xy = ae^x + be^{-x}$.

Now differentiating both sides with respect to x, we have

$$y + x\frac{dy}{dx} = ae^x - be^{-x}$$

Differentiating both sides of the equation once again, we get

Scanned with CamScanner

$$\frac{dy}{dx} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} = ae^x + be^{-x}.$$

$$\text{i.e., } x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy.$$

$$\text{i.e., } \frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0.$$

Ex. 2. Prove that if $y = \sin(m \sin^{-1} x)$,

$$(1 - x^2)\, y_2 - xy_1 + m^2 y = 0 \qquad \text{(B.Sc. 1988)}$$

$$y = \sin(m \sin^{-1} x).$$

$$\therefore \quad \sin^{-1} y = m \sin^{-1} x.$$

Differentiating both sides with respect to x, we get

$$\frac{1}{\sqrt{1 - y^2}}\frac{dy}{dx} = \frac{m}{\sqrt{1 - x^2}}.$$

Squaring and transposing, we have

$$(1 - x^2)\left(\frac{dy}{dx}\right)^2 = m^2(1 - y^2).$$

Differentiating the above equation with respect to x, we get

$$(1 - x^2)\, 2\frac{dy}{dx}\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = -2m^2 y\frac{dy}{dx}.$$

Cancelling the common factor $2\frac{dy}{dx}$ throughout, we get

$$(1 - x^2)\frac{d^2y}{dx^2} - \frac{dy}{dx} + m^2 y = 0.$$

Ex. 3. If $x = \sin\theta$, $y = \cos p\theta$, prove that

$$(1 - x^2)\, y_2 - xy_1 + p^2 y = 0. \qquad \text{(B.Sc. 1990)}$$

$$x = \sin\theta, \quad y = \cos p\theta.$$

(10)

$$\therefore \quad \frac{dx}{d\theta} = \cos\theta, \quad \frac{dy}{d\theta} = -p\sin p\theta$$

$$\therefore \quad \frac{dy}{dx} = -p \cdot \frac{\sin p\theta}{\cos\theta} = -p\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = p^2\frac{1-y^2}{1-x^2}.$$

$$\therefore \quad (1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2(1-y^2).$$

Differentiating the above equation and cancelling the common factor $2\frac{dy}{dx}$, we get $(1-x^2)\cdot\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$.

Exercises 14.

1. If $xy = ax^2 + \frac{b}{x}$, prove that

$$x^2\frac{d^2y}{dx^2} + 2\left(x\frac{dy}{dx} - y\right) = 0.$$

2. If $y = ax\cos mx$, prove that

$$x^2\left(\frac{d^2y}{dx^2} + m^2y\right) = 2\left(x\frac{dy}{dx} - y\right).$$

3. If $y = e^{-x}\cos x$, prove that

$$\frac{d^4y}{dx^4} + 4y = 0. \qquad \text{(B.Sc. 1990)}$$

4. If $y = x^2\cos x$, prove that

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (x^2+6)y = 0.$$

§ 2.1. Leibnitz formula for the n^{th} derivative of a product.

This formula expresses the n^{th} derivative of the product of two variables in terms of the variables themselves and their successive derivatives.

If u and v are functions of x, we have

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

i.e., $D(uv) = vDu + uDv.$

Differentiating again with respect to x, we get

$$D^2(uv) = D(v \cdot Du) + D(u \cdot Dv)$$
$$= vD^2u + 2Du \cdot Dv + u \cdot D^2v.$$

Similarly

$$D^3(uv) = v \cdot D^3u + 3D^2u \cdot Dv + 3Du \cdot D^2v + u \cdot D^3v.$$

However for this process may be continued, it will be seen that the numerical coefficients follow the same law as that of the Binomial Theorem and the indices of the derivatives correspond to the exponents of the Binomial Theorem. Hence

$$\frac{d^n}{dx^n}(uv) =$$

$$\frac{d^n u}{dx^n}v + {}^nC_1\frac{d^{n-1}u}{dx^{n-1}}\cdot\frac{dv}{dx} + {}^nC_2\frac{d^{n-2}u}{dx^{n-2}}\cdot\frac{d^2v}{dx^2}$$

$$+ \dots + {}^nC_r\cdot\frac{d^{n-r}u}{dx^{n-r}}\cdot\frac{d^rv}{dx^r} + \dots + {}^nC_1\frac{du}{dx}\cdot\frac{d^{n-1}v}{dx^{n-1}} + u\cdot\frac{d^nv}{dx^n}$$

§ 2.2. A complete formal proof by induction may be given as follows :

Assume the theorem to be true for some one value of n, i.e., suppose

$$D^n(uv) = u_nv + {}^nC_1u_{n-1}v_1 + {}^nC_2u_{n-2}v_2$$
$$+ \dots + {}^nC_{r-1}u_{n-r+1}v_{r-1} + {}^nC_ru_{n-r}v_r + \dots + uv_n.$$

12

Differentiating again we get

$D^{n+1}(uv) = (u_{n+1}v + u_n v_1) + n\,C_1\,(u_n v_1 + u_{n-1} v_2)$

$+ n\,C_2\,(u_{n-1} v_2 + u_{n-2} v_3) + \ldots n\,C_{r-1}\,(u_{n-r+2} v_{r-1} + u_{n-r+1} v_r)$

$+ n\,C_r\,(u_{n-r+1} v_r + u_{n-r} v_{r+1}) + \ldots + \ldots + (u_1 v_n + u v_{n+1})$

$= u_{n+1} v + (1 + n\,C_1)\,u_n v_1 + (nC_1 + n\,C_2)\,u_{n-1} v_2$

$+ \ldots + (n\,C_{r-1} + n\,C_r)\,u_{n-r+1} v_r + \ldots + u v_{n+1}$

Now $n\,C_{r-1} + n\,C_r = (n+1)\,C_r$ and so

$$1 + n\,C_1 = (n+1)\,C_1.$$

$$n\,C_1 + n\,C_2 = (n+1)\,C_2.$$

$$n\,C_2 + n\,C_3 = (n+1)\,C_3 \text{ and so on.}$$

$\therefore D^{n+1}(uv) = u_{n+1} v + (n+1)\,C_1\,u_n v_1 + \ldots$

$+ (n+1)\,C_r\,u_{n+1} v_r + \ldots + u v_{n+1}.$

Hence if the theorem be true for any value of n, it must be true for the next higher value $n + 1$. It has been seen that it is true for $n = 1$ and therefore it is true for $n = 2$ and therefore for $n = 3$ and so on for all values of n.

This theorem is particularly useful when one of the factors is a small integral multiple of x ; if this be taken as v in the proceding formula, its differential coefficients and the series will consist of only a few terms.

Examples.

Ex. 1. Find the n^{th} differential coefficient of $x^2 \log x$.

Taking $v = x^2$ and $u = \log x$,

$$\frac{d^n}{dx^n}(x^2 \log x) = \frac{d^n}{dx^n}(\log x)\,x^2 + n\,C_1 \,.\, \frac{d^{n-1}}{dx^{n-1}}(\log x)\,\frac{d}{dx}(x^2)$$

$$+ n\,C_2\,\frac{d^{n-2}}{dx^{n-2}}(\log x)\,\frac{d^2}{dx^2}(x^2).$$

All the other terms will be zero since the successive derivatives of x^2 after the second derivative vanish.

$$\therefore\ D^n\ (x^2 \log x) = \frac{(-1)^{n-1}\ (n-1)!}{x^n} x^2 +$$

$$n \frac{(-1)^{n-2}\ (n-2)!}{x^{n-1}}\ 2x + \frac{n\ (n-1)\ (-1)^{n-3}\ (n-3)!\ 2}{2\ \ x^{n-2}}$$

$$= \frac{(-1)^{n-3}\ 2\ (n-3)!}{x^{n-2}}.$$

Ex. 3. If $y = \sin (m \sin^{-1} x)$, prove that

$$(1 - x^2)\ y_2 - xy_1 + m^2 y = 0. \text{ and}$$

$$(1 - x^2)\ y_{n+2} - (2n+1)\ xy_{n+1} + (m^2 - n^2)\ y_n = 0$$

(Vide example 2 in § 1.6)

$$(1 - x^2)\ y_2 = xy_1 - m^2 y.$$

Taking then n^{th} derivative of each term by Leibnitz's Theorem, we have

$$y_{n+2} (1 - x^2) + n\ c_1\ y_{n+1}\ (-2x) + n\ c_2\ y_n\ (-2)$$

$$= y_{n+1}\ x + n\ c_1\ \ y_n - m^2 y_n$$

i.e., $y_{n+2}\ (1 - x^2)\ - 2nxy_{n+1} - n\ (n-1)\ y_n$

$$= xy_{n+1} +\ ny_n - m^2\ y_n$$

i.e., $(1 - x^2)\ y_{n+2} - (2n+1)\ xy_{n+1} + (m^2 - n^2)\ y_n = 0.$

Exercises 15.

1. Find the n^{th} differential coefficients of

(1) $x\ e^x$ (2) $x^2\ e^{3x}$

(3) $x \sin x$ (4) $x^2 \cos x.$

CHAPTER V

MAXIMA AND MINIMA

§ 1.1. We shall discuss in this chapter the maximum and minimum values of a function when it and its derivatives are continuous. We shall first define the maximum and minimum values of a function.

If a continuous function increases up to a certain value and then decreases, that value is called a *maximum value* of the function. Similarly, if a continuous function decreases up to a certain value and then increases, that value is called a *minimum value* of the function. We say that value $f(a)$ assumed by $f(x)$ at $x = a$ is a maximum if $f(x)$ in the immediate neighbourhood of $x = a$, i.e., if we can find an interval $(a - h, a + h)$ of values of x such that $f(a) > f(x)$ when $a - h < x < a$ and $a < x < a + h$, where h is an arbitrary small positive number. Similarly we define a minimum ; if in the interval $(a - h, a + h)$, $f(a) < f(x)$, $f(a)$ is said to be a minimum value of $f(x)$. Thus in the figure the points P correspond to maxima, the points Q to minima of the function $f(x)$ whose graph is shown below :

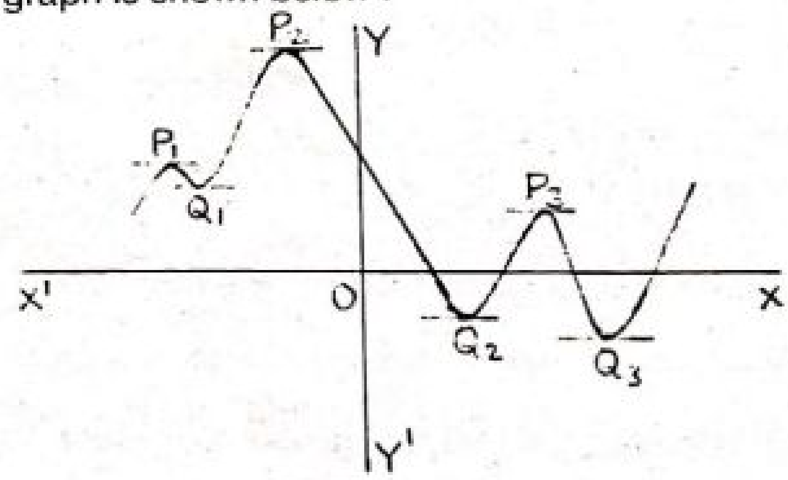


Fig. 8

It is to be noticed that (1) a maximum value is not necessarily the greatest value of the function can have; nor a minimum the least ; (2) the maxima and minima values occur alternately.

§ 1.2. Theorem 1. A necessary condition for a maximum or a minimum value of $f(x)$ at $x = a$ is that $f'(a) = 0$.

If $f(a)$ is a maximum value of $f(x)$, then as x increases from $a-h$ to a, $f(x)$ is increasing and therefore $f'(x)$ is positive. On the other hand, as x increases from a to $a+h$, $f(x)$ is decreasing and therefore $f'(x)$ is negative. Hence as x increases through a, $f'(x)$ must change from a positive to a negative value. Conversely, if as x increases through a, $f'(x)$ changes from a positive to a negative value, $f(a)$ will be a maximum value of $f(x)$.

Hence $f(a)$ will be maximum value of $f(x)$ if and only if $f'(x)$ changes from a positive to a negative value as x increases through a.

In the same way, it will be seen that if $f(a)$ will be a minimum value of $f(x)$ if and only if $f'(x)$ changes from a negative to a positive value as x increases through a.

$f'(x)$ is continuous and a continuous function can change sign only by passing through the value zero. Therefore, if $f(a)$ is turning value of $f(x)$, $f'(a)$ will be zero.

§ 1.3. Theorem 2. If $f'(a)=0$ and $f''(a)\neq 0$, then $f(x)$ has a maximum if $f''(a)<0$ and a minimum if $f''(a)>0$.

If $f(a)$ is a maximum value of $f(x)$, $f'(x)$ changes from a positive to a negative value as x increases through a.

Consider $f'(x)$ as a function of x. In the interval $(a-h, a+h)$, $f'(x)$ decreases continuously changing from a positive to a negative value. Hence its derivative $f''(x)$ is negative. Therefore at the maximum point $x=a$, $f''(a)$ is negative. Similarly when $f(x)$ attains a minimum value at $x=a$, $f''(a)$ is positive. For $f'(x)$ increases continuously in the interval $(a-h, a+h)$ changing from a negative to a positive value and hence its derivative $f''(a)$ is positive.

Rule for determining the maxima and minima values of $f(x)$ when $f(x)$ and $f'(x)$ are continuous.

The roots of the equation $f'(x)=0$ are, in general, the values of x which make $f(x)$ a maximum or a minimum. Let a be a root of $f'(x)=0$; then $f(a)$ will be a maximum value of $f(x)$ if $f''(a)$ is negative and a minimum value if $f''(a)$ is positive.

§ 14. A geometrical proof for the above two theorems.

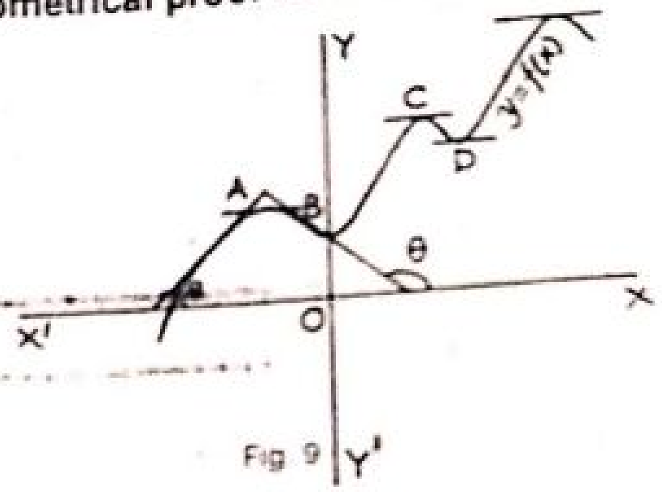


Fig 9

In this figure the ordinates of A and C represent the maximum values and the ordinates of B and D represent the minimum values of $f(x)$.

If the tangent at (x, y) on the curve makes angle θ with the positive direction of the x-axis, we have seen that $\tan\theta = \frac{dy}{dx}$

The tangents at A,B,C,D are parallel to the x-axis.

Therefore $\theta = 0$.

$\therefore$ At A,B,C,D, $\frac{dy}{dx} = 0$.

For the points just to the left of A and C on the curve, θ is an acute angle.

$$\therefore \tan\theta \text{ is } +\text{ve, i.e., } \frac{dy}{dx} \text{ is } +\text{ve}.$$

For the points to the right of A and C, θ is an obtuse angle.

$$\therefore \tan\theta \text{ is } -\text{ve, i.e., } \frac{dy}{dx} \text{ is } -\text{ve}.$$

Therefore in passing through a maximum value $\frac{dy}{dx}$ changes from positive to negative. In a similar manner, we can show that in passing through a minimum , $\frac{dy}{dx}$ changes from negative to positive.

So $\frac{d^2y}{dx^2}$ is negative at a maximum value and positive at a minimum value.

§ 1.5. The above conclusions, when $f(x)$ and its derivatives are continuous at a may be deduced from the theorem of mean value which will be proved later.

Proof :

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots\ldots$$

$$\ldots\ldots \frac{h^n}{n!}f^n(a+\theta h), \text{ where } 0 < \theta < 1.$$

$$\therefore\ f(a+h) - f(a) = h\left[f'(a) + \frac{h}{2!}f''(a) + \ldots\right.$$

$$\left.\ldots \frac{h^{n-1}}{n!}f^n(a+\theta h)\right] \qquad (1)$$

If $f(a)$ is maximum at $x = a$, then by definition, $f(a+h) - f(a)$ and therefore the right-hand side of the equation must be negative for sufficiently small values of h whether h is positive or negative.

$f'(a) + \frac{h}{2!}f''(a) + \ldots + \frac{h^{n-1}}{n!}f^n(a+\theta h)$ is a finite expression which does not tend to infinity as h tends to zero.

If $f'(a) \neq 0$, $f(a+h) - f(a)$ has one sign when h is positive and another when h is negative. The second member of (1) is of invariable sign for a maximum or a minimum since for a maximum $f(a+h) - f(a)$ is negative and for a minimum $f(a+h) - f(a)$ is positive.

$$\therefore\ f'(a) = 0.$$

If $f'(a) = 0$, then

(18)

$$f(a+h) - f(a) = h^2 \left[\frac{f''(a)}{2!} + \frac{h}{3!} f'''(a) + \dots \right.$$

$$\left. \dots \frac{h^{n-2}}{n!} f^n (a + \theta h) \right].$$

The sign of the expression within the brackets on the right-hand side is governed by $f''(a)$.

Since h^2 is always positive, $f(a+h) - f(a)$ is negative if $f''(a)$ is negative and $f(a+h) - f(a)$ is positive if $f''(a)$ is positive.

$\therefore$ If $f''(a) < 0$, $f(x)$ has a maximum at $x = a$.

If $f''(a) > 0$, $f(x)$ has a minimum at $x = a$.

Here we assume that $f(x)$, $f'(x)$ .... $f''(x)$ are continuous. In such cases as we are likely to meet with at present, the condition is generally satisfied.

Examples.

Ex. 1. Find the maxima and minima of the function

$2x^3 - 3x^2 - 36x + 10.$ (B.Sc. 1986)

Let $f(x)$ be $2x^3 - 3x^2 - 36x + 10.$

At the maximum or minimum, $f'(x) = 0$.

$$f'(x) = 6x^2 - 6x - 36$$
$$= 6(x-3)(x+2)$$

$x = 3$ and $x = -2$ give maximum or minimum.

To distinguish between the maximum and the minimum,

$$f''(x) = 6(2x-1).$$

When $x = 3$, $f''(x) = 6(6-1) = 30$ i.e., +ve

When $x = -2$, $f''(x) = 6(-4-1) = -30$, i.e., $-$ve

$\therefore$ $x = -2$ gives the maximum and $x = +3$ gives the minimum.

Maximum value $= f(-2) = 54.$

Minimum value $= f(3) = -71.$

Ex. 2. Find the maximum value of $\frac{\log x}{x}$ for positive values of x. (B.Sc. 1990)

Let $f(x)$ be $\frac{\log x}{x}$.

$$f'(x) = \frac{1 - \log x}{x^2}$$

$$f''(x) = \frac{-3 + 2\log x}{x^3}$$

At a maximum or a minimum, $f'(x) = 0$.

$$\therefore \; 1 - \log x = 0. \quad \therefore \; x = e.$$

$$f''(e) = \frac{-3 + 2\log e}{e^3} = -\frac{1}{e^3}, \text{ i.e., } -\text{ve.}$$

$\therefore x = e$ gives a maximum.

Maximum value of the function $f(e) = \frac{1}{e}$

Ex. 3. Show that the least value of

$a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x$ is $(a + b)^2$. (B.Sc. 1988)

Let $f(x)$ be $a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x$.

$$f'(x) = 2a^2 \sec^2 x \tan x - 2b^2 \operatorname{cosec}^2 x \cot x$$

$$= 2\,\frac{a^2 \sin^4 x - b^2 \cos^4 x}{\cos^3 x \sin^3 x}$$

$$f''(x) = 2\,\frac{4\sin^4 x \cos^4 x\,(a^2 \sin^2 x + b^2 \cos^2 x)}{\cos^6 x \sin^6 x}$$

$$- \frac{(a^2 \sin^4 x - b^2 \cos^4 x)\,\frac{d}{dx}(\sin^3 x \cos^3 x)}{\cos^6 x \sin^6 x}$$

At the maximum or minimum, $f'(x) = 0$.

20

$$\therefore \quad a^2 \sin^4 x - b^2 \cos^4 x = 0.$$

$$\text{Then } f''(x) = \frac{8 \sin^4 x \cos^4 x \,(a^2 \sin^2 x + b^2 \cos^2 x)}{\cos^6 x \sin^6 x}$$

$$= 8\,(a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x).$$

$$= +\text{ve expression.}$$

$\therefore$ $a^2 \sin^4 x - b^2 \cos^4 x = 0$ gives a minimum.

$$\therefore \quad \tan^2 x = \frac{b}{a}.$$

The least value of $f(x)$ is given when $\tan^2 x = \dfrac{b}{a}$.

$$\therefore \quad f(x) = a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x$$

$$= a^2 (1 + \tan^2 x) + b^2 (1 + \cot^2 x)$$

$$= a^2 \left(1 + \frac{b}{a}\right) + b^2 \left(1 + \frac{a}{b}\right)$$

$$= (a + b)^2$$

Ex. 4. The greatest value of $ax + by$ where x and y are positive and $x^2 + xy + y^2 = 3\kappa^2$ is $2\kappa\sqrt{a^2 - ab + b^2}$

(B.Sc.1990)

Let $u = ax + by$.

u attains a maximum or minimum when $\dfrac{du}{dx} = 0$

and $\dfrac{d^2 u}{dx^2}$ is $-$ ve or $+$ve.

$$a + b\frac{dy}{dx} = 0 \qquad (1)$$

$$x^2 + xy + y^2 = 3\kappa^2.$$

Differentiating the above equation, we get

(21)

$$(2x + y) + (x + 2y)\frac{dy}{dx} = 0 \qquad (2)$$

Equating the two values of $\frac{dy}{dx}$, we get $-\frac{a}{b} = -\frac{2x+y}{x+2y}$

Solving for y, $y = \frac{a-2b}{b-2a}x \qquad (3)$

Differentiating equation (2) once again, we get

$$2 + 2\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + (x + 2y)\frac{d^2y}{dx^2} = 0.$$

Substituting the values of $\frac{dy}{dx}$ and y from (1) and (3), we get

$$\frac{d^2y}{dx^2} = \frac{2}{3}\frac{a^2 - ab + b^2}{b^2}\frac{b - 2a}{x}$$

$\frac{d^2y}{dx^2}$ is negative for a maximum.

$$\frac{b-2a}{x} \text{ is } -\text{ve since } \frac{a^2 - ab + b^2}{b^2} \text{ is } +\text{ve}.$$

$$x^2 + y^2 + xy = 3\kappa^2.$$

Substituting the value for y from (3), we get

$$x\sqrt{a^2 - ab + b^2} = -\kappa(b - 2a) \qquad (4)$$

We take the negative sign since $\frac{b-2a}{x}$ is $-$ ve.

$$ax + by = ax + \frac{b(a-2b)}{(b-2a)}x$$

$$= -2(a^2 - ab + b^2)\frac{x}{b-2a}$$

$$= 2\kappa\sqrt{a^2 - ab + b^2} \text{ from (4)}$$

— x —

Scanned with CamScanner

15. Show that $\left(\kappa - \frac{1}{\kappa} - x\right)(4 - 3x^2)$ where κ is a positive constant, has one and only one maximum value and only one minimum value.

Example.

The bending moment at B at a distance x from one end of a beam of length l uniformly loaded is given by the formula $M = \frac{1}{2}\omega lx - \frac{1}{2}\omega x^2$, where ω = load per unit length. Show that the maximum bending moment is at the centre of the beam.

(B.Tech 1984)

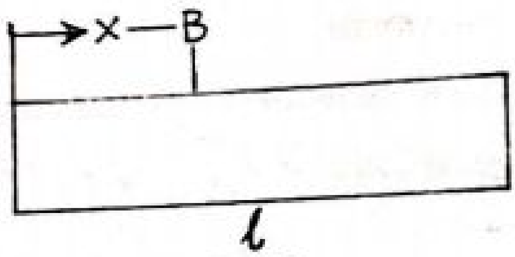


Fig. 10

$$M = \frac{1}{2}\omega lx - \frac{1}{2}\omega x^2.$$

M attains its maximum when $\frac{dM}{dx} = 0$ and $\frac{d^2M}{dx^2}$ is – ve.

$$\frac{dM}{dx} = \frac{1}{2}\omega l - \omega x$$

$$\frac{d^2M}{dx^2} = -\omega \ ; \ \frac{d^2M}{dx^2} \text{ is } - \text{ve.}$$

$$\frac{dM}{dx} = 0, \text{ where } \frac{1}{2}\omega l - \omega x. = 0, \text{ i.e., when } x = \frac{1}{2}l.$$

∴ The maximum bending moments is at the centre.

Exercises 20.

1. The velocity of waves of length λ on deep water is proportional to $\left(\frac{\lambda}{a} + \frac{a}{\lambda}\right)^{1/2}$, where a is a certain linear magnitude. Prove that the velocity is a minimum when $\lambda = a$.

the following examples, we have to find the relation between the variables from the data and then find the maxima or minima as the case may be.

Examples.

Ex. 1. From a given circular sheet of metal it is required to cut out a sector so that the remainder can be formed into a conical vessel of maximum capacity ; prove that the angle of the sector removed must by above 66°. (B.Sc. 1980)

Let θ be the angle of the sector ACB. Then $2\pi - \theta$ is the angle of the sector which has to be removed.

Length of arc ACB $= R\,\theta$, where R is radius of the circular disc.

Now the circumference of the base of the conical vessel must be the same as the length of the arc ACB.

Circumference of base $= R\,\theta$

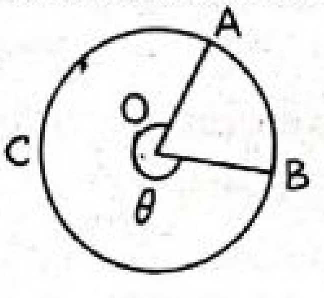


Fig. 11

Radius of base $= \dfrac{R\theta}{2\pi}$

Length of the slant height of the cone $= R$.

$\therefore$ Vertical height of the cone $= \left(R^2 - \dfrac{R^2\theta^2}{4\pi^2}\right)^{1/2}$.

Volume of the conical vessel

$$V = \frac{\pi}{3}\,\frac{R^2\theta^2}{4\pi^2}\left(R^2 - \frac{R^2\theta^2}{4\pi^2}\right)^{1/2};\ \text{i.e., } V = \frac{R^3}{12\pi}\left(\theta^4 - \frac{\theta^6}{4\pi^2}\right)^{1/2}$$

Now V will be the greatest when $\left(\theta^4 - \dfrac{\theta^6}{4\pi^2}\right)^{1/2}$ is greate.

That is when $\theta^4 - \dfrac{\theta^6}{4\pi^2}$ is a maximum.

(24)

$$\text{Let } f(\theta) = \theta^4 - \frac{\theta^6}{4\pi^2}$$

$$f'(\theta) = 4\theta^3 - \frac{3\theta^5}{2\pi^2}$$

$$f''(\theta) = 12\theta^2 - \frac{15\theta^4}{2\pi^2}$$

$f(\theta)$ is a maximum when $4\theta^3 - \dfrac{3}{2}\dfrac{\theta^5}{\pi^2} = 0.$

i.e., when $\theta = 0$ or $\theta = 2\pi \left(\dfrac{2}{3}\right)^{1/2}$

$\theta = 0$ is clearly inadmissible.

$\theta = 2\pi \left(\dfrac{2}{3}\right)^{1/2}$ makes $f''(\theta)$ negative and gives the maximum value for $f(\theta)$

The angle of the sector $= (2\pi - \theta)$ radians

$= 2\pi - 2\pi \left(\dfrac{2}{3}\right)^{1/2}$ radians

$= 1.153$ radians

$= 66^\circ\, 6'$ approximately.

Ex. 2. From a solid sphere, matter is scooped out so as to form a conical cup, with the vertex of the cup on the surface of the sphere. Find when the volume of the cup is a maximum.

(B.Sc. 1988)

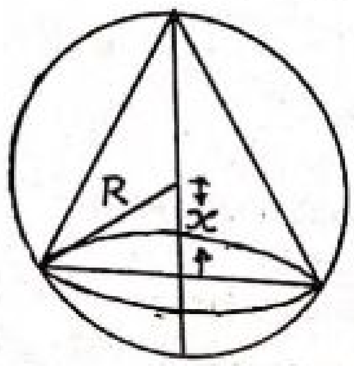


Fig. 12

25

Let x be the distance of the centre of the sphere from the base of the cone and R be the radius of the sphere.

Height of the cone $= R + x.$

Radius of the base of the cone $= \sqrt{R^2 - x^2}$

Volume of the cone . $V = \frac{\pi}{3}(R + x)(R^2 - x^2)$

$$\frac{dV}{dx} = \frac{\pi}{3}(R^2 - 2Rx - 3x^2)$$

$$\frac{d^2V}{dx^2} = \frac{\pi}{3}(-2R - 6x).$$

V is a maximum when $\frac{dV}{dx} = 0$ and $\frac{d^2V}{dx}$ is $-$ ve.

$\frac{dV}{dx} = 0$ when $R^2 - 2Rx - 3x^2 = 0$

i.e., $x = -R$ or $\frac{R}{3}$

$x = -R$ is clearly inadmissible ; $\frac{d^2V}{dx^2}$ for $x = \frac{R}{3}$ is negative.

$\therefore$ V is a maximum when $x = \frac{R}{3}$

Height of cone is $R + x = \frac{4R}{3}$

Radius of the base $= \sqrt{R^2 - x^2} = \frac{2\sqrt{2}}{3}R$

Volume of the cone $= \frac{\pi}{3} \cdot \frac{4R}{3} \cdot \frac{8R^2}{9} = \frac{32\pi R^3}{81}$

Ex. 3. Find the dimensions of a cylinderical vessel of greatest capacity which can be made from a given amount of sheet of metal (1) when the vessel has no lid and (2) when the vessel has a lid.

(1) *When the vessel has no lid.*

Let S be the area of sheet metal used without lid.

Scanned with CamScanner

S = surface area of vessel = $2\pi xy + \pi x^2$.

where x = radius of base and y = height.

Volume of vessel = $\pi x^2 y$.

$$V = \pi x^2 \left\{ \frac{S - \pi x^2}{2\pi x} \right\} = \frac{1}{2}(Sx - \pi x^3)$$

Then $\frac{dV}{dx} = \frac{1}{2}(S - 3\pi x^2)$; $\frac{d^2V}{dx^2} = -3\pi x$.

Now V is a maximum when $\frac{dV}{dx} = 0$ and $\frac{d^2V}{dx^2}$ is – ve.

$\frac{d^2V}{dx^2}$ is – ve when x is positive.

$$\frac{dV}{dx} = 0 \text{ when } x = \left(\frac{S}{3\pi}\right)^{1/2}$$

$\therefore$ V is a maximum when $x = \left(\frac{S}{3\pi}\right)^{1/2}$

i.e., when $S = 3\pi x^2$

i.e., $S = 2[\pi\}xy + \pi x^2$

$$3\pi x^2 = 2\pi xy + \pi x^2$$

i.e., $2\pi xy = 2\pi x^2$

$$\therefore y = x.$$

$\therefore$ Height of the vessel $= \left(\frac{S}{3\pi}\right)^{1/2}$

(2) *With lid.*

S = surface area of the vessel = $2\pi xy + 2\pi x^2$,

where x = radius of the base and y = height.

V, volume of vessel = $\pi x^2 y = \frac{\pi x^2 (S - 2\pi x^2)}{2\pi x}$

$$= \frac{1}{2}(Sx - 2\pi x^3).$$

Then $\frac{dV}{dx} = \frac{1}{2}(S - 6\pi x^2)$; $\frac{d^2V}{dx^2} = -6\pi x.$

Now V is a maximum when $\frac{dV}{dx} = 0$ and $\frac{d^2V}{dx^2}$ is – ve.

$\frac{dV}{dx} = 0$ when $S - 6\pi x^2 = 0$, i.e., $x = \left(\frac{S}{6\pi}\right)^{1/2}$

For this value of x, $\frac{d^2V}{dx^2}$ is – ve.

For this value of x, V is a maximum.

$$S = 2\pi xy + 2\pi x^2.$$

$$6\pi x^2 = 2\pi xy + 2\pi x^2.$$

$$y = 2x = 2\left(\frac{S}{6\pi}\right)^{1/2}$$

$\therefore$ Height of the vessel $= 2\left(\frac{S}{6\pi}\right)^{1/2}$.

Ex. 4. The cost of fuel in running an engine is proportional to the square of the speed and is Rs. 48 per hour for a speed of 16 m.p.h. Other costs amount to Rs. 300 per hour. What is the most economical speed ?

If v m.p.h. is the speed and Rs. c is the cost of fuel, $c = \kappa v^2$, where κ is a constant.

When $v = 16$, $c = 48$ $\therefore \kappa = \frac{3}{16}$

Hence the cost of fuel $= \frac{3}{16}v^2$

Total running cost per hour = Rs. $\left(300 + \frac{3}{16}v^2\right)$

If the distance travelled is s miles, the number of hours taken up is $\frac{s}{v}$ hours.

∴ Total cost for the journey is Rs. y, where

$$y = \frac{s}{v}\left(300 + \frac{3}{16}v^2\right)$$

For the economical speed

$$\frac{dy}{dv} = 0 \text{ and } \frac{d^2y}{dv^2} \text{ is } +\text{ve}.$$

$$\frac{dy}{dv} = s\left(-\frac{300}{v^2} + \frac{3}{16}\right) = 0. \qquad \therefore v = 40.$$

$$\frac{d^2y}{dv^2} = \frac{600s}{v^3} > 0.$$

Hence the most economical speed per hour is 40 m.p.h.

Exercises 21.

1. A rectangular sheet of metal has four equal square portions removed at the corners and the sides are then turned up so as to form a rectanular box. Show that when the volume contained in the box is a maximum, the depth will be $\frac{1}{6}\left\{a + b - \sqrt{a^2 - ab + b^2}\right\}$ where a and b are the sides of the original rectangle.

2. Show that, if the sum of the length of the hypotenuse and another side of a right-angled trangle is equal to a constant, its area is a maximum when the angle between those sides is 60°. Determine also the maximum area.

3. A window is in the shape of a rectangle surmounted by a semi-circle. If the perimeter of the window be a fixed legth l, find the maximum area.

4. Show that a conical tent of given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base.

circular cylinder of greatest volume which can be sent is 2 ft. long and 4 ft. in girth.

37. The corner of a rectangular sheet of paper is turned down just to reach the other edge of the page ; find when the length of the crease is a minimum ; also when the area of the part turned down is a minimum.

§ 2. Concavity and convexity, points of inflexion.

If in the neighbourhood of a point P on a curve is above the tangent at P [as in Fig. 13 (a) and (b)], it is said to be *concave upwards* ; if the curve is below the tangent at P [as in Fig. 13 (c) and (d)], it is said to be concave downwards or *convex upwards*.

If at a point P, a curve changes its concavity from upwards to downwards or *vice versa* [as in Fig. 13 (e) and (f)] , P is called a point of *inflexion*.

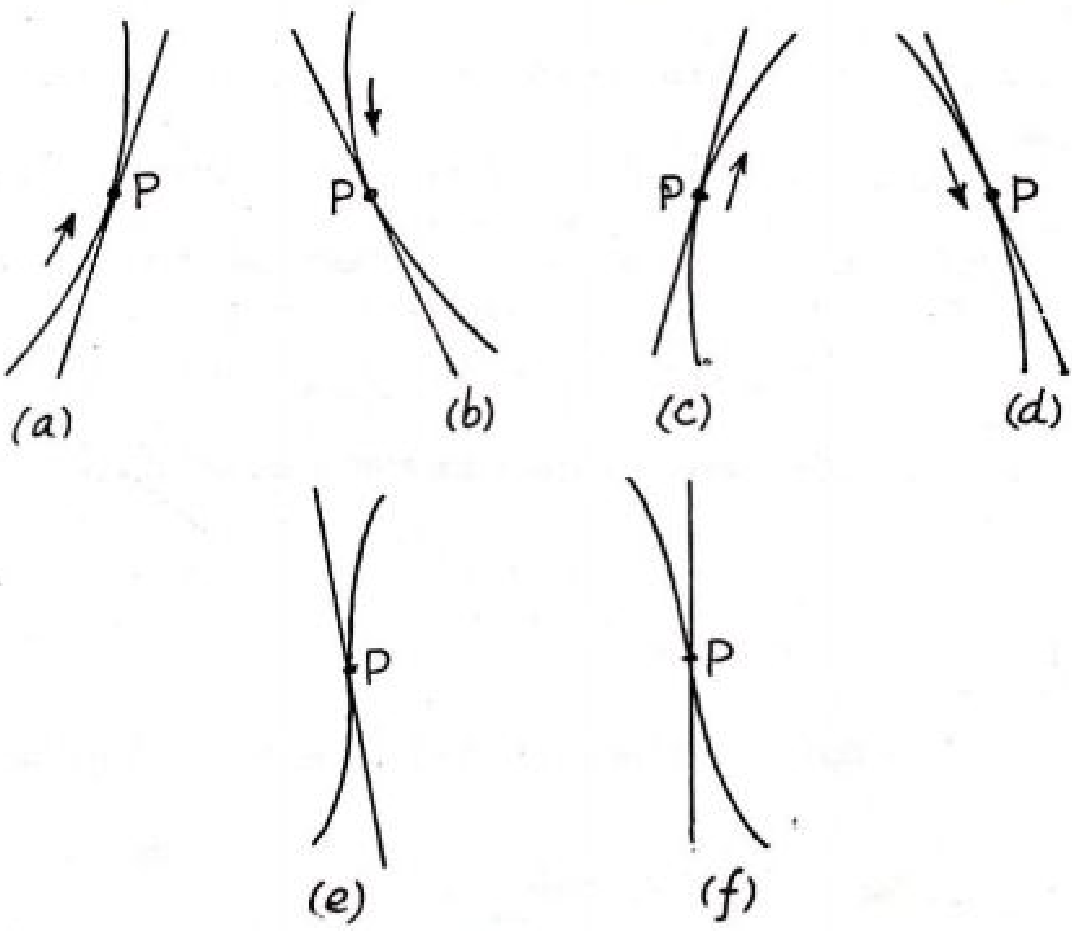


Fig. 13

From this definition, it is seen that the curve crosses its tangent at the point of inflexion and that a point of inflexion lies between a maximum and a minimum.

As the point on the curve in Fig. 13 (a) and (b) moves to the right (the direction of arrows), the tangent turns about its point of contact anti-clockwise and therefore the angle which it makes with the x-axis increases. So we get at all points in the neighbourhood of P on the curve when it is concave upwards ; the slope of the curve, i.e., $\frac{dy}{dx}$ increases as x increases. Therefore its differential coefficient is positive, i.e., $\frac{d^2y}{dx^2}$ is positive.

Similarly, if at all points in the neighbourhood of P, the curve is concave downwards, then the slope $\frac{dy}{dx}$ decreases as x increases. Therefore its differential coefficient $\frac{d^2y}{dx^2}$ is negative.

The concatity or convexity of a curve is determind from the sign of the second differential coefficient and, if it is negative, the curve is concave downwards or convex upwards. At the point of inflexion, the curve changes from concave upwards to convex upwards or vice versa. So $\frac{d^2y}{dx^2}$ changes sign and if it is continuous it is zero at that point. Hence the conditions for the point of inflexion are

(1) $\frac{d^2y}{dx^2} = 0$ at the point.

(2) $\frac{d^2y}{dx^2}$ changes its sign as x increases through the values at which $\frac{d^2y}{dx^2} = 0$, i.e., $\frac{d^3y}{dx^3} \neq 0$.

Examples.

Ex. 1. For what values of x is the curve $y = 3x^2 - 2x^3$ concave upwards and when is it convex upwards ?

$$y = 3x^2 - 2x^3$$

Then $\frac{dy}{dx} = 6x - 6x^2$.

$$\frac{d^2y}{dx^2} = 6 - 12x = -6(2x - 1).$$

If $x > \frac{1}{2}$, $\frac{d^2y}{dx^2}$ is negative and so convex upwards.

If $x < \frac{1}{2}$, $\frac{d^2y}{dx^2}$ is positive and so concave upwards.

If $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = -12$ and so there is a point of inflexion at $x = \frac{1}{2}$., i.e., at the point $(\frac{1}{2}, \frac{1}{2})$.

Ex. 2. Find the points of inflexion on the cubic $y = \frac{a^2x}{x^2 + a^2}$ and show that they lie on a straight line. (B.Sc. 1990)

$$y = \frac{a^2x}{x^2 + a^2}$$

Then $\frac{dy}{dx} = \frac{a^2(a^2 - x^2)}{(x^2 + a^2)^2}$ and $\frac{d^2y}{dx^2} = \frac{-2a^2x(3a^2 - x^2)}{(x^2 + a^2)^3}$.

At the points of inflexion, $\frac{d^2y}{dx^2} = 0$.

$\therefore\ x(3a^2 - x^2)^2 = 0$. $\qquad \therefore\ x = 0$ or $\pm\sqrt{3}\,a$.

$$\frac{d^3y}{dx^3} = \frac{-6a^2(x^4 + a^4 - 6a^2x^2)}{(x^2 + a^2)^4}.$$

(32)

At the points $x = 0$ or $\pm \sqrt{3}\,a$, $\dfrac{d^3y}{dx^3} \neq 0.$

When $x = 0$, $y = 0$; $x = \sqrt{3}\,a$, $y = \dfrac{\sqrt{3}\,a}{4}.$

$$x = -\sqrt{3}\,a\ ;\ y = -\frac{\sqrt{3}\,a}{4}.$$

The points of inflexion are

$$(0,0)\ (\sqrt{3}\,a\ , \frac{\sqrt{3}}{4}\,a)\ ,\ (-\sqrt{3}\,a\ , \frac{-\sqrt{3}}{4}\,a).$$

These three points of inflexion lie on the straight line $x = 4y$.

Exercises 22.

1. Find the points of inflexion in the following curves :-

(1) $y = x^4 - 6x^2 + 8x - 1$ (2) $y = x^3 - 9x^2 + 7x - 6$

(3) $y = \cos x$ (4) $y = a \sin x + b \cos x$

(5) $y = a \cos^2 x + b \sin^2 x$

(6) $y = \dfrac{x^3}{a^2 + x^2}$ (B.Sc. 1990)

(7) $xy^2 = a^2 (a - x)$ (8) $y = x^3 e^{-x}$

(9) $y = \dfrac{\log x}{x^{\sqrt{3}}}$, $(0 < x)$. (10) $y = c \sin (x/a)$ (B.Sc. 1988)

2. Show that the curve $y = \kappa \sin x$ cuts the x-axis at inflexional points.

3. Find the points of inflexion on the curve

$$y = (x - a)\ (x - b)\ (x - c).$$

4. Show that the curve $y = \dfrac{6x}{x^2 + 3}$ has three points of inflexion.

(33)

CHAPTER VIII

PARTIAL DIFFERENTIATION, ERRORS AND APPROXIMATIONS

§ 1.1 We have considered till now only functions of one variable but we come accross functions involving more thatn one variable. For example, the area of a rectangle is a function of two variables, the length and breadth of the rectangle.

If u be a function of two independent variables x and y, let us assume the functional relation as $u = f(x,y)$. Here x alone or y alone or both x and y simultaneously may be varied and in each case, a change in the value of u will result. Generally the change in the value of u will be different in each of these three cases. Since x and y are independent, x may be supposed to vary when y remains constant or the reverse.

The derivative of u with respect to x when x varies and y remains constant is called the partial derivative of u with respect to x and is denoted by the symbol $\frac{\partial u}{\partial x}$. We may then write

$$\frac{\partial u}{\partial x} = \underset{\Delta x \to 0}{Lt} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}.$$

Similarly, when x remains constant and y varies, the partial derivative of u with respect to y is

$$\frac{\partial u}{\partial y} = \underset{\Delta y \to 0}{Lt} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$\frac{\partial u}{\partial x}$ is also written as $\frac{\partial}{\partial x} f(x,y)$ or $\frac{\partial f}{\partial x}$.

Similarly $\frac{\partial u}{\partial y}$ is also written as $\frac{\partial}{\partial y} f(x,y)$ or $\frac{\partial f}{\partial y}$.

Successive partial derivatives.

Consider the function $u = f(x,y)$. Then in general $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are functions of both x and y and may be differentiated again

(31)

with respect to either of the independent variables giving rise to successive partiable derivatives. Regarding x alone as varying we denote the result by $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \ldots, \frac{\partial^n u}{\partial x^n}$ or when y alone varies, $\frac{\partial^2 u}{\partial y^2}, \frac{\partial^3 u}{\partial y^3}, \ldots, \frac{\partial^n u}{\partial y^n}$.

If we differentiate u with respect to x regarding y as constant and then this result is differentiated with respect to y regarding x as constant, we obtain $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$ which we denote by $\frac{\partial^2 u}{\partial y\,\partial x}$.

Similarly, if we differentiate u twice with respect to x and then once with respect to y, the result is denoted by the symbol $\frac{\partial^3 u}{\partial y\,\partial^2 x}$. The partial differential coefficient of $\frac{\partial u}{\partial y}$ with respect to x considering y as a constant is denoted by $\frac{\partial^2 u}{\partial x\,\partial y}$.

Generally, in the ordinary functions which we come across

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}.$$

§ 1.2. Function of function rule. This rule is very useful in partial differentiation.

Let z be a function of u where u is a function of two independent variables x and y.

Then $$\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y}.$$

Let x and y receive arbitrary increments Δx and Δy and let the corresponding increments in u and z be Δu and Δz respectively.

Then $$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta u}\frac{\Delta u}{\Delta x}.$$

Proceeding to the limit when $\Delta x \to 0$, $\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x}$

Scanned with CamScanner

(Note that the straight limit ' d ' is used in $\frac{dz}{dx}$ as z is a function of only one variable u while the curved ' ∂ ' is used in $\frac{\partial u}{\partial x}$ as u is a function of two independent variables).

Similarly the other result follows.

Examples.

Ex. 1. Find the partial differential coefficients of

$$u = \sin(ax + by + cz)$$

$$\frac{\partial u}{\partial x} = a\cos(ax + by + cz).$$

$$\frac{\partial u}{\partial y} = b\cos(ax + by + cz).$$

$$\frac{\partial u}{\partial z} = c\cos(ax + by + cz).$$

Ex. 2. If $u = \frac{xy}{x+y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$

$$\frac{\partial u}{\partial x} = \frac{(x+y)\,y - xy}{(x+y)^2} = \frac{y^2}{(x+y)^2}$$

Similarly $\frac{\partial u}{\partial y} = \frac{x^2}{(x+y)^2}$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{xy^2 + x^2y}{(x+y)^2} = \frac{xy}{x+y} = u.$$

Ex. 3. If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$$ (B.Sc. 1989)

$$\tan u = \frac{x^3+y^3}{x-y}$$

Differentiating w.r.t x alone,

$$\sec^2 u \frac{\partial u}{\partial x} = \frac{(x-y)\,3x^2 - (x^3+y^3)}{(x-y)^2}$$

$$= \frac{2x^3 - 3x^2y - y^3}{(x-y)^2}$$

Similarly, $\sec^2 u \dfrac{\partial u}{\partial y} = \dfrac{x^3 + 3xy^2 - 2y^3}{(x-y)^2}$

$$\therefore \quad \sec^2 u \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= \frac{x(2x^3 - 3x^2y - y^3) + y(x^3 + 3xy^2 - 2y^3)}{(x-y)^2}$$

$$= 2\,\frac{x^3+y^3}{x-y}$$

$$= 2\tan u.$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u\cos^2 u = \sin 2u.$$

Ex. 4. If $V = (x^2+y^2+z^2)^{-1/2}$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \qquad \text{(B.Sc. 1986)}$$

Differentiating V with respect to x alone, we get

$$\frac{\partial V}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\,2x.$$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

Differentiating once again with respect to x alone,

$$\frac{\partial^2 V}{\partial x^2} = \frac{3}{2}x(x^2+y^2+z^2)^{-5/2}\,2x - (x^2+y^2+z^2)^{-3/2}$$

$$= \frac{2x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly $\dfrac{\partial^2 V}{\partial y^2} = \dfrac{2y^2 - z^2 - x^2}{(x^2 + y^2 + z^2)^{5/2}}$

$$\frac{\partial^2 V}{\partial z^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

(V is said to satisfy Laplace's equation.)

Ex. 5. Illustrate the theorem that $\dfrac{\partial^2 u}{\partial x \, . \, \partial y} = \dfrac{\partial^2 u}{\partial y \, . \, \partial x}$ when u is equal to $\log \dfrac{x^2 + y^2}{xy}$. (B.Sc. 1988)

$$u = \log \frac{x^2 + y^2}{xy} = \log (x^2 + y^2) - \log x - \log y.$$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2} - \frac{1}{x}$$

$$\frac{\partial^2 u}{\partial y \, . \, \partial x} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2 + y^2} - \frac{1}{x}\right) = -\frac{4xy}{(x^2 + y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{1}{y}$$

$$\frac{\partial^2 u}{\partial x \, . \, \partial y} = \frac{\partial}{\partial x}\left(\frac{2y}{x^2 + y^2} - \frac{1}{y}\right) = -\frac{4xy}{(x^2 + y^2)^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial y \, . \, \partial x} = \frac{\partial^2 u}{\partial x \, . \, \partial y}$$

Exercises 32

1. If $u = \log (x^3 + y^3 + z^3 - 3xyz)$, show that

(i) $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3}{x + y + z}$. (B.Sc. 1986)

§ 1.3. Total differential coefficient.

If u be a continuous function of x and y if x and y receive small increments, Δx and Δy (which are usually quite independent of one another), u will receive in turn a small increment Δu. Then $\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y)$.

This quantity Δu is called the total increment of u.

$$\Delta u = f(x+\Delta x, y + \Delta y) - f(x,y+\Delta y)+f(x,y+\Delta y)-f(x,y).$$

Applying the theorem of mean value to each of the two differences on the right - hand side,

$$f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y) = f_x'(x + \theta_1 \Delta x,y+\Delta y)\Delta x,$$

$$f'(x, y + \Delta y) - f(x, y) = f_y'(x, y + \theta_2 \Delta y)\Delta y,$$

where f_x' and f_y' denote the partial differential coefficients with respect to x and y respectively and where θ_1 and θ_2 are positive fractions.

$$\therefore \Delta u = f_x'(x+\theta_1 \Delta x_1, y+\Delta y)\Delta x+f_y'(x,y+\theta_2 \Delta y)\Delta y.$$

If x and y and therefore also u are continuous functions of some other variable t and if Δx, Δy and Δu be the increments of x,y and u due to an increment Δt of t, dividing Δu by Δt, we get

$$\frac{\Delta u}{\Delta t} = f_x'(x+\theta_1 \Delta x,y+\Delta y)\frac{\Delta x}{\Delta t} + f_y'(x,y+\theta_2 \Delta y)\frac{\Delta y}{\Delta t}$$

Now let $\Delta t \to 0$.

Then $\dfrac{du}{dx} = f_x'(x,y)\dfrac{dx}{dt} + f_y'(x,y)\dfrac{dy}{dt}$.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}.$$

In the differential form, this can be written as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

du is called the total differential of u.

In the same way, if $u = f(x,y,z)$ and x, y, z are all functions of t, we get

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}.$$

And similarly if $u = f(x_1, x_2, \ldots\ldots x_n)$, where $x_1, x_2, \ldots\ldots x_3$ are known functions of a variable t, we have the relation.

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\cdot\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{dt} + \ldots\ldots + \frac{\partial u}{\partial x_n}\cdot\frac{dx_n}{dt}$$

or $$du = \frac{\partial x}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + \ldots\ldots + \frac{\partial u}{\partial x_n}dx_n.$$

§ 1.4. A special case.

If $u = f(x, y)$ where x and y are functions of t, we get

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}.$$

If we take t to be x, we get u as a function of x and y, where y is a function of x.

Since $\frac{dx}{dx}$ is now unity, this relation becomes

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}.$$

The quantities $\frac{du}{dx}$ and $\frac{\partial u}{\partial x}$ are quite distinct. For example, let $u = x^2 + 2xy + y^2$ and let y be a function of x.

Then $\frac{\partial u}{\partial x} = 2x + 2y$; $\frac{\partial u}{\partial y} = 2x + 2y$.

$\therefore \quad \frac{du}{dx} = 2x + 2y + (2x + 2y)\frac{dy}{dx}$ and the value of $\frac{dy}{dx}$ will depend on the relation between x and y.

§ 1.5. Implicit functions.

If the relation between x and y be given in the form $f(x,y) = c$, where c is a constant, then the total differential coefficient with respect to x is zero, since the differential coefficient of a constant is zero ; hence $\quad 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}.$

(40)

$$\therefore \quad \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

This gives an alternate method of finding the differential coefficient of y with respect to x when y is given as an implicit function of x.

Examples.

Ex. 1. Find $\frac{du}{dt}$ where $u = x^2+y^2+z^2$, $x = e^t$, $y = e^t \sin t$ and $z = e^t \cos t$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}.$$

$$= 2xe^t + 2y\,(e^t \sin t + e^t \cos t) + 2z\,(e^t \cos t - e^t \sin t)$$

$$= 2e^t\,(x + y \sin t + y \cos t + z \cos t - z \sin t)$$

$$= 2e^t\,(e^t + e^t \sin^2 t + e^t \sin t \cos t + e^t \cos^2 t - e^t \sin t \cos t)$$

$$= 2e^t \cdot 2e^t$$

$$= 4e^{2t}.$$

Ex. 2. Find $\frac{du}{dx}$ when $u = x^2 + y^2$ where $y = \frac{1-x}{x}$.

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$= 2x + 2y \frac{d}{dx}\left(\frac{1-x}{x}\right)$$

$$= 2x - \frac{2y}{x^2}$$

$$= 2x - \frac{2\,(1-x)}{x^3}$$

$$= \frac{2(x^4 + x - 1)}{x^3}.$$

Ex. 3. If $x^3 + y^3 + 3axy$, find $\frac{dy}{dx}$.

$x^3 + y^3 - 3axy = 0$, i.e., $f(x,y) = 0$.

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$\therefore \frac{dy}{dx} = -\frac{3x^2 - 3ay}{3y^2 - 3ax}$$

$$= -\frac{x^2 - ay}{y^2 - ax}$$

§ 1.6. Homogeneous functions.

Let us consider the function

$f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots + a_{n-1} x y^{n-1} + a_n y^n$

In this expression the sum of the indices of the variables x and y in each term is n. Such an expression is called a homogeneous function of degree n. This expression can be written as follows :-

$$f(x, y) = x^n \left(a_0 + a_1 \frac{y}{x} + a_2 \frac{y^2}{x^2} + \ldots + a_n \frac{y^n}{x^n}\right)$$

$$= x^n \text{ (a function of } \frac{y}{x})$$

$$= x^n F\left(\frac{y}{x}\right)$$

Similarly, a homogenous function of degree n consisting of m variables $x_1, x_2, \ldots x_m$ can be written as $x_r^n F\left(\frac{x_1}{x_r}, \frac{x_2}{x_r}, \ldots, \frac{x_m}{x_r}\right)$

Scanned with CamScanner

(42)

Euler's Theorem.

If $f(x, y)$ is a homogeneous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$$

This is known as Eulier's Theorem on homogenous functions.

Let us assume that

$$f(x, y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots a_n y^n$$

$$= x^n F\left(\frac{y}{x}\right)$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^n F\left(\frac{y}{x}\right)\right] = nx^{n-1} F\left(\frac{y}{x}\right) - x^n F'\left(\frac{y}{x}\right) \cdot \frac{y}{x^2}$$

$$= nx^{n-1} F\left(\frac{y}{x}\right) - x^{n-2} y F'\left(\frac{y}{x}\right).$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^n F\left(\frac{y}{x}\right)\right] = x^n F'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = x^{n-1} F'\left(\frac{y}{x}\right).$$

$$\therefore x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n F\left(\frac{y}{x}\right) - x^{n-1} y F'\left(\frac{y}{x}\right) + x^{n-1} y F'\left(\frac{y}{x}\right)$$

$$= nx^n F\left(\frac{y}{x}\right)$$

$$= nf.$$

In general if $f(x_1, x_2, \ldots x_m)$ is a homogeneous function of degree n, then $x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \ldots + x_m\frac{\partial f}{\partial x_m} = nf.$

Example 1.

Verify Euler's Theorem when $u = x^3+y^3+z^3+3xyz$. (B.Sc.1990)

$$\frac{\partial u}{\partial x} = 3x^2 + 3yz.$$

$$\frac{\partial u}{\partial y} = 3y^2 + 3zx.$$

$$\frac{\partial u}{\partial z} = 3z^2 + 3xy.$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x(3x^2 + 3yz) + y(3y^2 + 3zx)$$

$$+ z(3z^2 + 3xy)$$

$$= 3(x^3 + y^3 + z^3 + 3xyz)$$

$$= 3u.$$

Example 2.

If $u = \tan^{-1}\dfrac{x^3 + y^3}{x - y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$.

(B.Sc. 1990)

Now $\tan u = \dfrac{x^3 + y^3}{x - y} = x^2 \dfrac{1 + \left(\dfrac{y}{x}\right)^3}{1 - \left(\dfrac{y}{x}\right)} = x^2 f\left(\dfrac{y}{x}\right)$, which is a homogeneous function of degree 2.

Let $v = \tan u$. Then v is a homogeneous function of x and y of degree 2.

$$\therefore \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2v$$

i.e., $x\dfrac{\partial}{\partial x}(\tan u) + y\dfrac{\partial}{\partial y}(\tan u) = 2\tan u.$

i.e., $x\sec^2 u\dfrac{\partial u}{\partial x} + y\sec^2 u\dfrac{\partial u}{\partial y} = 2\tan u.$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = \sin 2u.$$

(See example 3, page ....).

§ 1.7. Partial derivatives of a function of two functions.

Let $V = F(u, v)$ where $u = f(x, y)$, $v = f_1(x, y)$ and x, y are independent variables.

If we write V in the form F { f (x,y) , f, (x,y) }, we can obtain $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}$ by the ordinary rules of partial differentiation but is usually done without substitution.

By definition since x, y are independent

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy \quad (1)$$

u is a function of x and y.

$$\therefore \quad du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad (2)$$

v is function of x and y.

$$\therefore \quad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \quad (3)$$

V is a function of u and v.

$$\therefore \quad dV = \frac{\partial V}{\partial u}du + \frac{\partial V}{\partial v}dv \quad (4)$$

Substituting the values of du and dv from (2) and (3) in (4), we get,

$$dV = \frac{\partial V}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + \frac{\partial V}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)$$

$$= \left(\frac{\partial V}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial V}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial y}\right)dy \quad (5)$$

Comparing (1) and (5), we get

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial V}{\partial v}\cdot\frac{\partial v}{\partial y}$$

These results may be expressed by saying that the operators

$\frac{\partial}{\partial x}$ and $\left(\frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\cdot\frac{\partial}{\partial v}\right)$ are equivalent.

(45)

Similarly $\frac{\partial}{\partial y} = \frac{\partial u}{\partial y} \cdot \frac{\partial}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial}{\partial v}$.

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)$$

$$= \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial V}{\partial x}\right)$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right)$$

$$= \left(\frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\right)\left(\frac{\partial V}{\partial y}\right)$$

In this way, it is possible to express higher partial derivatives.

Examples.

Ex. 1. If $z = f(x, y)$ and $x = r\cos\theta, y = r\sin\theta$, prove that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$ (B.Sc. 1990)

$$x = r\cos\theta.$$

$$\therefore \frac{\partial x}{\partial r} = \cos\theta ; \frac{\partial x}{\partial \theta} = -r\sin\theta.$$

$$y = r\sin\theta.$$

$$\therefore \frac{\partial y}{\partial r} = \sin\theta ; \frac{\partial y}{\partial \theta} = r\cos\theta.$$

Hence $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$

$$= \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}$$

and $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$

$$= -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y}$$

$$\therefore \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Hence the result.

Ex. 2. Transform $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$ into polar coordinates.

(B.Sc. 1993)

We have $x = r\cos\theta, y = r\sin\theta$

and $\quad r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}.$

$$\therefore \ 2r\frac{\partial r}{\partial x} = 2x. \qquad \therefore \ \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$$

and $\sec^2\theta\frac{\partial \theta}{\partial x} = -\frac{y}{x^2}. \qquad \therefore \ \frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}.$

$$\therefore \ \frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}$$

Thus $\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$

$$\therefore \ \frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)$$

$$= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial V}{\partial \theta}\right)$$

$$- \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$= \cos\theta\left[\cos\theta\frac{\partial^2 V}{\partial r^2} - \frac{\sin\theta}{r}\frac{\partial^2 V}{\partial r\,\partial\theta} + \frac{\sin\theta}{r^2}\frac{\partial V}{\partial\theta}\right]$$

$$-\frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial V}{\partial r} + \cos\theta\frac{\partial^2 V}{\partial\theta\partial r} - \frac{\cos\theta}{r}\frac{\partial V}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 V}{\partial\theta^2}\right]$$

$$\therefore\ \frac{\partial^2 V}{\partial x^2} = \cos^2\theta\frac{\partial^2 V}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\,\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2}$$

$$+\frac{\sin^2\theta}{r}\frac{\partial V}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial\theta}$$

assuming that $\frac{\partial^2 V}{\partial r\,.\,\partial\theta} = \frac{\partial^2 V}{\partial\theta\,.\,\partial r}$

To get $\frac{\partial}{\partial y}$, we note that we change θ in $\frac{\partial}{\partial x}$ to $\frac{\pi}{2} - \theta$

Hence $\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$

Similarly, $\frac{\partial^2 V}{\partial y^2}$ can be found from $\frac{\partial^2 V}{\partial x^2}$ by replacing θ by $\frac{\pi}{2} - \theta$. This gives

$$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta\frac{\partial^2 V}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\,\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2}$$

$$+\frac{\cos^2\theta}{r}\frac{\partial V}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial\theta}$$

$$\therefore\ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}$$

Alternate method.

We have shown that $\frac{\partial V}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial\theta}$

and $\frac{\partial V}{\partial y} = \sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial\theta}$

(48)

We have $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = \left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)\left(\dfrac{\partial V}{\partial x} + i\dfrac{\partial V}{\partial y}\right)$

$$\frac{\partial V}{\partial x} + i\frac{\partial V}{\partial y} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta} + i\left(\sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$= (\cos\theta + i\sin\theta)\frac{\partial V}{\partial r} + \frac{i}{r}(\cos\theta + i\sin\theta)\frac{\partial V}{\partial \theta}$$

$$= e^{i\theta}\left(\frac{\partial V}{\partial r} + \frac{i}{r}\frac{\partial V}{\partial \theta}\right)$$

Similarly $\dfrac{\partial V}{\partial x} - i\dfrac{\partial V}{\partial y} = e^{-1\theta}\left(\dfrac{\partial V}{\partial r} - \dfrac{i}{r}\dfrac{\partial v}{\partial \theta}\right)$

Hence $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = e^{-i\theta}\left(\dfrac{\partial}{\partial r} - \dfrac{i}{r}\dfrac{\partial}{\partial \theta}\right)e^{i\theta}\left(\dfrac{\partial V}{\partial r} + \dfrac{i}{r}\dfrac{\partial V}{\partial \theta}\right)$

$$= e^{-i\theta}\left[\frac{\partial}{\partial r}\left(e^{i\theta}\frac{\partial V}{\partial r}\right) + e^{i\theta}\frac{\partial}{\partial r}\left(\frac{i}{r}\frac{\partial V}{\partial \theta}\right)\right]$$

$$- e^{-i\theta}\cdot\frac{i}{r}\left[\frac{\partial}{\partial \theta}\left(e^{i\theta}\frac{\partial V}{\partial r}\right) + \frac{i}{r}\frac{\partial}{\partial \theta}\left(e^{i\theta}\frac{\partial V}{\partial \theta}\right)\right]$$

$$= e^{-i\theta}\left[e^{i\theta}\frac{\partial^2 V}{\partial r^2} + ie^{i\theta}\left(-\frac{1}{r^2}\frac{\partial V}{\partial \theta} + \frac{1}{r}\frac{\partial^2 V}{\partial r\partial \theta}\right)\right]$$

$$- \frac{i}{r}e^{-i\theta}\left[ie^{i\theta}\frac{\partial V}{\partial r} + e^{i\theta}\frac{\partial^2 V}{\partial \theta . \partial r} + \frac{i}{r}\left(ie^{i\theta}\frac{\partial V}{\partial \theta} + e^{i\theta}\frac{\partial^2 V}{\partial \theta^2}\right)\right]$$

$$= \frac{\partial^2 V}{\partial r^2} - \frac{i}{r^2}\frac{\partial V}{\partial \theta} + \frac{i}{r}\frac{\partial^2 V}{\partial r\,\partial \theta} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{i}{r}\frac{\partial^2 V}{\partial \theta\,\partial r} + \frac{i}{r^2}\frac{\partial V}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}$$

$$= \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} \quad \text{since } \frac{\partial^2 V}{\partial \theta . \partial r} = \frac{\partial^2 V}{\partial r . \partial \theta}.$$

— x — x — x —

Unit - 3 is Completed

Unit – IV

INTEGRAL CALCULUS

CHPTER 1

INTEGRATION

§ 1.1 We have so far considered the problem of differentiation, viz., being given $y = f(x)$, find $\frac{dy}{dx}$. Now we pass on to the process, called integration, which may be regarded as the inverse of differentiation.

The problem is : Given $\frac{dy}{dx} = f(x)$, find y in terms of x.

This process of finding y is called integration. We write symbolically that $y = \int f(x)\, dx$. $\int$ is the sign of integration and the above statement is read as : integral of $f(x)$ with respect to x or shortly "integral $f(x)$ dx"*. $f(x)$ is called the integrand, x is called the variable of integration. Hence $\int f(x)\, dx$ is called the indefinite integral of $f(x)$ with respect to x and is to be distinguished from the definite integral to be explained in the succeeding pages. Hence by definition, the problem of evaluating $\int f(x)\, dx$ is to find $F(x)$, a function of x whose derivative with respect to x shall be the integrand $f(x)$ i.e., $F'(x) = f(x)$.

An alternate method of defining an integral is to look upon it as the limit of a sum of a certain series. Many useful applications of Calculus depend on this method but is is not convenient as the former in evaluating integrals. So we shall start with the first definition and consider its application to calculation of the various forms of integrals and then study the second definition in detail.

§1.2. Take for example $\int 2x\, dx$. We know that $\frac{d}{dx}(x^2) = 2x$.

Hence by the first definition of an integral $\int 2x\, dx = x^2$. We may also add an arbitrary constant c to x^2 as $\frac{d}{dx}(x^2 + c) = 2x$.

---

* Strictly speaking, we find only the 'primitive' of f (x) but call it the 'integral of f (x) in a loose sense. There is a distinction between the two terms which we shall not go into here.

Scanned with CamScanner

Hence $\int 2x\,dx = x^2 + c$. As the arbitrary constant of integration is present, this integral is called an indefinite integral.

Similarly $\int \sec x \tan x\,dx = \sec x + c$

as $\frac{d}{dx}(\sec x + c) = \sec x \tan x$.

Thus in the above process, we depend on our knowledge of differentiation to guess at the function except for a numerical arbitrary constant which leads to the integrand by differentiation.

§ 2. The following list of formulae for integrals is based directly on the results of differentiation which have been studied earlier. It is necessary to commit them to memory.

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ for all values of n except when $n = -1$.

2. In the case when $n = -1$, $\int \frac{dx}{x} = \log x + c$.

(Hereafter, we shall take the constant c to be understood after the integral.)

3. $\int e^x\,dx = e^x$.

4. $\int \sin x\,dx = -\cos x$.

5. $\int \cos x\,dx = \sin x$.

6. $\int \sec^2 x\,dx = \tan x$.

7. $\int \operatorname{cosec}^2 x\,dx = -\cot x$

8. $\int \sec x \tan x\,dx = \sec x$

9. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x$

10. $\int \cosh x\,dx = \sinh x$.

11. $\int \sinh x\,dx = \cosh x$

12. $\int \frac{dx}{1+x^2} = \tan^{-1} x$, or $-\cot^{-1} x$

13. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$, or $-\cos^{-1} x$

14. $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x$, $\log(x + \sqrt{x^2-1})$

15. $\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x$, $\log(x + \sqrt{x^2+1})$

16. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x$, or $-\operatorname{cosec}^{-1} x$

§ 3. Before we proceed to systematic methods of integration, we give below a few important results which can be easily proved from the definition of integration.

(1) $\int c\,f(x)\,dx = c\int f(x)\,dx$, where c is a constant.

(2) $\int (u \pm v)\,dx = \int u\,dx \pm \int v\,dx$ where u and v are functions of x.

Exercises 1.

Integrate the following with respect to x :-

1. x^{-4}
2. $x^{2/3}$
3. $ax + \frac{b}{x^2}$
4. $\frac{ax^2 + bx + c}{x^3}$
5. $\frac{ax^{-2} + bx^{-1} + c}{x^{-4}}$
6. $\left(x + \frac{1}{x}\right)^2$
7. $(x^{2/3} - x^{-3/2})^2$.
8. $x^2(1-x)^2$.
9. $\frac{(x+1)^4}{x^2}$
10. $\frac{(1-x^2)^2}{x}$

Scanned with CamScanner

11. $(x^2 - x^{-3/2})^2$

12. $\frac{x+3}{x\sqrt{x}}$

13. $\frac{3x^2 + 4x - 5}{\sqrt{x}}$

14. $\frac{(x^2 + 4x)(2x - 3)}{x^3}$

15. $\tan^2 x$

16. $\cot^2 x$.

17. $(\tan x - 2\cot x)^2$

18. $\frac{1}{\sin^2 x \cos^2 x}$. $\left(Hint : \frac{1}{\sin^2 x \cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\right)$

19. $\frac{\sin^2 x}{1 + \cos x}$

20. $\frac{\cos^2 x}{1 - \sin x}$

21. $\frac{3}{\sqrt{1 - x^2}} + e^x + 8$.

22. $\sqrt{1 + \sin 2x}$

23. $\frac{1}{1+\sin x}$ $\left(Hint : \frac{1}{1+\sin x} = \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \tan x \sec x\right)$.

24. $\frac{1}{1 - \sin x}$

25. $\frac{1}{1 + \cos x}$.

26. $\frac{1}{1 - \cos x}$.

§ 4. Definite Integral.

Let $\int f(x)\,dx = F(x) + c$ where c is the arbitrary constant on integration. The value of the integral when $x = b$ is $F(b) + c$ and when $x = a$, the value is $F(a) + c$

Subtracting

$F(b) - F(a)$ = the value of the integral when $x = b$

– the value of the integral when $x = a$.

The symbol $\int_a^b f(x)\,dx$ denotes the value of the integral when $x = b$, minus the value of the integral when $x = a$ and is thus $F(b) - F(a)$. $\int_a^b f(x)\,dx$ is called the definite integral; a and b are called the limits of integration, a being the lower limit and b the upper limit.

Note :- $\int_a^b f(x)\,dx$ is a definite constant unlike $\int_a^x f(x)\,dx$ which is a function of the variable x. $\int_a^x f(x)\,dx$ is called an indefinite integral in quite a different sense. The upper limit here is x, a variable and not a constant. For this reason this integral is called an indefinite integral.

Rule to find $\int_a^b f(x)\,dx$.

Evaluate the indefinite integral of $f(x)$ with respect to x. Let it be $F(x)$. Subtract the value of $F(x)$ when $x = a$ from its value when $x = b$. The result obtained is $\int_a^b f(x)\,dx$.

Examples.

Ex. 1. $\int_1^2 \left(x^2 - 3x^{1/2} + \frac{1}{x^2}\right) dx$

$$= \left[\frac{x^3}{3} - 2x^{3/2} - \frac{1}{x}\right]_1^2$$

$$= \left(\frac{8}{3} - 4\sqrt{2} - \frac{1}{2}\right) - \left(\frac{1}{3} - 2 - 1\right)$$

$$= \frac{29}{6} - 4\sqrt{2}.$$

Scanned with CamScanner

Ex. 2. $\int_0^{\pi/6} \cos^2 \frac{x}{2}\, dx = \frac{1}{2}\int_0^{\pi/6} (1 + \cos x)\, dx$

$$= \frac{1}{2}\left[x + \sin x\right]_0^{\pi/6}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{6} + \sin\frac{\pi}{6}\right) - 0\right]$$

$$= \frac{\pi}{12} + \frac{1}{4}.$$

§ 5. Methods of Integration.

The various rules in the differential calculus enable us to differentiate almost any combination of the various ordinary functions. But it is not so with integration. In fact the integrals of some even fairly simple functions cannot be found in terms of the functions which are known to the students at this stage. For example, $(a + b\sin^2 x)^{1/2}$, $\sqrt{1 - x^3}$, $\sqrt{\sin x}$, $\frac{\cos x}{x}$ cannot be integrated in terms of functions which are known.

Corresponding to the various rules in the differential calculus for differentiating sums, products and functions of functions, we have more or less similar rules in the integral calculus. These give rise to the following methods of integration :-

(1) Substitution.

(2) Decomposition into a sum.

(3) Integration by parts.

(4) Successive reduction.

§ 6.1. The efficacy of the method of substitution depends on finding a suitable substitution to convert the given integral into a standard form. The form of the integrand often suggests the proper substitution.

To evaluate $\int f(x)\, dx$, we put $x = \varphi(t)$

$$\frac{dx}{dt} = \varphi'(t) \text{ or } dx = \varphi'(t)\, dt.$$

Then $\int f(x)\, dx = \int f\{\varphi(t)\}\, \varphi'(t)\, dt$

To prove this, $\frac{d}{dx}$ (left - hand side) = $f(x)$ by definition and $\frac{d}{dx}$ (right - hand side) = $\frac{d}{dt}$ (right - hand side) x $\frac{dt}{dx}$

$$= f\{\varphi(t)\}\, \varphi'(t) \times \frac{1}{\varphi'(t)}$$

$$= f(x)$$

Hence the result.

§ 6.2. Integrals of functions containing linear functions of x

i.e., $f(ax + b)$.

Put $ax + b = t$. $\therefore\ a\, dx = dt$.

$$\therefore \int f(ax + b)\, dx = \int f(t)\,.\, \frac{1}{a}\, dt$$

$$= \frac{1}{a}\int f(t)\, dt \text{ which can be evaluated}$$

Examples.

Ex. 1. (i) $\int (ax + b)^n\, dx\ (n \neq -1)$

Put $t = ax + b$, then $dt = a\, dx$.

$$\therefore \int (ax + b)^n\, dx = \frac{1}{a}\int t^n\, dt = \frac{t^{n+1}}{a(n+1)}$$

$$= \frac{(ax + b)^{n+1}}{a(n+1)}$$

Similarly

(ii) $\int \frac{dx}{ax + b} = \frac{1}{a}\log(ax + b)$

Scanned with CamScanner

(iii) $\int e^{ax+b}\,dx = \frac{1}{a} e^{ax+b}$

(iv) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)$

(v) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)$

(vi) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)$.

(vii) $\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)$.

(viii) $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)$.

(ix) $\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b)$.

Ex. 2. Evaluate $\int \frac{x^2}{(a+bx)^3}\,dx$

Put $a+bx = t$. $\quad b\,dx = dt$

$$\therefore \frac{x^2}{(a+bx)^3}\,dx = \int \frac{\left(\frac{t-a}{b}\right)^2 \frac{1}{b}\,dt}{t^3}$$

$$= \frac{1}{b^3}\int \frac{(t-a)^2}{t^3}\,dt$$

$$= \frac{1}{b^3}\int \left(\frac{1}{t} - \frac{2a}{t^2} + \frac{a^2}{t^3}\right) dt$$

$$= \frac{1}{b^3}\log t + \frac{2a}{b^3}\cdot\frac{1}{t} - \frac{a^2}{2b^3}\cdot\frac{1}{t^2}$$

$$= \frac{1}{b^3}\log(a+bx) + \frac{2a}{b^3}\cdot\frac{1}{a+bx} - \frac{a^2}{2b^3}\cdot\frac{1}{(a+bx)^2}$$

Ex. 3. Evaluate $\int \cos mx \cos nx\,dx$

Case (i). $m \neq n$.

$$\int \cos mx \cos nx\,dx = \frac{1}{2}\int \{\cos(m+n)x + \cos(m-n)x\}\,dx$$

$$= \frac{1}{2}\left\{\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right\}$$

Case (ii). $m = n$.

The integral is $\int \cos^2 mx\,dx = \frac{1}{2}\int (1+\cos 2mx)\,dx$

$$= \frac{1}{2}\left(x + \frac{\sin 2mx}{2m}\right)$$

Similarly

$$\int \sin mx \sin nx\,dx = -\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)}$$

if $m \neq n$.

Ex . 4. Evaluate $\int \sin^2 3x\,dx$.

$$\int \sin^2 3x\,dx = \frac{1}{2}\int (1-\cos 6x)\,dx = \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right)$$

Ex. 5. Evaluate $\int \cos^3 x\,dx$.

$$\int \cos^3 x\,dx = \int \frac{\cos 3x + 3\cos x}{4}\,dx$$

$$= \frac{1}{12}\sin 3x + \frac{3}{4}\sin x$$

Ex. 6. Evaluate $\int \sin^4 x\,dx$.

10

$$\int \sin^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx$$

$$= \int \left(\frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \right) dx$$

$$= \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x$$

Exercises 2.

Integrate the following expressions :-

1. x^3 , $(2 + x)^3$, $(3 - 2x)^3$, $(3x - 4)^3$, $(ax + b)^3$, $(b - ax)^3$.

2. x^n , $(x+a)^n$, $(3x+2)^n$, $(3 - 2x)^n$, $(ax - b)^n$, $(b-ax)^n$.

3. $\sqrt{x}$, $\sqrt{1 + x}$, $\sqrt{2 + 3x}$, $\sqrt{4 - 5x}$, $\sqrt{a + bx}$, $\left(a + \frac{bx}{c}\right)^{1/2}$.

4. $\frac{1}{x^4}$, $\frac{1}{(x+7)^4}$, $\frac{1}{(2x+3)^4}$, $\frac{1}{(4-3x)^4}$, $\frac{1}{(a+bx)^4}$, $\frac{1}{(a-bx)^4}$

5. $\frac{1}{x^{3/2}}$, $\frac{1}{(2x + 1)^{3/2}}$, $\frac{1}{(3x - 4)^{3/2}}$, $\frac{1}{(2 - 3x)^{3/2}}$,

$\frac{1}{(a + bx)^{3/2}}$, $\frac{1}{(a - bx)^{3/2}}$

6. $\frac{1}{2x}$, $\frac{1}{ax}$, $\frac{1}{3x + 7}$, $\frac{1}{2 - 7x}$, $\frac{1}{ax + b}$, $\frac{1}{b - ax}$.

7. e^{4x} , e^{3x+7} , e^{2-3x} , $e^{(x-4)/3}$, e^{ax+b}

8. $\sin 2x$, $\sin \frac{x}{2}$, $\sin (2x + 3)$, $\sin (3 - 2x)$

9. $\cos 3x$, $\cos \frac{x}{3}$, $\cos (3x + 2)$, $\cos (2 - 3x)$

10. $\sec^2 4x$, $\sec^2 \frac{x}{4}$, $\sec^2 (4x - 7)$, $\sec^2 (7 - 4x)$

§ 6.5. Integrals of functions of the form

$\int \{f(x)\}^n f'(x)\,dx.$

When $n \neq -1$, put $f(x) = t$, then $f'(x)\,dx = dt$.

$$\therefore \int \{f(x)\}^n f'(x)\,dx = \int t^n\,dt = \frac{t^{n+1}}{n+1}$$

$$= \frac{\{f(x)\}^{n+1}}{n+1}.$$

When $n = -1$, the integral reduces to

$$\int \frac{f'(x)}{f(x)}\,dx.$$

Putting $y = f(x)$, the above integral reduces to

$$\int \frac{dy}{y} = \log y = \log f(x).$$

Examples.

Ex. 1. $\int \sqrt{x^2 + a^2}\; x\,dx.$

Here the derivative of $x^2 + a^2$ is $2x$.

Hence put $x^2 + a^2 = t$. $\therefore$ $2x\,dx = dt$.

$$\int \sqrt{x^2 + a^2}\; x\,dx = \frac{1}{2}\int \sqrt{t}\,dt = \frac{1}{3}\,t^{3/2}$$

$$= \frac{1}{3}(x^2 + a^2)^{3/2}$$

Ex. 2. $\int \frac{\sin^{-1} x}{\sqrt{1 - x^2}}\,dx$. Putting $t = \sin^{-1} x$,

$$dt = \frac{1}{\sqrt{1 - x^2}}\,dx$$

$\therefore$ The integral reduces to $\int t\,dt = \frac{t^2}{2} = \frac{1}{2}(\sin^{-1} x)^2$.

Ex. 3. $\int \tan\theta\,d\theta = \int \frac{\sin\theta}{\cos\theta}\,d\theta$

$$= -\int \frac{dy}{y} \text{ on putting } y = \cos\theta$$

$$= -\log y$$

$$= -\log\cos\theta = \log(\sec\theta)$$

Ex. 4. $\int \cot\theta\,d\theta = \int \frac{\cos\theta}{\sin\theta}\,d\theta$

$$= \frac{dy}{y} \text{ on putting } y = \sin\theta$$

$$= \log y = \log\sin\theta.$$

Ex. 5. $\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx$

$$= \int \frac{dy}{y} \text{ where } y = \sec x + \tan x$$

$$= \log y = \log(\sec x + \tan x)$$

$$= \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

Ex. 6. $\int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x + \cot x)\,dx}{(\operatorname{cosec} x + \cot x)}$

$$= -\int \frac{d(\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)}$$

$$= -\log(\operatorname{cosec} x + \cot x)$$

$$= \log\tan\frac{x}{2}.$$

Scanned with CamScanner

Examples.

Ex. 1. $\int x^2\sqrt{1-4x^3}\,dx$. Putting $1-4x^3=y$,

$$-12x^2\,dx = dy$$

The integral $= -\int \frac{\sqrt{y}}{12}\,dy = -\frac{y^{3/2}}{18} = -\frac{(1-4x^3)^{3/2}}{18}$

Ex. 2. $\int \frac{e^x}{e^{x/2}-1}\,dx$

$= \int \frac{e^{x/2}\cdot e^{x/2}\,dx}{e^{x/2}-1}$. Put $y = e^{x/2}-1$;

$$dy = \frac{1}{2}e^{x/2}\,dx$$

$= \int \frac{(y+1)\,2\,dy}{y}$

$= 2\int\left(1+\frac{1}{y}\right)dy$

$= 2(y+\log y)$

$= 2\left\{e^{x/2}-1+\log(e^{x/2}-1)\right\}$

Ex. 3. $\int \frac{dx}{(1+e^x)(1+e^{-x})}$

$= \int \frac{e^x\,dx}{(1+e^x)^2}$

$= \int \frac{dy}{y^2}$, where $y = 1+e^x$

$= -\frac{1}{y} = -\frac{1}{1+e^x}$

Ex. 4. $\int \frac{dx}{\sin x\cos^2 x} = \int \frac{\sin^2 x+\cos^2 x}{\sin x\cos^2 x}\,dx$

$= \int \frac{\sin x}{\cos^2 x}\,dx + \int \frac{dx}{\sin x}$

$= \int \tan x\sec x\,dx + \int \frac{dx}{\sin x}$

$= \sec x + \log\tan\frac{x}{2}$

Ex. 5. $\int \frac{\tan x\,dx}{\sec x+\cos x}$

$= \int \frac{\sin x\,dx}{1+\cos^2 x}$

$= -\int \frac{dt}{1+t^2}$, where $t = \cos x$

$= -\tan^{-1}(t) = -\tan^{-1}(\cos x)$

Ex. 6. $\int \frac{2\cos x+3\sin x}{4\cos x+5\sin x}\,dx$.

$\frac{d}{dx}$ (denominator) $= -4\sin x+5\cos x$.

Putting the numerator

$2\cos x+3\sin x = l(4\cos x+5\sin x)+m(-4\sin x+5\cos x)$

and equating the coefficients of $\sin x$ and $\cos x$,

we have $4l+5m=2$ and $5l-4m=3$.

$\therefore\ l = \frac{23}{41}$ and $m = -\frac{2}{41}$

Hence the integral reduces to

$\frac{23}{41}\int dx - \frac{2}{41}\int \frac{d(4\cos x+5\sin x)}{4\cos x+5\sin x}\,dx$

$= \frac{23}{41}x - \frac{2}{41}\log(4\cos x+5\sin x)$.

Ex. 7. $\int \frac{dx}{1+\tan x}$

$$= \int \frac{\cos x}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{d(\cos x + \sin x)}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}x + \frac{1}{2}\log(\sin x + \cos x).$$

Exercises 6.

Integrate the following expression :-

1. $\frac{\sin(\log x)}{x}$

2. $\frac{\sec^2(\log x)}{x}$

3. $\frac{e^{\tan^{-1} x}}{1+x^2}$.

4. $\frac{\sin(\tan^{-1} x)}{1+x^2}$

5. $\frac{\sin x}{1+9\cos^2 x}$

6. $\frac{\cos x}{4+\sin^2 x}$

7. $e^{\sin^2 x + \cos x}(\sin 2x - \sin x)$

8. $\frac{\sin x}{\sqrt{2-\cos^2 x}}$

9. $\frac{x+2}{\sqrt{1-x^2}}$

10. $\frac{1}{\sqrt{1-x^2}}\,\frac{1}{(\sin^{-1} x)^2}$.

11. $\sin^4 x \cos^3 x$

12. $\frac{\cos^3 x}{\sqrt{\sin x}}$

13. $\frac{\sin^3 x}{\cos^4 x}$

14. $\frac{\cos^5 x}{\sin^2 x}$

15. $(\sin x)^{5/2}\cos^3 x$

Type i : $\int \dfrac{dx}{ax^2 + bx + c}$

Then dividing the denominator by the coefficient of x^2 and completing the square of the term which contains x. the integral reduces to one of the three forms just mentioned.

Examples.

Ex. 1. $\int \dfrac{dx}{x^2+2x+5} = \int \dfrac{dx}{(x+1)^2+2^2} = \dfrac{1}{2}\tan^{-1}\left(\dfrac{x+1}{2}\right)$

Ex. 2. $\int \dfrac{dx}{4x^2 - 4x + 2} = \dfrac{1}{4}\int \dfrac{dx}{x^2 - x + \frac{1}{2}}$

$$= \frac{1}{4}\int \frac{dx}{(x - \frac{1}{2})^2 + \frac{1}{4}}$$

$$= \frac{1}{4}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x - ½}{\frac{1}{2}}\right)$$

$$= \frac{1}{8}\tan^{-1}(2x - 1).$$

Ex. 3. $\int \dfrac{dx}{x^2 + 8x - 7} = \int \dfrac{dx}{(x + 4)^2 - 23}$

$$= \frac{1}{2\sqrt{23}}\log\frac{x + 4 - \sqrt{23}}{x + 4 + \sqrt{23}}.$$

Ex. 4. $\int \dfrac{dx}{3x^2 - 4x - 5} = \dfrac{1}{3}\int \dfrac{dx}{x^2 - \frac{4}{3}x - \frac{5}{3}}$

$$= \frac{1}{3}\int \frac{dx}{\left(x - \frac{2}{3}\right)^2 - \frac{19}{9}}$$

Scanned with CamScanner

$$= \frac{1}{\frac{3 \cdot 2\sqrt{19}}{3}} \log \frac{x - \frac{2}{3} - \frac{\sqrt{19}}{3}}{x - \frac{2}{3} + \frac{\sqrt{19}}{3}}$$

$$= \frac{1}{2\sqrt{19}} \log \frac{3x - 2 - \sqrt{19}}{3x - 2 + \sqrt{19}}$$

Ex. 5. $\int \frac{dx}{1 + x - x^2} = \int \frac{dx}{\frac{5}{4} - \left(x - \frac{1}{2}\right)^2}$

$$= \frac{1}{\frac{2\sqrt{5}}{2}} \log \frac{\frac{\sqrt{5}}{2} + x - \frac{1}{2}}{\frac{\sqrt{5}}{2} - \left(x - \frac{1}{2}\right)}$$

$$= \frac{1}{\sqrt{5}} \log \frac{\sqrt{5} - 1 + 2x}{\sqrt{5} + 1 - 2x}$$

Ex. 6. $\int \frac{dx}{1 - 6x - 9x^2} = \frac{1}{9} \int \frac{dx}{1/9 - 2/3\, x - x^2}$

$$= \frac{1}{9} \int \frac{dx}{2/9 - (x + 1/3)^2}$$

$$= \frac{1}{9 \cdot 2 \cdot \sqrt{(2/9)}} \log \frac{\sqrt{(2/9)} + x + 1/3}{\sqrt{(2/9)} - x - 1/3}$$

$$= \frac{1}{6\sqrt{2}} \log \frac{\sqrt{2} + 3x + 1}{\sqrt{2} - 3x - 1}$$

Type ii : $\int \frac{lx + m}{ax^2 + bx + c}\, dx.$

If $ax^2 + bx + c$ has no rational factors, express the numerator as A (derivative of the denominator) + B and integrate each part separately. The method is outlined below :

Examples.

Ex. 1. $\int \frac{2x + 3}{x^2 + x + 1}\, dx$

$$\frac{d}{dx}(x^2 + x + 1) = 2x + 1.$$

Hence $\int \frac{2x + 3}{x^2 + x + 1}\, dx$

$$= \int \frac{(2x + 1)\, dx}{x^2 + x + 1} + 2 \int \frac{dx}{x^2 + x + 1}$$

$$= \log(x^2 + x + 1) + 2 \int \frac{dx}{(x + 1/2)^2 + 3/4}$$

$$= \log(x^2 + x + 1) + \frac{4}{\sqrt{3}} \tan^{-1} \frac{x + 1/2}{\sqrt{3}/2}$$

$$= \log(x^2 + x + 1) + \frac{4}{\sqrt{3}} \tan^{-1} \frac{2x + 1}{\sqrt{3}}$$

Ex. 2. $\int \frac{x + 4}{6x - 7 - x^2}\, dx$

Here $\frac{d}{dx}(6x - 7 - x^2) = -2x + 6.$

$$\therefore\ x + 4 = -\frac{1}{2}(-2x + 6) + 7$$

$$\int \frac{x + 4}{6x - 7 - x^2}\, dx$$

$$= -\frac{1}{2} \int \frac{-2x + 6}{6x - 7 - x^2}\, dx + 7 \int \frac{dx}{-7 - (x^2 - 6x)}$$

$$= -\frac{1}{2} \log(6x - 7 - x^2) + 7 \int \frac{dx}{2 - (x - 3)^2}$$

$$= -\frac{1}{2} \log(6x - 7 - x^2) + \frac{7}{2\sqrt{2}} \log \frac{\sqrt{2} + x - 3}{\sqrt{2} - x + 3}$$

20. Show that $\int_{-\infty}^{+\infty} \frac{dx}{a + 2bx + cx^2} = \frac{\pi}{\sqrt{ac - b^2}}$

21. Show that $\int_0^1 \frac{dx}{1 + 2x \cos\theta + x^2} = \frac{\theta}{2 \sin\theta}$.

§ 7.4. Rule (c) . If the denominator can be resolved into rational factors of the first or second degree, the method of partial fractions is to be used.

Examples.

Ex. 1. $\int \frac{dx}{x^2 - a^2}$

Let $\frac{1}{x^2 - a^2} = \frac{A}{x - a} + \frac{B}{x + a}$.

Then $1 \equiv A(x + a) + B(x - a)$

Putting $x = a$, $A = \frac{1}{2a}$ and $x = -a$, $B = -\frac{1}{2a}$.

$$\therefore \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \int \frac{dx}{x - a} - \frac{1}{2a} \int \frac{dx}{x + a} = \frac{1}{2a} \log \frac{x - a}{x + a}$$

Ex. 2. Similarly $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \int \frac{dx}{a - x} + \frac{1}{2a} \int \frac{dx}{a + x}$

$$= -\frac{1}{2a} \log(a - x) + \frac{1}{2a} \log(a + x)$$

$$= \frac{1}{2a} \log \frac{a + x}{a - x}$$

Ex. 3. $\int \frac{x^3}{(x - 1)(x - 2)} dx$.

[Here the degree of the numerator is higher than that of the denominator. Hence rule (a) is to be applied.]

Scanned with CamScanner

20

By division $\dfrac{x^3}{(x-1)(x-2)} = x + 3 + \dfrac{7x-6}{(x-1)(x-2)}$

Let $\dfrac{7x-6}{(x-1)(x-2)} \equiv \dfrac{A}{x-1} + \dfrac{B}{x-2}$

$\therefore\ 7x - 6 \equiv A(x-2) + B(x-1)$

Putting x = 1 and 2 in turn, A = − 1 and B = 8.

$\therefore \int \dfrac{x^3}{(x-1)(x-2)}\,dx = \int \left(x+3 - \dfrac{1}{x-1} + \dfrac{8}{x-2}\right) dx$

$= \dfrac{x^2}{2} + 3x - \log(x-1) + 8\log(x-2)$

Ex. 4. $\int \dfrac{3x+1}{(x-1)^2(x+3)}\,dx.$

Assume $\dfrac{3x+1}{(x-1)^2(x+3)} \equiv \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+3}$

$\therefore\ 3x + 1 \equiv A(x-1)(x+3) + B(x+3) + C(x-1)^2$

Put $x = 1$, B = 1 ; put $x = -3$, $C = -\dfrac{1}{2}$.

Put $x = 0$, $-3A + 3B + C = 1$ hence $A = \dfrac{1}{2}$

$\int \dfrac{3x+1}{(x-1)^2(x+3)}\,dx$

$= \dfrac{1}{2}\int \dfrac{dx}{x-1} + \int \dfrac{dx}{(x-1)^2} - \dfrac{1}{2}\int \dfrac{dx}{x+3}$

$= \dfrac{1}{2}\log(x-1) - \dfrac{1}{x-1} - \dfrac{1}{2}\log(x+3)$

$= \dfrac{1}{2}\log\dfrac{x-1}{x+3} - \dfrac{1}{x-1}$

21

Ex. 5. $\int \dfrac{2dx}{(1-x)(1+x^2)}$

Let $\dfrac{2}{(1-x)(1+x^2)} = \dfrac{A}{1-x} + \dfrac{Bx+C}{1+x^2}$

$\therefore\ 2 \equiv A(1+x^2) + (Bx + C)(1-x).$

Put $x = 1$, A = 1 ; put $x = 0$, A + C = 2, ∴ C = 1.

Put $x = -1$, 2A + 2(− B + C) = 2, ∴ B = 1

Hence the integrals is $\int \dfrac{dx}{1-x} + \int \dfrac{x+1}{x^2+1}\,dx$

$= -\log(1-x) + \int \dfrac{x}{x^2+1}\,dx + \int \dfrac{dx}{x^2+1}$

$= -\log(1-x) + \dfrac{1}{2}\log(x^2+1) + \tan^{-1} x$

Exercises 9.

Integrate

1. $\dfrac{1}{(x+1)(x+2)}$
2. $\dfrac{2x+3}{(2x+1)(1-3x)}$
3. $\dfrac{1}{2-3x+x^2}$
4. $\dfrac{x}{(x-1)(x-2)(x-3)}$
5. $\dfrac{x^2+11x+14}{(x+3)(x^2-4)}$
6. $\dfrac{x^2+1}{(x^2-1)(2x+1)}$
7. $\dfrac{10x-21}{(2x-3)(2x+5)}$
8. $\dfrac{1-4x^2}{x(1-4x)}$
9. $\dfrac{x^2-1}{x^2-4}$
10. $\dfrac{x}{(x-1)^2(x+2)}$

37. $\dfrac{1}{3\sin x + \sin 2x}$

38. Evaluate $\displaystyle\int_1^{\infty} \frac{dx}{x^2(x+1)}$

39. Evaluate $\displaystyle\int_0^1 \frac{dx}{1+x^3}$

40. Show that $\displaystyle\int_0^{\pi/2} \frac{\sin x \cos x \, dx}{\cos^2 x + 3\cos x + 2} = \log \frac{9}{8}$

41. Evaluate $\displaystyle\int \frac{dx}{\cos x - \cos^2 x}$

§ 7.5. Special cases.

(1) In certain cases a substitution materially shortens the work. This is especially so if some power of x, say, x^{n-1}, is a factor of the numerator and the rest of the fraction is a rational function of x^n.

Examples.

Ex. 1. $\displaystyle\int \frac{x^2\, dx}{x^6 + 2x^3 + 2}$

In this case since the numerator x^2 is $\frac{1}{3}$ of the d.c. of x^3,

put $\quad x^3 = t$. $\therefore\ 3x^2\, dx = dt$.

$$\therefore \int \frac{x^2\, dx}{x^6 + 2x^3 + 2} = \frac{1}{3}\int \frac{dt}{t^2 + 2t + 2}$$

$$= \frac{1}{3}\int \frac{dt}{(t+1)^2 + 1}$$

$$= \frac{1}{3}\tan^{-1}(t+1)$$

$$= \frac{1}{3}\tan^{-1}(x^3+1).$$

Scanned with CamScanner

Ex. 2. $\int \frac{dx}{x(x^3+1)}$

$$= \int \frac{x^2\,dx}{x^3(x^3+1)};\ \text{put } x^3 = t.\ \therefore\ 3x^2\,dx = dt.$$

$$= \frac{1}{3}\int \frac{dt}{t(t+1)}$$

$$= \frac{1}{3}\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt$$

$$= \frac{1}{3}\{\log t - \log(t+1)\}$$

$$= \frac{1}{3}\log\frac{t}{t+1}$$

$$= \frac{1}{3}\log\frac{x^3}{x^3+1}.$$

(2) In fractions in which there is no odd power of x and in which the denominator can be broken up into factors of the form $x^2 \pm a^2$, it is not necessary to resolve the denominator into linear factors. The partial fraction corresponding to each factor $x^2 + a^2$ or $x^2 - a^2$ should be obtained regarding x^2 as the variable.

Example. $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$

$$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{A}{x^2+a^2} + \frac{B}{x^2+b^2}$$

$$= \frac{1}{a^2-b^2}\left(\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right)$$

$$\therefore \int \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

$$= \frac{1}{a^2-b^2}\int \frac{dx}{x^2+b^2} - \frac{1}{a^2-b^2}\int \frac{dx}{x^2+a^2}$$

$$= \frac{1}{a^2-b^2}\cdot\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) - \frac{1}{a^2-b^2}\cdot\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{1}{a^2-b^2}\left(\frac{1}{b}\tan^{-1}\frac{x}{b} - \frac{1}{a}\tan^{-1}\frac{x}{a}\right)$$

(3) Sometimes it is more convenient to break up the denominator completely into linear factors although this may introduce imaginary numbers. After resolution into partial fractions or after integration, the pairs of terms corresponding to conjugate roots can be combined and reduced to real form by the help of De Moivre's theorem.

(4) Very often expressions involving $x^2 + a^2$ can be integrated more conveniently by the substitution $x = a\tan\theta$.

Examples.

Ex. 1. $\int \frac{dx}{(1+x^2)^2}$

Putting $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$.

$$\therefore \int \frac{dx}{(1+x^2)^2} = \int \frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^2}$$

$$= \int \cos^2\theta\,d\theta$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta$$

$$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\cdot\frac{x}{1+x^2}$$

Ex. 2. $\int \frac{x\,dx}{(x^2+2x+2)^2}$

Scanned with CamScanner

$$\frac{x}{(x^2+2x+2)^2} = \frac{x}{\{(x+1)^2+1\}^2}$$

Put $x + 1 = \tan\theta$, $\therefore dx = \sec^2\theta \, d\theta$

$$\therefore \int \frac{x\,dx}{(x^2+2x+2)^2} = \int \frac{(\tan\theta - 1)\sec^2\theta\, d\theta}{(\tan^2\theta+1)^2}$$

$$= \int (\tan\theta - 1)\cos^2\theta\, d\theta$$

$$= \int (\sin\theta\cos\theta - \cos^2\theta)\, d\theta$$

$$= \int \frac{1}{2}\sin 2\theta\, d\theta - \int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= -\frac{1}{4}\cos 2\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta$$

$$= \frac{1}{4}\frac{(x+1)^2-1}{(x+1)^2+1} - \frac{1}{2}\tan^{-1}(x+1) - \frac{1}{4}\frac{2(x+1)}{1+(x+1)^2}$$

$$= \frac{x^2+2x}{4(x^2+2x+2)} - \frac{(x+1)}{2(x^2+2x+2)} - \frac{1}{2}\tan^{-1}(x+1)$$

$$= \frac{x^2-2}{4(x^2+2x+2)} - \frac{1}{2}\tan^{-1}(x+1).$$

(5) Integrals of the form $\int \frac{(ax^2+b)\,dx}{x^4+cx^2+1}$.

We have $ax^2 + b = \frac{a+b}{2}(x^2+1) + \frac{a-b}{2}(x^2-1)$. Hence the integral can be written as the sum of the two integrals.

$$I_1 = \frac{a+b}{2}\int \frac{(x^2+1)\,dx}{x^4+cx^2+1}$$

and $I_2 = \frac{a-b}{2}\int \frac{(x^2-1)\,dx}{x^4+cx^2+1}$

I_1 can be written as $\frac{a+b}{2}\int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2+c+\frac{1}{x^2}}$ and this integral is evaluated by the substitution $x - \frac{1}{x} = t$.

I_2 can be written as $\frac{a-b}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{x^2+c+\frac{1}{x^2}}$ and this integral is evaluated by the substitution $x + \frac{1}{x} = t$.

Exercises 10.

Integrate

1. $\frac{1}{x(x^n+1)}$
2. $\frac{1}{x(x^5+1)}$
3. $\frac{1}{x(x^2+1)^3}$
4. $\frac{1}{x(2x^7+1)}$
5. $\frac{2x}{(x^2+1)(x^2+3)}$
6. $\frac{x^4}{(x^2-a^2)(x^2-b^2)}$
7. $\frac{x^2}{(x^2+1)^2}$
8. $\frac{x^4}{(x^2+1)^2}$
9. $\frac{1}{x^2(1+x^2)^2}$
10. $\frac{x^3}{(a^2+x^2)^2}$
11. $\frac{1}{(x^2+4x+5)^2}$
12. $\frac{x^5}{(x^2+a^2)^2}$
13. $\frac{1}{(a^2+b^2x^2)^2}$

Scanned with CamScanner

14. Prove that

$$\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{\pi}{2(b+c)(c+a)(a+b)}$$

15. Integrate $\frac{x^2}{(x^2-1)(x^2+2)}$

16. $\frac{x^2+4}{(x^2+1)(x^2+3)}$

17. Prove that $\int_0^a \frac{a^2-x^2}{(a^2+x^2)^2}\,dx = \frac{1}{2a}$

18. $\frac{x^2+1}{x^4-x^2+1}$

19. $\frac{x^2-1}{x^4+x^2+1}$

20. $\int_0^{\pi/4} \sqrt{\tan\theta}\;d\theta.$

21. $\sqrt{\tan x} + \sqrt{\cot x}$

§ 8. Integration of irrational functions.

Many irrational expressions can be rationalised by a suitab change of variable as will be explained later on.

It has already been shown that

(1) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}.$

(2) $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}$, or $\log\left(x+\sqrt{x^2+a^2}\right)$

(3) $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}$ or $\log\left(x+\sqrt{x^2-a^2}\right).$

Let us consider the allied integrals.

(1) $\int \sqrt{a^2-x^2}\,dx$. Put $x = a\sin\theta$, then $dx = a\cos\theta\,d\theta$.

$\therefore \int \sqrt{a^2-x^2}\,dx = a^2\int \cos^2\theta\,d\theta$

$$= \frac{a^2}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)$$

$$= \frac{a^2}{2}(\theta + \sin\theta\cos\theta)$$

$$= \frac{a^2}{2}\left\{\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\left(1-\frac{x^2}{a^2}\right)^{1/2}\right\}$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{2}$$

(2) $\int \sqrt{a^2+x^2}\,dx$. Put $x = a\sinh\theta$, $dx = a\cosh\theta\,d\theta$.

$\therefore \int \sqrt{a^2+x^2}\,dx = a^2\int \cosh^2\theta\,d\theta$

$$= \frac{a^2}{2}\int (1+\cosh 2\theta)\,d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{\sinh 2\theta}{2}\right)$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\sinh\theta\cosh\theta$$

$$= \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{a^2}{2}\cdot\frac{x}{a}\left(1+\frac{x^2}{a^2}\right)^{1/2}$$

$$= \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x\sqrt{a^2+x^2}}{2}$$

(3) $\int \sqrt{x^2-a^2}\,dx$. Put $x = a\cosh\theta$, $dx = \sinh\theta\,d\theta$.

$\therefore \int \sqrt{x^2-a^2}\,dx = a^2\int \sinh^2\theta\,d\theta$

$$= \frac{a^2}{2}\int (\cosh 2\theta - 1)\,d\theta$$

$$= \frac{a^2}{2}\left[\frac{\sinh 2\theta}{2} - \theta\right]$$

$$= \frac{a^2}{2}\sinh\theta\cosh\theta - \frac{a^2}{2}\theta$$

$$= \frac{a^2}{2}\frac{x}{a}\left(\frac{x^2}{a^2} - 1\right)^{1/2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$

$$= \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$

Case (i). Integration of the form $\frac{1}{\sqrt{a x^2 + bx + c}}$.

Divide the expression under the root by the numerical value of the coefficient of x^2 and complete the square of the terms which contain x, the integral reduces to one of the forms above.

Examples.

Ex. 1. $\int \frac{dx}{\sqrt{2 - 3x + x^2}}$

$$= \int \frac{dx}{\left\{\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}\right\}^{1/2}} = \cosh^{-1}(2x - 3).$$

Ex. 2. $\int \frac{dx}{\sqrt{3x - x^2 - 2}}$

$$= \int \frac{dx}{\left\{\frac{1}{4} - \left(x - \frac{3}{2}\right)^2\right\}^{1/2}} = \sin^{-1}(2x - 3)$$

Ex. 3. $\int \frac{dx}{\sqrt{x(3 - 2x)}} = \int \frac{dx}{\sqrt{3x - 2x^2}}$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\left(\frac{3}{2}x - x^2\right)^{1/2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\left\{\left(\frac{3}{4}\right)^2 - \left(x - \frac{3}{4}\right)^2\right\}^{1/2}}$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{x - \frac{3}{4}}{\frac{3}{4}} = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x - 3}{3}\right)$$

Ex. 4. $\int \frac{dx}{\sqrt{3x^2 + x - 2}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\left(x^2 + \frac{x}{3} - \frac{2}{3}\right)^{1/2}}$

$$= \frac{1}{\sqrt{3}}\int \frac{dx}{\left\{\left(x + \frac{1}{6}\right)^2 - \frac{1}{36} - \frac{2}{3}\right\}^{1/2}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\left\{\left(x + \frac{1}{6}\right)^2 - \left(\frac{5}{6}\right)^2\right\}^{1/2}}$$

$$= \frac{1}{\sqrt{3}}\cosh^{-1}\frac{x + \frac{1}{6}}{\frac{5}{6}} = \frac{1}{\sqrt{3}}\cosh^{-1}\frac{6x + 1}{5}$$

Case (ii) . Integration of the form $\frac{px + q}{\sqrt{a x^2 + bx + c}}$

Put the numerator equal to A (d.c. of the expression under the radical sign) + B where A and B are constants.

i.e., $px + q = A(2ax + b) + B$

The values of A and B can easily be determined . The radical breaks up into two parts, in one of which the numerator is the differential coefficient of $a x^2 + bx + c$ and in the other the numerator does not involve x. The method of integration of the two parts is illustrated by the following examples.

Examples.

Ex. 1. $\int \frac{x}{\sqrt{x^2 + x + 1}}\,dx.$

Let us assume that $x = A\,(\text{d.c. of } x^2 + x + 1) + B$

$$= A\,(2x + 1) + B.$$

$$\therefore A = \frac{1}{2},\; B = -\frac{1}{2}$$

$$\therefore \int \frac{x}{\sqrt{x^2 + x + 1}}\,dx = \int \frac{\frac{1}{2}(2x+1) - \frac{1}{2}}{\sqrt{x^2 + x + 1}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2 + x + 1}}\,dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2 + x + 1}}$$

$$= \sqrt{x^2 + x + 1} - \frac{1}{2}\int \frac{dx}{\sqrt{(x + {}^1\!/_2)^2 + 3/4}}$$

$$= \sqrt{x^2 + x + 1} - \frac{1}{2}\sinh^{-1}\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \sqrt{x^2 + x + 1} - \frac{1}{2}\sinh^{-1}\frac{2x + 1}{\sqrt{3}}$$

Ex. 2. $\int \dfrac{6x + 5}{\sqrt{6 + x - 2x^2}}\,dx.$

Let $6x + 5 = A\,(\text{d.c. of } 6 + x - 2x^2) + B$

$$= A\,(1 - 4x) + B.$$

$$\therefore A = -\frac{3}{2};\; B = \frac{13}{2}.$$

$$\therefore \int \frac{6x + 5}{\sqrt{6 + x - 2x^2}}\,dx = \frac{13}{2}\int \frac{-\frac{3}{2}(1 - 4x) + \frac{13}{2}}{\sqrt{6 + x - 2x^2}}\,dx$$

$$= -\frac{3}{2}\int \frac{1 - 4x}{\sqrt{6 + x - 2x^2}}\,dx + \frac{13}{2}\int \frac{dx}{\sqrt{6 + x - 2x^2}}$$

$$= -3\sqrt{6 + x - 2x^2} + \frac{13}{2\sqrt{2}}\int \frac{dx}{\sqrt{3 + \frac{1}{2}x - x^2}}$$

$$= -3\sqrt{6 + x - 2x^2} + \frac{13}{2\sqrt{2}}\int \frac{dx}{\left\{\frac{49}{16} - \left(x - \frac{1}{4}\right)^2\right\}^{1/2}}$$

$$= -3\sqrt{6 + x - 2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\frac{x - \frac{1}{4}}{\frac{7}{4}}$$

$$= -3\sqrt{6 + x - 2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\frac{4x - 1}{7}.$$

Ex. 3. $\int \dfrac{3x - 2}{\sqrt{4x^2 - 4x - 5}}\,dx.$

$$3x - 2 = A\,(8x - 4) + B.$$

$$\therefore A = \frac{3}{8},\; B = -\frac{1}{2}.$$

$$\therefore \int \frac{3x - 2}{\sqrt{4x^2 - 4x - 5}}\,dx = \int \frac{3\,(8x - 4)}{8\sqrt{4x^2 - 4x - 5}}\,dx - \frac{1}{2}\int \frac{dx}{\sqrt{4x^2 - 4x - 5}}$$

$$= \frac{3}{4}\sqrt{4x^2 - 4x - 5} - \frac{1}{4}\int \frac{dx}{\left\{\left(x - \frac{1}{2}\right)^2 - \frac{3}{2}\right\}^{1/2}}$$

$$= \frac{3}{4}\sqrt{4x^2 - 4x - 5} - \frac{1}{4}\cosh^{-1}\frac{2x - 1}{\sqrt{6}}$$

Ex. 4. $\int \left(\dfrac{5 - x}{2 - x}\right)^{1/2} dx = \int \dfrac{5 - x}{\sqrt{(x - 2)(5 - x)}}\,dx$

-33

$$= \int \frac{5 - x}{\sqrt{-10 + 7x - x^2}}\, dx$$

which can easily be found as

$$\sqrt{10 - 7x - x^2} + \frac{7}{2} \sin^{-1} \left(\frac{2x - 7}{3} \right)$$

Exercises 11.

Integrate

1. $\dfrac{1}{\sqrt{2x - x^2}}$

2. $\dfrac{2x}{\sqrt{3 + 4x - x^2}}$

3. $\dfrac{2x - 3}{\sqrt{2x^2 - 7x + 5}}$

4. $\dfrac{1 - 4x}{\sqrt{x^2 - 2x + 4}}$

5. $\dfrac{x + 1}{\sqrt{x(x - 2)}}$

6. $\dfrac{x + 1}{\sqrt{2x^2 + x - 3}}$

7. $\dfrac{3x - 4}{\sqrt{3x^2 + 4x + 7}}$

8. $\left(\dfrac{x - 1}{2x + 3} \right)^{1/2}$ (Hint : Mu
Nr and Dr. by $\sqrt{x - 1}$

9. $\left(\dfrac{3 - 2x}{1 - x} \right)^{1/2}$

10. $\dfrac{1}{\sqrt{8 + 3x - x^2}}$

11. $\dfrac{2x - 1}{\sqrt{x^2 + 5x + 6}}$

12. $\dfrac{1}{\sqrt{(x - \alpha)(\beta - x)}}$

13. $\dfrac{2 + x}{\sqrt{x^2 - 1}}$

14. $\dfrac{1}{\sqrt{(a - x)(b + x)}}$

15. $\dfrac{2x - 4}{\sqrt{3x^2 + 4x + 7}}$

16. $\dfrac{2x}{\sqrt{6 - 5x^2 - x^4}}$

(Put x^2 =

Ex. 4. $\int \frac{dx}{x^3\sqrt{x^2-9}}$

[Put $x = 3\sec\theta$; $dx = 3\sec\theta\tan\theta\, d\theta$.]

$$\text{Integral} = \int \frac{3\sec\theta\tan\theta\, d\theta}{27\sec^3\theta\sqrt{9\sec^2\theta - 9}} = \int \frac{d\theta}{27\sec^2\theta}$$

$$= \frac{1}{27}\int \cos^2\theta\, d\theta$$

$$= \frac{\theta}{54} + \frac{1}{108}\sin 2\theta = \frac{\theta}{54} + \frac{\sin\theta\cos\theta}{54}$$

$$= \frac{1}{54}\sec^{-1}\left(\frac{x}{3}\right) + \frac{1}{18}\frac{\sqrt{x^2-9}}{x^2}$$

Case (ix). The expressions

$\sqrt{(x-\alpha)(\beta-x)}$, $\frac{1}{\sqrt{(x-\alpha)(\beta-x)}}$ and $\left(\frac{x-\alpha}{\beta-x}\right)^{1/2}$ where $\beta > \alpha$ are all rationalised by the substitution $x = \alpha\cos^2\theta + \beta\sin^2\theta$.

Examples.

Ex. 1. $\int \sqrt{(x-3)(7-x)}\, dx$.

Put $x = 3\cos^2\theta + 7\sin^2\theta$;

$$dx = (-6\cos\theta\sin\theta + 14\sin\theta\cos\theta)\, d\theta$$

$$= 8\sin\theta\cos\theta\, d\theta.$$

$$x - 3 = 3\cos^2\theta + 7\sin^2\theta - 3$$

$$= 7\sin^2\theta - 3\sin^2\theta = 4\sin^2\theta$$

$$7 - x = 7 - 3\cos^2\theta - 7\sin^2\theta = 4\cos^2\theta$$

$$\therefore \int \sqrt{(x-3)(7-x)}\, dx = \int \sqrt{4\sin^2\theta\, 4\cos^2\theta}\; .\; 8\sin\theta\cos\theta\, d\theta$$

$$= 32\int \sin^2\theta\cos^2\theta\, d\theta = 8\int \sin^2 2\theta\, d\theta$$

$$= 4\int (1 - \cos 4\theta)\, d\theta = 4\theta - \sin 4\theta$$

$$= 4\theta - 2\sin 2\theta \,.\, \cos 2\theta$$

$$= 4\theta - 4\sin\theta\cos\theta\,(2\cos^2\theta - 1)$$

$$= 4\sin^{-1}\left(\frac{x-3}{4}\right)^{1/2} - \sqrt{(x-3)(7-x)}\left\{\frac{7-x}{2} - 1\right\}$$

$$= 4\sin^{-1}\left(\frac{1}{2}\sqrt{x-3}\right) - \frac{5-x}{2}\sqrt{(x-3)(7-x)}$$

Ex. 2. $\int \left(\frac{5-x}{x-2}\right)^{1/2} dx$. Put $x = 2\sin^2\theta + 5\cos^2\theta$.

$$dx = -6\sin\theta\cos\theta\, d\theta.$$

$$\text{Integral} = \int \left(\frac{3\sin^2\theta}{3\cos^2\theta}\right)^{1/2} (-6\sin\theta\cos\theta)\, d\theta$$

$$= -6\int \sin^2\theta\, d\theta = -3\int (1 - \cos 2\theta)\, d\theta$$

$$= -3\left[\theta - \frac{\sin 2\theta}{2}\right] = -3\theta + 3\sin\theta\cos\theta$$

$$= -3\sin^{-1}\left(\frac{1}{3}\sqrt{5-x}\right) + \sqrt{(5-x)(x-2)}$$

Ex. 3. Evaluate $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

$(\beta > \alpha)$.

Putting $x = \alpha\sin^2\theta + \beta\cos^2\theta$;

$$dx = 2(\alpha - \beta)\sin\theta\cos\theta\, d\theta.$$

$x - \alpha = (\beta - \alpha)\cos^2\theta$ and $\beta - x = (\beta - \alpha)\sin^2\theta$.

The integral reduces to $-2\int d\theta = -2\theta$

$$= -2\sin^{-1}\left(\frac{\beta - x}{\beta - \alpha}\right)^{1/2}$$

In particular $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

$$= -2\left[\sin^{-1}\left[\frac{\beta - x}{\beta - \alpha}\right]\right]_\alpha^\beta = \pi.$$

Case (x). Sometimes rationalisation of the denominator may aid integration.

Examples.

Ex. 1. $\int \frac{dx}{x + \sqrt{x^2 - 1}} = \int \{x - \sqrt{x^2-1}\}\, dx$

$$= \frac{1}{2}x^2 - \int \sqrt{x^2 - 1}\, dx$$

$$= \frac{1}{2}x^2 - \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\cosh^{-1} x.$$

Ex. 2. $\int \frac{dx}{\sqrt{x} + \sqrt{1+x}} = \int (\sqrt{1+x} - \sqrt{x})\, dx$

$$= \frac{2}{3}(1+x)^{3/2} - \frac{2}{3}x^{3/2}$$

Exercises 13.

1. $\frac{1}{x^2\sqrt{1-x^2}}$
2. $\frac{1}{x^2\sqrt{1+x^2}}$
3. $\frac{1}{x^2\sqrt{a^2-x^2}}$
4. $\frac{x^3}{\sqrt{a^2-x^2}}$
5. $\frac{x^2}{\sqrt{x^2+1}}$
6. $\frac{x^2}{(1+x^2)^{3/2}}$
7. $\frac{x}{(x^2+1)^{3/2}}$
8. $\sqrt{(x+1)(4-x)}$
9. $\left(\frac{x-1}{2-x}\right)^{1/2}$
10. $\frac{\sqrt{a^2-x^2}}{x^4}$

37

29. $\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$

30. $\int_0^4 \frac{dx}{\sqrt{x(1+x)}}$

31. $\int_0^a x \left(\frac{a^2-x^2}{a^2+x^2}\right)^{1/2} dx$

32. $\int_0^2 \left(\frac{2+x}{2-x}\right)^{1/2} dx.$

33. $\int_0^a x \left(\frac{a+x}{a-x}\right)^{1/2} dx.$

34. $\frac{1}{(x^2-a^2)^{3/2}}$

35. Prove that $\int_2^3 \sqrt{(x-2)(3-x)}\, dx = \frac{\pi}{8}.$

36. Evaluate $\int_2^3 [(x-2)(3-x)]^{3/2}\, dx.$

Integrate

37. $\frac{1}{x^3(x^2-1)^{1/2}}$

38. $\frac{x^2}{(1-x^2)^{5/2}}$

39. $\int_a^b \left(\frac{b-x}{x-a}\right)^{1/2} dx$

40. $\frac{x+1}{\sqrt{1+x^2}}$

41. $\frac{\sqrt{x^2-a^2}}{x}$

42. $\frac{1}{(1-x)\sqrt{1-x^2}}$

43. $\sqrt{e^x-1}$

44. $\int_0^{\frac{1}{\sqrt{2}}} \frac{x^2}{(1-x^2)^{1/2}} dx$

45. $\int \frac{\sin x}{\sin 4x} dx$ and $\int \frac{\sqrt{1+x^2}}{1-x^2} dx.$

§ 9. Type $\int \frac{dx}{a+b\cos x}$

Put $t = \tan\frac{x}{2}$; $dt = \frac{1}{2}\sec^2\frac{x}{2}dx = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right) dx$

Scanned with CamScanner

i.e., $dx = \frac{2\,dt}{1+t^2}$; $\cos x = \frac{1-t^2}{1+t^2}$

Let $I = \int \frac{dx}{a + b\cos x} = \int \frac{2\,dt}{(a+b)+(a-b)t^2}$

Two cases arise.

Case (i) Let $a > b$.

$$I = \frac{2}{a-b}\int \frac{dt}{\frac{a+b}{a-b}+t^2} = \frac{2}{\left(\frac{a+b}{a-b}\right)^{1/2}(a-b)}$$

$$.\tan^{-1} t\left(\frac{a-b}{a+b}\right)^{1/2} \quad \text{(by §6.3)}$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left\{\left(\frac{a-b}{a+b}\right)^{1/2}\tan\frac{x}{2}\right\}$$

Case (ii). Let $a < b$.

$$I = 2\int \frac{dt}{a+b-(b-a)t^2} = \frac{2}{(b-a)}\int \frac{dt}{\frac{a+b}{b-a}-t^2}$$

$$= \frac{2}{2(b-a)}\left(\frac{b-a}{b+a}\right)^{1/2}\log\frac{t+\left(\frac{b+a}{b-a}\right)^{1/2}}{\left(\frac{a+b}{b-a}\right)^{1/2}-t} \quad \text{(by §6.3)}$$

$$= \frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b-a}\tan\frac{x}{2}+\sqrt{b+a}}{-\sqrt{b-a}\tan\frac{x}{2}+\sqrt{b+a}}.$$

[**Note:-** The above substitution can be used when the denominator of the integrand is of the first degree in $\cos x$ and $\sin x$.]

Examples.

Ex. 1. Evaluate $\int_0^{\pi} \frac{dx}{5+4\cos x}$

Putting $t = \tan\frac{x}{2}$, the integral reduces to

$$\int_0^{\infty} \frac{2\,dt}{9+t^2} = \frac{2}{3}\left\{\tan^{-1}\left(\frac{t}{3}\right)\right\}_0^{\infty} = \frac{\pi}{3}$$

(The limits of the definite integral must be changed when the variable x is changed to t. When $x = 0$, $t = 0$ and $x = \pi$, $t \to \infty$)

Ex. 2. Evaluate $\int \frac{dx}{a\cos x + b\sin x + c}$

Let $a = r\cos\alpha$ and $b = r\sin\alpha$.

The auxiliary constants r and α are thus given by $r = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\frac{b}{a}$. Hence the integral becomes

$$\int \frac{dx}{r\cos(x-\alpha)+c} = \int \frac{dy}{r\cos y + c}, \text{ where } y = x - \alpha.$$

This reduces to the type considered.

Ex. 3. $\int_0^{\pi/2} \frac{dx}{9\cos x + 12\sin x}$

Putting $t = \tan\frac{x}{2}$ and noting that $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$, the integral reduces to $\frac{2}{3}\int_0^1 \frac{dt}{3+8t-3t^2}$ as the limits for t change to 0 to 1 when x takes the values 0 and $\frac{\pi}{2}$. Hence the integral is

$$\frac{2}{3}\int_0^1 \frac{dt}{(3-t)(3t+1)} = \frac{1}{15}\int_0^1 \left\{\frac{3}{3t+1}+\frac{1}{3-t}\right\}dt$$

Scanned with CamScanner

40

$$= \frac{1}{15}\left[\log\frac{3t+1}{3-t}\right]_0^1 = \frac{1}{15}\left(\log 2 - \log\frac{1}{3}\right) = \frac{\log 6}{15}$$

Exercises 14.

Evaluate

1. $\int \frac{dx}{4 + 5\cos x}$

2. $\int \frac{dx}{13 + 12\cos x}$

3. $\int \frac{dx}{12 + 13\cos x}$

4. $\int_0^{\pi} \frac{dx}{13 + 5\cos x}$

5. $\int \frac{dx}{4 + 5\sin x}$

6. $\int_0^{\pi/4} \frac{d\theta}{1 + \cos a \sin 2\theta}$ $(0 < a < \pi/2)$

7. $\int \frac{dx}{1 + e\cos x}$ $(e \lessgtr 1)$

8. $\int \frac{dx}{3 - 4\cos x}$

9. $\int \frac{dx}{2\cos x + 3\sin x}$

10. $\int \frac{dx}{\sin x + \sqrt{3}\cos x}$

11. $\int \frac{dx}{1 + 3\sin x + 4\cos x}$

12. $\int \frac{dx}{1 + \sin x + \cos x}$

13. Show that

(a) $\int_0^{\pi} \frac{d\theta}{5 + 3\cos\theta} = \frac{\pi}{4}$

(b) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + 2\cos\theta} = \frac{1}{\sqrt{3}}\log(2 + \sqrt{3})$

(c) $\int_0^{\pi} \frac{dx}{a^2 - 2ab\cos x + b^2} = \frac{\pi}{a^2 - b^2}$, $(a > b > 0)$.

41

14. By means of the substitution $\tan\frac{x}{2} = \left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{u}{2}$, evaluate $\int \frac{dx}{(1 + e\cos x)^2}$.

15. $\int \frac{d\theta}{\sin\theta(1 + \sin\theta)}$

16. $\int_0^{\pi/2} \frac{dx}{5 + 4\sin x}$

§ 10. Evaluate $\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

Multiplying numerator and denominatory by $\sec^2 x$, the integral reduces to $\int \frac{\sec^2 x\, dx}{a^2 + b^2\tan^2 x}$

$= \int \frac{dt}{a^2 + b^2 t^2}$ on putting $\tan x = t$; $\sec^2 x\, dx = dt$

$= \frac{1}{ab}\tan^{-1}\left(\frac{bt}{a}\right) = \frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right)$

Exercises 15.

Integrate

1. $\frac{1}{1 + 7\cos^2 x}$

2. $\frac{1}{2\sin^2 x + 3\cos^2 x}$

3. Show that $\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{ab}$

4. Show that $\int_{\pi/4}^{3\pi/4} \frac{d\theta}{2\cos^2\theta + 1} = \int_{\pi/4}^{3\pi/4} \frac{\sec^2\theta\, d\theta}{2 + \sec^2\theta} = \frac{2\pi}{3\sqrt{3}}$.

5. Show that $\int_0^{\pi/2} \frac{\sec^2 x\, dx}{(\sec x + \tan x)^n} = \frac{n}{n - 1}$ (n being positive and greater than 1). [Hint: Put $z = \sec x + \tan x$.]

Scanned with CamScanner

6. Show that $\int_0^{\pi/2} \frac{dx}{1 + a^2 \cos^2 x + b^2 \sin^2 x}$

$$= \frac{\pi}{2} \frac{1}{\sqrt{(1 + a^2)(1 + b^2)}}$$

7. Show that

$$\int \frac{dx}{1 - \sin^4 x} = \frac{1}{2} \tan x + \frac{1}{2\sqrt{2}} \tan^{-1} (\sqrt{2} \tan x)$$

§ 11. Properties of definite integrals.

1. $\int_a^b f(x)\, dx = - \int_b^a f(x)\, dx$. This is obvious from the definition of a definite integral.

2. $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ where c is some value of x between a and b.

Let $\int f(x)\, dx = F(x)$

Then $\int_a^b f(x)\, dx = F(b) - F(a)$.

The R.H.S. $= F(c) - F(a) + F(b) - F(c)$

$= F(b) - F(a)$. Hence the result.

3. $\int_{-a}^{+a} f(x)\, dx = 2 \int_0^a f(x)\, dx$, if $f(x)$ is an even function of x.

If $f(x)$ is even. $f(x) = f(-x)$.

$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx$ by (2)

$= \int_{-a}^{0} f(-x)\, dx + \int_0^a f(x)\, dx$

$= - \int_a^0 f(y)\, dy + \int_0^a f(x)\, dx$ (by putting $y = -x$ in the first integral)

$= \int_0^a f(y)\, dy + \int_0^a f(x)\, dx = 2 \int_0^a f(x)\, dx$

as in a definite integral we can replace the variable y by x.

4. If $f(x)$ is an odd function of x, $\int_{-a}^{a} f(x)\, dx = 0$.

If $f(x)$ is odd, $f(x) = -f(-x)$.

$$\therefore \int_{-a}^{a} f(x)\, ax = \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx$$

$$= - \int_{-a}^{0} f(-x)\, dx + \int_0^a f(x)\, dx$$

$$= + \int_a^0 f(y)\, dy + \int_0^a f(x)\, dx$$

$$= - \int_0^a f(x)\, dx + \int_0^a f(x)\, dx$$

$$= 0.$$

5. $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$.

In $\int_0^a f(a - x)\, dx$, put $a - x = y$

R.H.S. $= - \int_a^0 f(y)\, dy = \int_0^a f(y)\, dy = \int_0^a f(x)\, dx$.

This result is very useful in evaluating many integrals.

Examples.

Ex. 1. Prove that $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx.$

Let $f(x) = \sin^n x$. Here $a = \frac{\pi}{2}$.

$$\therefore f(a - x) = \sin^n\left(\frac{\pi}{2} - x\right) = \cos^n x.$$

By § 11.5 the result follows.

Ex. 2. $\int_0^{\frac{\pi}{2}} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + \cos x^{3/2}}\,dx = \frac{\pi}{4}.$

Let I be the value of this integral and $f(x)$ denote the integrand $\frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}$.

$$\therefore I = \int_0^{\frac{\pi}{2}} f(x)\,dx.$$

$$f(a - x) = \frac{(\cos x)^{3/2}}{(\cos x)^{3/2} + (\sin x)^{3/2}} \text{ as } a = \frac{\pi}{2} \text{ here.}$$

Also $\quad I = \int_0^{\frac{\pi}{2}} f(a - x)\,dx.$

Adding (1) and (2).

$$2I = \int_0^{\frac{\pi}{2}} \frac{(\sin x)^{3/2} + (\cos x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}\,dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

Hence $I = \frac{\pi}{4}$.

Ex. 3. $\int_0^{\pi/4} \log(1 + \tan\theta\,d\theta = \frac{\pi}{8}\log 2.$

Let $f(\theta) = \log(1 + \tan\theta)$. Here $a = \frac{\pi}{4}$.

$$\therefore f\left(\frac{\pi}{4} - \theta\right) = \log\left\{1 + \tan\left(\frac{\pi}{4} - \theta\right)\right\}.$$

$$= \log\left\{1 + \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}\right\} = \log\frac{2}{1 + \tan\theta}$$

$$I = \int_0^{\pi/4} \log(1 + \tan\theta)\,d\theta$$

and $\quad I = \int_0^{\pi/4} \log\frac{2}{1 + \tan\theta}\,d\theta$ by § 11.5

Adding, $2I = \int_0^{\pi/4} \log 2\,d\theta = \log 2\,[\theta]_0^{\pi/4}$

$$= \frac{\pi}{4}\log 2.$$

Hence the result.

Ex. 4. $\int_0^{\pi} \theta\sin^3\theta\,d\theta = \frac{2\pi}{3}.$

$f(\theta) = \theta\sin^3\theta$. Here $a = \pi$.

$\therefore \quad f(a - \theta) = (\pi - \theta)\sin^3\theta.$

Hence $I = \int_0^{\pi} \theta\sin^3\theta\,d\theta$ and $I = \int_0^{\pi} (\pi - \theta)\sin^3\theta\,d\theta$ by §11.5

Adding, $2I = \pi\int_0^{\pi} \sin^3\theta\,d\theta$

$$= \pi\int \sin^2\theta\,(-dy) \text{ putting } \cos\theta = y\,;\ -\sin\theta\,d\theta = dy$$

$$= -\pi \int_{1}^{-1} (1 - y^2)\, dy = -\pi \left[y - \frac{y^3}{3} \right]_{1}^{-1}$$

$$= -\pi \left[-1 + \frac{1}{3} - 1 + \frac{1}{3} \right] = \frac{4\pi}{3}.$$

Hence $\mathrm{I} = \frac{2\pi}{3}$.

Ex. 5. $\int_0^{2a} f(x)\, dx = 2 \int_0^{a} f(x)\, dx$ if $f(2a - x) = f(x)$.

and $= 0$ if $f(2a - x) = -f(x)$.

$$\int_0^{2a} f(x)\, dx = \int_0^{a} f(x)\, dx + \int_a^{2a} f(x)\, dx$$

In the second integral, put $2a - x = y$; $dx = -dy$

When $x = a$, $y = a$ and $x = 2a$, $y = 0$.

Hence $\int_a^{2a} f(x)\, dx = -\int_a^{0} f(2a - y)\, dy = \int_0^{a} f(2a - y)\, dy$

$$= \int_0^{a} f(2a - x)\, dx.$$

$\therefore \int_0^{2a} f(x)\, dx = \int_0^{a} f(x)\, dx + \int_0^{a} f(2a - x)\, dx$ from (1).

If $f(2a - x) = f(x)$, $\int_0^{2a} f(x)\, dx = 2 \int_0^{a} f(x)\, dx$.

If $f(2a - x) = -f(x)$, $2 \int_0^{a} f(x)\, dx = 0$.

Cor. $\int_0^{\pi} f(\sin x)\, dx = 2 \int_0^{\frac{\pi}{2}} f(\sin x)\, dx$.

Ex. 6. Evaluate $\mathrm{I} = \int_0^{\frac{\pi}{2}} \log \sin x\, dx$

$\mathrm{I} = \int_0^{\frac{\pi}{2}} \log \sin \left(\frac{\pi}{2} - x \right) dx = \int_0^{\frac{\pi}{2}} \log \cos x\, dx$ (by §11.5).

Hence $2\mathrm{I} = \int_0^{\frac{\pi}{2}} \log \sin x\, dx + \int_0^{\frac{\pi}{2}} \log \cos x\, dx$

$$= \int_0^{\frac{\pi}{2}} \log (\sin x \cos x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} (\log \sin 2x - \log 2)\, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x\, dx - \frac{\pi}{2} \log 2.$$

Put $2x = z$; $dx = \frac{1}{2} dz$; then

$$\int_0^{\frac{\pi}{2}} \log \sin 2x\, dx = \frac{1}{2} \int_0^{\pi} \log \sin z\, dz = \frac{1}{2} \int_0^{\pi} \log \sin x\, dx$$

$$= \frac{1}{2} \times 2 \int_0^{\frac{\pi}{2}} \log \sin x\, dx \text{ (by 5 Cor.)}$$

$$= \int_0^{\frac{\pi}{2}} \log \sin x\, dx.$$

Thus, $2\mathrm{I} = \mathrm{I} - \frac{\pi}{2} \log 2$.

Scanned with CamScanner

25. If $f(x) = f(a + x)$, show that

(i) $\int_0^{na} f(x)\,dx = n \int_0^a f(x)\,dx$

(ii) $\int_a^{na} f(x)\,dx = (n - 1) \int_0^a f(x)\,dx.$

§ 12. Integration by parts.

If u and v are functions of x,

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ by the product rule.

Integraing both sides with respect to x

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$\therefore\ uv = \int u\,dv + \int v\,du.$

Hence $\int u\,dv = uv - \int v\,du.$

Note : - The success of this method depends on the proper choice of u and v ; the auxiliary integral $\int v\,du$ must be easier to integrate than the given integral.

Examples.

Ex. 1. $\int x e^x\,dx.$

Writing $dv = e^x\,dx$ and $u = x$, $v = \int e^x\,dx = e^x$.

$\therefore \int x e^x\,dx = \int x d(e^x) = \int u\,dv = uv - \int v\,du$

$= x e^x - \int e^x\,dx = x e^x - e^x = e^x(x - 1).$

Ex. 2. $\int x \sin 2x\,dx.$

Here $dv = \sin 2x\,dx$, i.e., $v = \int \sin 2x\,dx = \frac{-\cos 2x}{2}$ and $u = x$.

$\therefore \int x\sin 2x\,dx = \int x d\left(\frac{-\cos 2x}{2}\right) = \int u\,dv = uv - \int v\,du$

$= -\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x\,dx = -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}$

Ex. 3. $\int x^n \log x\,dx$. Put $u = \log x$ and $dv = x^n\,dx$.

i.e., $v = \int x^n\,dx = \frac{x^{n+1}}{n+1}$

$\therefore \int x^n \log x\,dx = \int \log x d\left(\frac{x^{n+1}}{n+1}\right)$

$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^{n+1}\frac{1}{x}dx.$

$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^n\,dx.$

$= \frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2}$

Ex. 4. $\int \sin^{-1} x\,dx$

Put $u = \sin^{-1} x$ and $dv = dx$, i.e., $v = x$

$\int \sin^{-1} x\,dx = \int u\,dv = uv - \int u\,du = x\sin^{-1} x - \int \frac{x\,dx}{\sqrt{1-x^2}}$

$= x\sin^{-1} x - \int \sin\theta\,d\theta$ on putting $x = \sin\theta$,

$= x\sin^{-1} x + \cos\theta = x\sin^{-1} x + \sqrt{1 - x^2}.$

Ex. 5. $\int \tan^{-1} x\,dx$. [Here $u = \tan^{-1} x$ and $v = x$]

$= x\tan^{-1} x - \int \frac{x\,dx}{1 + x^2} = x\tan^{-1} x - \frac{1}{2}\log(1 + x^2)$

Ex. 6. $\int x^2 \tan^{-1} x\,dx$. [Here $u = \tan^{-1} x$; $dv = x^2\,dx$.

$\therefore\ v = x^3/3$.]

$= \int \tan^{-1} x\,d\left(\frac{x^3}{3}\right) = \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x^3}{1 + x^2}dx$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\left(x - \frac{x}{1+x^2}\right)dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\left\{\frac{x^2}{2} - \frac{1}{2}\log(1+x^2)\right\}.$$

(B.Sc. 1994)

Ex. 7. $\int (\log x)^2\, dx.$

Here $u = (\log x)^2$ and $v = x$.

$$\therefore \int (\log x)^2 dx = x(\log x)^2 - \int x \,.\, 2\log x \,.\, \frac{1}{x}\,dx$$

$$= x(\log x)^2 - 2\int \log x\, dx$$

$$= x(\log x)^2 - 2\left(x\log x - \int x\,.\,\frac{1}{x}\,dx\right)$$

$$= x(\log x)^2 - 2x\log x + 2x.$$

Ex. 8. $\int \sqrt{a^2+x^2}\,dx.$

[Here $u = \sqrt{a^2+x^2}$ and $v = x$.]

$$\text{Integral} = x\sqrt{a^2+x^2} - \int \frac{x^2\,dx}{\sqrt{a^2+x^2}}$$

$$= x\sqrt{a^2+x^2} - \int \frac{a^2+x^2-a^2}{\sqrt{a^2+x^2}}\,dx$$

$$= x\sqrt{a^2+x^2} - \int\sqrt{a^2+x^2}\,dx + a^2\int\frac{dx}{\sqrt{a^2+x^2}}$$

$$= x\sqrt{a^2+x^2} - \int\sqrt{a^2+x^2}\,dx + a^2\sinh^{-1}\frac{x}{a}.$$

Transposting, we get

$$\int\sqrt{a^2+x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2+x^2} + a^2\sinh^{-1}\frac{x}{a}\right)$$

Ex. 9. $\int \frac{x+\sin x}{1+\cos x}\,dx.$

$$I = \int\frac{x\,dx}{1+\cos x} + \int\frac{\sin x\,dx}{1+\cos x}$$

$$= \int\frac{x\,dx}{2\cos^2\left(\frac{x}{2}\right)} + \int\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx$$

$$= \int x\,d\left(\tan\frac{x}{2}\right) + \int\tan\frac{x}{2}\,dx.$$

$$= x\tan\frac{x}{2} - \int\tan\frac{x}{2}\,dx + \int\tan\frac{x}{2}\,dx = x\tan\frac{x}{2}$$

Ex. 10. $\int e^x\frac{x+1}{(x+2)^2}\,dx$

$$= \int e^x\frac{x+2-1}{(x+2)^2}\,dx = \int\frac{e^x}{x+2}\,dx - \int\frac{e^x}{(x+2)^2}\,dx$$

$$= \int\frac{1}{x+2}\,d(e^x) - \int\frac{e^x}{(x+2)^2}\,dx$$

$$= \int\frac{e^x}{x+2} + \int\frac{e^x}{(x+2)^2}\,dx - \int\frac{e^x\,dx}{(x+2)^2} = \frac{e^x}{x+2}$$

Ex. 11. $\int e^x(\sin x+\cos x)\,dx = \int e^x\sin x\,dx + \int e^x\cos x\,dx.$

$$= \int\sin x\,d(e^x) + \int e^x\cos x\,dx$$

$$= \sin x\,e^x - \int e^x\cos x\,dx + \int e^x\cos x\,dx = \sin x\,e^x$$

Exercies 17.

Integrate

1. $\log x.$

2. $x^3\log x$

3. $x\log(x+1)$

4. $\cos^{-1}\frac{x}{a}$

32. Evaluate $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx.$

33. If for $x > 0$, $\frac{d^2y}{dx^2} = \log x$ and if the values of y and $\frac{dy}{dx}$ when $x = 1$ are both zero, express y in terms of x.

Evaluate

34. $\int_0^{\pi/2} x \cot x \, dx$

35. $\int \tan^{-1}\left(\frac{1-x}{1+x}\right)^{1/2} dx.$

36. $\int \frac{x^2+1}{(x+1)^2} e^x \, dx.$

37. $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx.$

§ 1.3. Reduction formulae.

§ 13.1. $I_n = \int x^n e^{ax} dx$, where n is a positive integer.

Here $dv = e^{ax} dx$, *i.e.*, $v = \int e^{ax} dx = \frac{e^{ax}}{a}$ and $u = x^n$.

$$\therefore \ I_n = \int x^n d\left(\frac{e^{ax}}{a}\right) = \frac{e^{ax}}{x} x^n - \frac{n}{a}\int e^{ax} x^{n-1} dx$$

$$= \frac{e^{ax}}{a} x^n - \frac{n}{a} I_{n-1}.$$

The auxiliary integral is of the same type as the given integral but with index n reduced by 1. Such a formula is called a reduction formula and by successive applications, we can evaluate I_n

The ultimate integral is obviously $\int e^{ax} dx = \frac{e^{ax}}{a}$.

§ 13.2. $I_n = \int x^n \cos ax \, dx$ (n a positive integer).

$$I_n = \int x^n \cos ax dx = \int x^n d\left(\frac{\sin ax}{a}\right) \left[\text{Here } u = x^n \text{ and } v = \frac{\sin ax}{a}\right]$$

$$= \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1} \sin ax \, dx$$

$$= \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1}\, d\left(\frac{-\cos ax}{a}\right)$$

$$= \frac{x^n \sin ax}{a} + \frac{n}{a^2}x^{n-1}\cos ax - \frac{n(n-1)}{a^2}\int x^{n-2}\cos ax\,dx$$

$$= \frac{x^n \sin ax}{a} + \frac{n}{a^2}x^{n-1}\cos ax - \frac{n(n-1)}{a^2}\, I_{n-2}$$

The ultimate integral is either $\int x\cos ax\,dx$ or $\int \cos ax\,dx$ according as n is odd or even.

(i) $\int x\cos ax\,dx = \int x d\left(\frac{\sin ax}{a}\right) = \frac{x\sin ax}{a} - \frac{1}{a}\int \sin ax\,dx$

$$= \frac{x\sin ax}{a} + \frac{1}{a^2}\cos ax.$$

(ii) $\int \cos ax\,dx = \frac{\sin ax}{a}$

Exercises 18.

Integrate

1. $x^2 e^{-x}$
2. $x^3 e^{2x}$
3. $e^x (x-1)^2$
4. $x \cos 2x$
5. $x^2 \sin 3x$
6. $x^3 \cos (x+a)$
7. Establish a reduction formula for $\int x^n \sin ax\,dx$; hence find

$$\int_0^{\pi/2} x^3 \sin x\,dx.$$

8. If $I_n = \int_0^{\pi/2} x^n \cos x\,dx$, show that

$I_n + n(n-1)\,I_{n-2} = \left(\frac{\pi}{2}\right)^n$. Evaluate $\int_0^{\pi/4} x^3 \cos^2 x\,dx.$

9. Evaluate $\int_0^1 x^n e^x\,dx$.

§ 13.3. $I_n = \int \sin^n x\,dx$ (n being a positive integer)

$I_n = \int \sin^{n-1} x \sin x\,dx = \int \sin^{n-1} x d(-\cos x)$

$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx$

$= -\sin^{n-1} x\cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x\,dx$

$= -\sin^{n-1} x\cos x + (n-1)\, I_{n-2} - (n-1)\, I_n$

$\therefore\ n\,I_n = -\sin^{n-1} x\cos x + (n-1)\, I_{n-2}$

The ultimate integral is $\int \sin x\,dx$ or $\int dx$ according as n is odd or even, i.e., $-\cos x$ or x.

Corollary

$$\int_0^{\pi/2} \sin^n x\,dx = \left[-\frac{\sin^{n-1}x\cos x}{n}\right]_0^{\pi/2} + \frac{(n-1)}{n}\int_0^{\pi/2}\sin^{n-2} x\,dx$$

$= \frac{(n-1)}{n}\int_0^{\pi/2}\sin^{n-2} x\,dx$ as the first term vanishes at both limits.

$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\int_0^{\pi/2}\sin^{n-4} x\,dx$$

$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\cdots$$

If n is even, the ultimate integral is $\int_0^{\pi/2} dx = (x)_0^{\pi/2} = \frac{\pi}{2}$

If n is odd, the ultimate integral is

$$\int \sin x\,dx = (-\cos x)_0^{\pi/2} = 1$$

$$\therefore \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-1}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} \text{ when } n \text{ is even and}$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}, \text{ when } n \text{ is odd.}$$

Examples.

Ex. 1. $\int_0^{\pi/2} \sin^6 x\,dx = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}.$

Ex. 2. $\int_0^{\pi/2} \sin^7 x\,dx = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} = \frac{48}{105} = \frac{16}{35}$

Ex. 3. In $\int \sin^n x\,dx$, if n be an odd positive integer, we can directly integrate without using the reduction formula. For instance, let us find $\int \sin^5 x\,dx$.

Put $y = \cos x$: $dy = -\sin x\,dx$.

$$\int \sin^5 x\,dx = -\int \sin^4 x\,dy = -\int (1-y^2)^2\,dy$$

$$= -\int (1-2y^2+y^4)\,dy$$

$$= -y + \frac{2y^3}{3} - \frac{y^5}{5} = -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5}$$

Ex. 4. Evaluate $\int_0^{\pi/2} x(1-x^2)^{1/2}\,dx$

Put $x = \sin\theta$; $dx = \cos\theta\,d\theta$.

When $x = 0$, $\theta = 0$ and $x = 1$, $\theta = \frac{\pi}{2}$.

The integral becomes

$$\int_0^{\pi/2} \sin\theta\cos^2\theta\,d\theta = \int \cos^2\theta\,d(-\cos\theta) = \left[\frac{-\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{1}{3}$$

§ 13. 4. $I_n = \int \cos^n x\,dx$ (n being a positive integer).

$$I_n = \int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx$$

$$= \int \cos^{n-1} x\,d(\sin x)$$

$$= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx$$

$$= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore n I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

The ultimate integral is $\int \cos x\,dx$ or $\int dx$, i.e., $\sin x$ or x according as n is odd or even.

Corollary.

$$\int_0^{\pi/2} \cos^n x\,dx = \left(\frac{\cos^{n-1} x \sin x}{n}\right)_0^{\pi/2} + \frac{(n-1)}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx$$

$$= \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx \text{ as the first term vanishes at both limits.}$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\int_0^{\pi/2} \cos^{n-4} x\,dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots$$

The ultimate integral is

$$\int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1 \text{ when } n \text{ is odd.}$$

The ultimate integral is

$$\int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}, \text{ when } n \text{ is even.}$$

Scanned with CamScanner

Thus $\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{1}{2}\,\frac{\pi}{2}$, if n is even

and $= \frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{2}{3}$, if n is odd.

Examples.

Ex. 1. $\int_0^{\pi/2} \cos^8 x\,dx = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\,\frac{\pi}{2} = \frac{35\pi}{256}$.

Ex. 2. $\int_0^{\pi/2} \cos^5 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$

Ex. 3 In $\int \cos^n x\,dx$, if n be an odd positive integer, we can directly integrate employing the reduction formula.

For example, take $\int \cos^7 x\,dx$.

Put $y = \sin x$; $dy = \cos x\,dx$

$$\int \cos^7 x\,dx = \int \cos^6 x \cos x\,dx = \int (1-y^2)^3\,dy$$

$$= \int (1 - 3y^2 + 3y^4 - y^6)\,dy = y - y^3 + \frac{3y^5}{5} - \frac{y^7}{7}$$

$$= \sin x - \sin^3 x + \frac{3\sin^5 x}{5} - \frac{\sin^7 x}{7}.$$

§ 13. 5. $I_{m,n} = \int \sin^m x \cos^n x\,dx$ (m,n being positive integers)

$$I_{m,n} = \int \sin^m x \cos^{n-1} x\,d(\sin x)$$

$$= \int \cos^{n-1} x\,d\left(\frac{\sin^{m+1} x}{m+1}\right)$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} - \frac{1}{m+1}\int \sin^{m+1} x\,d(\cos^{n-1} x)$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2} x \cos^{n-2} x\,dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1}$$

$$+ \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x\,(1 - \cos^2 x)\,dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x\,dx$$

$$- \frac{n-1}{m+1}\int \sin^m x \cos^n x\,dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}.$$

$$\therefore\ (m+n)\,I_{m,n} = \cos^{n-1} x \sin^{m+1} x + (n-1)\,I_{m,n-2} \quad \text{.... (a)}$$

Here, the power of $\cos x$ has been reduced by 2. We may, by a similar argument, arrive at the reduction formula in the form

$$(m+n)\,I_{m,n} = -\sin^{m-1} x \cos^{n+1} x + (m-1)\,I_{m-2,n} \quad \text{.... (b)}$$

Here, the power of $\sin x$ has been reduced by 2.

To apply this formula, we note two cases.

Case (i). Let m or n be an odd integer, say n.

Applying the formula (a) successively, the ultimate integral is

$$I_{m,1} = \int \sin^m x \cos x\,dx = \frac{\sin^{m+1} x}{m+1}.$$

If however, m is odd, we can use (b) and the ultimate integral is

$$I_{1,m} = \int \sin x \cos^n x\,dx = -\frac{\cos^{n+1} x}{n+1}.$$

If both m and n are odd, reduce the smaller index.

Note :- When either m or n or both are odd, we can integrate $\sin^m x \cos^n x$ directly without recourse to a reduction formula

For example, take

(1) $\int \sin^6 x \cos^3 x\, dx$. Put $y = \sin x$; $dy = \cos x\, dx$

$$\int \sin^6 x \cos^3 x\, dx = \int y^6 (1 - y^2)\, dy = \frac{y^7}{7} - \frac{y^9}{9}$$

$$= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9}.$$

(2) $\int \sin^9 x \cos^5 x\, dx$. Put $\sin x = y$; $\cos x\, dx = dy$.

$$\int \sin^9 x \cos^5 x\, dx = \int y^9 (1 - 2y^2 + y^4)\, dy$$

$$= \frac{y^{10}}{10} - \frac{y^{12}}{6} + \frac{y^{14}}{14} = \frac{\sin^{10} x}{10} - \frac{\sin^{12} x}{6} + \frac{\sin^{14} x}{14}$$

Case (ii). Let both m and n be even +ve integers.

Let $n < m$. Applying (a), the ultimate integral is

$$I_{m,0} = \int \sin^m x\, dx$$

which has been discussed in §13.3.

Corollary.

$\int_0^{\pi/2} \sin^m x \cos^n x\, dx$ (m,n being positive integers).

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx$$

$$= \left[\frac{\cos^{n-1} x \sin^{m+1} x}{m+n}\right]_0^{\pi/2} + \frac{n-1}{m+n} \cdot \int_0^{\pi/2} \sin^m x \cos^{n-2} x\, dx$$

$$= \frac{n-1}{m+n} \int_0^{\pi/2} \sin^m x \cos^{n-2} x\, dx$$ as the first term vanishes at both limits

$$= \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \int_0^{\pi/2} \sin^m x \cos^{n-4} x\, dx$$

$$\frac{n-1}{m+n} \frac{n-3}{m+n-2} \frac{n-5}{m+n-4} \cdots I_{m,1} \text{ or } I_{m,0}$$

according as n is odd or even.

(i) If n is odd, $I_{m,1} = \int_0^{\pi/2} \sin^m x \cos x\, dx$

$$= \left[\frac{\sin^{m+1} x}{m+1}\right]_0^{\pi/2} = \frac{1}{m+1}.$$

When n is odd,

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{n-1}{m+n} \frac{n-3}{m+n-2} \cdots \frac{2}{m+3} \frac{1}{m+1}.$$

(ii) If n is even,

$$I_{m,0} = \int_0^{\pi/2} \sin^m x\, dx = \frac{m-1}{m} \frac{m-3}{m-2} \cdots \frac{1}{2} \frac{\pi}{2}$$ by §13.3. Cor.

when m is even

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx$$

$$= \frac{n-1}{m+n} \frac{n-3}{m+n-2} \cdots \frac{1}{m+1} \frac{m-1}{m} \frac{m-3}{m-2} \cdots \frac{1}{2} \frac{\pi}{2}$$

Examples.

Ex. 1. $\int_0^{\pi/2} \sin^6 x \cos^5 x\, dx = \frac{4}{11} \cdot \frac{2}{9} \cdot \frac{1}{7}$ by (i)

$$= \frac{8}{693}.$$

Ex. 2. $\int_0^{\pi/2} \sin^6 x \cos^4 x\, dx = \frac{5}{10} \cdot \frac{3}{8} \cdot \frac{1}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \frac{\pi}{2} = \frac{3\pi}{512}.$

§ 13.6. $I_n = \int \tan^n x\, dx$ (n being a positive integer)

(B.Sc. 1994)

Scanned with CamScanner

61

$$I_n = \int \tan^{n-2} x \tan^2 x\, dx$$
$$= \int \tan^{n-2} x (\sec^2 x - 1)\, dx$$
$$= \int \tan^{n-2} x\, d(\tan x) - \int \tan^{n-2} x\, dx$$
$$= \frac{\tan^{n-1}}{n-1} - I_{n-2}$$

(i) When n is even, the ultimate integral is $\int dx = x$

(ii) When n is odd, the ultimate integral is

$\int \tan x\, dx = \log \sec x$ (Vide § 6.5, Ex. 3)

Examples.

Ex. 1. $\int \tan^4 x\, dx = \frac{\tan^3 x}{3} - \int \tan^2 x\, dx$ by putting $n = 4$ in the formula for I_n

$$= \frac{\tan^3 x}{3} - \int (\sec^2 x - 1)\, dx$$
$$= \frac{\tan^3 x}{3} - \tan x + x$$

Ex. 2. $\int_0^{\pi/4} \tan^3 x\, dx = \left[\frac{\tan^2 x}{2}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan x\, dx$

by putting $n = 3$

$$= \frac{1}{2} + [\log \cos x]_0^{\pi/4} = \frac{1}{2} + \log \frac{1}{\sqrt{2}} = \frac{1}{2}(1 - \log 2)$$

§ 13. 7. $I_n = \cot^n x\, dx$ (n being a positive integer).

$$\int \cot^n x\, dx = \int \cot^{n-2} x \cot^2 x\, dx$$
$$= \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1)\, dx$$
$$= \int \cot^{n-2} x\, d(-\cot x) - \int \cot^{n-2} x\, dx$$

62

$$= -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

The ultimate integral is $\int dx$ or $\int \cot x\, dx$, i.e., x or $\log \sin x$ according as n is even or odd.

§ 13. 8. $I_n = \int \sec^n x\, dx$ (n being a positive integer).

$$\int \sec^n x\, dx = \int \sec^{n-2} x\, d(\tan x)$$
$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x\, dx$$
$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1)\, dx$$
$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x\, dx + (n-2) \int \sec^{n-2} x\, dx$$
$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$
$$\therefore (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

(i) If n be an odd integer, the ultimate integral is

$\int \sec x\, dx = \log(\tan x + \sec x)$. (Vide § 6.5 Ex. 5)

(ii) If n be an even integer, the ultimate integral is $\int dx = x$.

Examples.

Ex. 1. $I = \int \sec^3 x\, dx = \int \sec x\, d(\tan x)$

$$= \sec x \tan x - \int \tan^2 x \sec x\, dx$$
$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx$$
$$= \sec x \tan x - I + \log(\sec x + \tan x)$$
$$\therefore 2I = \sec x \tan x + \log(\sec x + \tan x).$$

Ex. 2. $\int \sec^6 x\, dx = \int \sec^4 x\, d(\tan x) = \int (1 + t^2)^2\, dt$ (where $t = \tan x$);

$$= \int (1 + 2t^2 + t^4)\, dt = t + \frac{2t^3}{3} + \frac{t^5}{5}$$

$$= \tan x + \frac{2\tan^3 x}{3} + \frac{\tan^5 x}{5}$$

§ 13. 9. $I_n = \int \operatorname{cosec}^n x\, dx$ (n being a positive integer).

$$I_n = \int \operatorname{cosec}^n x\, dx = -\int \operatorname{cosec}^{n-2} x\, d(\cot x)$$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x\, dx$$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-2} x \,.\, (\operatorname{cosec}^2 x - 1)\, dx$$

$$= -\operatorname{cosec}^{n-2} \cot x - (n-2)\, I_n + (n-2)\, I_{n-2}$$

$$\therefore (n-1)\, I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2)\, I_{n-2}$$

(i) If n be an odd integer, the ultimate integral is

$\int \operatorname{cosec} x\, dx = -\log(\operatorname{cosec} x + \cot x)$ [Vide § 6.5. Ex. 6 .]

(ii) If n be an even integer, ultimate integral is $\int dx = x$.

Examples.

Ex. 1. $\int \operatorname{cosec}^4 x\, dx = -\int \operatorname{cosec}^2 x\, d(\cot x)$

$$= -\int (1 + y^2)\, dy, \text{ where } y = \cot x$$

$$= -y - \frac{y^3}{3} = -\cot x - \frac{\cot^3 x}{3}$$

Ex. 2. $\int \operatorname{cosec}^5 x\, dx$

Putting $n = 5$ in the above formula for I_n

$$\int \operatorname{cosec}^5 x\, dx = -\frac{\operatorname{cosec}^3 x \cot x}{4} + \frac{3}{4}\int \operatorname{cosec}^3 x\, dx$$

$$= -\frac{\operatorname{cosec}^3 x \cot x}{4} - \frac{3}{8}\operatorname{cosec} x \cot x$$

$$-\frac{3}{8}\log(\operatorname{cosec} x + \cot x)$$

§ 13. 10. $I_{m,n} = \int x^m (\log x)^n\, dx$ (where m and n are positive integers).

Hence or otherwise evaluate $\int x^4 (\log x)^3\, dx$.

$$I_{m,n} = \int (\log x)^n\, d\left(\frac{x^{m+1}}{m+1}\right)$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1}\int x^m (\log x)^{n-1}\, dx$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}.$$

The ultimate integral is $I_{m,0} = \int x^m\, dx = \dfrac{x^{m+1}}{m+1}$.

$$\int (\log x)^3 x^4\, dx = \int (\log x)^3\, d\left(\frac{x^5}{5}\right)$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{5}\int (\log x)^2 x^4\, dx$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{5}\int (\log x)^2\, d\left(\frac{x^5}{5}\right)$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{25}x^5(\log x)^2 + \frac{6}{25}\int x^4 (\log x)\, dx$$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{25}x^5(\log x)^2 + \frac{6}{25}\left[\frac{x^5}{5}\log x - \frac{x^5}{25}\right]$$

$$= x^5\left\{\frac{1}{5}(\log x)^3 - \frac{3}{25}(\log x)^2 + \frac{6}{125}\log x - \frac{6}{625}\right\}.$$

CHAPTER 6

FOURIER SERIES

§ 1. Consider the following trigonometric series

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots$$

where the a's and b's are constants and x a variable.

We see that every term except the first term has a period of 2π and consequently any function represented by a series of the above form in an interval of length 2π will also be periodic with period 2π. If the series converges in any closed interval, say $\lambda \leq x \leq \lambda + 2\pi$, then the series is convergent for every real value of x since the series represented by the function is periodic.

§ 2. Suppose that a given function $f(x)$ can be expressed as a trigonometric series as

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots \qquad \ldots(1)$$

Let us assume that the series is uniformly convergent in the interval $\lambda \leq x \leq \lambda + 2\pi$.

Then the series can be integrated term by term. To determine the a's and b's in the series, the following identities have to be used:

(i) $\int_{\lambda}^{\lambda+2\pi} \cos nx\, dx = 0$ where n is an integer.

(ii) $\int_{\lambda}^{\lambda+2\pi} \sin nx\, dx = 0$ where n is an integer.

(iii) $\int_{\lambda}^{\lambda+2\pi} \cos mx \cos nx\, dx = 0$ if $m \neq n$ and m and n are integers.

(iv) $\int_{\lambda}^{\lambda+2\pi} \sin mx \sin nx\, dx = 0$ if $m \neq n$ and m and n are integers.

(v) If $m = n$ and m and n are integers, then

$$\int_{\lambda}^{\lambda+2\pi} \cos mx \cos nx\, dx = \int_{\lambda}^{\lambda+2\pi} \cos^2 mx\, dx = \pi$$

$$\int_{\lambda}^{\lambda+2\pi} \sin mx \sin nx\, dx = \int_{\lambda}^{\lambda+2\pi} \sin^2 mx\, dx = \pi$$

$$\int_{\lambda}^{\lambda+2\pi} \sin mx \cos nx\, dx = \frac{1}{2}\int_{\lambda}^{\lambda+2\pi} \sin 2mx\, dx = 0$$

If we integrate both sides of the equation (1), we have

$$\int_{\lambda}^{\lambda+2\pi} f(x)\, dx = \int_{\lambda}^{\lambda+2\pi} \frac{a_0}{2}\, dx = \pi a_0$$

$$\therefore \quad a_0 = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\, dx \qquad \ldots(2)$$

If both sides of the equation (1) are multiplied by $\cos nx$ and integrating term by term from λ to $\lambda + 2\pi$, we see that all the terms on the right side vanish except the term containing a_n.

$$\therefore \text{ We have } \int_{\lambda}^{\lambda+2\pi} f(x) \cos nx\, dx = a_n \pi$$

$$\therefore \quad a_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x) \cos nx\, dx \qquad \ldots(3)$$

Similarly, multiplying both sides of the equation (1) by $\sin nx$ and integrating we have

$$b_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x) \sin nx\, dx \qquad \ldots(4)$$

In (3), if $n = 0$ is substituted, a_0 is obtained.

Hence we have the result that if $f(x)$ can be expressed as a trigonometric series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

then $$a_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x) \cos nx \, dx \quad (n = 0, 1, 2, \dots)$$

$$b_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x) \sin nx \, dx \quad (n = 1, 2, \dots)$$

Note :- (1) The constant term in the series is taken as $\frac{a_0}{2}$ instead of a_0, for the formula for finding a_n is valid when $n = 0$ as well as when n is a positive integer.

(2) If we construct the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

from $f(x)$, by means of these relations, the series is called a *Fourier series* for $f(x)$.

(3) We cannot conclude that the Fourier series for $f(x)$ will converge to and represent $f(x)$. What our analysis has shown is merely that if $f(x)$ has an expression of the form (1), then the coefficients of the terms in the series are given by the formulae.

(4) The convergence of the Fourier series and if convergent under what conditions it will represent the function which generates it are broad questions under investigation.

(5) But Dirichlet has formulated certain conditions known as Dirichlet conditions under which certain functions posses valid Fourier expansions.

These conditions guarantee that the Fourier expansion of $f(x)$ will converget to $f(x)$ at all points of continuity. The conditions are

(i) $f(x)$ must never become infinite in the defined interval.

(ii) $f(x)$ must be single-valued.

(iii) $f(x)$ must have at most a finite number of maxima and minima in the interval of a definition.

(iv) $f(x)$ must have at most, a finite number of discontinuities (including infinities) in the interval of definition.

Note. These conditions are not necessary ; but it is not easy to give more general conditions without a deep study of the subject.

(6) It can be shown that at any point of discontinuity (say, $x = a$) where a function is represented by a Fourier series, the value of the Fourier series is

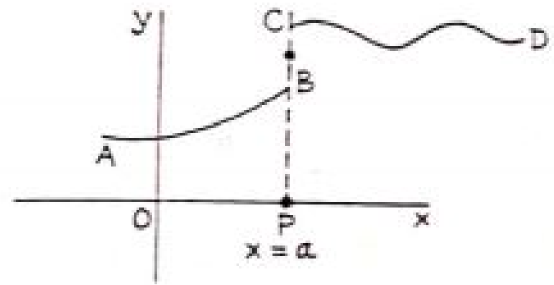


Fig. 3

$$\frac{1}{2}[f(a+0) + f(a-0)].$$

The function is discontituous at $x = a$. If $f(x)$ is expressed at a Fourier series, the value of the series at $x = a$ is $\frac{1}{2}$ (PB + PC).

(7) Generally $f(x)$ is expanded in the interval from 0 to 2π or in the interval $-\pi$ to π.

Putting $\lambda = 0$, in the interval $\lambda \le x \le \lambda + 2\pi$, we get the interval

$$0 \le x \le 2\pi.$$

Putting $\lambda = -\pi$, in the interval $\lambda \le x \le \lambda + 2\pi$, we get the interval $-\pi \le x \le \pi$.

Scanned with CamScanner

Examples.

Ex.1. Show that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}$ in the interval $(-\pi \le x \le \pi)$.

Deduce that (i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$.

Then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{+\pi} = \frac{2\pi^2}{3}$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n}\right]_{-\pi}^{\pi} - \frac{2}{\pi}\int_{-\pi}^{\pi}\frac{x \sin nx}{n}\, dx$$

$$= -\frac{2}{n\pi}\left\{\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos nx}{n}\, dx\right\}$$

$$= \frac{4}{n^2}\cos n\pi = \frac{(-1)^n 4}{n^2}.$$

When n is odd, $a_n = \frac{-4}{n^2}$.

When n is even, $a_n = \frac{4}{n^2}$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x^2\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{2}{n}\int_{-\pi}^{\pi} x\cos nx\, dx\right\}$$

$$= \frac{2}{n\pi}\int_{-\pi}^{\pi} x\cos nx\, dx$$

$$= \frac{2}{n\pi}\left\{\left[\frac{x\sin nx}{n}\right]_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}\sin nx\, dx\right\} = 0.$$

(This could have been inferred as the integrand is an odd function and $\int_{-\pi}^{\pi} f(x)\, dx = 0$ where $f(x)$ is odd.)

$$\therefore\ x^2 = \frac{\pi^2}{3} + \sum\frac{(-1)^n 4\cos nx}{n^2}$$

$$= \frac{\pi^2}{3} + 4\sum\frac{(-1)^n\cos nx}{n^2}$$

When $x = 0$, we have

$$0 = \frac{\pi^2}{3} + 4\sum\frac{(-1)^n}{n^2}$$

$$= \frac{\pi^2}{3} + 4\left\{-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots\right\}.$$

$$\therefore\ \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \dots = \frac{\pi^2}{12} \qquad \dots(1)$$

Put $x = \pi$, we have

$$\pi^2 = \frac{\pi^2}{3} + 4\sum\frac{(-1)^n\cos n\pi}{n^2}$$

$$= \frac{\pi^2}{3} + 4\sum\frac{1}{n^2}.$$

Scanned with CamScanner

$$\therefore \sum \frac{1}{n^2}, \text{ i.e., } \frac{1}{1^2} + \frac{1}{2^2} + \ldots = \frac{\pi^2}{6}. \qquad \ldots (2)$$

Adding (1) and (2) and dividing it by 2, we get the result

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

Ex.2. Express $f(x) = \frac{1}{2}(\pi - x)$ as a Fourier series with period 2π, to be valid in the interval 0 to 2π.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2}(\pi - x)\, dx$$

$$= -\frac{1}{4\pi} \left[(\pi - x)^2 \right]_0^{2\pi} = 0$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (\pi - x) \cos nx\, dx$$

$$= \frac{1}{2\pi} \left\{ \left[\frac{(\pi - x)\sin nx}{n} \right]_0^{2\pi} + \int_0^{2\pi} \frac{\sin nx}{n}\, dx \right\}$$

$$= 0.$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (\pi - x) \sin nx\, dx$$

$$= \frac{1}{2\pi} \left\{ \left[\frac{-(\pi - x)\cos nx}{n} \right]_0^{2\pi} - \int_0^{2\pi} \frac{\cos nx}{n}\, dx \right\}$$

$$= \frac{1}{n}.$$

$$\therefore \frac{1}{2}(\pi - x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots$$

In this series if we put $x = \frac{\pi}{2}$, we get the well-known result

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}.$$

Ex.3. A function $f(x)$ is defined within the range $(0, 2\pi)$ by the relations

$f(x) = x$ in the range $(0, \pi)$

$= 2\pi - x$ in the range $(\pi, 2\pi)$

Express $f(x)$ as a Fourier series in the range $(0, 2\pi)$.

If we draw the curve $f(x)$ in the range $(0, 2\pi)$, the shape of the curve is as shown in the figure.

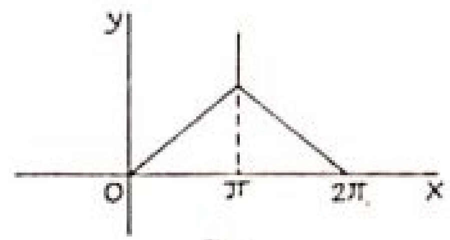


Fig. 4

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

Scanned with CamScanner

$$= \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} (2\pi - x)\,dx\right\}$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} - \frac{1}{\pi}\left[\frac{(2\pi - x)^2}{2}\right]_{\pi}^{2\pi}$$

$$= \pi.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right\}$$

$$+ \frac{1}{\pi}\left\{\left[\frac{(2\pi - x)\sin nx}{n}\right]_{\pi}^{2\pi} + \int_{\pi}^{2\pi}\frac{\sin nx}{n}\,dx\right\}$$

$$= -\frac{1}{\pi}\int_0^{\pi}\frac{\sin nx}{n}\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}\frac{\sin nx}{n}\,dx$$

$$= \frac{1}{n^2\pi}[\cos nx]_0^{\pi} - \frac{1}{n^2\pi}[\cos nx]_{\pi}^{2\pi}$$

$$= \frac{1}{n^2\pi}(\cos n\pi - 1) - \frac{1}{n^2\pi}(1 - \cos n\pi)$$

$$= \frac{2\cos n\pi}{n^2\pi} - \frac{2}{n^2\pi}$$

$$= \frac{2(-1)^n - 2}{n^2\pi}.$$

Similarly it can be shown that $b_n = 0$.

$$\therefore\quad f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2(-1)^n - 2}{n^2\pi}\cos nx$$

$$= \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos nx$$

When n is even, $1 - (-1)^n = 0$ and

When n is odd, $1 - (-1)^n = 2$.

$$\therefore\quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right).$$

When $x = 0$, $f(x) = 0$.

$$\therefore\quad 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right).$$

$$\therefore\quad 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

If we put $x = \pi$, we get the same result.

Ex.4. Find in the range $-\pi$ to π, a Fourier series for

$y = 1 + x \qquad 0 < x < \pi$

$y = -1 + x \qquad -\pi < x < 0$

Let $y = \dfrac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} y\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-1 + x)\,dx + \frac{1}{\pi}\int_0^{\pi}(1 + x)\,dx$$

$$= \frac{1}{\pi}\left[\frac{(x-1)^2}{2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{(1+x)^2}{2}\right]_0^{\pi}$$

$$= 0.$$

Scanned with CamScanner

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-1+x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(1+x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\left[\frac{(x-1)\sin nx}{n}\right]_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\sin nx}{n}\,dx\right\}$$

$$+ \frac{1}{\pi}\left\{\left[\frac{(x+1)\sin nx}{n}\right]_{0}^{\pi} - \int_{0}^{\pi}\frac{\sin nx}{n}\,dx\right\}$$

$$= \frac{1}{n^2}[\cos nx]_{-\pi}^{0} + \frac{1}{n^2}[\cos nx]_{0}^{\pi}$$

$$= 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-1+x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(1+x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\left[\frac{-(x-1)\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0}\cos nx\,dx\right\}$$

$$+ \frac{1}{\pi}\left\{\left[\frac{-(1+x)\cos nx}{n}\right]_{0}^{\pi} + \frac{1}{n}\int_{0}^{\pi}\cos nx\,dx\right\}$$

$$= \frac{2}{n\pi} - \frac{2(\pi+1)}{n\pi}\cos n\pi$$

$$= \frac{2}{n\pi} - \frac{2(\pi+1)(-1)^n}{n\pi}.$$

When n is even, $b_n = \frac{2}{n\pi} - \frac{2(\pi+1)}{n\pi} = -\frac{2}{n}$.

When n is odd, $b_n = \frac{2}{n\pi} + \frac{2(\pi+1)}{n\pi}$

$$= \frac{2(\pi+2)}{n\pi}.$$

$$\therefore\ y = \frac{2(\pi+2)}{\pi}\sin x - \frac{2}{2}\sin 2x + \frac{2(\pi+2)}{3\pi}\sin 3x$$

$$- \frac{2}{4}\sin 4x + \frac{2(\pi+2)}{5\pi}\sin 5x - \ldots$$

When $x = \frac{\pi}{2}$, $y = 1 + \frac{\pi}{2} = \frac{2+\pi}{2}$.

When $x = \frac{\pi}{2}$, the right side of the series becomes

$$\frac{2(\pi+2)}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\right).$$

$$\therefore\ \left(1 - \frac{1}{3} + \frac{1}{5}\ldots\right)\frac{2(\pi+2)}{\pi} = \frac{2+\pi}{2}.$$

$$\therefore\ 1 - \frac{1}{3} + \frac{1}{5} \ldots = \frac{\pi}{4}.$$

Exercises 35.

1. Determine the Fourier expansion of the following functions in the intervals noted against them :-

(i) $f(x) = x$ $\qquad -\pi < x < \pi$.

(ii) $f(x) = \pi^2 - x^2$ $\qquad -\pi < x < \pi$.

(iii) $f(x) = \frac{(\pi - x)^2}{4}$ $\qquad 0 < x < 2\pi$.

2. Show that in the range 0 to 2π, the expansion of e^x as a Fourier series is

$$e^x = \frac{e^{2\pi} - 1}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2+1} - \sum_{n=1}^{\infty}\frac{n\sin nx}{n^2+1}\right\}.$$

Scanned with CamScanner

Deduce that $1 - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{32}$.

14. If $f(x) = x$ in $\left(0 < x < \frac{\pi}{2}\right)$

$= \pi - x$ in $\left(\frac{\pi}{2} < x < \frac{3\pi}{2}\right)$

$= x - 2\pi$ in $\left(\frac{3\pi}{2} < x < 2\pi\right)$,

prove that $f(x) = \frac{4}{\pi}\left\{\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} \dots\right\}$.

15. If $f(x) = a$ in $0 < x < \frac{\pi}{2}$

$= 0$ in $\frac{\pi}{2} < x < \frac{3\pi}{2}$

$= a$ in $\frac{3\pi}{2} < x < 2\pi$,

show that $f(x)$ can be expressed as a Fourier series in cosines only and prove that

$$f(x) = \frac{a}{2} + \frac{2a}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} \dots\right).$$

§ 3. Even and odd functions.

If $f(x) = f(-x)$, then $f(x)$ is said to be even function.

If $f(x) = -f(-x)$, then $f(x)$ is said to be an odd function.

The functions x^2, $x^4 + 3x^2 + 2\cos x, \dots$ are examples of even functions and x^3, $2x^3 + 3x$, $\sin 2x, \dots$ are examples of odd functions. If we actually draw the graphs of some odd functions and some even functions, we will note that graphs of even functions are symmetrical with respect to the y-axis and the graphs of odd functions are symmetrical with respect to the origin.

§ 3.1. Properties of odd and even functions.

(i) $\int_{-a}^{+a} f(x)\,dx = 0$ if $f(x)$ is odd.

(ii) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if $f(x)$ is even.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

In the first integral on the right side, put $x = -y$.

Then $\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-y)(-dy) = \int_{0}^{a} f(-y)\,dy$

$$= \int_{0}^{a} f(-x)\,dx$$

$$\therefore \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x) + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} [f(-x) + f(x)]\,dx$$

If $f(x)$ is odd, $f(-x) = -f(x)$.

Hence if $f(x)$ is odd, $\int_{-a}^{a} f(x)\,dx = 0$.

If $f(x)$ is even, $f(-x) = f(x)$.

$\therefore \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if $f(x)$ is even.

§ 3.2. These properties of odd and even functions can be used to shorten the computation when we have to find the Fourier series of either an even or odd function for the interval $-\pi < x < \pi$.

If $f(x)$ be expanded as a Fourier series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

we have $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \, (n = 0, 1, 2, \ldots).$

$$b_n = \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad (n = 1, 2, \ldots).$$

Case i. $f(x)$ is an odd function, then $f(x) \cos nx$ is also an odd function.

$$\therefore \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0.$$

Hence $a_n = 0$.

$f(x) \sin nx$ is even function.

$$\therefore \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 2\int_{0}^{\pi} f(x) \sin nx \, dx.$$

Hence $b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx.$

Case ii. If $f(x)$ is an even function, then $f(x) \sin nx$ is an odd function and hence $\int_{-\pi}^{\pi} f(x) \sin nx = 0.$

$$\therefore b_n = 0.$$

$f(x) \cos nx$ is an even function.

$$\therefore \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 2\int_{0}^{\pi} f(x) \cos nx \, dx.$$

$$\therefore a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx.$$

Hence we get the results that

(i) If $f(x)$ is an even function, $f(x)$ can be expanded as a series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

in the interval $(-\pi < x < \pi)$ where

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx \, (n = 0, 1, 2, \ldots);$$

(ii) If $f(x)$ is an odd function, $f(x)$ can be expanded as a series of the form

$$\sum_{n=1}^{\infty} b_n \sin nx$$

in the interval $(-\pi < x < \pi)$ where

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx.$$

Examples.

Ex.1. Express $f(x) = x \, (-\pi < x < \pi)$ as a Fourier series with period 2π.

$f(x) = x$ is an odd function.

Hence in the expansion, the cosine terms are absent.

$$\therefore x = \sum b_n \sin nx$$

where $b_n = \frac{2}{\pi} \int_{0}^{\pi} x \sin nx \, dx$

$$= \frac{2}{\pi}\left[\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} \, dx\right]$$

$$= -\frac{2}{n} \cos n\pi = -\frac{2}{n}(-1)^n$$

$$= \frac{(-1)^{n-1} 2}{n}.$$

$$\therefore x = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right).$$

Ex.2. If $f(x) = -x$ in $-\pi < x < 0$

$= x$ in $0 \le x < \pi$

expand $f(x)$ as Fourier series in the interval $-\pi$ to π.

Deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

We easily see that $f(x)$ is an even function. By drawing the graph of the function and noting that it is symmetrical with respect to the y-axis.

$$\therefore \; f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx \quad (n = 0, 1, 2, \dots)$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi} x\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx$$

$$= \frac{2}{\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\, dx\right\}$$

$$= \frac{2}{n^2\pi}\left[\cos nx\right]_0^{\pi} = \frac{2}{n^2\pi}(\cos n\pi - 1)$$

$$= \frac{2}{n^2\pi}\{(-1)^n - 1\}.$$

When n is odd, $a_n = -\frac{4}{n^2\pi}$.

When n is even, $a_n = 0$.

$$\therefore \; f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$$

When $x = 0$, $f(x) = 0$.

$$\therefore \; 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right).$$

$$\therefore \; \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

§ 4. Half range fourier series.

It is often convenient to obtain a Fourier expansion of a function to hold for a range which is half the period of the Fourier series, that is to expand $f(x)$ in the range $(0, \pi)$ in a Fourier series of period 2π. In the half range $f(x)$ can be expanded as a series containing cosines alone or sines alone.

The folowing identities are very useful in this connection :-

(i) $\int_0^{\pi} \cos mx\, dx = 0$ if m is an integer.

(ii) $\int_0^{\pi} \cos mx \cos nx\, dx = 0$ if $m = n$ and m and n are integers.

(iii) $\int_0^{\pi} \sin mx \sin nx\, dx = 0$ if $m \ne n$ and m and n are integers.

(iv) $\int_0^{\pi} \cos mx \cos nx\, dx = \int_0^{\pi} \cos^2 mx\, dx$ if $m = n$

$$= \frac{\pi}{2}.$$

(v) $\int_0^{\pi} \sin mx \sin nx\, dx = \int_0^{\pi} \sin^2 mx\, dx$ if $m = n$

$$= \frac{\pi}{2}.$$

§ 5.1. Development in cosine series.

Let $f(x)$ be expanded as a series containing cosines only and let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad ...(1)$$

If we integrate both sides of (1) between limits 0 and π, then

$$\int_0^{\pi} f(x)\,dx = \int_0^{\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} a_n \int_0^{\pi} \cos nx\,dx$$

$$= \frac{a_0 \pi}{2}.$$

$$\therefore \; a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

If we multiply both sides of the equation (1) by $\cos nx$ and integrate between 0 and π, then

$$\int_0^{\pi} f(x)\cos nx\,dx = a_n \frac{\pi}{2}.$$

Since all the terms except the term containing a_n vanish.

$$\therefore \; a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

§ 5.2. Development in sine series.

Let $f(x)$ be expanded as a series containing sines only and let

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Multiply both sides of the above equation by $\sin nx$ and integrate from 0 to π.

Then $\int_0^{\pi} f(x)\sin nx\,dx = b_n \frac{\pi}{2}$ since all the terms except the term containing b_n vanish.

$$\therefore \; b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Examples.

Ex.1. Find a sine series for $f(x) = c$ in the range 0 to π.

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} c \sin nx\,dx$$

$$= \frac{2c}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2c}{n\pi}(1 - \cos n\pi)$$

$$= \frac{2c}{n\pi}[1 - (-1)^n].$$

When n is even, $b_n = 0$.

When n is odd, $b_n = \frac{4c}{n\pi}$.

Hence $c = \frac{4c}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x \ldots\right)$.

Putting $x = \frac{\pi}{2}$, $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Ex.2. If $f(x) = x$ when $0 < x < \frac{\pi}{2}$

$= \pi - x$ when $x > \frac{\pi}{2}$,

expand $f(x)$ as a sine seires in the interval $(0, \pi)$.

Scanned with CamScanner

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi}\int_0^{\pi/2} x \sin nx \, dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx$$

$$= \frac{2}{\pi}\left\{\left[\frac{-x\cos nx}{n}\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos nx}{n} dx\right\}$$

$$+ \frac{2}{\pi}\left\{\left[\frac{-(\pi - x)\cos nx}{n}\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \frac{\cos nx}{n} dx\right\}$$

$$= \frac{2}{n^2\pi}[\sin nx]_0^{\pi/2} - \frac{2}{n^2\pi}[\sin nx]_{\pi/2}^{\pi}$$

$$= \frac{2}{n^2\pi}\sin\frac{n\pi}{2} + \frac{2}{n^2\pi}\sin\frac{n\pi}{2}$$

$$= \frac{4}{n^2\pi}\sin\frac{n\pi}{2}.$$

When n is even, $b_n = 0$.

When n is odd and is of the form $4p + 1$, $b_n = \frac{4}{n^2\pi}$.

When n is odd and is of the form $4p - 1$, $b_n = -\frac{4}{n^2\pi}$.

$$\therefore \; b_2 = b_4 = b_6 = \ldots = 0.$$

$$b_1 = \frac{4}{1^2\pi}, \; b_5 = \frac{4}{5^2\pi}, \; b_9 = \frac{4}{9^2\pi}, \ldots$$

$$b_3 = \frac{-4}{3^2\pi}, \; b_7 = -\frac{4}{7^2\pi}.$$

$$\therefore \; f(x) = \frac{4}{\pi}\left\{\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} \ldots\right\}$$

Ex.3. Find a cosine series in the range 0 to π for

$$f(x) = x \quad \left(0 < x < \frac{\pi}{2}\right)$$

$$= \pi - x \quad \left(\frac{\pi}{2} < x < \pi\right).$$

Let $f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx$

where $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$

$$\therefore \; a_0 = \frac{2}{\pi}\int_0^{\pi/2} x \, dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x) \, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi/2} - \frac{2}{\pi}\left[\frac{(\pi - x)^2}{2}\right]_{\pi/2}^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi/2} x \cos nx \, dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x) \cos nx \, dx$$

$$= \frac{2}{\pi}\left\{\left[\frac{x \sin nx}{n}\right]_0^{\pi/2} - \frac{1}{n}\int_0^{\pi/2} \sin nx \, dx\right\}$$

$$+ \frac{2}{\pi}\left\{\left[\frac{(\pi - x)\sin nx}{n}\right]_{\pi/2}^{\pi} + \frac{1}{n}\int_{\pi/2}^{\pi} \sin nx \, dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\left[\cos nx\right]_0^{\pi/2}\right\}$$

$$+ \frac{2}{\pi}\left\{-\frac{\pi}{2n}\sin\frac{n\pi}{2} - \frac{1}{n^2}\left[\cos nx\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{1}{n^2} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{1}{n^2}\cos n\pi + \frac{1}{n^2}\cos\frac{n\pi}{2}\right\}$$

Scanned with CamScanner

$$= \frac{2}{\pi}\left\{\frac{-1-(-1)^n+2\cos\frac{n\pi}{2}}{n^2}\right\}.$$

When n is odd, $a_n = 0$,

When n is even and is of the form $4p$, $a_n = 0$.

When n is even and is of the form $4p+2$, $a_n = -\frac{8}{n^2\pi}$.

$$\therefore \quad a_1 = a_3 = a_5 \ldots = 0$$

$$a_4 = a_8 = a_{12} \ldots = 0$$

$$a_2 = -\frac{8}{2^2\pi} = -\frac{2}{1^2\pi}$$

$$a_6 = -\frac{8}{6^2\pi} = -\frac{2}{3^2\pi}$$

$$a_{10} = -\frac{8}{10^2\pi} = -\frac{2}{5^2\pi}.$$

$$\therefore \quad f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\right)$$

Exercises 36.

1. If the function $y = x$ in the range 0 to π is expanded as a sine series, show that it is equal to

$$2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \ldots\right).$$

2. Expand $\frac{\pi x}{8}(\pi - x)$ in a sine series valid when $0 \le x < \pi$.

3. Find a sine series for

$$f(x) = x, \quad 0 < x < \frac{\pi}{2}$$

$$= 0, \quad \frac{\pi}{2} < x < \pi.$$

8. Find a Fourier cosines series corresponding to the function $f(x) = x$ defined in the interval $(0, \pi)$.

9. Find the Fourier sine series and the Fourier cosine series corresponding to the function, $f(x) = \pi - x$ when $0 < x < \pi$ defined in the interval 0 to π.

10. Show that, when $0 < x < \pi$

$$f(x) = \sin 2x + \frac{1}{2}\sin 4x + \frac{1}{4}\sin 8x + \frac{1}{5}\sin 10x \ldots$$

$$= \frac{2}{\sqrt{3}}\left(\cos x - \frac{1}{5}\cos 5x + \frac{1}{7}\cos 7x - \frac{1}{11}\cos 11x \ldots\right)$$

where $f(x) = \frac{\pi}{3} \quad \left(0 < x < \frac{\pi}{3}\right)$

$= 0 \quad \left(\frac{\pi}{3} < x < \frac{2\pi}{3}\right)$

$= -\frac{\pi}{3} \quad \left(\frac{2\pi}{3} < x < \pi\right).$

11. Expand x^3 and x in Fourier sine series valid when $-\pi < x < \pi$ and hence find the value of the sum of the series $\sin x - \frac{1}{2^3}\sin 2x + \frac{1}{3^3}\sin 3x - \frac{1}{4^3}\sin 4x + \ldots$ for all values of x.

12. Expand $x\sin x$ as a Fourier cosine series in the range $0 < x < \pi$.

Deduce that $\frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots = \frac{\pi}{4}$.

§ 6. Change of interval.

In practice we often require to find a Fourier series for an interval which is not of length π or 2π. In many problems, the period of the function is to be expanded is not 2π but some other interval, say $2l$.

Suppose we have to expand $f(x)$ in the interval $-l$ to l as a Fourier series.

Let $X = \frac{\pi x}{l}$ i.e., $x = \frac{lX}{\pi}$.

When $x = -l$, $X = -\pi$ and

when $x = l$, $X = \pi$.

Hence the function becomes $f\left(\frac{lX}{\pi}\right)$ where $-\pi < X < \pi$. $f\left(\frac{lX}{\pi}\right)$ can be expanded as Fourier series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nX + b_n \sin nX)$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lX}{\pi}\right)\cos nX\, dX \quad (n = 0, 1, 2, \ldots)$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lX}{\pi}\right)\sin nX\, dX \quad (n = 1, 2, \ldots).$

Reverting back to the original variable x

$$x = \frac{lX}{\pi}, \quad \therefore \quad dx = \frac{l\,dX}{\pi}.$$

When $X = \pi$, $x = l$; when $X = -\pi$, $x = -l$.

$$\therefore \quad a_n = \frac{1}{\pi}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\cdot\frac{\pi\,dx}{l}$$

$$= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

Similarly $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right).$$

§ 6.1. It can be shown, in a similar manner, that

(i) if $f(x)$ is an even function, $f(x)$ can be expanded as a Fourier series consisting of cosine terms only in the interval of length $2l$.

Scanned with CamScanner

In that case, $f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos \frac{n\pi x}{l}$ where

$$a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$$

(ii) If $f(x)$ is an odd function, $f(x)$ can be expanded as a fourier series consisting of sine terms only in the interval of length $2l$.

In that case, $f(x) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l}$ where

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

(iii) $f(x)$ can be expanded as a sine series in half range $(0, l)$ with period $2l$ of the form

$$\sum b_n \sin \frac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$

(iv) $f(x)$ can be expanded as a cosine series in half range $(0, l)$ with period $2l$ of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where $a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l}\, dx \; (n = 0, 1, 2, \ldots)$.

Examples.

Ex.1. In the range $(0, 2l)$ $f(x)$ is defined by the relations

$f(x) = 0$ when $0 < x < l$

$\quad = a$ when $l < x < 2l$,

expand $f(x)$ as a Fourier series of period $2l$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right)$$

where $a_n = \frac{1}{l}\int_0^{2l} f(x) \cos \frac{n\pi x}{l}\, dx = \frac{1}{l}\int_l^{2l} a \cos \frac{n\pi x}{l}\, dx$

$$= \frac{a}{l}\left[\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_l^{2l} = \frac{a}{n\pi}(\sin 2n\pi - \sin n\pi)$$

$$= 0.$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\, dx = \frac{1}{l}\int_l^{2l} a\, dx = a.$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx = \frac{1}{l}\int_l^{2l} a \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= -\frac{a}{l}\left[\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_l^{2l} = -\frac{a}{n\pi}(\cos 2n\pi - \cos n\pi)$$

$$= \frac{a}{n\pi}[(-1)^n - 1].$$

Hence $b_n = 0$ when n is even and $b_n = -\frac{2a}{n\pi}$ when n is odd.

$$\therefore f(x) = \frac{a}{2} - \frac{2a}{\pi}\left[\sin \frac{\pi x}{l} + \frac{1}{3}\sin \frac{3\pi x}{l} + \frac{1}{5}\sin \frac{5\pi x}{l} + \ldots\right]$$

Ex.2. Find a Fourier series with period 3 to represent $f(x) = 2x - x^2$ in the range $(0, 3)$

Hence $2l = 3$. $\therefore l = \frac{3}{2}$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right)$$

Scanned with CamScanner

When n is even, $a_n = 0$.

$$a_0 = \frac{2}{c}\int_0^c (c - x)\, dx = c.$$

$$\therefore\ f(x) = \frac{c}{2} + \frac{2c}{\pi^2}\left[\frac{\cos\frac{\pi x}{c}}{1^2} + \frac{\cos\frac{3\pi x}{c}}{3^2} + \dots\right].$$

Exercises 37.

1. If $f(x) = \pi x \quad (0 \le x < 1)$

$= \pi(2 - x) \quad (1 \le x \le 2)$,

show that in the range (0,2)

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_0^{\infty}\frac{\cos(2n+1)\pi x}{(2n+1)^2}.$$

2. If $f(x) = 0 \quad$ when $-l < x < 0$

$= 1 \quad$ when $0 < x < l$,

express $f(x)$ as a Fourier series in the range $-l < x < l$.

3. Express $f(x) = x$ as Fourier series with interval 2 in the range $-1 < x < 1$.

4. Express $f(x) = x^2$ as a Fourier series with interval 2 in the range $-1 < x < 1$.

5. Express in the form of Fourier series the function

$f(x) = x \quad 0 < x < a$

$= a \quad a < x < 2a$.

6. If $f(x) = \frac{x}{l}$ when $(0 < x < l)$

$= \frac{2l - x}{l}$ when $(l < x < 2l)$.

prove that $f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \ldots\right)$.

7. Find a half range sine series for

$$f(x) = \frac{2x}{l} \qquad 0 < x < \frac{l}{2}$$
$$= \frac{2}{l}(l - x) \qquad \frac{l}{2} < x < l.$$

8. If $f(x) = \frac{l}{4} - x \qquad \left(0 < x < \frac{l}{2}\right)$

$$= x - \frac{3}{4}l \qquad \left(\frac{l}{2} < x < l\right).$$

prove that over the range $0 < x < l$

$$f(x) = \frac{2l}{\pi^2}\left\{\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{6\pi x}{l} + \frac{1}{5^2}\cos\frac{10\pi x}{l} + \ldots\right\}.$$

9. Expand $f(x) = x$ as a half range cosine series in the interval $0 < x < l$.

10. Expand $f(x) = a\left(1 - \frac{x}{l}\right)$ in the range $(0, l)$ in a half range sine series and also in a half range cosine series.

11. Given that $f(x) = l \qquad$ when $0 < x < \frac{l}{2}$

$$= \frac{2d}{l}(l - x) \quad \text{when } \frac{l}{2} < x < l,$$

prove that $f(x) = \frac{8d}{\pi^2}\left[\sin\frac{\pi x}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l} + \frac{1}{5^2}\sin\frac{5\pi x}{l} \ldots\right]$.

12. Find a sine series and a cosine series of period 4 which represents the function

$$f(x) = 2x \qquad \text{when } 0 < x < 1$$
$$= 4 - 2x \qquad \text{when } 1 < x < 2.$$

§ 7. Combination of series.

Some of the known Fourier expansions may be easily combined to yield Fourier expansions of any linear and quadratic functions of x over the half range interval $0 < x < l$.

The following half range expansions in the interval $0 < x < l$ will be found useful :-

(i) $1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \ldots\right)$.

(ii) $x = \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} \ldots\right)$.

(iii) $x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} \ldots\right)$.

(iv) $x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left(\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} \ldots\right)$.

(v) $$x^2 = \frac{2l^2}{\pi^3}\left\{(\pi^2 - 4)\sin\frac{\pi x}{l} - \frac{\pi^2}{2}\sin\frac{2\pi x}{l} + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin\frac{3\pi x}{l} - \frac{\pi^2}{4}\sin\frac{4\pi x}{l} + \left(\frac{\pi^2}{5} - \frac{4}{5^3}\right)\sin\frac{5\pi x}{l} - \frac{\pi^2}{6}\sin\frac{6\pi x}{l} \ldots\right.$$

How the above series can be combined to give Fourier expansions in half range is illustrated in the follwing examples.

Examples.

Ex.1. Find a sine and a cosine series for the function

$f(x) = 3x - 2$ in the interval $0 < x < 4$.

We have in the interval $0 < x < 4$

$$x = \frac{8}{\pi}\left(\sin\frac{\pi x}{4} - \frac{1}{2}\sin\frac{2\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} \ldots\right)$$

Scanned with CamScanner

$$2 = \frac{8}{\pi}\left(\sin\frac{\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} + \frac{1}{5}\sin\frac{5\pi x}{4}\ \dots\right).$$

$\therefore\ 3x - 2$

$$= \frac{24}{\pi}\left(\sin\frac{\pi x}{4} - \frac{1}{2}\sin\frac{2\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4}\ \dots\right)$$

$$- \frac{8}{\pi}\left(\sin\frac{\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} + \frac{1}{5}\sin\frac{5\pi x}{4}\ \dots\right)$$

$$= \frac{8}{\pi}\left(2\sin\frac{\pi x}{4} - \frac{3}{2}\sin\frac{2\pi x}{4} + \frac{2}{3}\sin\frac{3\pi x}{4} - \frac{3}{4}\sin\frac{4\pi x}{4} + \frac{2}{5}\sin\frac{5\pi x}{4}\ \dots\right.$$

In the interval $0 < x < 4$

$$x = 2 - \frac{16}{\pi^2}\left(\cos\frac{\pi x}{4} + \frac{1}{3^2}\cos\frac{3\pi x}{4} + \frac{1}{5^2}\cos\frac{5\pi x}{4}\ \dots\right).$$

$$\therefore\ 3x - 2 = 4 - \frac{48}{\pi^2}\left(\cos\frac{\pi x}{4} + \frac{1}{3^2}\cos\frac{3\pi x}{4} + \frac{1}{5^2}\cos\frac{5\pi x}{4}\ \dots\right).$$

Ex.2. Express in the interval $0 < x < l$ as a half range cosine and sine series the function $f(x) = x(l - x)$.

In the interval $0 < x < l$, we have

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l}\ \dots\right).$$

$$x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left(\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l}\ \dots\right).$$

$$\therefore\ x(l - x) = lx - x^2$$

$$= \frac{l^2}{2} - \frac{4l^2}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l}\ \dots\right).$$

$$- \frac{l^2}{3} + \frac{4l^2}{\pi^2}\left(\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l}\ \dots\right)$$

$$= \frac{l^2}{6} - \frac{4l^2}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{4^2}\cos\frac{4\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l}\ \dots\right).$$

In the interval $0 < x < l$, we have

$$x = \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l}\ \dots\right).$$

$$x^2 = \frac{2l^2}{\pi^3}\left\{(\pi^2 - 4)\sin\frac{\pi x}{l} - \frac{\pi^2}{2}\sin\frac{2\pi x}{l} + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin\frac{3\pi x}{l} - \frac{\pi^2}{4}\sin\frac{4\pi x}{l}\ \dots\right\}.$$

$$\therefore\ x(l - x) = lx - x^2$$

$$= \sin\frac{\pi x}{l}\left\{\frac{2l^2}{\pi} - \frac{2l^2}{\pi^3}(\pi^2 - 4)\right\}$$

$$+ \sin\frac{2\pi x}{l}\left\{-\frac{l^2}{\pi} + \frac{l^2}{\pi}\right\}$$

$$+ \sin\frac{3\pi x}{l}\left\{\frac{2l^2}{3\pi} - \frac{2l^2}{\pi^3}\left(\frac{\pi^3}{3} - \frac{4}{3^3}\right)\right\}$$

$$+ \sin\frac{4\pi x}{l}\left\{-\frac{2l^2}{4\pi} + \frac{2l^2}{4\pi}\right\} + \dots$$

$$= \frac{8l^2}{\pi^3}\left\{\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \dots\right\}.$$

Exercies 38.

1. Find a sine series and a cosine series for the function

   $f(x) = 2x - 4,\quad 0 < x < 4.$

2. Find a sine series and cosine series for the function

   $f(x) = 3x - 9,\quad 0 < x < 6.$

3. Expand $f(x) = x - 1,\quad 0 < x < \pi$ as a cosine series and a sine series by combining suitable series.

FOURIER TRANSFORMS

9. Complex form of Fourier Integral Formula

§9.1 Theorem If $f(x)$ is piece wise continuously differentiable in (-l, l) then $f(x)$ has the complex Fourier series expansion

$$f(x) = \sum_{1}^{\infty} C_n \, e^{-\frac{i n \pi x}{l}}$$

where the coefficients C_n are given by

$$C_n = \frac{1}{2l} \int_{-l}^{l} f(x) \, e^{-\frac{i n \pi x}{l}} \, dx$$

Multiply $f(x)$ by $e^{\frac{i m \pi x}{l}}$ and integrate from $-l$ to l

$$\therefore \int_{-l}^{l} f(x) \, e^{-\frac{i m \pi x}{l}} \, dx$$

$$= \sum_{n=-\infty}^{\infty} \int_{-l}^{l} C_n \, e^{\frac{i(m-n)\pi x}{l}} \, dx$$

$$= \sum_{n=-\infty}^{\infty} C_n \int_{-l}^{l} \left\{ \cos \frac{(m-n)\pi x}{l} \, i + \sin \frac{(m-n)\pi x}{l} \right\} dx$$

$$= \sum_{n=-\infty}^{\infty} C_n \int_{-l}^{l} \cos \frac{(m-n)\pi x}{l} \, dx$$

$$+ \; i \sum_{n=-\infty}^{\infty} C_n \int_{-l}^{l} \sin \frac{(m-n)\pi x}{l} \, dx$$

If $m = n$, $\int_{-l}^{l} \cos \frac{(m-n)\pi x}{l} \, dx = 2l$ and

$$\int_{-l}^{l} \sin \frac{(m-n)\pi x}{l} \, dx = 0$$

Hence $\int_{-l}^{l} f(x) \; e^{\frac{i m \pi x}{l}} \, dx = 2lC_n$ when $m = n$

$\therefore \quad C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{\frac{in\pi x}{l}}\, dx$

9.2 Fourier Integral Theorem

If $f(x)$ is piece wise continuously diferentiable and absolutely integrable in the entire line (x axis) then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i(t-x)s}\, dt\, ds$$

By § 9.1

$$f(x) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2l}\int_{-l}^{l} f(t)\, e^{\frac{in\pi t}{l}}\, dt\right) e^{\frac{-in\pi x}{l}}\, dx$$

$$= \sum_{n=-\infty}^{\infty}\frac{1}{2l}\int_{-l}^{l} f(t)\, e^{\frac{in\pi(t-x)}{l}}\, dt$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\delta s\int_{-l}^{l} f(t)\, e^{in(t-x)\delta s}\, dt$$

by putting $\frac{\pi}{l} = \delta s$

Letting $l \to \infty$ i.e. $\delta s \to 0$ and changing summation to definite integral, we get the Fourier integral (since $n\,\delta s \to s$)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i(t-x)s}\, dt\, ds$$

This is the **complex form** of the Fourier Integral

9.3. $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i(t-x)s}\, dt\, ds$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\left\{\frac{1}{\sqrt{2\pi}} f(t)\, e^{ist}\, dt\right\} e^{-ixs}\, ds$$

If $F(s) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}} f(t)\, e^{ist}\, dt$

then $f(x) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}} F(s)\, e^{-ixs}\, ds$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-ixs}\, ds.$$

$F(s)$ is called the **Fourier transform** of $f(x)$

Cor. $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left[\cos(st) + i\sin(st)\right] dt$$

We note the following results

(i) $F(s)$ is a complex function if $f(t)$ is a real function.

(ii) $F(s)$ is a real function if $f(t)$ is a real even function.

(iii) $F(s)$ is a purely imaginary function if $f(t)$ is a real odd function.

(iv) If $F\{f(x)\} = F(s)$ then $f(x) = F^{-1}\{F(s)\}$

where F^{-1} denotes the inverse of F.

10. Properties of Fourier transform

1. $F\{af(x) + b\varphi(x)\} = aF\{f(x)\} + bf\{\varphi(x)\}$

$$F\{af(x) + b\varphi(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[af(x) + b\varphi(x)\right] e^{isx}\, dx$$

$$= a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx + b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x)\, e^{isx}\, dx$$

$$= aF\{f(x)\} + bF\{\varphi(x)\}$$

2. $F\{f(x-a)\} = e^{ias} F(s)$

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{is(a+t)}\, dt$$ by putting $x - a = t$

$$= \frac{1}{\sqrt{2\pi}} e^{isa} \int_{-\infty}^{\infty} f(t)\ e^{ist}\ dt$$

$$= \frac{1}{\sqrt{2\pi}} e^{isa}\ F(s)$$

3. $F\left\{e^{iax} f(x)\right\} = F(s+a)$ where $F(s) = F\{f(x)\}$

$$F\left\{e^{iax} f(x)\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\ e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ e^{i(s+a)x}\ dx$$

$$= F(s+a)$$

4. $F\{f(ax)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right) \qquad a \neq 0$

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\ e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{i\frac{s}{a}t} \frac{dt}{a} \text{ by putting } ax = t$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right) \text{ if } a \text{ is positive.}$$

Let a be negative $= -b$ where b is positive.

$$F\{f(ax)\} = F\{f(-bx)\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-bx)\ e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} f(t)\ e^{i\left(\frac{s}{-b}\right)t} \frac{dt}{-b}$$

$$= -\frac{1}{\sqrt{2\pi}} \left(-\frac{1}{b}\right) \int_{-\infty}^{\infty} f(t)\ e^{i\left(\frac{s}{-b}\right)t}\ dt$$

$$= \frac{1}{b} F\left(-\frac{s}{b}\right)$$

$$= \frac{1}{|b|} F\left(\frac{s}{a}\right)$$

Hence $F\{f(ax)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right)$ for $a > 0$ and $a < 0$

Cor $F\{f(-x)\} = F(-s)$

5. $F\left\{x^n f(x)\right\} = (-i)^n \frac{d^n}{ds^n} F\left\{f(x)\right\}$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ e^{ixs}\ dx$$

$$\frac{d}{ds}\left[F(s)\right] = \frac{1}{\sqrt{2\pi}} \frac{d}{ds} \int_{-\infty}^{\infty} f(x)\ e^{ixs}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial}{\partial s}\left[f(x)\ e^{ixs}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ix)\ f(x)\ e^{ixs}\ dx$$

$$= \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(fx)\ e^{ixs}\ ds$$

$$= iF\left\{xf(x)\right\}$$

$$\therefore F\{xf(x)\} = (-i)\frac{d}{ds}[F(s)] \text{ since } \frac{1}{i} = -i.$$

Continuing this process n times we get

$$F\left\{x^n f(x)\right\} = (-i)^n \frac{d^n}{ds^n}\left[F(s)\right]$$

$$= (-i)^n \frac{d^n}{ds^n}\left[F\left\{f(x)\right\}\right]$$

Scanned with CamScanner

6. $F\left\{\frac{d^n}{dx^n} f(x)\right\} = (-is)^n F(s)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$\therefore \frac{d}{dx} f(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left[F(s)\, e^{-isx}\right] ds$$

$$= \frac{1}{\sqrt{2\pi}} (-is) \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= (-is) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= (-is)\, f(x)$$

$$\therefore F\left\{\frac{d}{dx} f(x)\right\} = (-is)\, F(s)$$

Continuing this process n times, we get

$$F\left\{\frac{d^n}{dx^n} f(x)\right\} = (-is)^n F(s)$$

7. $F\{F(x)\} = f(-s)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

Interchanging x and s, we get

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x)\, e^{-isx}\, dx$$

Changing s into $-s$, we get

$$f(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x)\, e^{isx}\, dx$$

$$= F\{F(x)\}$$

8. $F\{F(-x)\} = f(s)$

By the previous property we get

$$f(-s) = F\{F(x)\}$$

By corollary of property **4**, we get

$$F\{F(-x)\} = f(s)$$

9. $F\left\{\overline{f(x)}\right\} = \overline{F(-s)}$ where $\overline{f(x)}$ stands for the complex conjugate of $f(x)$.

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

Taking the complex conjugate on both sides, we get

$$\overline{F(-s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)\, e^{-isx}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{isx}\, dx$$

$$= F\left\{\overline{f(x)}\right\}.$$

10. $F\left\{\overline{f(-x)}\right\} = \overline{F(s)}$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ixs}\, dx$$

$$\overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)\, e^{ixs}}\, dx$$

Scanned with CamScanner

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{-ixs}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \overline{f(-t)}\, e^{its} (-dt)$$

by putting $x = -t$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-t)}\, e^{its}\, dt$$

$$= F\left\{\overline{f(x)}\right\}$$

Fourier cosine and Fourier sine Transforms

11.1 Fourier Cosine Transform

Let $f(x)$ be defined for all $x > 0$ and

Let $f_+(x) = f(x)$ for $x \geq 0$

$= f(-x)$ for $x < 0$

Hence $f_+(x)$ is an even function.

The Fourier cosine transform of $f(x)$ is defined as the Fourier transform of $f_+(x)$ and is denoted by $F_c(s)$.

Thus $F_c(s) = F\{f_+(x)\}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_+(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_+(x) \left[\cos(sx) + i\sin(sx)\right] dx$$

Since $f_+(x)$ and $\cos(sx)$ are even and $\sin(sx)$ is odd

$$\int_{-\infty}^{\infty} f_+(x)\cos(sx)\, dx = 2\int_0^{\infty} f_+(x)\cos(sx)\, dx$$

$$= 2\int_0^{\infty} f(x)\cos(sx)\, dx$$

and $\int_{-\infty}^{\infty} f_+(x)\sin(sx)\, dx = 0$

Hence $F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\cos(sx)\, dx$ ... (1)

$$F^{-1}\{F_c(s)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(s)\, e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(s)\cos(sx)\, dx$$

Hence $f_+(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_c(s)\cos(sx)\, dx$

i.e. $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s)\cos(sx)\, dx$ ... (2)

Equations (1) and (2) constitute Fourier cosine tansform pair.

11.2 Fourier sine transform

Let $f(x)$ be defined for all $x \geq 0$

and let $f_-(x) = f(x)$ when $x \geq 0$

$= -f(-x)$ when $x < 0$

Here $f_-(x)$ is an odd function and

$$F\{f_-(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_-(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_-(x)\left[\cos(sx) + i\sin(sx)\right] dx$$

Since $f_-(x)$ and $f_-\cos(sx)$ are odd and $f_-(x)\sin(sx)$ is even

$$\int_{-\infty}^{\infty} f_-(x)\cos(sx)\, dx = 0 \text{ and}$$

$$\int_{-\infty}^{\infty} f_-(x)\sin(sx)\, dx = 2\int_0^{\infty} f_-(x)\sin(sx)\, dx$$

Scanned with CamScanner

$$\therefore F\left\{f_-(x)\right\} = i\sqrt{\frac{2}{\pi}}\int_0^\infty f_-(x)\sin(sx)\,dx$$

The Fourier sine transform of $f(x)$, is defined as the imaginary part of $F\left\{f_-(x)\right\}$ and is denoted by $F_s\left\{f(x)\right\}$

$$\text{Hence } F_s\left\{f(x)\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f_-(x)\sin(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(sx)\,dx \qquad \ldots (3)$$

By Fourier inverse formula, we have

$$f_-(x) = F^{-1}\left\{F_s(s)\right\}$$

$$= \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_s(s)\,e^{-isx}\,ds$$

$$= \frac{-2(i)^2}{\sqrt{2\pi}}\int_0^\infty F_s(s)\sin(sx)\,ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin(sx)\,ds$$

But for $x > 0 \quad f_-(x) = f(x)$

$$\text{Hence } f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin(sx)\,ds \qquad \ldots (4)$$

Equations (3) and (4) constitute Fourier sine transform pair.

12. Properties of F_c and F_s

1. (a) $F_c\left\{a f(x) + b\,\varphi(x)\right\} = a F_c\left\{f(x)\right\} + b F_c\left\{\varphi(x)\right\}$

   (b) $F_s\left\{a f(x) + b\,\varphi(x)\right\} = a F_s\left\{f(x)\right\} + b F_s\left\{\varphi(x)\right\}$

where a and b are constants. These two properties can be proved from the definition of Fourier cosine and Fourier sine transforms.

2. $F_c\left\{f(x)\cos(ax)\right\} = \frac{1}{2}\left[F_c(s+a) + F_c(s-a)\right]$

$$F_c\left\{f(x)\cos(ax)\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(ax)\cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,\frac{1}{2}\left[\cos(s+a)x + \cos(s-a)x\right]dx$$

$$= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)x\,dx\right\}$$

$$= \frac{1}{2}\left\{F_c(s+a) + F_c(s-a)\right\}$$

Similarly the following three properties can be proved.

3 (i) $F_c\left\{f(x)\sin ax\right\} = \frac{1}{2}\left[F_s(a+s) + F_s(a-s)\right]$

(ii) $F_s\left\{f(x)\cos ax\right\} = \frac{1}{2}\left[F_s(s+a) - F_s(s-a)\right]$

(iii) $F_s\left\{f(x)\sin ax\right\} = \frac{1}{2}\left[F_c(s-a) - F_c(s+a)\right]$

4. $F_c\left\{f(ax)\right\} = \frac{1}{a}F_c\left(\frac{s}{a}\right)$

$$F_c\left\{f(ax)\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(ax)\cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\left(\frac{s}{a}t\right)\frac{dt}{a} \quad \text{putting } ax = t$$

$$= \frac{1}{a}F_c\left(\frac{s}{a}\right)$$

5. Similarly it can be proved that

$$F_s\left\{f(ax)\right\} = \frac{1}{a}F_s\left(\frac{s}{a}\right)$$

6. $F_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}}\, f(0) + s\,F_s(s)$ if $f(x) \to 0$ as $x \to \infty$.

$$F_c\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[f(x)\cos(sx)\right]_0^\infty + \int_0^\infty f(x)\,s\sin(sx)\,dx$$

by integrating by parts

$$= -\sqrt{\frac{2}{\pi}}\,f(0) + s\,F_s(s)$$

7. Similarly it can be proved that

$$F_s\{f'(x)\} = -s\,F_c(s)$$

8. $F_c\{f''(x)\} = -\sqrt{\frac{2}{\pi}}\,f'(0) - s^2 F_c(s)$

$$F_c\{f''(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f''(x)\cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[f'(x)\cos(sx)\right]_0^\infty + s\int_0^\infty f'(x)\sin(sx)\,dx\right\}$$

$$= -\sqrt{\frac{2}{\pi}}\,f'(0) + s\,F_s\{f'(x)\} \text{ assuming } f'(x) = 0$$

when $x \to \infty$

$$= -\sqrt{\frac{2}{\pi}}\,f'(0) + s\left[-s\,F_c(s)\right]$$

$$= -\sqrt{\frac{2}{\pi}}\,f'(0) - s^2 F_c(s)$$

9. Similarly it can be proved that

$$F_s\{f''(x)\} = \sqrt{\frac{2}{\pi}}\,s\,f(0) - s^2 F_s(s)$$

10. $F_s\{F_s(x)\} = f(s)$

$$f(x) = F_s^{-1}\{F_s(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin(sx)\,ds$$

Interchanging x and s we get

$$f(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(x)\sin(sx)\,dx$$

$$= F_s\{F_s(x)\}$$

13. **Parsival's Identity**

$$\int_0^\infty |F_c(s)|^2\,ds = \int_0^\infty |F_s(s)|^2\,ds = \int_0^\infty |f(x)|^2\,dx$$

Let us prove the identities

(i) $\int_0^\infty F_c(s)\,G_c(s)\,dx = \int_0^\infty f(x)\,g(x)\,dx$

(ii) $\int_0^\infty F_s(s)\,G_s(s)\,dx = \int_0^\infty f(x)\,g(x)\,dx$

$$\int_0^\infty F_c(s)\,G_c(s)\,ds$$

$$= \int_0^\infty F_c(s)\left[\sqrt{\frac{2}{\pi}}\int_0^\infty g(t)\cos(st)\,dt\right]ds$$

$$= \int_0^\infty g(t)\cos(st)\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\,ds\right\}dt$$

$$= \int_0^\infty g(t)\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty F_c\cos(st)\,ds\right\}dt$$

$$= \int_0^\infty g(t)\,f(t)\,dt$$

$$= \int_0^\infty g(x)\,f(x)\,dx$$

Scanned with CamScanner

Similarly it can be proved that

$$\int_0^{\infty} F_s(s)\, G_s(s)\, ds = \int_0^{\infty} g(x)\, f(x)\, dx$$

Putting $g(x) = f(x)$ we get

$$\int_0^{\infty} | F_c(s) |^2 ds = \int_0^{\infty} | F_s(s) |^2 ds = \int_0^{\infty} | f(x) |^2 dx$$

14. Convolution: The convolution of two functions $f(x)$ and

$g(x)$ is defined by $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$

and is denoted by $f^* g$

14.1 Convolution Theorem

$F\left\{ f^* g \right\} = F(s)\, G(s)$ where $F(s)$ and $G(s)$ are the Fourier transform of $f(x)$ and $g(x)$ respectively

$$F\left\{ f^* g \right\} = F\left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, F\left\{ g(x-t) \right\} dt$$

since $f(t)$ is independent of x

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, G(s)\, dt$$

since $F\left\{ f(x-a) \right\} = e^{ias} F(s)$ by **10 (2)**

and hence $F\left\{ g(x-t) \right\} = e^{ist}\, G(s)$

$$\therefore\ F\left\{ f^* g \right\} = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-0}^{\infty} e^{ist} f(t)\, dt \right\} G(s)$$

$$= F(s)\, G(s)$$

15. Parsival's Identity

A function $f(x)$ and its Fourier transform $F(s)$ satisfy the identity

$$\int_{-\infty}^{\infty} \left| f(x) \right|^2 dx = \int_{-\infty}^{\infty} \left| F(s) \right|^2 ds$$

$$F(s)\, G(s) = F\left\{ f^* g \right\}$$

$$\therefore\ F^{-1}\left\{ F(s)\, G(s) \right\} = f^* g$$

i.e. $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, G(s)\, e^{-isx}\, ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$

Putting $x = 0$ and cancelling $\frac{1}{\sqrt{2\pi}}$ on both sides we get

$$\int_{-\infty}^{\infty} F(s)\, G(s)\, ds = \int_{-\infty}^{\infty} f(t)\, g(-t)\, dt$$

Let $g(t) = \overline{f(-t)}$, then $g(-t) = \overline{f(t)}$

$$G(s) = F\left\{ g(t) \right\} = F\left\{ \overline{f(-t)} \right\}$$

$= \overline{F(s)}$ **by property 2 (10)**

$$\therefore\ \int_{-\infty}^{\infty} F(s)\, G(s)\, ds = \int_{-\infty}^{\infty} F(s)\, \overline{F(s)}\, ds$$

$$= \int_{-\infty}^{\infty} \left| F(s) \right|^2 ds$$

Also $\int_{-\infty}^{\infty} f(t)\, g(-t)\, dt = \int_{-\infty}^{\infty} f(t)\, \overline{f(t)}\, dt$

$$= \int_{-\infty}^{\infty} \left| f(t) \right|^2 dt$$

Hence $\int_{-\infty}^{\infty} \left| F(s) \right|^2 ds = \int_{-\infty}^{\infty} \left| f(t) \right|^2 dt$

Example 1. Find $F_c\left\{e^{-ax}\right\}$ and $F_s\left\{e^{-ax}\right\}$

$$F_c\left\{e^{-ax}\right\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\,L\left\{\cos(sx)\right\}$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}$$

$$F_s\left\{e^{-ax}\right\} = \sqrt{\frac{2}{\pi}}\,L\left\{\sin(sx)\right\}$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2}$$

Example 2. Prove that (i) $F_c\left\{x f(x)\right\} = \frac{d F_s}{d s}$

(ii) $F_s\left\{x f(x)\right\} = -\frac{d F_c}{d s}$

Hence or otherwise determine $F_c\left\{x e^{-ax}\right\}$ and $F_s\left\{x e^{-ax}\right\}$

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(sx)\,dx$$

$$\frac{d}{ds}\left\{F_s(s)\right\} = \sqrt{\frac{2}{\pi}}\,\frac{d}{ds}\int_0^{\infty} f(x)\sin(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\partial}{\partial s}\left[f(x)\sin(sx)\right]dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} x\,f(x)\cos(sx)\,dx$$

$$= F_c\left\{x f(x)\right\}$$

$$F_c(s) = \int_0^{\infty} f(x)\cos(sx)\,dx$$

$$\frac{d}{ds}\left\{F_c(s)\right\} = \sqrt{\frac{2}{\pi}}\,\frac{d}{ds}\int_0^{\infty} f(x)\cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\partial}{\partial s}\left[f(x)\cos(sx)\right]dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\left[-x\sin(sx)\right]dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^{\infty} x\,f(x)\sin(sx)\,dx$$

$$= -F\left\{x f(x)\right\}$$

Putting e^{-ax} in the first result, we get

$$F_c\left\{x e^{-ax}\right\} = \frac{d}{ds}F_s\left\{e^{-ax}\right\}$$

$$= \frac{d}{ds}\sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2} \qquad \text{by Ex.1}$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{a^2-s^2}{(s^2+a^2)^2}$$

Again $F_s\left\{x\,f(x)\right\} = -\frac{d}{ds}\left[F_c\left\{f(x)\right\}\right]$

Putting $f(x) = e^{-ax}$, we get

$$F_s\left\{e^{-ax}x\right\} = -\frac{d}{ds}F_c\left\{e^{-ax}\right\}$$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{2as}{(s^2+a^2)^2}$$

Scanned with CamScanner

Example 3. Find $F_c\left\{\dfrac{1}{1+x^2}\right\}$ and $F_s\left\{\dfrac{x}{1+x^2}\right\}$

In example 1, we have shown that

(i) $F_c\left\{e^{-ax}\right\} = \sqrt{\dfrac{2}{\pi}}\,\dfrac{a}{a^2+s^2}$

(ii) $F_s\left\{e^{-ax}\right\} = \sqrt{\dfrac{2}{\pi}}\,\dfrac{s}{s^2+a^2}$

Putting $a = 1$, we get $F_c\left\{e^{-x}\right\} = \sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{1+s^2}$

We have proved in property **12 (10)**

$$F_c\left\{F_c\right\} = f(s)$$

$$\therefore\ F_c\left\{\sqrt{\frac{2}{\pi}}\,\frac{1}{1+x^2}\right\} = e^{-s}$$

$$\text{i.e. } F_c\left\{\frac{1}{1+x^2}\right\} = \sqrt{\frac{\pi}{2}}\,e^{-s}$$

Again $F_c\left\{\dfrac{x}{x^2+1}\right\} = -\dfrac{d}{ds}\,F_c\left\{\dfrac{1}{1+x^2}\right\}$ By example 2

$$= -\frac{d}{ds}\sqrt{\frac{\pi}{2}}\,e^{-s}$$

$$= \sqrt{\frac{\pi}{2}}\,e^{-s}$$

Example 4. Show that $F_c\left\{\dfrac{1}{\sqrt{x}}\right\} = F_s\left\{\dfrac{1}{\sqrt{x}}\right\} = \dfrac{1}{\sqrt{s}}$

$$F_c\left\{\frac{1}{\sqrt{x}}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{\sqrt{x}}\cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\text{ Real part of}\int_0^\infty \frac{1}{\sqrt{x}}\,e^{-isx}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\text{ Real part of}\int_0^\infty \sqrt{\frac{is}{t}}\,e^{-t}\,\frac{dt}{is}$$

by putting $isx = t$

$$= \sqrt{\frac{2}{\pi}}\text{ Real part of}\ \frac{1}{\sqrt{is}}\int_0^\infty t^{-1/2}\,e^{-t}\,dt$$

$$= \sqrt{\frac{2}{\pi}}\text{ Real part of}\ \frac{1}{\sqrt{is}}\,\Gamma\left(\frac{1}{2}\right)$$

$$= \sqrt{\frac{2}{\pi}}\text{ Real part of}\ \frac{\sqrt{\pi}}{\sqrt{s}\,\sqrt{i}}$$

$$= \frac{\sqrt{2}}{\sqrt{s}}\text{ Real part of}\ \frac{\sqrt{2}}{1+i}$$

$$= \frac{2}{\sqrt{s}}\text{ Real part of}\ \frac{1-i}{2}$$

$$= \frac{1}{\sqrt{s}}$$

Again $F_s\left(\dfrac{1}{\sqrt{x}}\right) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty \dfrac{1}{\sqrt{x}}\sin(sx)\,dx$

$$= -\sqrt{\frac{2}{\pi}}\text{ Imaginary part of}\int_0^\infty \frac{1}{\sqrt{x}}\,e^{-isx}\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\text{ Imaginary part of}\ \frac{\sqrt{2\pi}}{\sqrt{s}}\ \frac{1-i}{2}\ \text{as proved above}$$

$$= \frac{1}{\sqrt{s}}$$

Hence $F_c\left\{\dfrac{1}{\sqrt{x}}\right\} = F_s\left\{\dfrac{1}{\sqrt{x}}\right\} = \dfrac{1}{\sqrt{s}}$

Example 5. Find $F_c\left\{e^{-a^2x^2}\right\}$ and $F_s\left\{x\,e^{-a^2x^2}\right\}$ and show that $e^{x^2/2}$ is self reciprocal under Fourier cosine transform and $x\,e^{\frac{-x^2}{2}}$ is self reciprocal under Fourier sine transform

$$F_c\left\{e^{-a^2x^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-a^2x^2}\cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}} \text{ Real part of } \int_0^\infty e^{-a^2x^2} e^{isx}\,dx$$

$$= \sqrt{\frac{2}{\pi}} \text{ Real part of } \int_0^\infty e^{-a^2x^2 + isx}\,dx$$

$$= \sqrt{\frac{2}{\pi}} \text{ Real part of } e^{-\frac{s^2}{4a^2}}\int_0^\infty e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$

$$= \sqrt{\frac{2}{\pi}} \text{ Real part of } e^{-\frac{s^2}{4a^2}}\int_0^\infty e^{-t^2}\frac{dt}{a}$$

by putting $ax - \frac{is}{2a} = t$

$$= \sqrt{\frac{2}{\pi}} \text{ Real part of } \frac{e^{-\frac{s^2}{4a^2}}}{a} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2}}.$$

By **example 2,** $F_s\left\{x\,f(x)\right\} = -\frac{d}{ds}F_c\left\{f(x)\right\}$

Putting $f(x) = e^{-a^2x^2}$ we get

$$F_s\left\{xe^{-a^2x^2}\right\} = -\frac{d}{ds}F_c\left\{e^{-a^2x^2}\right\}$$

$$= -\frac{d}{ds}\left(\frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2}}\right)$$

$$= \frac{e^{-\frac{s^2}{4a^2}}\,2s}{4a^2\,a\sqrt{2}}$$

$$= \frac{s\,e^{-\frac{s^2}{4a^2}}}{2\sqrt{2}\,a^3}$$

If $a = \frac{1}{\sqrt{2}}$, then $F_c\left\{e^{-\frac{s^2}{2}}\right\} = e^{-\frac{s^2}{2}}$

and $F_s = \left\{xe^{-\frac{x^2}{2}}\right\} = s\,e^{-\frac{s^2}{2}}$

Hence the result.

Example 6 Find the Fourier Cosine transform for F (x) if

$$f(x) = 1 \text{ when } |x| < 1$$
$$= 0 \text{ when } |x| > 1$$

Deduce that (i) $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

(ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

$$F_c\left\{f(x)\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 \cos(sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left.\frac{\sin(sx)}{s}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\sin s}{s}$$

From the Fourier cosine inverse formula

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s) \cos(sx)\, ds$$

$\therefore$ In thise case $1 = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin s}{s} \cos(sx)\, ds$

i.e. $\sqrt{\frac{\pi}{2}} \sqrt{\frac{\pi}{2}} = \int_0^{\infty} \frac{\sin s}{s} \cos(sx)\, ds$

Putting x = 0, we get $\int_0^{\infty} \frac{\sin s}{s}\, ds = \frac{\pi}{2}$

From Parseval's identity, we get

$$\int_0^{\infty} |f(x)|^2\, dx = \int_0^{\infty} |F_c(s)|^2\, ds$$

Since $f(x) = 1$ in the interval 0 to 1

and $f(x) = 0$ when $x > 1$, we get

$$\int_0^1 dx = \int_0^{\infty} \frac{2}{\pi} \left(\frac{\sin s}{s}\right)^2 ds$$

Hence $\int_0^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{2}$

$$\therefore \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 = \frac{\pi}{2}$$

Example 7. Solve the integral equation

$$\frac{1}{2} \int_{-\infty}^{\infty} f(t)\, e^{-|x-t|}\, dt = h(x)$$

where h(x) is a given function

We get from convolution theorem that

$$\text{if } f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

then $F\{f * g\} = F(s)\, G(s)$

In this case take $g(t) = e^{-t}$

Then $G(s) = \sqrt{\frac{2}{\pi}} \frac{1}{s^2+1}$

$$\therefore F\{f * e^{-t}\} = \sqrt{\frac{2}{\pi}} F(s) \frac{1}{s^2+1}$$

i.e $F\left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-(x-t)}\, dt\right\} = \sqrt{\frac{2}{\pi}} F(s) \frac{1}{s^2+1}$

i.e $F\left\{\frac{1}{\sqrt{2\pi}}\, 2h(x)\right\} = \sqrt{\frac{2}{\pi}} F(s) \frac{1}{s^2+1}$

$$\sqrt{\frac{2}{\pi}} H(s) = \sqrt{\frac{2}{\pi}} F(s) \frac{1}{s^2+1}$$

$$\therefore F(s) = (s^2+1)\, H(s)$$

Taking inverse transform we get

$$\begin{aligned} f(x) &= F^{-1}\{(s^2+1)\, H(s)\} \\ &= F^{-1}\{s^2 H(s)\} + F^{-1}\{H(s)\} \\ &= F^{-1}\{-(is)^2 H(s)\} + h(x) \\ &= -h''(x) + h(x) \end{aligned}$$

by property **§ 2 (6).**

Unit - IV is Completed