GOVERNMENT ARTS COLLEGE - ARIYALUR

PG & RESEARCH DEPT. OF MATHEMATICS

M.Sc., MATHEMATICS

CORE COURSE XI

HANDLING STAFF:

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TOPOLOGY

[P16MA33]

Objectives

1. To study the concepts concerned with properties that are preserved under continuous deformations of objects.
2. To train the students to develop analytical thinking and the study of continuity and connectivity.

UNIT I TOPOLOGICAL SPACES:

Topological spaces - Basis for a topology - The order topology - The product topology on X x Y - The subspace topology - Closed sets and limit points.

UNIT II CONTINUOUS FUNCTIONS :

Continuous functions - the product topology - The metric topology.

UNIT III CONNECTEDNESS:

Connected spaces- connected subspaces of the Real line - Components and local connectedness.

UNIT IV COMPACTNESS:

Compact spaces - compact subspaces of the Real line - Limit Point Compactness – Local Compactness.

UNIT V COUNTABILITY AND SEPARATION AXIOMS:

The countability Axioms - The separation Axioms - Normal spaces - The Urysohn Lemma - The Urysohn metrization Theorem - The Tietz extension theorem.

TEXT BOOK

James R. Munkres, Topology (2nd Edition) Pearson Education Pvt. Ltd., New Delhi-2002 (Third Indian Reprint).

UNIT – I Chapter 2: Sections 12 to 17
UNIT – II Chapter 2 : Sections 18 to 21 (Omit Section 22)
UNIT – III Chapter 3 : Sections 23 to 25.
UNIT – IV Chapter 3 : Sections 26 to 29.
UNIT – V Chapter 4 : Sections 30 to 35.

REFERENCES

1 J. Dugundji, Topology, Prentice Hall of India, ,New Delhi, 1975.
2 George F.Sinmons, Introduction to Topology and Modern Analysis, McGraw Hill Book co.1963.
3 J.L. Kelly, General Topology, Van Nostrand, Reinhold Co., New York
4 L.Steen and J.Seeback, Counter examples in Topology, Holt, Rinehart and Winston, New York, 1970.

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UNIT - I

TOPOLOGICAL SPACES

05/08/2020

Defn. : Topology → On set X is a collection $\mathcal{T}$ of subsets of X with foll. properties
1) ϕ & $X \in \mathcal{T}$
2) $\cup$ (any subcollection) $\mathcal{T} \in \mathcal{T}$
3) $\cap$ (finite ") "

A set X with $\mathcal{T}$ → Topological space → $(X, \mathcal{T})$
↓ simply denoted as X.

$U \subseteq X$ is open set of X if $U \in \mathcal{T}$.

∴ Topological space is open.

There are many possible topologies on X:

Ex: $X = \{a, b, c\}$

i) 

→ contains only X & ϕ

ii) 

iii) 

→ $X, \phi, \{a, b\}, \{b\}$ & $\{b, c\}$

iv) 

v) 

vi) 

vii) 

viii) 

ix) 
→ contains every subset of X.

Note :- Not every collection of subsets of X is a topology on X.

Discrete topology $\rightarrow$ If collection of all subsets of X is a topo. on X.

Indiscrete (or) trivial topo. $\rightarrow$ collection $\supset$ only X & ϕ.

Finite complement topo. $\rightarrow$ X be a set, $\mathcal{T}_f \rightarrow$ coll. of all subsets U of X $\ni$: $X - U$ either is finite or all of X.

Note: Both X & $\phi \in \mathcal{T}_f$.

$\because$ $X - X = \phi$ is finite & $X - \phi = X$

Defn : $\mathcal{T}, \mathcal{T}'$ on X.

Finer $\rightarrow$ If $\mathcal{T}' \supseteq \mathcal{T} \Rightarrow \mathcal{T}'$ is finer than $\mathcal{T}$

" $\mathcal{T}' \supset \mathcal{T} \Rightarrow$ " " strictly finer than $\mathcal{T}$.

Coarser $\rightarrow$ Vise versa of finer.

$\mathcal{T}$ is comparable with $\mathcal{T}'$ $\Rightarrow$ If $\mathcal{T}' \supset \mathcal{T}$ (or) $\mathcal{T} \supset \mathcal{T}'$

Basis for a topology:-

Defn: **Basis** $\rightarrow$ X is a set, basis for top. on X is a collection $\mathcal{B}$ of subsets of $X \ni$:

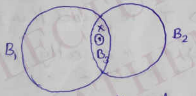
i) $\forall\, x \in X$, $\exists$ at least one basis elt. $B \supset x$.

ii) If $x \in B_1 \cap B_2$, $\exists\, B_3 \ni x \ni: B_3 \subset B_1 \cap B_2$.

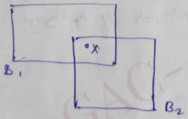
Defn: Topology $\mathcal{T}$ generated by $\mathcal{B}$ $\rightarrow$ subset U of X is open in X if $\forall\, x \in U$, $\exists\, B \in \mathcal{B} \ni: x \in B$ & $B \subset U$.

Note: Each basis elt. itself an elt. of $\mathcal{T}$.

Ex: $\mathcal{B}$ $\rightarrow$ Collection of all circular regions in the plane (interior of circles)



Ex: $\mathcal{B}$ $\rightarrow$ Collection of all rectangular regions in the plane (interior of □les)



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RESULT:

P.T the collection of open set generated by a basis $\mathcal{B}$ is a topology. ④

Proof: $X \to$ any set, $\mathcal{T}$ - topo on X, $\mathcal{B} \to$ basis for $\mathcal{T}$ on X.

consider the indexed family $\{U_\alpha\}_{\alpha \in \mathcal{J}} \in \tilde{\mathcal{J}}$

We S.T. $U = \bigcup_{\alpha \in \mathcal{J}} U_\alpha \in \mathcal{T}$

Given, $x \in U$ then $x \in U_\alpha$

$\because$ U_α is open, $\exists$ basis $B \in \mathcal{B} \ni: x \in B \subset U_\alpha$

$\therefore$ $x \in B$ and $B \subset U$

$\Rightarrow U$ is open

Let $U_1, U_2 \in \mathcal{T}$, we S.T $U_1 \cap U_2 \in \mathcal{T}$

Let $x \in U_1 \cap U_2$

Choose $B_1 \ni x \ni: B_1 \subset U_1$ & $B_2 \ni x \ni: B_2 \subset U_2$

$\therefore$ We have, $B_3 \ni x \ni: B_3 \subset B_1 \cap B_2$

$\Rightarrow x \in B_3$, $B_3 \subset U_1 \cap U_2$

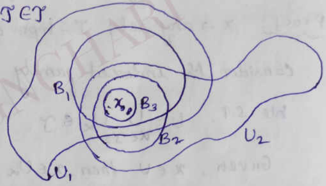
$\therefore U_1 \cap U_2 \in \mathcal{T}$

By induction $U_1 \cap U_2 \cap \dots \cap U_n$ of $\mathcal{T} \in \mathcal{T}$.

This is trivial for $n=1$

Suppose, it is true for $n-1$.

We prove it for n



Now, $(U_1 \cap U_2 \cap \dots \cap U_n) = (U_1 \cap U_2 \cap \dots \cap U_{n-1}) \cap U_n$

$\Rightarrow U_1 \cap U_2 \cap \dots \cap U_{n-1} \in \mathcal{T}$

$\therefore (U_1 \cap U_2 \cap \dots \cap U_{n-1}) \cap U_n \in \mathcal{T}$

$\therefore$ True for n.

Hence the result.

Lemma 1: Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all union of elements of $\mathcal{B}$.

Proof: Given, a collection of elts of $\mathcal{B}$ are also elts of $\mathcal{T}$.

$\because \mathcal{T}$ is a top. $\Rightarrow \bigcup \mathcal{T} \in \mathcal{T}$

Conversely, given $U \in \mathcal{T}$

Choose, $\forall x \in U$ an elt $B_x \in \mathcal{B} \ni : x \in B_x \subset U$.

Then, $U = \bigcup_{x \in U} B_x \subseteq \mathcal{B}$

$\therefore U \subseteq \mathcal{B} \in \mathcal{T}$

Hence $U = \bigcup$ elts. of $\mathcal{B}$

Lemma 2: Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of $X \ni$: for each open set U of X and each x in U, there is an elt. C of $\mathcal{C} \ni: x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

Proof: T.P $\mathcal{C}$ is a basis.

i) Given, $x \in X \ \exists \ C \in \mathcal{C} \ni: x \in C \subset X$.

ii) Let $x \in C_1 \cap C_2$, where $C_1, C_2 \in \mathcal{C}$

$\because C_1, C_2$ are open $\Rightarrow C_1 \cap C_2$ is open

$\therefore \exists \ C_3 \subset C_1 \cap C_2$

$\Rightarrow x \in C_3 \subset C_1 \cap C_2$

$\therefore \mathcal{C}$ is a basis.

Let $\mathcal{T} \to$ coll. of open sets of X. We've t.s.t., $\mathcal{T}'$ generated by $\mathcal{C} = \mathcal{T}$. , i.e., $\mathcal{T}' = \mathcal{T}$.

If $U \in \mathcal{T}$ & $x \in U$ then $x \in C \subset U$ & $C \in \mathcal{C} \Rightarrow U \in \mathcal{T}'$ [By defn.]

Conversely, if $W \in \mathcal{T}' \Rightarrow W = \bigcup_{\alpha} C_\alpha \ \forall \ C_\alpha \in \mathcal{C}$.

$\because$ Each elt. of $C \in \mathcal{T}$ & $\mathcal{T}$ is topo.

$\Rightarrow W \in \mathcal{T}$, $\therefore \mathcal{T}' = \mathcal{T}$

Hence the lemma.

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Lemma 3:- Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively on X. Then the following are equivalent:

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$

(2) $\forall x \in X$, and each basis elt. $B \in \mathcal{B}$ containing x, there is a basis elt. $B' \in \mathcal{B}'$ $\ni: x \in B' \subset B$.

Proof: $(2) \Rightarrow (1)$

Given $U \in \mathcal{T}$, we've to S.T $U \in \mathcal{T}'$.

Let $x \in U$

$\because \mathcal{B}$ generates $\mathcal{T}$, $\exists\, B \in \mathcal{B} \ni: x \in B \subset U$.

By (2) $\exists\, B' \in \mathcal{B}' \ni: x \in B' \subset B$

Then $x \in B' \subset U$

$\therefore U \in \mathcal{T}'$ [By defn] $\Rightarrow \mathcal{T}' \supset \mathcal{T}$

$(1) \Rightarrow (2)$

Given $x \in X$ & $B \in \mathcal{B}$ with $x \in B$.

Now, by defn & (1) $B \in \mathcal{T}$ & $\mathcal{T} \subset \mathcal{T}'$

$\because B \in \mathcal{T}'$

$\because \mathcal{T}'$ is generated by $\mathcal{B}'$ $\exists\, B' \in \mathcal{B}' \ni: x \in B' \subset B$.

Defn: Standard Topo.:- If $\mathcal{B}$ is the collection of all open intervals in the real line, $(a, b) = \{x \mid a < x < b\}$, the topology generated by $\mathcal{B}$ is called the standard topology on the real line $\mathbb{R}$.

Defn: If $\mathcal{B}'$ is the collection of all half-open intervals of the form $[a,b) = \{x \mid a \le x < b\}$, where $a < b$, the topology generated by $\mathcal{B}'$ is called the lower limit topology on $\mathbb{R}$ and is denoted by $\mathbb{R}_l$.

K-topology: Let K denote the set of all nos. of the form $1/n$ for $n \in \mathbb{Z}_+$ and let $\mathcal{B}''$ be the collection of all open intervals (a,b), along with all sets of the form $(a,b) - K$. The topo. generated by $\mathcal{B}''$ is called K-topology on $\mathbb{R}$.

When $\mathbb{R}$ is given this topology is denoted by $\mathbb{R}_K$.

Relation b/w Standard, $\mathbb{R}_l$, $\mathbb{R}_K$ topologies:-

Lemma 4:- The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

Proof: Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ be the topologies of $\mathbb{R}$, $\mathbb{R}_l$ and $\mathbb{R}_K$ respectively.

Given a basis elt. (a,b) for $\mathcal{T}$ & $x \in (a,b)$.

Also, the basis elt. $[x,b)$ for $\mathcal{T}' \ni x$ and lies in (a,b).

On the other hand, given basis elt. $[x,d)$ for $\mathcal{T}'$, there is no open interval (a,b) that contains x and lies in $[x,d)$.

Thus $\mathcal{T}'$ is strictly finer than $\mathcal{T}$.

Given, a basis elt. (a,b) for $\mathcal{T}$ & $x \in (a,b)$, this same interval is basis elt. for $\mathcal{T}'' \ni x$.

On the other hand, given basis elt. $B = (-1,1) - K$ for $\mathcal{T}''$ & 0 of B, there is no open interval that contains 0 and lies in B.

Thus $\mathcal{T}''$ is strictly finer than $\mathcal{T}$.

T.P: $\mathbb{R}$, $\mathbb{R}_l$ & $\mathbb{R}_K$ are not comparable

By definition of $\mathbb{R}$, $\mathbb{R}_l$ and $\mathbb{R}_K$ topologies we've that $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology, but we cannot arrive $\mathcal{T}' \subset \mathcal{T}''$ or $\mathcal{T}'' \subset \mathcal{T}'$.

Hence $\mathbb{R}_l$ & $\mathbb{R}_K$ are not comparable.

Defn: A subbasis $\mathcal{S}$ for a topo. on X is a collection of subsets of X whose union equals X.

The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

08/08/2020

The Order Topology

Set $X : a < b$, $a, b \in X$. Then the intervals are

$(a, b) = \{x : a < x < b\} \to$ open

$(a, b] = \{x : a < x \leq b\}$ →] → Half open

$[a, b) = \{x : a \leq x < b\}$ →]

$[a, b] = \{x : a \leq x \leq b\} \to$ closed

Defn: Let X be a set with a simple order relation and has more than one elt. Let $\mathcal{B}$ be the collection of all sets as follows:

i) All (a, b) in X

ii) All $[a_0, b)$, where a_0 is the smallest ~~of~~ elt. of X

iii) All $(a, b_0]$, " b_0 " " largest elt. of X.

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the order topology.

Examples:-

1) The standard topology on $\mathbb{R} \to$ Order topology

2) The set $\mathbb{R} \times \mathbb{R}$

3) $\mathbb{Z}_+$ form an ordered set with a smallest elt.

The order topo. on $\mathbb{Z}_+$ is the discrete topology $\forall$ one-pt. set is open:

If $n > 1$, then one-pt. set $\{n\} = (n-1, n+1)$ is a basis elt.

If $n = 1$, " " " $\{1\} = [1, 2)$ is a basis elt.

4) The set $X = \{1,2\} \times \mathbb{Z}_+$ is an ordered set with a smallest elt.

Let a_n be $1 \times n$, & b_n be $2 \times n$ and X be $a_1, a_2, \dots ; b_1, b_2, \dots$

The order topology on X is not the discrete topology.

Definition: If X is an ordered set and $a \in X$ then the following subsets of X ~~det~~ are called the rays determined by a.

$(a, +\infty) = \{x : x > a\}$, $(-\infty, a) = \{x : x < a\}$ → Open Rays

$[a, +\infty) = \{x : x \geq a\}$, $(-\infty, a] = \{x : x \leq a\}$ → Closed Rays

The Product Topology on $X \times Y$

Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Theorem 1: If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection

$$\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$ is a basis for the topology of $X \times Y$.

Proof: Let W be open set of $X \times Y$ and a point $x \times y \in W$.

By defn. of prod. top. $\exists$ a basis elt. $U \times V \ni: x \times y \in U \times V \subset W$.

$\because$ $\mathcal{B}$ and $\mathcal{C}$ are basis for X and Y respectively.

Choose $B \in \mathcal{B} \ni: x \in B \subset U$.

$C \in \mathcal{C} \ni: y \in C \subset V$

$\Rightarrow x \times y \in B \times C \subset W$

Thus $\mathcal{D} = \{ B \times C : B \in \mathcal{B} \ \& \ C \in \mathcal{C} \}$ is a basis for $X \times Y$. [By Lemma 2]

Hence the thm.

Note :- Standard topology on $\mathbb{R}$: the order topology.

Product of this topology with itself is called the standard topology on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

The basis elts. of this topo. of $\mathbb{R}^2$ are the products $(a,b) \times (c,d)$ of open intervals in $\mathbb{R}$.

Defn: Let $\pi_1 : X \times Y \to X$ be defined by $\pi_1(x,y) = x$;

let $\pi_2 : X \times Y \to Y$ " " " $\pi_2(x,y) = y$

The maps π_1 & π_2 are called the projections of $X \times Y$ onto X and Y respectively.

Note:

1) π_1 & π_2 are surjective

2) If U is an open subset of X, then $\pi_1^{-1}(U)$ is the set $U \times Y$ which is open in $X \times Y$.

   If V is open in Y, then $\pi_2^{-1}(V) = X \times V$ is open in $X \times Y$

3) $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = (U \times Y) \cap (X \times V) = U \times V$.

Theorem 2: The collection $\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$ is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}$ be the product topology on $X \times Y$ and $\mathcal{T}'$ be the topology generated by $\mathcal{S}$.

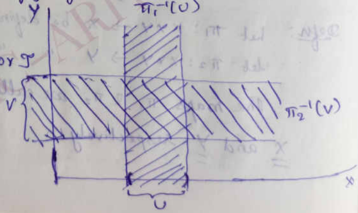
$\because$ Every elt. of $\mathcal{S} \in \mathcal{T}$.

We've arbitrary unions of finite intersection of elements of $\mathcal{S}$.

$\because$ $\mathcal{T}'$ is the topology by $\mathcal{S}$

Thus $\mathcal{T}' \subset \mathcal{T}$

Now, every basis elt $U \times V$ for $\mathcal{T}$ is finite $\cap$tion of elts of $\mathcal{S}$.



$\because U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$

$\therefore U \times V \in \mathcal{T}' \Rightarrow \mathcal{T} \subset \mathcal{T}'$

Hence the theorem

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5 # The Subspace Topology

Defn :- Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection $\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$ is a topology on Y, called the subspace topology.

With this topo, Y is called a subspace of X; its open sets consists of all intersections of open sets of X with Y.

(x) RESULT: P.T $\mathcal{T}_Y$ is a topology.

Proof: By defn. of topo.

i) $\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$

We've $\phi = Y \cap \phi$ & $Y = Y \cap X$

$\forall$ ϕ & X in $\mathcal{T}$ we've ϕ & Y in $\mathcal{T}_Y$

ii) $(U_1 \cap Y) \cap \dots \cap (U_n \cap Y) = (U_1 \cap \dots \cap U_n) \cap Y \in \mathcal{T}_Y \in \mathcal{T}$

iii) $\bigcup_{\alpha \in J} (U_\alpha \cap Y) = (\bigcup_{\alpha \in J} U_\alpha) \cap Y \in \mathcal{T}$

Hence $\mathcal{T}_Y$ is a topology.

Lemma 1 :- If $\mathcal{B}$ is a basis for the topology of X then the collection $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topo on Y.

Proof: Let $\mathcal{B}$ be a basis for topo. of X & $\mathcal{T}_Y$ be the subspace topo.

Given $U \in X$ & $y \in U \cap Y$

Choose $B \in \mathcal{B} \ni : y \in B \subset U$

Then $y \in B \cap Y \subset U \cap Y$

Hence the collection $\mathcal{B}_Y$ is a basis for the sub space topo on Y.

Note: $X \to$ top. space & $Y \to$ subspace of X.

Then the set U is open in Y (or open relative to Y)

if $U \in \mathcal{T}_Y \Rightarrow U \subseteq Y$

U is open in X if $U \in$ top. of X.

Lemma 2: Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof:

Since U is open in Y, $U = Y \cap V$ for some set V open in X.

Since Y and V are open in $X \Rightarrow Y \cap V$ is open in X.

Hence U is open in X.

Theorem 1: If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topo. $A\times B$ inherits as a subspace of $X\times Y$.

Proof: Let A and B be subspace of X and Y respectively.

Let U be open set in X and V be open set in Y.

The set $U\times V$ is the general basis elt. for $X\times Y$.

$\therefore (U\times V)\cap(A\times B)$ is the gen. basis elt. for subsp. top. on $A\times B$

Now, $(U\times V)\cap(A\times B) = (U\cap A)\times(V\cap B)$

$\because (U\cap A)$ & $(V\cap B)$ are general open sets for the subspace topologies on A and B respectively.

$\therefore (U\cap A)\times(V\cap B)$ is the gen. basis elt. for the product topo on $A\times B$.

Hence basis for subsp. top. on $A\times B$ = basis for ~~subsp.~~ product top. on $A\times B$

Hence the topologies are same.

Note:-

1. Let $I=[0,1]$. Then the set $I\times I$ on the plane $\mathbb{R}\times\mathbb{R}$ order topo. will be called the Ordered square and denoted by I_o^2.
2. Let X be an ordered set then a subset Y of X is convex in X if for each pair of points $a<b$ of Y, the entire interval (a,b) of pts. of X lies in Y.
3. Intervals & rays in X are convex in X.

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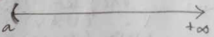
Theorem 2: Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

Proof: Consider the ray $(a, +\infty)$ in X.



If $a \in Y$, then $(a, +\infty) \cap Y = \{x \mid x \in Y \text{ and } x > a\} \rightarrow (1)$

(1) is an open ray of the ordered set Y.

If $a \notin Y$, then a is either a lower bound on Y or [$\because Y$ is convex]
an upper " " "

i.e., If a is lower bound $\Rightarrow (a, +\infty) \cap Y = Y$

" " " upper " $\Rightarrow (a, +\infty) \cap Y = \phi$

Consider the ray $(-\infty, a)$ in X.

If $a \in Y$, then $(-\infty, a) \cap Y = \{x \mid x \in Y \text{ and } x < a\} \rightarrow (2)$

(2) is an open ray of the ordered set Y.

We've If a is lower bound $\Rightarrow (-\infty, a) \cap Y \neq \phi$

" " " upper " $\Rightarrow (-\infty, a) \cap Y = Y$

$\therefore$ (1) & (2) form a subbasis for the subspace topo. on Y, and since each is open in the order topo., the order topo. contains the subspace topo.

T.P :- the reverse, we've any open ray of $Y = \cap^{tion}$ of an open ray of X with Y.

$\Rightarrow$ It's open in the subspace topo. on Y.

Since, open rays of Y are subbasis for the order topo on Y, this topo is contained in the subspace topo.

Closed Sets & Limit Points

Defn: A subset A of a topo. space X is said to be closed if the set $X-A$ is open.

Examples:

1) The subset $[a,b]$ of $\mathbb{R}$ is closed:
$\because \mathbb{R}-[a,b] = (-\infty, a) \cup (b, +\infty)$ is open.

2) The subset $[a, \infty)$ of $\mathbb{R}$ is closed
$\because \mathbb{R} - [a, \infty) = (-\infty, a)$ is open

Note :- The subset $[a,b) \in \mathbb{R}$ is neither open nor closed.

3) The set $\{x \times y \mid x \geq 0 \ \& \ y \geq 0\}$ in the plane $\mathbb{R}^2$ is closed:
$\because \mathbb{R}^2 - \{x \times y \mid x \geq 0 \ \& \ y \geq 0\} = (-\infty, 0) \times \mathbb{R} \cup \mathbb{R} \times (-\infty, 0)$ is open.

4) In finite complement topo on a set $X \rightarrow$ the closed sets of $X \supset$ itself & finite subsets.

5) In Discrete topo. on the set $X \rightarrow$ every set is open $\Rightarrow$ every set is closed.

6) $Y = [0,1] \cup (2,3)$ of $\mathbb{R}$ are closed:
$[0,1] = (-1/2, 3/2) \cap Y$ is open

Theorem: 1 Let X be a topo. space. Then the foll. conditions hold.

i) ϕ & X are closed

ii) Arbit. $\cap$tions of closed sets are closed.

iii) Finite $\cup$ns. of " " " "

Proof:

i) $\because$ $X - \phi = X$ is open $\Rightarrow \phi$ is closed

$X - X = \phi$ " " $\Rightarrow X$ " "

ii) Given a collection of closed set $\{A_\alpha\}_{\alpha \in J}$

Applying DeMorgan's law,

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha) \rightarrow (1)$$

Since the sets $X - A_\alpha$ are open [By dfn.],

R.H.S. of (1) is arbit. $\cup$ of open sets

$\therefore$ " $\cap A_\alpha$ is closed.

iii) If A_i is closed for $i = 1, 2, \ldots, n$

Applying Demorgan's law

$$X - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X - A_i) \rightarrow (2)$$

Since the sets $X - A_i$ are open [By dfn.]

R.H.S of (2) is the finite $\cap$tion of open sets

$\therefore \bigcup_{i=1}^{n} A_i$ is closed.

—x—

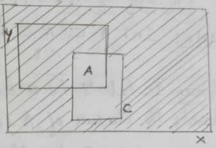
5M **Theorem 2**: Let Y be a subspace of X. Then a set A is closed with Y iff it equals the intersection of closed set of X with Y.

Proof: Let Y be a subspace of X.

Assume that $A = C \cap Y$, where C is closed in X

$\Rightarrow X - C$ is open in X.

$\therefore (X-C) \cap Y$ is open in Y



But $(X-C) \cap Y = Y - A$.

Hence $Y - A$ is open in Y

$\therefore A$ is closed in Y.

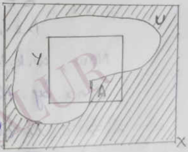
Conversely, assume that A is closed in Y.

$\Rightarrow Y - A$ is open in Y

By defn. $Y - A = \cap^{\text{tion}}$ of open set U of X with Y.

$\because$ Set $X - U$ is closed in X & $A = Y \cap (X - U)$

$\therefore A = \cap^{\text{tion}}$ of closed set of X with Y



Hence the theorem

Note: If Y is a subspace of X and A is closed in Y, Y is closed in X then A is closed in X.

Defn:- Given a subset A of a topo. space X

Interior of A → $\cup^{on}$ of all open sets contained in A. Denoted as Int A

Closure of A → $\cap^{tion}$ " " closed sets containing A. Denoted as Cl A (or) $\bar{A}$

Note: $\text{Int } A \subset A \subset \bar{A}$, if A is open, $A = \text{Int } A$ &

" " " closed $A = \bar{A}$

Theorem 3:- Let Y be a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

Proof: Let B be the closure of A in Y.

The set $\bar{A}$ is closed in X $\Rightarrow \bar{A} \cap Y$ is closed in Y.

Since, $\bar{A} \cap Y \supset A$, by defn.

$B = \cap^{tion}$ of all closed subsets of Y containing A

i.e., $B \subset (\bar{A} \cap Y)$.

Now, w.k.t B is closed in Y

Hence by thm: 2, $B = C \cap Y$, where C is closed in X.

$\Rightarrow$ C is closed set of $X \supset A$.

$\therefore \bar{A} \subset C$

$\Rightarrow (\bar{A} \cap Y) \subset (C \cap Y)$

$\therefore \bar{A} \cap Y \subset B$

Hence $B = \bar{A} \cap Y$

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Theorem 4 :- Let A be a subset of the topo. spa. X.

i) Then $x \in \bar{A}$ iff every open set U containing x intersects A.

ii) Supposing the topo. of X is given by a basis, then $x \in \bar{A}$ iff every ~~elt.~~ basis elt. B containing x intersects A.

Proof: Consider the statement i)

$x \notin \bar{A} \iff \exists$ an open set $U \ni x$ & doesn't intersect A.

Now, if $x \notin \bar{A} \Rightarrow U = X - \bar{A} \ni x$

$\Rightarrow U$ is open & U doesn't int. A.

Conversely, if $\exists$ an open set $U \ni x$ & $U \notin \cap^{\text{tion}} A$.

$\Rightarrow X - U$ is closed set containing A.

$\therefore$ The set $X - U \supset \bar{A}$ [By defn. of $\bar{A}$]

$\therefore x \notin \bar{A}$

Consider the statement ii)

If every open set containing x intersects A, then every basis elt. $B \ni x$ [$\because B$ is open]

Conversely, if every basis elt. $\ni x$ $\cap^{\text{ts}}$ A

$\Rightarrow$ every open set $U \ni x$.

Hence the thm.

Defn:- The open set $U \supset x$, then U is a neighbourhood of x.

Note :- $A \subseteq X$, then $x \in \bar{A} \Leftrightarrow$ every nbhd. of x $\cap$ts. A.

Limit Points :-

Defn: If $A \subseteq X$ and if x is a pt. of X, then x is a limit point (or cluster pt. (or) pt. of accumulation) of A if every nbhd $\cap$ts. A in some pt. other than x itself.

Theorem 5: Let A be a subset of the topological space X; let A' be the set of all lt. pts. of A. Then $\bar{A} = A \cup A'$.

Proof: T.P: $\bar{A} = A \cup A'$

If $x \in A' \Rightarrow$ Every nbhd. of x $\cap$ts A.

i.e., $\therefore x \in \bar{A}$

Hence $A' \subset \bar{A}$

$\because$ By defn., $A \subset \bar{A} \Rightarrow A \cup A' \subset \bar{A}$

Now, let $x \in \bar{A}$

We've t.s.t. $x \in A \cup A'$

If $x \subset A$ then it's trivial that $x \in A \cup A'$

Suppose $x \notin A$, i.e., $x \notin A$.

Since, $x \in \bar{A}$, w.k.t, every nbhd. U of x $\cap$ts A.

Since, $x \notin A$, the set U $\cap$ts. A in a pt. diff. from x.

$\Rightarrow x \in A'$, $\therefore x \in A \cup A'$

$\therefore A \cup A' \supset \bar{A}$

Hence $\bar{A} = A \cup A'$

Corollary:- A subset of a topological space is closed iff it contains all its lt. pts.

Proof: The set A is closed $\Leftrightarrow A = \bar{A}$

By thm, we've $A' \subseteq A$.

Hausdorff Spaces:-

Defn: A topo. sp. X is called a Hausdorff space if for each pair x_1, x_2 of distinct pts. of X, $\exists$ nbhds U_1 & U_2 of x_1 & x_2 respectively that are disjoint.

Theorem 6: Every finite pt. set in a Hausdorff space X is closed.

Proof: Let X be Haus. sp..

It suffices t.s.t every one-pt. set $\{x_0\}$ is closed.

If $x \in X$ & $x \neq x_0$

$\Rightarrow x$ & x_0 have disjoint nbhds U & V respectively.

since U doesn't $\cap^{t}$ $\{x_0\} \Rightarrow x \notin cl\{x_0\}$

$\Rightarrow cl\{x_0\} = \{x_0\}$

Hence $\{x_0\}$ is closed.

Note:- The real line $\mathbb{R}$ in the finite complement topo is not a Hausd sp., but it's a sp. in which finite pt. sets are closed and is called the T_1 axiom.

Theorem 7:- Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the pt. x is a lt. pt. of A iff every nbhd of x contains cntly. many pts. of A.

Proof: If every nbhd of x $\cap$ts A in cntly many pts.

$\therefore$ x $\cap$ts A in some pts. other than x itself.

$\therefore$ x is a lt. pt. of A.

Conversely, suppose x is a lt. pt. of A & some nbhd U of x $\cap$ts. A in only finitely many pts.

$\Rightarrow U$ $\cap$ts. $A-\{x\}$ in finitely many pts.

Let $\{x_1, \ldots, x_m\}$ be the pts. of $U \cap (A-\{x\})$

The set $X-\{x_1, \ldots, x_m\}$ is an open set of X.

$\because$ the finite pt. set $\{x_1, \ldots, x_m\}$ is closed

$\Rightarrow U \cap (X-\{x_1, \ldots, x_m\})$ is a nbhd. of x that $\cap$ts the set $A-\{x\}$ not at all.

This $\Rightarrow \Leftarrow$ to x is a lt. pt. of A.

Hence the thm.

Theorem 8:- If X is a Hausd. spc., then a sequence of pts. of X converges to at most one pt. of X.

Proof : Let X be a Hausd sp.

Suppose x_n is a seq. of pts. of X that converges to x.

If $y \neq x$, let U & V be disjoint nbhds. of x & y respectively.

Since $U \supset x_n$ $\forall$ but finitely many values of n

$\Rightarrow x_n \notin V$.

$\therefore$ x_n cannot conv. to y.

Note:- If the seq. x_n of pts. of the Hausd. sp X conv to the pt. x of X, then $x_n \to x$, i-e, x is the lt of x_n.

Assignment :-

1) P.T every simply ordered set is a Hausdorff space in the order topology.

2) P.T product of two Hausdorff space is a Hausd. sp..

3) P.T subsp. of a Hausd. sp. is a Hausd. sp.

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UNIT – II

(27)

CONTINUOUS FUNCTIONS

Defn: $X, Y \to$ top. sps. $f: X \to Y$ is cont. if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Theorem 1: Let X & Y be top. sp; let $f: X \to Y$. Then the fol. are equiv:

i) f is cont. ii) $\forall$ subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$.

iii) $\forall$ closed set B of Y, the set $f^{-1}(B)$ is closed in X.

iv) $\forall x \in X$ & each nbhd. V of $f(x)$, there's a nbhd. U of $x \ni: f(U) \subset V$.

Note: If cond (iv) holds for pt. $x \in X$, then f is cont. at the pt. x.

Proof: w.s.t. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) & (i) $\Rightarrow$ (iv) $\Rightarrow$ (i)

(i) $\Rightarrow$ (ii) Assume f is cont..

Let $A \subseteq X$ & w.s.t if $x \in \bar{A} \Rightarrow f(x) \in \overline{f(A)}$.

Let V be a nbhd of $f(x)$.

$\Rightarrow f^{-1}(V)$ is an open set of $X \ni x$ & $f^{-1}(V) \cap A = y$.

$\Rightarrow V \cap f(A) = f(y)$

$\therefore f(x) \in \overline{f(A)}$

(ii) $\Rightarrow$ (iii) Let B be closed in Y & let $A = f^{-1}(B)$.

T.P: A is closed in X & t.s.t $\bar{A} = A$.

We've $f(A) = f(f^{-1}(B)) \subset B$

$\therefore$ if $x \in \bar{A}$, $f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B$.

$\Rightarrow x \in f^{-1}(B) = A$

Thus $\bar{A} \subset A \Rightarrow \bar{A} = A$.

(iii) ⇒ (i) Let V be an open set of Y.

$$B = Y - V \Rightarrow f^{-1}(B) = f^{-1}(Y) - f^{-1}(V)$$
$$= X - f^{-1}(V)$$

Now, B is a closed set of Y. $\Rightarrow f^{-1}(B)$ is closed in X [By hypo]

$\therefore f^{-1}(V)$ is open in X.

(i) ⇒ (iv) Let $x \in X$ & V be a nbhd. of $f(x)$.

Then $U = f^{-1}(V)$ is a nbhd. of $x \ni: f(U) \subset V$.

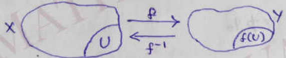
(iv) ⇒ (i) Let V be an open set of Y & x be a pt. of $f^{-1}(V)$.

Then $f(x) \in V \Rightarrow \exists$ a nbhd. U_x of $x \ni: f(U_x) \subset V$.

$\Rightarrow U_x \subset f^{-1}(V)$

$\therefore f^{-1}(V) = \cup U_x$ is open.

Defn:- $X, Y \to$ topo. sp. & $f: X \to Y$ be a bijection (1-1 & onto). If both f & $f^{-1}: Y \to X$ are cont., then f is called a homeomorphism.



Theorem: 2 (Rules for constructing continuous fns.):-

Let X, Y & Z be top. sp.

a) (Constant fn.) If $f: X \to Y$ maps all of X into the single pt. y_0 of Y, then f is cont.

b) (Inclusion) If A is a subsp. of X, the inclusion fn. $j: A \to X$ is cont.

c) (Composites) If $f: X \to Y$ & $g: Y \to Z$ are cont., then the map $g \circ f: X \to Z$ is cont.

d) (Restricting the domain) If $f: X \to Y$ is cont., & if A is a subsp. of X, then the restricted fn. $f|A : A \to Y$ is cont.

e) (Restricting or expanding the range) Let $f: X \to Y$ be cont. If Z is a subsp. of Y containing the image set $f(X)$, then the fn. $g: X \to Z$ obtained by restricting the range of f is cont. If Z is a space having Y as a subsp., then the fn. $h: X \to Z$ obtained by expanding the range of f is continuous.

f) (Local formulation of continuity) The map $f: X \to Y$ is cont. if X can be written as the un. of open sets U_α $\ni$: $f|U_\alpha$ is cont. for each α.

Proof:

a) Let $f(x) = y_0$ $\forall x \in X$ & V be open in Y.

$\therefore f^{-1}(V) = X$ if $V \ni y_0$ &

$f^{-1}(V) = \phi$ if $V \not\ni y_0$

$\Rightarrow f^{-1}(V)$ is open.

$\therefore f$ is cont.

b) Let $A \subseteq X$ & $j: A \to X$

If U is open in $X \Rightarrow j^{-1}(U) = U \cap A$ is open in A

$\therefore j$ is cont.

c) If U is open in $Z \Rightarrow g^{-1}(U)$ is open in Y & $f^{-1}(g^{-1}(U))$ is open in X.

But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open

$\therefore g \circ f$ is cont.

d) $f|A = (j : A \to X) \circ (f : X \to Y)$.

i.e., $f|A = j \circ f$

$\therefore f|A$ is cont..

e) Let $f : X \to Y$ be cont.

If $f(X) \subset Z \subset Y$, w.s.t $g : X \to Z$ is cont.

Let B be open in Z. Then, $B = Z \cap U$ for some $U \in Y$

$\because Z \supset$ entire $f(X) \Rightarrow f^{-1}(U) = g^{-1}(B)$ is open.

$\therefore g$ is cont..

Now, t.s.t ~~h is cont~~. $h : X \to Z$ is cont.

If Z has Y as a subsp., then

$h = (f : X \to Y) \circ (j : Y \to Z)$

$= f \circ j$

$\therefore h$ is cont.

f) By hypo. $X = \bigcup U_\alpha \ni : f|U_\alpha$ is cont. $\forall \alpha$.

Let V be an open set in Y.

Then, $f^{-1}(V) \cap U_\alpha = (f|U_\alpha)^{-1}(V)$

Since, $f|U_\alpha$ is cont. $\Rightarrow f^{-1}(V) \cap U_\alpha$ is open in U_α & hence open in X.

But $f^{-1}(V) = \bigcup_\alpha (f^{-1}(V) \cap U_\alpha)$

$\therefore f^{-1}(V)$ is also open in X.

Hence f is cont.

Theorem 3: (The pasting lemma):- Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be cont. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$.

Proof: Let $X = A \cup B$, where A & B are closed in X.

Let $f : A \to Y$ & $g : B \to Y$ be cont. & $f(x) = g(x)\ \forall x \in A \cap B$

Let $h : X \to Y$ be defnd as $h(x) = f(x)$ if $x \in A$ &

$h(x) = g(x)$ if $x \in B$

Let C be a closed subset of Y.

Now, $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

Since, f is cont.

$f^{-1}(C)$ is closed in A $\Rightarrow f^{-1}(C)$ is closed in X

$g^{-1}(C)$ " " " B $\Rightarrow g^{-1}(C)$ " " " "

$\therefore h^{-1}(C)$ is closed in X.

Theorem 4: (Maps into products):- Let $f : A \to X \times Y$ be gn. by the eqn. $f(a) = (f_1(a), f_2(a))$. Then f is cont. iff the fns. $f_1 : A \to X$ & $f_2 : A \to Y$ are cont. The maps f_1 and f_2 are called the coordinate functions of f.

18/08/20 T.P.: $f_1 : A \to X$ & $f_2 : A \to Y$ are continuous.

Proof: Let $\pi_1 : X \times Y \to X$ & $\pi_2 : X \times Y \to Y$ be projections.

Now, $\pi_1^{-1}(U) = U \times Y$ & $\pi_2^{-1}(V) = X \times V$ are open if U & V are open.

Also, $\forall\, a \in A$, $f_1(a) = \pi_1(f(a))$ & $f_2(a) = \pi_2(f(a))$

$\therefore$ f is cont. $\Rightarrow f = f_1 \circ f_2$

$\Rightarrow f_1$ & f_2 are cont.

Conversely, suppose f_1 & f_2 are continuous.

W.S.T. $\forall$ basis elt. $U \times V \in X \times Y$ its inv. img. $f^{-1}(U \times V)$ is open.

Now, $a \in f^{-1}(U \times V) \Leftrightarrow f(a) \in U \times V$

i.e., $\Leftrightarrow f_1(a) \in U$ & $f_2(a) \in V$

$\therefore f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$ is open.

The Product Topology :-

Defn: ① $\{X_\alpha\}_{\alpha \in J}$ → indexed family of top. spaces. The collection of all sets of the form $\prod_{\alpha \in J} U_\alpha$ is a basis for a topo. on the product space $\prod_{\alpha \in J} X_\alpha$, where U_α is open in X_α $\forall\, \alpha \in J$. The topo. generated by this basis is called box topology.

② $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ & $\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$ is called the projection mapping associated with the index β.

③ The collection $S_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}$ & $S = \bigcup_{\beta \in J} S_\beta$. The topo. generated by the subbasis S is called the product topology. Here, $\prod_{\alpha \in J} X_\alpha$ is called a product space.

Theorem 1 :- (Comparison of the box & product topologies) :-
The box topo. on $\prod X_\alpha$ has a basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topo. on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α $\forall \alpha$ and U_α equals X_α except for finitely many values of α.

Proof : It's clear that for finite products $\prod_{\alpha=1}^{n} X_\alpha$ the box & product topo. are same.

Also, the box topo. is finer than the product topo..

Theorem 2 :- If topo. on each sp. X_α is given by a basis $\mathcal{B}_\alpha$. The coll. of all sets of the form $\prod_{\alpha \in J} B_\alpha$, $B_\alpha \in \mathcal{B}_\alpha$ $\forall \alpha$ will be a basis for the box top. on $\prod_{\alpha \in J} X_\alpha$. The coll. of the same form, $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α & $B_\alpha = X_\alpha$ $\forall$ remaining indices will be a basis for the product topo. $\prod_{\alpha \in J} X_\alpha$.

Proof : proof by ex..

Let $\mathbb{R}^n$ be the euclidean n-space.

Basis for $\mathbb{R} \supset$ all open int. in $\mathbb{R}$.

$\therefore$ Basis for top. of $\mathbb{R}^n = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$

Since $\mathbb{R}^n$ is finite $\Rightarrow$ This's satisfied for both box & product topo..

Thm 3:- $A_\alpha \to$ subsp. of $X_\alpha \ \forall\ \alpha \in J$. Then $\prod A_\alpha$ is a subsp. of $\prod X_\alpha$ if both products are gn the box top., or if both products are gn. the prod top..

Thm.4:- $\forall$ space X_α is a Haus. sp, then $\prod X_\alpha$ is a Haus. sp. in both the box & prod. topo.

Thm 5:- Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subset X_\alpha \ \forall\ \alpha$. If $\prod X_\alpha$ is gn. either the prod. (or) the box topo, then $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$.

Proof: Let $x = (x_\alpha) \in \prod \overline{A_\alpha}$

w.s.t $x \in \overline{\prod A_\alpha}$

Let $U = \prod U_\alpha$ be a basis elt. of either the box or prod. top. cotains x.

Since, $x_\alpha \in \overline{A_\alpha}$, we can choose $y_\alpha \in U_\alpha \cap A_\alpha \ \forall\ \alpha$.

$\Rightarrow y = (y_\alpha) \in U \ \& \ \prod A_\alpha$

$\because U$ is arbitrary $\Rightarrow x \in \overline{\prod A_\alpha}$

Conversely, suppose $x = (x_\alpha) \in \overline{\prod A_\alpha}$ in either topo..

w.s.t for any given index β, we've $x_\beta \in \overline{A_\beta}$

Let V_β be an arbit. open set of $X_\beta \ni x_\beta$.

$\because \pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ in either top. & $\pi_\beta^{-1}(V_\beta) \ni y = (y_\alpha) \in \prod A_\alpha$

$\Rightarrow y_\beta \in V_\beta \cap A_\beta \Rightarrow x_\beta \in \overline{A_\beta}$

Thm. 6:- Let $f: A \to \prod_{\alpha \in J} X_\alpha$ be gn. by the eqn. $f(a) = (f_\alpha(a))_{\alpha \in J}$, where $f_\alpha : A \to X_\alpha \ \forall\ \alpha$. Let $\prod X_\alpha$ have the product topo. Then the fn. f is continuous iff each fn. f_α is cont..

Proof: Let π_β be the projection of the product onto its βth factor.

T.P: π_β is conti.

If U_β is open in X_β, the set $\pi_\beta^{-1}(U_\beta)$ is a subbasis elt. for the product topo. on X_α.

Now, suppose $f: A \to \prod X_\alpha$ is conti. $\Rightarrow f_\beta = \pi_\beta \circ f$ is cont.

Conversely, suppose each coordinate fn. f_α is conti.

T.P: f is cont.

It suffices t.p.t the inv. img. under f of each subbasis elt. is open in A.

Subbasis elt. of the prod. top. on $\prod X_\alpha$ is $\pi_\beta^{-1}(U_\beta)$

Now, $f^{-1}(\pi_\beta^{-1}(U_\beta)) = f_\beta^{-1}(U_\beta)$ $[\because f_\beta = \pi_\beta \circ f]$

$\because$ f_β is cont., this set is open in A.

The Metric Topology:-

Defn: A metric on a set X is a fn. $d: X \times X \to \mathbb{R}$ having the foll. prop:

i) $d(x,y) \geq 0 \; \forall x,y \in X$ & $d(x,y) = 0 \Leftrightarrow x = y$

ii) $d(x,y) = d(y,x) \; \forall x,y \in X$

iii) (Δ^{lar} inequality): $d(x,y) + d(y,z) \geq d(x,z) \; \forall x,y,z \in X$.

9.08.20

Defn: Given $\epsilon > 0$, the set $B_d(x,\epsilon) = \{y \mid d(x,y) < \epsilon\} \; \forall y \ni: d(x,y) < \epsilon$ is called the ϵ-ball centered at x and is denoted by $B(x,\epsilon)$.

Defn: If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x,\epsilon)$ for $x \in X$ & $\epsilon > 0$ is a basis for a top. on X, called the metric topo. induced by d.

RESULT:- A set U is open in the metric topo. induced by d iff $\forall\, y \in U$, $\exists\, \delta > 0 \ni : B_d(y, \delta) \subset U$.

Proof: If $\forall\, y \in U$, $\exists\, \delta > 0 \ni : B_d(y,\delta) \subset U \Rightarrow U$ is open.

Conversely, if U is open $\Rightarrow U \supset B = B_d(x, \epsilon) \ni y$ & $B \supset$ a basis elt. $B_d(y, \delta)$ centred at y.

Ex:1 Set X defnd by $d(x,y) = 1$ if $x \neq y$ & $d(x,y) = 0$ if $x = y$.

Soln: d is metric.

Topo. induced by this is the discrete topo..

$B(x, 1)$ is the basis elt..

Ex:2 The standard metric on $\mathbb{R}$ defnd. by $d(x,y) = |x-y|$.

Soln: d is metric, the top. induces is the same as order top.. i.e., $(a,b) = B(x, \epsilon)$ where $x = (a+b)/2$ & $\epsilon = (b-a)/2$. Conversely, each $B(x, \epsilon) = (x-\epsilon, x+\epsilon)$.

Defn:- X is a top. sp. & X is metrizable if $\exists$ a metric d on the set X that induces the top. of X.

Metric sp. is metrizable space X if metric d gives the topo. of X.

Defn.:- $X \to$ met. sp with metric d. $A \subseteq X$ is bnd if $\exists\, M \ni : d(a_1, a_2) \leq M$ $\forall$ pair $a_1, a_2 \in A$.

$A \to$ bnd. & non-empty, then $\operatorname{diam} A = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$.

Thm.:1 Let X be a met. sp. with met. d. Define $\bar{d} : X \times X \to \mathbb{R}$ by $\bar{d}(x,y) = \min\{d(x,y), 1\}$. Then $\bar{d}$ is a metric that induces the same topo. as d.

The metric $\bar{d}$ is called the standard bounded metric corresponding to d.

Proof:- Let X be a met. sp. with metric d.

$\bar{d} : X \times X \to \mathbb{R}$ is defined as $\bar{d}(x,y) = \min\{d(x,y), 1\} \to (1)$

First two conditions for a metric is trivial.

To check Δ^{le} inequality : $\bar{d}(x,z) \le \bar{d}(x,y) + \bar{d}(y,z) \to (2)$

If either $d(x,y) \ge 1$ (or) $d(y,z) \ge 1$ then

R.H.S. of (2) is atleast 1.

Since L.H.S of (2) is atmost 1.

The inequality (2) holds.

Now, consider $d(x,y) < 1$ & $d(y,z) < 1$.

We've $d(x,z) \le d(x,y) + d(y,z) = \bar{d}(x,y) + \bar{d}(y,z)$

$\because \bar{d}(x,z) \le d(x,z)$ by defn., the Δ^{le} ineq. holds for $\bar{d}$.

W.k.t. the coll. of ϵ-balls with $\epsilon < 1$ forms a basis for met. top.

Every basis elt $\ni x$ contains such an ϵ-ball centered at x.

$\therefore$ the coll. of ϵ-balls with $\epsilon < 1$ under d & $\bar{d}$ are same

$\Rightarrow d$ and $\bar{d}$ induce the same top. on X.

Defn: $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, norm of x is $\|x\| = (x_1^2 + \dots + x_n^2)^{1/2} = \sqrt{x_1^2 + \dots + x_n^2}$

Defn: Euclidean metric d on $\mathbb{R}^n$ is

$$d(x,y) = \|x - y\| = [(x_1 - y_1)^2 + \dots + (x_n - y_n)^2]^{1/2}$$

Defn:- Square metric ρ is $\rho(x,y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$

Lemma 2: Let $d, \bar{d}$ be two metrics on the set X; let $\mathcal{T}$ & $\mathcal{T}'$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ iff for each x in X and each $\epsilon > 0$, $\exists\ \delta > 0 \ni: B_{d'}(x, \delta) \subset B_d(x, \epsilon)$.

Proof: Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$.

Given basis elt. $B_d(x, \epsilon)$ for $\mathcal{T}$ $\exists$ a basis elt. B' for $\mathcal{T}' \ni: x \in B' \subset B_d(x, \epsilon)$

$\therefore$ We can find ~~$B_d(x)$~~ a ball $B_{d'}(x, \delta)$ within B'.

Hence $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$.

Conversely, $B_{d'}(x, \delta) \subset B_d(x, \epsilon) \rightarrow (*)$

w.p.t. $\mathcal{T}'$ is finer than $\mathcal{T}$

Given $B \in \mathcal{T} \ni x$, we can find a ball $B_d(x, \epsilon)$ within B.

$(*) \Rightarrow \exists\ \delta \ni: B_{d'}(x, \delta) \subset B_d(x, \epsilon) \Rightarrow \mathcal{T}'$ is finer than $\mathcal{T}$.

Thm: 3 The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Proof: Let $x = (x_1, \dots, x_n)$ & $y = (y_1, \dots, y_n)$ be two pts. of $\mathbb{R}^n$.

$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\, \rho(x, y) \rightarrow (1)$

$\rho(x, y) \leq d(x, y) \Rightarrow B_d(x, \epsilon) \subset B_\rho(x, \epsilon)\ \forall\, x, \epsilon. \rightarrow (2)$

$d(x, y) \leq \sqrt{n}\, \rho(x, y) \Rightarrow B_\rho(x, \epsilon/\sqrt{n}) \subset B_d(x, \epsilon)\ \forall\, x, \epsilon. \rightarrow (3)$

From (2) & (3) the two metric topo. are the same.

Now, t.s.t. the prod. top. is same as given by metric ρ.

Let $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be a basis for the prod topo. &

& let $x = (x_1, \dots, x_n) \in B$

$\forall\, i$, $\exists$ an $\epsilon_i \ni: (x_i-\epsilon_i, x_i+\epsilon_i) \subset (a_i, b_i)$

Choose $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\} \Rightarrow B_\rho(x,\epsilon) \subset B$.

$\therefore$ ρ-topo. is finer than the prod. topo.

Conversely, let $B_\rho(x,\epsilon)$ be a basis for ρ-topo.

Given $y \in B_\rho(x,\epsilon)$ we've to find a basis elt. B for the prod. topo. $\ni: y \in B \subset B_\rho(x,\epsilon)$.

Now, $B_\rho(x,\epsilon) = (x_1-\epsilon, x_1+\epsilon) \times \dots \times (x_n-\epsilon, x_n+\epsilon)$ is itself a basis elt. for the prod. topo.

2M **Defn:** $J \to$ index set, $\mathbf{x} = (x_\alpha)_{\alpha\in J}$ & $\mathbf{y} = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, the metric $\bar{\rho}$ on $\mathbb{R}^J$

$$\bar{\rho}(\mathbf{x},\mathbf{y}) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\},$$ where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$; then $\bar{\rho}$ is called the uniform metric on $\mathbb{R}^J$ & the topo. it induces is called the uniform topology.

Relation between uniform topo. & the prod. top. & box topo.:-

Thm 4:- The uniform topo. on $\mathbb{R}^J$ is finer than the product topo. & coarser than the box topo.; these three topo. are all different if J is ∞.

Proof: Suppose $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ & a prod. top. basis elt. $\prod U_\alpha$ about $\mathbf{x}$.

Let $\alpha_1, \dots, \alpha_n$ be the indices for which $U_\alpha \neq \mathbb{R}$.

$\because U_{\alpha_i}$ is open in $\mathbb{R} \Rightarrow \forall\, i$ choose $\epsilon_i > 0 \ni: B_{\bar{d}}(x_{\alpha_i}, \epsilon_i) \subset U_{\alpha_i}$

Let $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\} \Rightarrow B_{\bar{\rho}}(\mathbf{x}, \epsilon) \subset \prod U_\alpha$

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If $z \in \mathbb{R}^J \ni : \bar{\rho}(x,z) < \epsilon \Rightarrow \bar{d}(x_\alpha, z_\alpha) < \epsilon \ \forall \alpha$

$\therefore z \in \prod U_\alpha$.

Hence uniform topo. is finer than the product topo..

On the other hand, let $B(x,\epsilon) \in \bar{\rho}$ metric. Then the box nbhd.

$$U = \prod (x_\alpha - \tfrac{1}{2}\epsilon, x_\alpha + \tfrac{1}{2}\epsilon) \text{ of } x \subset B.$$

If $y \in U$, then $\bar{d}(x_\alpha, y_\alpha) < \epsilon/2 \ \forall \alpha$

$\therefore \bar{\rho}(x,y) \leq \epsilon/2$

Thm 5:- Let $\bar{d}(a,b) = \min\{|a-b|, 1\}$ be the standard metric on $\mathbb{R}$. If x and y are two pts. of $\mathbb{R}^{\omega}$, define $D(x,y) = \sup\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\}$. Then D is a metric that induces the prod. top. on $\mathbb{R}^{\omega}$.

Proof: Let $\bar{d}(a,b) = \min\{|a-b|, 1\}$ on $\mathbb{R}$.

If $x, y \in \mathbb{R}^{\omega}$, define $D(x,y) = \sup\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\}$

First two conditions for a metric is trivial.

T.P Δ^{le} ineq.:- we've $\forall i$, $\frac{\bar{d}(x_i,z_i)}{i} \le \frac{\bar{d}(x_i,y_i)}{i} + \frac{d(y_i,z_i)}{i}$

$$\le D(x,y) + D(y,z)$$

$$\Rightarrow \sup\left\{\frac{\bar{d}(x_i,z_i)}{i}\right\} \le D(x,y) + D(y,z)$$

T.P: D induces prod. top. on $\mathbb{R}^{\omega}$

Let U be open in met. top. & let $x \in U$

We find an open set V in the prod. top. $\ni: x \in V \subset U$.

Choose ϵ-ball $B_D(x,\epsilon)$ lying in U.

Choose N large enough that $1/N < \epsilon$

V is a basis elt. for prod. top.

$$V = (x_1-\epsilon, x_1+\epsilon) \times \cdots \times (x_N-\epsilon, x_N+\epsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots$$

T.P: $V \subset B_D(x,\epsilon)$: Given any y in $\mathbb{R}^{\omega}$

$$\frac{\bar{d}(x_i,y_i)}{i} \le \frac{1}{N} \text{ for } i \ge N.$$

$$\therefore D(x,y) \le \max\left\{\frac{\bar{d}(x_1,y_1)}{1}, \ldots, \frac{\bar{d}(x_N,y_N)}{N}, \frac{1}{N}\right\}$$

If $y \in V \Rightarrow D(x,y) < \epsilon \Rightarrow V \subset B_D(x,\epsilon)$

Conversely, $U = \prod_{i \in \mathbb{Z}_+} U_i$ for the prod. top., where U_i is open in $\mathbb{R}$ for $i = \alpha_1, \dots, \alpha_n$ and $U_i = \mathbb{R}$ $\forall$ other indices i.

Given $x \in U$, we find an open set V of met. top. $\ni: x \in V \subset U$.

Choose $(x_i - \epsilon_i, x_i + \epsilon_i)$ in $\mathbb{R} \subset U_i$ for $i = \alpha_1, \dots, \alpha_n$, choose each $\epsilon_i \le 1$

$\Rightarrow \epsilon = \min\{\epsilon_i / i \mid i = \alpha_1, \dots, \alpha_n\}$

T.P: $x \in B_D(x, \epsilon) \subset U$

Let $y \in B_D(x, \epsilon)$. Then $\forall i$, $\frac{\bar{d}(x_i, y_i)}{i} \le D(x, y) < \epsilon$

Now, if $i = \alpha_1, \dots, \alpha_n$ then $\epsilon \le \epsilon_i / i$

$\Rightarrow \bar{d}(x_i, y_i) < \epsilon_i \le 1$

$\Rightarrow |x_i - y_i| < \epsilon_i$

$\therefore y \in \prod U_i$

Thm: 6 Let $f : X \to Y$; let X & Y be metrizable with metrics d_X & d_Y respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ & given $\epsilon > 0$, $\exists$ $\delta > 0$ $\ni$: $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$.

Proof: suppose $f : X \to Y$ is cont.

T.P: $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$

Given x & ϵ, consider $f^{-1}(B(f(x), \epsilon))$ open in X & $\ni x$.

Also contains δ-ball $B(x, \delta)$.

If $y \in \delta$-ball $\Rightarrow f(y) \in \epsilon$-ball centered at $f(x)$.

$\therefore d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$.

Conversely, suppose ϵ-δ cond. is satisfied.

T.P: f is cont.

Let V be open Y.

W.s.t $f^{-1}(V)$ is open in X.

Let $x \in f^{-1}(V)$

Since $f(x) \in V$, $\exists$ ϵ-ball $B(f(x), \epsilon) \subset V$.

By ϵ-δ condition, $\exists$ δ-ball $B(x, \delta) \ni f(B(x,\delta)) \subset B(f(x), \epsilon)$.

Then $B(x, \delta)$ is a nbhd of $x \in f^{-1}(V)$

$\therefore f^{-1}(V)$ is open.

Hence f is conti.

Lemma 7 (The sequence lemma):- Let X be a top. sp., let $A \subset X$. If there is a sequence of pts. of A converging to x, then $x \in \bar{A}$; the converse holds if X is metrizable.

Proof: Suppose $x_n \to x$, where $x_n \in A$.

Then every nbhd. U of x contains a pt. of A

$\therefore x \in \bar{A}$.

Conversely, suppose X is metrizable & $x \in \bar{A}$.

Let d be a metric for the top. of X.

$\forall$ +ve int. n take nbhd. $B_d(x, 1/n)$ and choose

$x_n = B_d(x, 1/n) \cap A$.

T.P: $x_n \to x$

Any open set U of x $\supset$ ϵ-ball $B_d(x, \epsilon)$.

Choose N so that $1/N < \epsilon$, then $U \supset x_i$ $\forall$ $i \geq N$.

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Thm 8:- Let $f: X \to Y$. If f is cont., then for every conv. seq. $x_n \to x$ in X, the seq. $f(x_n)$ conv. to $f(x)$. The converse holds if X is metrizable.

Proof: Assume f is continuous.

Given $x_n \to x$, w.s.t. $f(x_n) \to f(x)$.

Let V be a nbhd. of $f(x)$.

$\Rightarrow f^{-1}(V)$ is a nbhd of x

$\therefore$ There's an $N \ni: x_n \in f^{-1}(V)$ for $n \geq N$.

$\Rightarrow f(x_n) \in V$ for $n \geq N$

Conversely, let $A \subseteq X$, w.s.t $f(\bar{A}) \subset \overline{f(A)}$.

If $x \in \bar{A}$ then $\exists$ a seq. x_n of pts. of A convs to x [By seq. lemma]

By assumption $f(x_n) \to f(x)$.

$\because f(x_n) \in f(A) \Rightarrow f(x) \in \overline{f(A)}$.

Hence $f(\bar{A}) \subset \overline{f(A)}$

Thm:9 If X is a topo. sp., & if $f, g : X \to \mathbb{R}$ are cont. fns., then $f+g, f-g$ & $f \cdot g$ are cont.. If $g(x) \neq 0 \; \forall \; x$, then f/g is cont..

Proof: Let X be a top. sp. & $f, g : X \to \mathbb{R}$ are cont..

The map $h : X \to \mathbb{R} \times \mathbb{R}$ defined by

$h(x) = f(x) \times g(x)$

$\because f(x) : X \to \mathbb{R}$ & $g(x) : X \to \mathbb{R}$ are cont..

$\Rightarrow h(x)$ is cont..

$f+g = h \circ (+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R})$

$\therefore f+g$ is cont..

Ily $f-g$, $f \cdot g$ & f/g are cont.

$|||^{ly}$ $f-g$, $f\cdot g$ & f/g are cont.

Defn :- Let $f_n : X \to Y$ be a seq. of fns. from the set X to the metric space Y. Let d be the metric for Y. The seq. (f_n) converges uniformly to the fn. $f : X \to Y$ if given $\epsilon > 0$, $\exists$ an int. $N \ni$:
$d(f_n(x), f(x)) < \epsilon$ $\forall$ $n > N$ & all $x \in X$.

Thm 10 (Uniform Limit Theorem) :- Let $f_n : X \to Y$ be a seq. of cont. fns. from the topo. sp. X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.

Proof: Let V be open in Y

Let $x_0 \in f^{-1}(V)$.

We've t.f. a nbhd. U of $x_0 \ni$: $f(U) \subset V$.

Let $y_0 = f(x_0)$.

Choose ϵ so that ϵ-ball $B(y_0, \epsilon) \subset V$.

By uniform conv., choose $N \ni$: $\forall$ $n \geq N$ & all $x \in X$,

$d(f_n(x), f(x)) < \epsilon/3$.

By continuity of f_N, choose a nbhd. U of $x_0 \ni$: f_N carries U into the $\epsilon/3$ ball in Y centered at $f_N(x_0)$.

W.S.T. f carries U into $B(y_0, \epsilon)$ & hence into X.

Now, if $x \in U$, then

$d(f(x), f_N(x)) < \epsilon/3$ [By choice of N]
$d(f_N(x), f_N(x_0)) < \epsilon/3$ [" " " U]
$d(f_N(x_0), f(x_0)) < \epsilon/3$ [" " " N]

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Adding & by Δ^{le} ineq. we've $d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) < \epsilon/3 + \epsilon/3 + \epsilon/3$

$\therefore d(f(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) < \epsilon$

$\Rightarrow d(f(x), f(x_0)) < \epsilon$

Hence f is cont..

P.T $\mathbb{R}^\omega$ in the box topo. is not metrizable.

soln. W.S.T. the seq. lemma doesn't hold for $\mathbb{R}^\omega$.

Let $A \subseteq \mathbb{R}^\omega$: $A = \{(x_1, x_2, \ldots) \mid x_i > 0 \ \forall \ i \in \mathbb{Z}_+\}$

Let $0 = (0, 0, \ldots)$ be origin in $\mathbb{R}^\omega$.

T.P: In box topo., $0 \in \bar{A}$

If $B = (a_1, b_1) \times (a_2, b_2) \times \ldots$ is any basis elt. cing. 0

then B $\cap^{ts}$ A.

$\therefore$ The pt. $(\frac{1}{2}b_1, \frac{1}{2}b_2, \ldots) \in B \cap A$.

We've t.s.t there's no seq. of pts. of $A \to 0$.

Let (a_n) be a seq. of pts. of A, where

$a_n = (x_{1n}, x_{2n}, \ldots, x_{in}, \ldots)$

Every coordinate x_{in} is +ve.

$\therefore$ We can construct a basis elt. B' for the box topo. on $\mathbb{R}$

by $B' = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \ldots$

$\Rightarrow B' \supset$ origin 0. & $B' \not\supset (a_n)$

$\because a_n \notin B'$ bcz. $x_{nn} \notin (-x_{nn}, x_{nn})$

Hence (a_n) can't conv. to 0 in the box topo..

$\therefore 0 \in \bar{A}$

Hence $\mathbb{R}^\omega$ in the box top. is not metrizable.

25/08/20

CONNECTEDNESS

Defn: $X \to$ top. sp.. A separation of X is a pair U, V of disjoint non-empty open subsets of X whose union is X. The space X is said to be connected if there doesn't exist a separation of X.

Note :- A space X is connected ~~is the~~ iff the only subsets of X that are both open & closed in X are the empty set & X itself.

Lemma 1:- If Y is a subspace of X, a separation of Y is a pair of disjoint non-empty sets A & B whose union is Y, neither of which contains a lt. pt. of the other. The space Y is connected if $\exists$ no separation of Y.

Proof: Suppose A & B form a separation of Y, i.e., $A \cup B = Y$ & $A \cap B = \phi$.

$\Rightarrow A$ is both closed & open in Y

$\therefore \bar{A} (\in Y) = \bar{A} \cap Y$.

$\because A$ is closed in $Y \Rightarrow A = \bar{A} \cap Y$ (or) $\bar{A} \cap B = \phi$.

$\because \bar{A} = \cup A$ & its lt. pts., $B \ni^{ns}$ no lt. pts of A.

$|||^{ly}$, $A \ni^{ns}$ no lts. pts. of B.

Conversely, suppose $A \cap B = \phi$, $A, B \neq \phi$ & $A \cup B = Y$, neither of which contains a lt. pt. of the other.

$\Rightarrow \bar{A} \cap B = \phi$ & $A \cap \bar{B} = \phi$

$\because \bar{A} \cap Y = A$ & $\bar{B} \cap Y = B$

Thus both A & B are closed in Y & since $A = Y - B$ & $B = Y - A$, they're open in Y as well.

SM
Lemma 2 :- If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D. (48)

Proof: ∵ C & D are both open in X $\Rightarrow$ $C \cap Y$ & $D \cap Y$ are open in Y.

$\Rightarrow$ $C \cap Y$ & $D \cap Y$ are disjoint &

$(C \cap Y) \cup (D \cap Y) = Y$ if $C \cap Y$ & $D \cap Y$ are ~~disjoint~~ non-empty.

$\Rightarrow$ $C \cap Y$ & $D \cap Y$ form a separation of Y.

∴ We h/e $C \cap Y = \phi$ (or) $D \cap Y = \phi$

Hence Y must lie entirely in C or in D.

SM
Thm: 3 The union of a collection of connected subspaces of X that have a pt. in common is connected.

Proof: Let $\{A_\alpha\}$ be a collec. of con. subsp. of a sp. X.

Let $p \in \cap A_\alpha$.

T.P: The sp. $Y = \cup A_\alpha$ is connected.

Suppose $Y = C \cup D$ is a separation of Y.

Then $p \in C$ (or) $p \in D$.

If $p \in C$ ~~A_α~~

Since A_α is connected $\Rightarrow$ p must lie entirely in C or D.

∴ p lies entirely in C.

Hence $A_\alpha \subset C$ $\forall \alpha$

$\Rightarrow$ $\cup A_\alpha \subset C$

$\Rightarrow \Leftarrow$ to $D \neq \phi$.

Thm: 4 Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.

Pf: Let A be a connected subspace of X.

Let $A \subset B \subset \bar{A}$

Suppose $B = C \cup D$ is a separation of B.

Then A must lie entirely in C or in D.

Suppose $A \subset C \Rightarrow \bar{A} \subset \bar{C}$

Since $\bar{C}$ & D are disjoint, B can't intersect D.

$\Rightarrow\Leftarrow$ to D is non-empty subset of B.

2M 5M **Thm: 5:-** The image of a connected space under a continuous map is connected.

Pf:- Let $f : X \to Y$ be cont.

Let X be connected.

T.P: the image space $Z = f(X)$ is connected.

Consider a cont. surjective map $g : X \to Z$.

Suppose, $Z = A \cup B$ is a separation of Z, where A & B are open in Z

$\Rightarrow g^{-1}(A)$ & $g^{-1}(B)$ are disjoint

$g^{-1}(A) \cup g^{-1}(B) = X$

$\because g$ is cont.. $g^{-1}(A), g^{-1}(B)$ are open in X.

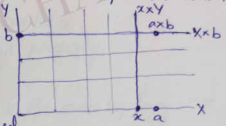
Also, since g is surjective, $g^{-1}(A)$ & $g^{-1}(B) \neq \phi$.

$\therefore$ They form a separation of X.

$\Rightarrow\Leftarrow$ to X is connected.

10m T-2 Thm: 6 A finite cartesian product of connected spaces is connected. (50)

Pf: First, w.p. for the product of two connected spaces X and Y.



Choose, base pt. $a \times b \in X \times Y$

The horizontal slice $X \times b$ is connected [Homeomorphic with X]

The vertical slice $x \times Y$ is connected [Homeomorph with Y].

Gives, T-shaped space $T_x = (X \times b) \cup (x \times Y)$ is connected

Here $x \times b$ is the common pt.

Now, form $\bigcup_{x \in X} T_x$ of all T-shaped spaces.

$\therefore \bigcup_{x \in X} T_x$ is connected and have $a \times b$ as common pt.

Also, $\bigcup_{x \in X} T_x = X \times Y \Rightarrow X \times Y$ is connected

By induction, $X_1 \times X_2 \times \cdots \times X_n$ is homeomorphic with

$(X_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$

Hence the thm.

26/08/20

Connected Subspaces of the Real Line

Defn: A simply ordered set L having more than one elt. is called a linear continuum if the following holds:

(i) L has the least upper bound property.

(ii) If $x<y$, $\exists!\ z \ni : x<z<y$.

Thm 1:- If L is a linear continuum in the order topology, then L is connected and so are intervals and rays in L.

Proof: Let L be a Linear continuum in the order topo..

Let Y be a subspace of L.

W.k.t., Y of L is convex if $\forall$ pair of pts. a, b of Y with $a<b$, the entire interval $[a,b]$ of pts. of L lies in Y.

W.p.t:- If Y is a convex subspace of L, then Y is connected.

Suppose $Y = A \cup B$, where A & B are disjoint non-empty sets open in Y.

Choose $a \in A$ & $b \in B$.

Suppose $a<b$.

The interval $[a,b]$ of pts. of L is contained in Y.

Hence $[a,b] = A_0 \cup B_0$, where

$A_0 = A \cap [a,b]$ & $B_0 = B \cap [a,b]$ are open in $[a,b]$ in the subspace top. which is the same as the order topo..

$\because$ $a \in A_0$ & $b \in B_0 \Rightarrow$ A_0 & B_0 are non-empty.

Thus, A_0 & B_0 form a separation of $[a, b]$.

Let $c = \sup A_0$.

W.S.T. c belongs neither to A_0 nor to B_0,

which contradicts to $[a, b] = A_0 \cup B_0$.

Case 1:- Suppose $c \in B_0$.

Then $c \neq a$, so either $c = b$ or $a < c < b$.

$\because$ B_0 is open in $[a, b] \Rightarrow$ some interval $(d, c] \subset B_0$

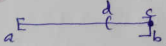
If $c = b$, we've $\Rightarrow\Leftarrow$

i.e., d is a smaller upper bound on A_0 than c.

If $c < b$, we've $(c, b]$ doesn't intersect A_0.

Then, $(d, b] = (d, c] \cup (c, b]$ doesn't intersect A_0.

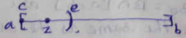
Again, d is a smaller upper bound on A_0 than c $\Rightarrow\Leftarrow$

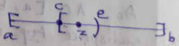


Case 2:- Suppose $c \in A_0$

Then $c \neq b \Rightarrow$ either $c = a$ or $a < c < b$.

$\because$ A_0 is open in $[a, b] \Rightarrow$ some interval $[c, e) \subset A_0$





By order property (ii) of L, we can choose a pt. z of $L \ni: c < z < e$.

Then $z \in A_0 \Rightarrow \Leftarrow$ to c is an upper bound for A_0.

Hence the thm..

10M

Cor: The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.

5M

Thm: - (Intermediate Value thm.). Let $f: X \to Y$ be cont., where X's a connected space and Y is an ordered set in the order topology. If a and b are two pts. of X and if r is a pt. of Y lying between $f(a)$ and $f(b)$, then $\exists$ a pt. c of $X \ni: f(c) = r$.

Pf: Let $f: X \to Y$ be cont.

Let X be a con. sp. & Y be an ordered set in the order topo.

Let $a, b \in X$ & $r \in Y$ lying b/w $f(a)$ & $f(b)$.

The set $A = f(X) \cap (-\infty, r)$ & $B = f(X) \cap (r, +\infty)$ are disjoint

Since $f(a) \in A$, $f(b) \in B \Rightarrow A$ & B are non-empty.

Since $f(x)$ nts. with the open ray in Y

$f(a)$ & $f(b)$ are open in $f(X)$.

If there's no pt. $c \in X \ni: f(c) = r$,

then $f(X) = A \cup B$.

$\Rightarrow A$ & B form a separation of $f(X)$

$\Rightarrow \Leftarrow$ to the image of a connected space under a continuous map is connected.

27-08-20 (54)

Defn:- Given pts. x & y of space X, a path in X from x to y is a continuous map $f:[a,b]\to X$ of some closed interval in the real line into X, $\ni$: $f(a)=x$ and $f(b)=y$.

A space X is said to be path connected if every pair of pts. of X can be joined by a path in X.

Result:

2M P.T a path-connected space X is connected.

Suppose $X=A\cup B$ is a separation of X.

Let $f:[a,b]\to X$ be any path in X.

$\Rightarrow$ The set $f([a,b])$ is connected

$\therefore$ $f([a,b])$ lies entirely in either A or B.

$\therefore$ There's no path in X joining a pt. of A to a pt. of B.

Which $\Rightarrow\Leftarrow$ to X is path connected.

Note:- The converse doesn't hold.

Remark:-

1) The unit ball B^n in $\mathbb{R}^n$ is $B^n=\{x \mid \|x\|\le 1\}$ $\to$ Path connected

2) Every open ball $B_d(x,\epsilon)$ & every closed ball $\bar{B}_d(x,\epsilon)$ in $\mathbb{R}^n$ $\to$ Path conn.

3) Punctured euclidean space is the space $\mathbb{R}^n-\{0\}$, where 0 is the origin in $\mathbb{R}^n$. $\to$ path con. if $n>1$

4) Unit sphere S^{n-1} in $\mathbb{R}^n$ is $S^{n-1}=\{x \mid \|x\|=1\}$. $\to$ Path con. if $n>1$

5) The ordered square I_o^2 is connected but not path connected.

Components & Local Connectedness

Defn: An equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the components (or connected components) of X.

Thm:1 The components of X are connected disjoint subspaces of X whose union is X, $\ni$: each non-empty connected subspace of X intersects only one of them.

Pf: Being equivalence classes, the components of X are disjoint & their union is X.

Each connected subspace A of X $\cap^{ts}$ only one of them.

~~If A $\cap$ts components c_1 and c_2 of X~~

Let c_1, c_2 be the components of X

If $A \cap c_1 = x_1$ and $A \cap c_2 = x_2 \Rightarrow x_1 \sim x_2$ [by defn]

This can't happen unless $c_1 = c_2$.

To show the component C is connected

Choose x_0 of C.

$\forall\, x \in C$ w.k.t $x_0 \sim x$

$\Rightarrow$ there's a connected subspace $A_x \supset x_0, x$.

By, $A_x \subset C$

$\therefore C = \bigcup_{x \in C} A_x$.

$\because$ the subspaces A_x are connected & have x_0 in common, their union is connected.

Defn: The equivalence relation on the space X by defining $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the path components of X.

Thm: 2 The path components of X are path-connected disjoint subspaces of X whose union is X, $\ni$: each $\neq \phi$ path-connected subsp. of X $\cap$ts only one of them.

5M

Thm : 3 A space x is locally conn iff for every open set U of x each components of U is open in x. (57)

Defn :- A sp. x is ~~local~~ locally connected at x if for every nbhd. U of x, there is a connected nbhd. V of x $\supset$ned in U.

If x is locally connected at each of its pts., it's locally connected.

2M x is locally path connected at x if for every nbhd U of x, there $\exists$ is a path-connected nbhd V of x $\supset$ned in U.

If x is locally path connected at each of its pts., then it's locally path connected.

Thm : 3 :-

Proof : Suppose x is locally connected.

Let U be an open set in x, C be a component of U.

If $x \in C$, choose a connected nbhd. V of $x \ni : V \subset U$.

$\because$ V is conn., V lies entirely in C of U.

$\therefore$ C is open in x.

Conversely, suppose components of open sets in x are open.

Given, $x \in X$ and a nbhd. U of x, let C $\supset$ing x.

Now, C is connected [$\because$ C is open in x]

$\therefore$ x is locally connected at x.

Thm : 4 :- A space x is loc. path. con iff $\forall$ open set U of x, each path comp. of U is open in x. [Proof : III to Thm 3]

Thm: 5 :- If X is a top. sp., each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.

Pf: Let X be a top. sp.

Let C be a comp. of X & $x \in C$.

Let P be the path comp. of X $\supseteq^{ing}$ x

Since P is connected, $P \subset C$.

T.S.T :- If X is loc. path connected, $P = C$.

Suppose $P \subsetneq C$.

Let Q be the $\cup^{on}$ of all path components of X that are different from P and $\cap^{ts}$ C

Each ~~of them~~ of path components of Q lies in C.

$\Rightarrow C = P \cup Q$

Each path component of X is open in X. [$\because X$ is loc. path con.]

$\therefore P$ & Q are open in X gives a separation of C.

This $\Rightarrow \Leftarrow$ to C is conn..

Hence $P \subset C \Rightarrow P = C$

UNIT-IV

COMPACTNESS

28-08-20

Defn: A collection $\mathcal{A}$ of subsets of a space X is said to cover of X or covering of X, if $\cup^{on}$ of elts. of $\mathcal{A} = X$.

If the elts. are open subsets of X, then $\mathcal{A}$ is open covering of X.

Defn: A sp. X is compact if every open cov of $\mathcal{A}$ of X cns. a finite subcoll. that also covers X.

Ex:

1) The real line $\mathbb{R}$ is not compact.

Soln. $\mathcal{A} = \{(n, n+2) \mid n \in \mathbb{Z}\}$ cns no finite subcoll. that covers $\mathbb{R}$.

2) The subsp. $X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$ of $\mathbb{R}$ is compact.

Note:- If Y is subsp. of X, a collection $\mathcal{A}$ of subsets of X is said to cover Y if the union of its elts. cns Y.

Lemma 1 :- Let Y be a subspace of X. Then Y is compact iff every covering of Y by sets open in X contains a finite subcollection covering Y.

Pf: Suppose Y is compact & $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by sets open in X.

Then $\{A_\alpha \cap Y \mid \alpha \in J\}$ is a covering of Y by sets open in Y.

Hence finite subcoll. $\{A_{\alpha_1} \cap Y, \ldots, A_{\alpha_n} \cap Y\}$ covers Y.

$\Rightarrow \{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ is a subcol. of $\mathcal{A}$ that covers Y.

Conversely,

T.P: Y is compact.

Let $\mathcal{A}' = \{A'_\alpha\}$ be a covering of Y by sets open in Y.

$\forall \alpha$, choose A_α open in $X \ni: A'_\alpha = A_\alpha \cap Y$.

$\therefore \mathcal{A} = \{A_\alpha\}$ is a covering of Y by sets open in X.

By hypo., some finite subcoll. $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ covers Y.

$\Rightarrow \{A'_{\alpha_1}, \ldots, A'_{\alpha_n}\}$ is a subcoll. of $\mathcal{A}'$ that covers Y.

Thm 2:- Every closed subspace of a compact space is compact.

Pf: Let Y be a closed subspace of a compact space X.

Given a covering $\mathcal{A}$ of Y by sets open in X.

Let us form a open covering $\mathcal{B}$ of X

i.e., $\mathcal{B} = \mathcal{A} \cup \{X - Y\}$.

some finite subcollection of $\mathcal{B}$ covers X.

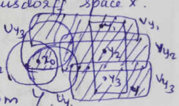
If subcollection contains $X - Y$ then discard $X - Y$.

The resulting collection is a finite subcoll. of $\mathcal{A}$ that covers Y.

Thm 3: Every compact subspace of a Hausdorff space is closed.

Pf: Let Y be a compact subspace of the Hausdorff space X.

w.p.t :- $X - Y$ is open $\Rightarrow$ Y is closed.



Let $x_0 \in X - Y$

we show $\exists$ a nbhd of x_0 disjoint from Y.

For each $y \in Y$

Choose disjoint nbhds U_y of x_0 and V_y of y_0.

The collection $\{V_y \mid y \in Y\}$ is a covering of Y by sets open in X.

$\therefore$ finitely many of them $V_{y_1}, \dots, V_{y_n}$ cover Y.

The open set $V = V_{y_1} \cup \dots \cup V_{y_n} \supset Y$ and

V is disjoint from the open set

$U = U_{y_1} \cap \dots \cap U_{y_n}$

If $z \in V$, then $z \in V_{y_i}$ for some i

Hence $z \notin U_{y_i} \Rightarrow z \notin U$.

Then U is a nbhd of x_0 disjoint from Y.

Hence the thm.

Lemma 4: If Y is a compact subsp. of the Hausdorff space X and x_0 is not in Y, then $\exists$ disjoint open sets U and V of X containing x_0 and Y respectively.

(*) SM Thm 5: The image of a compact space under a continuous map is compact.

Pf) Let $f: X \to Y$ be continuous.

Let X be compact.

Let $\mathcal{A}$ be a covering of the set $f(X)$ by sets open in Y.

Then the collection $\{f^{-1}(A) \mid A \in \mathcal{A}\}$ covering X.

The collection are open in X [$\because f$ is cont.]

Hence, finitely many of them say,

$f^{-1}(A_1), \dots, f^{-1}(A_n)$ cover X.

$\Rightarrow$ The sets $A_1, A_2, \dots, A_n$ cover $f(X)$.

Thm 6: Let $f: X \to Y$ be a bijective continuous fn.. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Pf: Let $f: X \to Y$ be bijective cont. fn..

W.p.t. images of closed sets of X under f are closed in Y,

This will prove continuity of f^{-1}.

If A is closed in X

$\Rightarrow A$ is compact [By Thm. 2]

$\therefore f(A)$ is compact [By Thm: 5]

$\therefore Y$ is Hausdorff $\Rightarrow f(A)$ is closed in Y [By thm: 3]

29.08.20

10M Thm. 7:- The product of finitely many compact spaces is compact.

Pf: By induction

Step 1: Let X, Y be given spaces and Y is compact.

Suppose $x_0 \in X$ and N is an open set of $X \times Y \supset$ slice $x_0 \times Y$ where $x_0 \times Y \in X \times Y$.

Now, we prove :- There is a nbhd W of x_0 in $X \ni$: N contains the entire set $W \times Y$.

First, we cover $x_0 \times Y$ by basis elts $U \times V$ lying in N.

The space $x_0 \times Y$ is compact [$\because x_0 \times Y$ homeomo to Y]

$\therefore x_0 \times Y$ is covered by $U_1 \times V_1, \ldots, U_n \times V_n$

Define, $W = U_1 \cap \ldots \cap U_n$.

W is open and $x_0 \in W$ [$\because U_i \times V_i \cap^{ts} x_0 \times Y$]

We prove, $U_i \times V_i$ which covers $x_0 \times Y$ cover the tube $W \times Y$.

Let $x \times y \in W \times Y$.

Consider $x_0 \times y \in x_0 \times Y$.

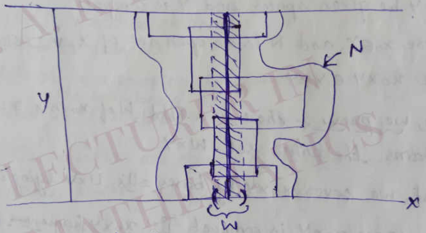
Now, $x_0 \times y \in U_i \times V_i$ for some $i \Rightarrow y \in V_i$

But $x \in U_j \ \forall j$ [$\because x \in W$]

$\therefore x \times y \in U_i \times V_i$

$\because$ all sets $U_i \times V_i$ lie in N and they cover $W \times Y$,

$\Rightarrow$ Tube $W \times Y$ lies in N also.



Step 2:- Let X and Y be compact spaces.

Let $\mathcal{A}$ be open covering of $X \times Y$.

Given $x_0 \in X$, the slice $x_0 \times Y$ is compact

$\therefore$ covered by $A_1, \dots, A_m$ of $\mathcal{A}$,

$N = A_1 \cup \dots \cup A_m$ is open $\supset x_0 \times Y$

Open set $N \supset$ tube $W \times Y$ about $x_0 \times Y$, [By step-1]
where W is open in X.

$\Rightarrow W \times Y$ is covered by $A_1, \dots, A_m$ of $\mathcal{A}$.

Thus, for each x in X, we can choose a nbhd. W_x of x $\ni$: tube $W_x \times Y$ can be covered by finitely many elts. of $\mathcal{A}$. (64)

The collec. of all nbhds. W_x is an open covering of X

$\therefore$ by compactness of X, $\exists$ $\{W_1, \dots, W_k\}$ covering X.

$$\therefore X \times Y = W_1 \times Y \cup \dots \cup W_k \times Y$$

Hence $X \times Y$ is covered.

2M, 5M

Lemma: 8 (The Tube Lemma):- Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ $\supseteq$ing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a nbhd. of x_0 in X.

Imp **Defn**:- A collection $\mathcal{C}$ of subsets of X is said to have the finite intersection property if for every finite subcollection $\{C_1, \dots, C_n\}$ of $\mathcal{C}$ then $C_1 \cap \dots \cap C_n$ is nonempty.

Thm: 9 Let X be a top. sp.. Then X is compact iff for every collection $\mathcal{C}$ of closed sets in X having the finite $\cap^{tion}$ prop., the $\bigcap_{C \in \mathcal{C}} C$ of all the elts. of $\mathcal{C}$ is nonempty.

Pf: Given a collection $\mathcal{A}$ of subsets of X,

Let $\mathcal{C} = \{X - A \mid A \in \mathcal{A}\}$ be their complements. $\to$ (1)

Then the foll. holds:

i) $\mathcal{A}$ is a coll. of open sets iff $\mathcal{C}$ is a coll. of closed sets.

ii) The coll. $\mathcal{A}$ covers X iff $\bigcap_{C \in \mathcal{C}} C$ of all elts. of $\mathcal{C}$ is ϕ.

iii) The finite subcol. $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ covers X iff the intersection of the corresponding elts. $C_i = X - A_i$ of $\mathcal{C}$ is ϕ.

(i) is trivial

(ii) & (iii) from DeMorgan's law:

$$X - \left(\bigcup_{\alpha \in J} A_\alpha\right) = \bigcap_{\alpha \in J} (X - A_\alpha).$$

Now, the pf. of the thm by taking contrapositive & the complement & is complement is equivalent to

"Given any coll. $\mathcal{A}$ of open subsets of X, if $\mathcal{A}$ covers X, then some finite subcollection of $\mathcal{A}$ covers X". $\to$ (2)

(2) is equivalent to its contrapositive

"Given any coll. $\mathcal{A}$ of open sets, if no finite subcollection of $\mathcal{A}$ covers X, then $\mathcal{A}$ doesn't cover X". $\to$ (3)

From ① and by (i)-(iii) we've ③ is equivalent to

"Given any coll. $\mathcal{C}$ of closed sets, if every finite $\cap^{tion}$ of elts of $\mathcal{C}$ is non-empty, then the intersection of all the elts. of $\mathcal{C}$ is nonempty".

Compact Subspaces of the Real Line

10M

Thm 1:- Let X be a simply ordered set having the least upper bound property. In the order topo., each closed intervals in X is compact.

Pf: Step 1:- Given $a<b$, let $\mathcal{A}$ be a covering of $[a,b]$ by sets open in $[a,b]$ in the subspace top..

T.P:- Existence of finite subcoll. of $\mathcal{A}$ covering $[a,b]$

First we prove :- If $x \in [a,b]$, $x \neq b$, then $\exists$ $y > x$ of $[a,b]$ $\ni$: int. $[x,y]$ can be covered by at most two elts. of $\mathcal{A}$

If x has an immediate successor y in X.

Then $[x,y] \supset x$ and y.

$\therefore$ $[x,y]$ is covered by at most two elts.

If x has no immediate successor in X.

Choose $A \in \mathcal{A}$ & $x \in A$.

Since, $x \neq b$ & A is open, $A \ni$ an interval of the form $[x,c)$ for some $c \in [a,b]$

Choose $y \in (x, c)$

Then $[x, y]$ is covered by the single elt. A of $\mathcal{A}$.

01.09.20

Step 2:- Let C be the set of all pts. $y > a$ of $[a, b] \ni$:

$[a, y]$ covered by finitely many elts. of $\mathcal{A}$.

For $x = a$, apply step 1,

$\Rightarrow \exists$ at least one such y.

$\therefore C \neq \phi$.

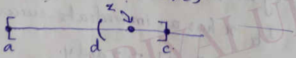
Let c be the l.u.b of C, then $a < c \leq b$.

Step 3:- T.P $c \in C$, i.e., the ~~int~~ $[a, c]$ can be covered by finitely many elts. of $\mathcal{A}$.

Choose $A \in \mathcal{A} \ni c$.

$\because$ A is open, $A \supset (d, c]$ for some $d \in [a, b]$.

If $c \notin C$, $\exists$ $z \in C$ in (d, c)



$\because$ $z \in C$, $[a, z]$ can be covered by finitely many, say n elts of $\mathcal{A}$.

Now, $[z, c] \subset A \in \mathcal{A}$

Hence, $[a, c] = [a, z] \cup [z, c]$ can be covered by $n+1$ elts. of $\mathcal{A}$.

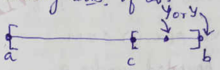
Thus $c \in C$, $\Rightarrow \Leftarrow$ to assumption.

Step 4:- T.P: $c = b$

Suppose $c < b$

For $x = c$, apply step 1,

$\Rightarrow \exists\, y > c$ of $[a,b] \ni: [c,y]$ can be covered by finitely many elts. of $\mathcal{A}$.



By step 3, $[a,y] = [a,c] \cup [c,y]$ can be covered by finitely many elts. of $\mathcal{A}$.

i.e., $y \in C$, $\Rightarrow \Leftarrow$ to c is u.b on C.

Cor:-2 Every closed interval in $\mathbb{R}$ is compact.

Thm 3:- A subsp. A of $\mathbb{R}^n$ is compact iff it is closed and is bounded in the euclidean metric d or the square metric ρ.

Pf: Let A be the subsp. of $\mathbb{R}^n$. It suffices to consider only the metric ρ,

$$\rho(x,y) \le d(x,y) \le \sqrt{n}\,\rho(x,y)$$

$\Rightarrow A$ is bnd. under d iff it's bnd. under ρ.

Suppose A is compact. ~~Then by thm~~ Then by the thm. ~~Since~~ 'Every compact subsp. of a Hausdorff sp. is closed',

$\Rightarrow$ A is closed.

Consider the collection of open sets

$\{B_\rho(0,m) \mid m \in \mathbb{Z}_+\}$ whose union is all of $\mathbb{R}^n$.

Some finite subcoll. covers A

$\Rightarrow A \subset B_\rho(0,M)$ for some M.

$\therefore$ For any two pts. x and y of A,

We've $\rho(x,y) \le 2M$.

Thus A is bounded under ρ.

Conversely, suppose A is closed and bounded under ρ.

Suppose $\rho(x,y) \le N$ $\forall$ pair $x, y \in A$.

Choose, $x_0 \in A$ and let $\rho(x_0, 0) = b$.

By Δ^{le} ineq. $\rho(x,0) \le N+b$ $\forall$ $x \in A$.

If $P = N+b$, then A is a subset of the cube $[-P, P]^n$ which is compact.

Since A is closed $\Rightarrow$ A is compact.

Thm: A (Extreme Value Theorem):- Let $f: X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then $\exists$ pts. c and d in $X \ni$: $f(c) \le f(x) \le f(d)$ for every $x \in X$.

Pf: Let $f: X \to Y$ be continuous and X be compact.

Given, the set $A = f(X)$ is compact.

T.P:- A has a largest elt. M & a smallest elt. m. Then since $m, M \in A$, we've $m = f(c)$ & $M = f(d)$ for some $c, d \in X$.

If A has no largest elt., then the collection

$\{(-\infty, a) \mid a \in A\}$ forms an open covering of A.

Since A is compact, some finite subcoll.

$\{(-\infty, a_1), \dots, (-\infty, a_n)\}$ covers A.

~~If a_i is the largest elts of the $a_1, \dots, a_n$~~

If a_i is the largest of the elts. $a_1, \dots, a_n$, then a_i belongs to none of these sets which is $\Rightarrow\Leftarrow$ to the fact that they cover A.

IIIly if A has no smallest elt.,

$\therefore$ $f(c) \le f(x) \le f(d) \ \forall x \in X$

Hence the thm..

Defn: Let (X, d) be a metric sp., let A be a nonempty subset of X. For each $x \in X$, the distance from x to A is defined by
$$d(x, A) = \inf \{d(x, a) \mid a \in A\}$$

Note: The diameter of a bounded subset A of a metric space (X, d) is the no. $\sup \{d(a_1, a_2) \mid a_1, a_2 \in A\}$.

Lemma: 5 (The Lebesgue number lemma) :- Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ $\ni$: $\forall$ subset of X having diameter $< \delta$, $\exists$ an elt. of $\mathcal{A}$ containing it.

The no. δ is called a Lebesgue number for the covering $\mathcal{A}$.

Pf: Let $\mathcal{A}$ be an open covering of X.

If X itself is an elt. of $\mathcal{A}$, then any +ve no. is a Lebesgue no. for $\mathcal{A}$.

So, assume X is not an elt. of $\mathcal{A}$.

Choose a finite subcoll. $\{A_1, \dots, A_n\}$ of $\mathcal{A}$ that covers X.

For each i, set $C_i = X - A_i$ and define $f : X \to \mathbb{R}$

by letting $f(x)$ be the average nos. $d(x, c_i)$.

i.e., $f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, c_i)$

T.P:- $f(x) > 0 \ \forall x$.

Given $x \in X$

Choose i so that $x \in A_i$.

Choose ϵ so the ϵ-nbhd. of x lies in A_i.

Then $d(x, c_i) \geq \epsilon \Rightarrow f(x) \geq \epsilon/n$.

T.P:- δ is the ~~leg~~ Lebesgue no..

$\because f$ is cont., it has a minimum value δ.

Let B be a subset of X of diameter $< \delta$.

Choose $x_0 \in B$, then B lies in the δ-nbhd. of x_0.

Now, $\delta \leq f(x_0) \leq d(x_0, c_m)$,

where $d(x_0, c_m)$ is the largest of the nos. $d(x_0, c_i)$.

Then δ-nbhd. of $x_0 \subset A_m = X - C_m$ of $\mathcal{A}$.

$\therefore \delta$ is the Lebesgue no.

Defn: If X is a sp.., a pt. $x \in X$ is said to be an isolated pt. of X if the one-point set $\{x\}$ is open in X.

Defn: A fn. $f : (X, d_X) \to (Y, d_Y)$ is uniformly cont. if given $\epsilon > 0$, $\exists \ \delta > 0 \ \ni: \forall$ pair of pts. x_0, x_1 of X,

$d_X(x_0, x_1) < \delta \Rightarrow d_Y(f(x_0), f(x_1)) < \epsilon$.

Thm: 5:- (Uniform continuity thm.):- Let $f: X \to Y$ be a cont. map of the compact metric sp. (X, d_X) to the metric space (Y, d_Y). Then f is uniformly cont.

Pf: Given $\epsilon > 0$, take open covering of Y by balls $B(y, \epsilon/2)$

Let $\mathcal{A}$ be the open covering of X by $f^{-1}(B(y, \epsilon/2))$.

Choose δ as the Lebesgue no. for the covering $\mathcal{A}$.

Then if $x_1, x_2 \in X \ni : d_X(x_1, x_2) < \delta$,

the set $\{x_1, x_2\}$ has diameter $< \delta$.

$\Rightarrow$ The image ~~set~~ $\{f(x_1), f(x_2)\}$ lies in some $B(y, \epsilon/2)$.

$\Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon$.

Thm: 6:- Let X be a nonempty compact Hausdorff space. If X has no isolated pts., then X is uncountable.

Pf: Step 1:- T.P:- Given any open set $U \in X$, $U \neq \phi$ and any $x \in X$, $\exists$ a open set $V \subset U$, $V \neq \phi \ni : x \notin \bar{V}$.

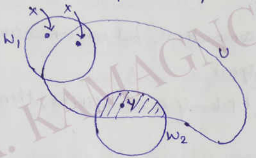
Choose $y \in U$, $y \neq x$, this is possible if $x \in U$ [$\because x$ is not an iso.pt of X]

and it is possible if $x \notin U$ [$\because U \neq \phi$]

Now, choose disjoint open sets W_1 about x and W_2 about y. (T4)

Then $V = W_2 \cap U$ is open and $V \subset U$.

Since, $V \ni y \Rightarrow V \neq \phi$ and $\bar{V} \not\ni x$.



Step 2 :- T.P :- Given $f : \mathbb{Z}_+ \to X$, f is not surjective gives X is uncountable

Let $x_n = f(n)$

By ~~step~~ Applying step 1 to $U = X$, $U \neq \phi$, choose $V_1 \subset X$, $V_1 \neq \phi \ni : \bar{V}_1 \not\ni x_1$.

In general, given V_{n-1} open and nonempty.

Choose V_n, $V_n \neq \phi \ni : V_n \subset V_{n-1}$ and $\bar{V}_n \not\ni x_n$.

Consider, $\bar{V}_1 \supset \bar{V}_2 \supset \cdots$ of nonempty closed sets of X.

Since X is compact, $\exists_1\ x \in \bigcap \bar{V}_n$.

Now, $x \neq x_n$ for any n

Since $x \in \bar{V}_n$ and $x_n \notin \bar{V}_n$

Cor: Every closed interval in $\mathbb{R}$ is uncountable.

Limit Point Compactness :-

Defn :- A space X is said to be limit point compact if every infinite subset of X has a lt. pt.

Thm 1 :- Compactness $\Rightarrow$ lt. pt. compactness, but not ~~conver~~ conversely.

Pf: Let X be a compact space,

Given $A \subset X$, we've t.p.t. if A is infinite, then A has a lt. pt..

We prove the contrapositive, — if A has no lt. pt., then A must be finite.

Suppose A has no lt. pt.

Then $A \supset^{ns}$ all its lt. pts.,

so A is closed.

Also, $\forall\, a \in A$, we can choose a nbhd U_a of $a \ni$:

$U_a \cap^{ts} A$ in the pt. a alone.

The space X is covered by open set $X - A$ & open set U_a

$\because X$ is compact, X can be covered by finitely many of these sets.

Since $X - A$ doesn't $\cap^{t}$ A & each $U_a \supset^{ns}$ only one pt. of A

$\Rightarrow$ The set A must be finite.

Defn:- Let X be a top. sp.. If (x_n) is a seq. of pts of X, and if $n_1 < n_2 < \dots < n_i < \dots$ is an increasing sequence of +ve terms integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a subsequence of the sequence (x_n). (76)

The space X is said to be sequentially compact if every seq. of pts. of X has a convergent subseq..

Theorem 2: Let X be a metrizable space. Then the following are equivalent (i) X is compact (ii) X is lt. pt. compact (iii) X is sequentially compact.

Pf: (i) $\Rightarrow$ (ii) : Proof of Thm 1.

(ii) $\Rightarrow$ (iii) :-

Assume X is a lt. pt. compact.

Given a sequence (x_n) of points of X,

Consider $A = \{x_n \mid n \in \mathbb{Z}_+\}$

If A is finite, then $\exists$ a pt. $x \ni: x = x_n$ for infinitely many values of n.

Here subsequence of (x_n) is constant.

$\therefore$ (x_n) converges trivially.

Also, if A is infinite, then A has a lt. pt. x.

Define a subseq. $(x_n) \to x$ as follows :-

Choose n_1 so that $x_{n_1} \in B(x,1)$.

Suppose +ve integer n_{i-1} is given.

Since, $B(x, 1/i) \cap A$ in infinitely many pts.,

choose $n_i > n_{i-1} \ni : x_{n_i} \in B(x, 1/i)$

Then the subsequence $x_{n_1}, x_{n_2}, \ldots$ converges to x.

(iii) $\Rightarrow$ (i) :-

First w.s.t :- If X is sequentially compact then the Lebesgue no. lemma holds for X.

Let $\mathcal{A}$ be an open covering of X.

Assume there's no $\delta > 0 \ni :$ each set of diameter $< \delta$ has an elt. of $\mathcal{A}$ containing it.

We prove the contradiction.

By our assumption $\Rightarrow \forall$ +ve int. n $\exists$ a set of diameter $< \frac{1}{n}$ $\notin \mathcal{A}$

Let C_n be such a set.

Choose $x_n \in C_n, \forall n$.

By hypo., some subseq. (x_{n_i}) of (x_n) converges to a

i.e., $(x_{n_i}) \to a$

Now, $a \in A \in \mathcal{A}$

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$\therefore$ A is open, choose $\epsilon > 0 \ni : B(a, \epsilon) \subset A$.

If i is large enough that $1/n_i < \epsilon/2$

$\Rightarrow C_{n_i}$ lies in the $\epsilon/2$ - nbhd of x_n

If i also chosen large enough that $d(x_{n_i}, a) < \epsilon/2$,

$\Rightarrow C_{n_i}$ lies in the ϵ-nbhd. of a.

i.e., $C_{n_i} \subset A \Rightarrow \Leftarrow$ to hypo..

Second w.s.t :- If X is sequentially compact, then given $\epsilon > 0$, $\exists$ a finite covering of X by open ϵ-balls.

We prove by contradiction :-

Assume $\exists\ \epsilon > 0 \ni : X$ can't be covered by finitely many ϵ-balls.

Construct a sequence of pts. x_n of X as follows :-

Choose x_1 be any pt. of X.

Note that $B(x_1, \epsilon)$ is not all of X.

Choose $x_2 \in X$ but $x_2 \notin B(x_1, \epsilon)$.

In general, given $x_1, \ldots, x_n$

Choose $x_{n+1} \notin B(x_1, \epsilon) \cup \ldots \cup B(x_n, \epsilon)$

[$\because$ The Balls doesn't cover X]

$\therefore d(x_{n+1}, x_i) \geq \epsilon$ for $i = 1, \ldots, n$.

$\therefore (x_n)$ have no convergent subseq.

$\Rightarrow \Leftarrow$ to our hypo.

Finally w.s.t :- If X is sequentially compact, then X is compact.

Let $\mathcal{A}$ be an open covering of X.

Since X is sequentially compact, $\mathcal{A}$ has a Lebesgue no. δ.

Let $\epsilon = \delta/3$

By sequential compactness of X, we can find a finite covering of X by open ϵ-balls.

Each of these balls has diameter atmost $2\delta/3$.

So, each balls lies in an elt. of $\mathcal{A}$.

Choosing one such elt. of $\mathcal{A}$ for each of these ϵ balls,

we obtain a finite subcollection $\mathcal{A}$ that covers X.

04-09-20

Local Compactness

Defn:- A space x is locally compact at x if there is some compact subspace C of x that contains a nbhd of x. If X is tog.

If X is locally compact at each of its points, X is said to be locally compact.

Note:- 1) Compact space is locally compact.

2) $\mathbb{R}$ is locally compact.

Soln. $x \in (a,b) \Rightarrow x \subset$ compact subsp. $[a,b]$.

3) The space $\mathbb{R}^n$ is locally compact.

Soln: $x \in$ some $(a_1,b_1) \times \dots \times (a_n,b_n) \subset [a_1,b_1] \times \dots \times [a_n,b_n]$

4) Every simply ordered set X having l.u.b property is locally compact.

Soln: Given a basis elt. for X say x_1

$x_1 \subset$ in a closed int. of X which is compact.

Thm:1 Let X be a space. Then X is locally compact Hausdorff iff $\exists$ a space Y satisfying the following conditions:-

(i) X is a subspace of Y

(ii) The set $Y - X$ consists of a single point

(iii) Y is a compact Hausdorff space.

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

Pf: Step 1:- I.P uniqueness.

Let Y and Y' be two spaces satisfying those conditions

Define $h : Y \to Y'$ by

Let h map point $p \in Y - X$ to the point $q \in Y' - X$

Let h equal the identity of X.

W.S.T : If U is open in Y, then $h(U)$ is open in Y'

By symmetry $\Rightarrow$ h is a homeomorphism.

First, if $U \not\ni p$

$\Rightarrow h(U) = U$

$\because$ U is open in Y and contained in X,

$\Rightarrow$ U is open in X.

Also, X is open in Y'

$\Rightarrow$ U is open in Y'

Second, suppose $U \ni p$.

$\because$ $C = Y - U$ is closed in Y

$\Rightarrow$ C is compact as a subspace of Y.

$\because C \subset X \Rightarrow C$ is a compact subspace of Y'. (82)

$\because Y'$ is Hausdorff, C is closed in Y

$\Rightarrow h(U) = Y' - C$ is open in Y'

Step 2:- Suppose X is locally compact Hausdorff and construct the space Y.

Let the points $\notin X$ be denoted by ∞ and is adjoint to X, and adjoin to X, forming the set $Y = X \cup \{\infty\}$

Now, define the collection of open sets of Y consists of (i)

1) all sets U open in X

2) all sets of the form $Y - C$, where C is a compact subspace of X.

To check this collection is a topology on Y.

The ~~{φ} set~~ empty set is a set of type (1)

The space Y is a set of type (2)

Checking $\cap^{tion}$:-

$U_1 \cap U_2$ is of type (1)

$(Y - C_1) \cap (Y - C_2) = Y - (C_1 \cup C_2)$ " " " (2)

$U_1 \cap (Y - C_1) = U_1 \cap (X - C)$ " " " (1)

[$\because C_1$ is closed in X]

Checking Union :- i.e., $\cup^{\text{on}}$ of any open sets is open.

$\bigcup U_\alpha = U$. is of type (1)

$\bigcup (Y - C_\beta) = Y - (\bigcap C_\beta) = Y - C$ " " " (2)

$(\bigcup U_\alpha) \cup (\bigcup (Y - C_\beta)) = U \cup (Y - C) = Y - (C - U)$ is of type(2)

Since, $C - U$ is closed subspace of C

$\therefore$ C is compact.

Now, w.s.t. X is a subspace of Y.

Given any open set of Y, w.s.t. its $\cap^{\text{tion}}$ with Y is open in X.

If U is of type(1), then $U \cap X = U \rightarrow (1)$

If $Y - C$ is of type(2), then $(Y - C) \cap X = X - C \rightarrow (2)$

Both (1) & (2) are open in X.

Conversely, any set open in X is a set of type (1) and therefore open in Y by definition.

T.S.T :- Y is compact

Let $\mathcal{A}$ be an open covering of Y.

$\mathcal{A} \supset$ open set $Y - C$ of type (2).

[$\because$ None of the open sets of type (1) $\supset^{\text{ns}}$ the pt. ∞]

Take all elements of $\mathcal{A}$ different from $Y-C$ and intersect them with X. (84)

They form a collection of open sets of X covering C.

Because C is compact, finitely many of them cover C

$\therefore$ the finite collection of elements of $\mathcal{A}$ and the ~~the elements of $Y-C$~~ elements of $Y-C$, cover all of Y.

T.S.T :- Y is Hausdorff

Let $x, y \in Y$

If $x, y \in X$, $\exists$ disjoint open sets $U \ni x$ and $V \ni y$.

Also, if $x \in X$ and $y = \infty$

we can choose a compact set $C \in X$ and $C \supset$ nbhd U of x.

$\Rightarrow U$ and $Y-C$ are disjoint nbhds of x and ∞ in Y.

05/09/20

Step 3:- T.P the converse:-

Suppose space Y satisfies conditions (i)-(iii) exists.

Then X is a Hausd sp., [$\because X$ is a subsp. of Haus.p.sp Y]

Given $x \in X$, w.s.t. X is locally compact at x.

Choose disjoint open sets $U \ni x$ and $V \supset Y - X$ of Y.

Then $C = Y - V$ is closed in Y

$\Rightarrow$ C is a compact subspace of Y.

$\because C \subset X \Rightarrow C$ is a compact subspace of X

$\therefore$ C contains the nbhd U of x.

08/09/20, 09/09/20

Defn:- If Y is a compact Hausd. space & X is a proper subspace of Y whose closure equals Y, then Y is said to be a compactification of X.

If $Y - X$ equals a single point, then Y is called the one-pt. compactification of X.

Thm: 2 Let X be a Hausdorff space. Then X is locally compact iff given x in X, and given a nbhd U of x, there is a nbhd V of $x \ni: \bar{V}$ is compact & $\bar{V} \subset U$.

Pf: Let X be a Hausd. space.

If $x \in X$ and given a nbhd U of x $\exists$ a nbhd V of $x \ni:$ $\bar{V}$ is compact and $\bar{V} \subset U$.

$\Rightarrow$ Local Compactness

$\therefore$ $C = \bar{V}$ is locally compact set $\ni^{ing}$ a nbhd. of x.

T.P the converse, suppose X is locally compact.

Let $x \in X$ and let U be a nbhd. of x.

Take one-pt. compactification Y of X

Let $C = Y - U$.

Then C is closed in $Y \Rightarrow C$ is a compact subspace of Y.

Applying lemma that 'If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then $\exists$ disjoint open sets U & V of X containing x_0 & Y, respectively'.

Here choose disjoint opent sets $V \ni x$ and $W \supset C$

Then $\bar{V}$ of V in Y is compact.

Also, $\bar{V}$ is disjoint from C

$\therefore \bar{V} \subset U$.

Cor: 3:- Let X be locally compact Hausdorff; let A be a subsp. of X. If A is closed in X or open in X, then A is locally compact.

Pf: Suppose A is closed in X.

Given $x \in A$, let C be a compact subspace of X, C c^ing nbhd U of x in X.

$\Rightarrow C \cap A$ is closed in C

Thus $C \cap A$ is compact and

$C \cap A$ contains the nbhd. $U \cap A$ of x in A.

Suppose A is open in X.

Given $x \in A$,

Applying Thm 2:- Choose a nbhd V of x in $X \ni$: $\overline{V}$ is compact and $\overline{V} \subset A$.

Then $C = \overline{V}$ is a compact subsp. of A containing V of x in A.

Cor:4 :- A space X is homeomorphic to an open subspace of a compact Hausdorff space iff X is locally compact Hausdorff.

Pf:

Assignment

10/09/20

UNIT-V

COUNTABILITY & SEPARATION AXIOMS

Defn:- A space X is said to have a countable basis at x if there is a countable collection $\mathcal{B}$ of neighborhoods of x $\ni$: each nbhd. of x $\supseteq$ at least one of the elts. of $\mathcal{B}$.

A space that has a countable basis at each of its pts. is said to satisfy the first countability axiom (or) first-countable.

Note:- Every metrizable space satisfies 1^{st} countable axiom.

Thm 1:- Let X be a topological space.

a) Let A be a subset of X. If there is a sequence of pts. of A converging to x, then $x \in \bar{A}$; the converse holds if X is first-countable.

b) Let $f: X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the seq. $f(x_n)$ converges to $f(x)$. The converse holds if X is 1^{st} countable.

Proof: The seq. lemma 1: - U-2
Thm: 8 - U-2

Defn:- If a space X has a countable basis for its topology, then X is said to satisfy <u>second countability axiom</u> (or) <u>second countable</u>.

<u>Note</u>: Second countability axiom $\Rightarrow$ First countability axiom.

<u>Result 1</u>:- The real line $\mathbb{R}$ has a countable basis.

Soln:- The coll. of all open int. (a,b) with rational end pts.

$\mathbb{R}^n$ has a countable basis the coll. of all products of intervals having rational end pts.

Also, $\mathbb{R}^\omega$ has a countable basis the coll. of all products of ~~intervals having rational en~~ $\prod_{n\in\mathbb{Z}_+} U_n$, where U_n is an open int. with rational end pts. for finitely many values of n & $U_n = \mathbb{R}$ $\forall$ other values of n.

Res: 2:- In the uniform topo., $\mathbb{R}^\omega$ satisfies 1st countability axiom but doesn't satisfy 2nd.

Soln. T.S.T:- If X is a sp. having a countable basis $\mathcal{B}$, then any discrete subsp. A of X must be countable.

Choose $\forall\ a\in A$, a basis elt. B_a that $\cap^{ts}$. A in pt. a alone.

If $a, b \in A$ & $a \neq b$, the sets B_a & B_b are different.

$\because$ $B_a \ni a$ & $B_b \not\ni b$ $\Rightarrow$ map $a \to B_a$ is an injection of A into $\mathcal{B}$.

$\therefore$ A must be countable.

Now, the subsp A of $\mathbb{R}^{\omega}$ consisting of all sequences of 0's & 1's is uncountable.

$\because$ $\bar{\rho}(a,b) = 1$ for any $a, b \in A$

$\Rightarrow$ A has discrete topo.

$\therefore$ In the uniform topo. $\mathbb{R}^{\omega}$ doesn't have a countable basis.

11/09/2020

Thm: 2:- A subsp. of a 1^{st}-count. sp. is 1^{st}-count & a countable product of 1^{st} count sp. is 1^{st}-count. A subsp. of a 2^{nd} count. sp is 2^{nd}-count., & a countable product of 2^{nd}-count. sp. is 2^{nd}-count.

Pf: Consider the second countability axiom.

If $\mathcal{B}$ is a countable basis for X, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable bases for the subspace A of X.

If $\mathcal{B}_i$ is a countable basis for the space X_i, then the collection of all $\prod U_i$, where $U_i \in \mathcal{B}_i$ for finitely many values of i and $U_i = X_i$ $\forall$ other values of i, is a countable basis for $\prod X_i$.

Consider the first countability axiom
The proof is similar to 2nd countability axiom.

Defn:- A subset A of a space X is said to be dense in X if $\bar{A} = X$.

Thm 3:- Suppose that X has a countable basis. Then:

a) Every open covering of X contains a countable subcollection covering X.

b) $\exists$ a countable subset of X that is dense in X.

Pf: Let $\{B_n\}$ be a countable basis for X.

a) Let $\mathcal{A}$ be an open covering of X.
For each +ve int. n, choose $A_n \in \mathcal{A}$ and $A_n \supset B_n$.

The collection $\mathcal{A}'$ of the sets A_n is countable. [$\because \mathcal{A}'$ is indexed with a subset J of +ve int.]

Also, $\mathcal{A}'$ covers X:

Given a pt. $x \in X$

Choose $A \in \mathcal{A}$ & $A \supset x$.

Since A is open $\exists$ a basis $B_n \ni: x \in B_n \subset A$.

$\because B_n$ lies in an element of $\mathcal{A}$

$\Rightarrow$ the index $n \in$ set J

$\therefore A_n$ is defined.

$\because A_n \supset B_n \Rightarrow A_n \supset x$.

Thus $\mathcal{A}'$ is a countable subcollection of $\mathcal{A}$ that covers X.

b) From each nonempty basis elt B_n,

Choose a pt. x_n.

Let D be the set consisting of the pts. x_n;

Then D is dense in X:

Given any pt. x of X, every basis elt. containing x $\cap^{ts}$. D, $\therefore x \in \overline{D}$.

Defn:- A space for which every open covering contains a countable subcovering is called a <u>Lindelöf space</u>.

A space having a countable subset is said to be separable.

Result:-

1) The product of two Lindelöf spaces need not be Lindelöf.

Soln. Let $\mathbb{R}_l$ be Lindelöf space.

W.S.T $\mathbb{R}_l \times \mathbb{R}_l = \mathbb{R}_l^2$ is not Lindelöf.

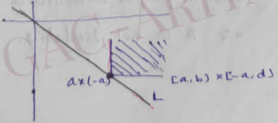
$\mathbb{R}_l^2$ has basis all sets of the form $[a,b) \times [c,d)$

We consider the subspace

$$L = \{x \times (-x) \mid x \in \mathbb{R}_l\}$$

$\therefore$ L is closed in $\mathbb{R}_l^2$.

$\mathbb{R}_l^2$ is covered by the open set $\mathbb{R}_l^2 - L$ and by all basis elts of the form $[a,b) \times [-a,d)$.



Each of these open sets intersects L in at most one point.

Since L is uncountable, no countable subcollection covers $\mathbb{R}_l^2$.

2) A subspace of a Lindelöf space need not be Lindelöf.

Soln. The ordered square I_o^2 is compact

$\therefore I_o^2$ is Lindelöf, trivially.

Now, consider the subspace $A = I \times (0,1)$

Let A is the ~~dis~~ union of disjoint sets

$U_x = \{x\} \times (0,1)$

Each U_x is open in A.

This collection of sets is uncountable and no proper subcollection covers A.

$\therefore$ A is not Lindelöf.

12/09/20

The Separation Axioms :-

Defn:- Suppose that one-pt. sets are closed in X. Then X is said to be regular if for each pair consisting of a pt. x and a closed set B disjoint from x, $\exists$ disjoint open sets containing x and B respectively.

Defn: A space X is said to be normal if for each pair A, B of disjoint closed sets of X, $\exists$ disjoint open sets $\supseteq$ing. A & B respectively.

Lemma 1:- Let X be a topological space. Let one-pt. sets in X be closed.

a) X is regular iff given a pt. x of X and a nbhd. U of x, there is a nbhd. V of x $\ni: \bar{V} \subset U$.

b) X is normal iff given a closed set A and an open set U containing A, there's an open set V containing $A \ni: \bar{V} \subset U$.

Pf:

a) Suppose X is regular & suppose the pt. x & the nbhd. U of x are given.

Let $B = X - U \Rightarrow B$ is closed.

By hypothesis, $\exists$ disjoint open sets $V \supset x$ & $W \supset B$.

Since ~~y ∈ B~~ $y \in B$, the set W is a nbhd. of y disjoint from V.

$\Rightarrow \overline{V}$ is disjoint from B.

$\therefore \overline{V} \subset U$.

T.P: The converse.

Suppose the pt. x & the closed set $B \not\ni x$.

Let $U = X - B$.

By hypo., $\exists$ a nbhd. V of $x \ni \overline{V} \subset U$.

The open sets V & $X - \overline{V}$ are disjoint open sets containing x & B

The open set $V \supset x$ & $X - \overline{V} \supset B$.

Thus X is regular.

b)

Thm:2:- a) A subsp. of a Haus. sp. is Haus.; a product of Haus. sp. is Haus..

b) A subsp. of a regular space is regular, a product of regular spaces is regular.

Pf: a) Let X be a Hausdorff.

Let Y be subsp. of X.

Let $x, y \in Y$

If U is a nbhd. of x and V is a nbhd. of y in X and U and V are disjoint.

Then $U \cap V$ & $V \cap Y$ are disjoint nbhds. of x & y in Y.

$\therefore$ Sub sp. of Haus. is Haus. p.

Let $\{X_\alpha\}$ be a family of Haus. sp.

Let $x = (x_\alpha)$ & $y = (y_\alpha)$ be disjoint pts. of the product space $\prod X_\alpha$.

Since $x \neq y$, there's some index $\beta \ni: x_\beta \neq y_\beta$

Choose disjoint open sets $U \supset x_\beta$ & $V \supset y_\beta$ in X_β.

Then the sets $\pi_\beta^{-1}(U) \supset x$ & $\pi_\beta^{-1}(V) \supset y$ are disjoint open sets in $\prod X_\alpha$.

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b) Let Y be a subspace of the regular space X.

Then one-pt. sets are closed in Y.

Let $x \in Y$ and let B be a closed subset of Y disjoint from x.

Now, $\bar{B} \cap Y = B$, where $\bar{B}$ is closure of B in X.

$\therefore x \notin \bar{B}$

By regularity of X,

Choose disjoint open sets $U \ni x$ and $V \supset \bar{B}$.

Then $U \cap Y \ni x$ and $V \cap Y \supset B$ are disjoint open sets in Y.

Let $\{X_\alpha\}$ be a family of regular spaces

Let $X = \prod X_\alpha$.

$\therefore X$ is Hausdorff [By (a)]

$\Rightarrow$ one-pt. sets are closed in X.

Let $\mathbf{x} = (x_\alpha)$ be a pt. of X

Let U be a nbhd. of $\mathbf{x}$ in X.

Choose a basis elt. $\prod U_\alpha$ about $\mathbf{x} \subset U$.

Choose, $\forall \alpha$, a nbhd. V_α of x_α in $X_\alpha \ni : \bar{V}_\alpha \subset U_\alpha$

$\Rightarrow U_\alpha = X_\alpha$, choose $V_\alpha = X_\alpha$.

Then $V = \prod V_\alpha$ is a nbhd. of x in X.

Since, $\overline{V} = \prod \overline{V}_\alpha \Rightarrow \overline{V} \subset \prod U_\alpha \subset U$

$\therefore X$ is regular.

Note:-

1. The space $\mathbb{R}_K$ is Hausdorff but not regular.
2. " " $\mathbb{R}_\ell$ is normal.
3. The space $\mathbb{R}_\ell^2$ is not normal.

Normal Spaces:-

Thm: 1:- Every regular space with a countable basis is normal.

Pf:. Let X be a regular space with a countable basis $\mathcal{B}$.

Let A & B be disjoint closed subsets of X.

Each pt. $x \in A$ has a nbhd. U not intersecting B.

By regularity,

choose a nbhd. V of x & $\bar{V}$ lies in U.

choose an elt. of $\mathcal{B}$ containing x and contained in V.

By choosing such a basis elt. $\forall\, x \in A$, we construct a countable covering of A by open sets whose closures doesn do not intersect B.

Since the covering A is countable, indexing it with ive integers and denoted by $\{U_n\}$

Similarly, choose a countable collection $\{V_n\}$ of open sets covering B, $\ni$: each set $\bar{V}_n$ is disjoint from A.

The sets $U = \cup U_n \supset A$ & $V = \cup V_n \supset B$ need not be disjoint.

For disjoint open sets,

Given n, define

$$U_n' = U_n - \bigcup_{i=1}^{n} \bar{V}_i \quad \& \quad V_n' = V - \bigcup_{i=1}^{n} \bar{U}_i$$

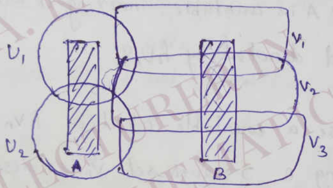
$\therefore$ Each set U_n' is open & V_n' is open.

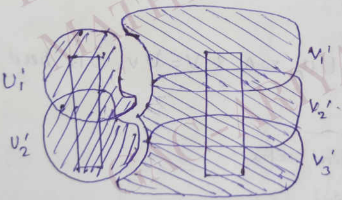
The collection $\{U_n'\}$ covers A,

Since $x \in A$ & $x \in U_n$ for some n &

$x \notin \bar{V}_i$.

Similarly, the collection $\{V_n'\}$ covers B.





Finally, T.P: $U' = \bigcup_{n \in \mathbb{Z}_+} U_n'$ and $V' = \bigcup_{n \in \mathbb{Z}_+} V_n'$ are disjoint.

If $x \in U' \cap V'$, then $x \in U'_j \cap V'_k$ for some j & k.

Suppose $j \leq k$.

By defn. of U'_j we've $x \in U_j$

since $j \leq k \Rightarrow$ by defn. of V'_k we've $x \notin \overline{U_j}$

This is a contradiction.

Similarly if $j \geq k$ we've a contradiction.

Hence U' & V' are disjoint.

$\therefore$ It is normal.

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Thm. 2:- Every metrizable space is normal.

Pf: Let X be a metrizable space with metric d.

Let A & B be disjoint closed subsets of X.

For each $a \in A$, choose ϵ_a so that the ball $B(a, \epsilon_a)$ does not intersect B.

|||ly, $\forall\, b \in B$, choose ϵ_b so that the ball $B(b, \epsilon_b)$ does not intersect A.

Define. $U = \bigcup_{a \in A} B(a, \epsilon_a/2)$ & $V = \bigcup_{b \in B} B(b, \epsilon_b/2)$

Then $U \supset A$ & $V \supset B$ are open sets.

T.P : U & V are disjoint.

If $z \in U \cap V$, then

$$z \in B(a, \epsilon_a/2) \cap B(b, \epsilon_b/2)$$

for some $a \in A$, $b \in B$.

By Δ^{le} ineq. t.s.t $d(a,b) < (\epsilon_a + \epsilon_b)/2$.

If $\epsilon_a \le \epsilon_b \Rightarrow d(a,b) < \epsilon_b$

$\therefore B(b, \epsilon_b) \ni a$

If $\epsilon_b \le \epsilon_a \Rightarrow d(a,b) < \epsilon_a$

$\therefore B(a, \epsilon_a) \ni b$

Hence $d(a,b) < (\epsilon_a + \epsilon_b)/2$

Thm 3:- Every compact Hausdorff space is normal.

Pf:- Let X be a compact Hausdorff sp.

T.P. X is Regular:-

If $x \in X$ & B is closed set in X, $B \not\ni x$,

$\Rightarrow B$ is compact.

This gives, $\exists$ disjoint open sets

$U \supset x$ and $V \supset B$.

T.P. X is Normal:-

Given disjoint closed sets A & B in X.

Choose $\forall a \in A$, disjoint open sets

$U_a \supset a$ & $V_a \supset B$.

By regularity,

~~Since A is com~~

The collection $\{U_\alpha\}$ covers A

Since A is compact

$\Rightarrow A$ may be covered by finitely many sets $U_{a_1}, U_{a_2}, \ldots, U_{a_m}$

Then $U = U_{a_1} \cup \cdots \cup U_{a_m} \supset A$ & $V = V_{a_1} \cap \cdots \cap V_{a_m} \supset B$ are disjoint.

Thm 4:- Every well-ordered set X is normal in the order topology.

Pf:- Let X be a well-ordered set.

T.P : Every interval $(x, y]$ is open in X.

If X has a largest elt y.

$(x, y]$ is a basis elt. about y.

If y is not the largest elt. of X,

then $(x, y] = (x, y')$, where y' is the immediate successor of y.

Now, let A & B be disjoint closed sets in X.

Assume neither A nor B $\ni$ the smallest elt a_0 of X

For each $a \in A$, $\exists$ a basis elt. about 'a' disjoint from B.

$B \ni$ some int. $(x, a]$.

Choose $\forall$ $a \in A$, an int. $(x_a, a]$ disjoint from B.

III^{ly}, for each $b \in B$, choose $(y_b, b]$ disjoint from A.

The sets $U = \bigcup_{a \in A} (x_a, a] \supset A$ & $V = \bigcup_{b \in B} (y_b, b] \supset B$ are open.

T.P: U & V are disjoint

Let $z \in U \cap V$

Then $z \in (x_a, a] \cap (y_b, b]$ for some $a \in A$ & $b \in B$.

Assume $a < b$, then

if $a \leq y_b$, the two intervals are disjoint.

If $a > y_b$, we've $a \in (y_b, b]$

$\Rightarrow \Leftarrow$ to $(y_b, b]$ is disjoint from A.

A similar contradiction occurs if $b < a$.

Finally, assume A & B are disjoint closed sets in X and

$A \supset$ smallest elt. a_0 of A.

The set $\{a_0\}$ is both open & closed in X.

$\therefore \exists$ disjoint open sets ~~U & V~~ ~~U~~ U & V

containing the closed sets $A - \{a_0\}$ & B respectively.

Then $U \cup \{a_0\}$ & V are disjoint open sets

containing A & B respectively.

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S.T a closed subspace of a normal space is normal.

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The Urysohn Lemma :-

Thm : 1 :- (Urysohn lemma) Let X be a normal sp.; let A & B be disjoint closed subsets of X. Let $[a,b]$ be a closed interval in the real line. Then $\exists$ a continuous map $f : X \to [a,b] \ni :$ $f(x) = a \ \forall x \in A$ & $f(x) = b \ \forall x \in B$.

Pf :-

Step 1 :- Let P be the set of all rational nos in $[0,1]$.

We define, $\forall p \in P$ an open set U_p of $X \ni :$ whenever $p < q$, we've $\bar{U}_p \subset U_q$

Thus, the sets U_p will be simply ordered. [By inclusion]

$\because$ P is countable, we can use induction to define the sets U_p.

Arrange the elts of P in an infinite seq.

Let us suppose 1 & 0 as first two elts of seq.

Now define the sets U_p, as follows :

i) $U_1 = X - B$.

ii) $\because$ A is closed, $A \subset U_1$ & by normality of X, choose $U_0 \ni : A \subset U_0$ & $\bar{U}_0 \subset U_1$.

In general, let P_n be the set consisting of the first n rational nos in the seq..

Suppose U_p is defined for all rational nos. $p \in P_n$ satisfies $p < q \Rightarrow \bar{U}_p \subset U_q \longrightarrow (*)$

Let r be the next rational no. in the seq..

We've ~~to~~ to define U_r.

Consider, $P_{n+1} = P_n \cup \{r\}$

P_{n+1} is finite subset of $[0,1]$

Since in a finitely simply ordered set, every elt. other than smallest & largest has an immediate predecessor & an successor.

We've 0 is the smallest & 1 is the largest of P_{n+1} & r is neither 0 nor 1.

So, r has a immediate predecessor $p \in P_{n+1}$ & successor $q \in P_{n+1}$.

By our definition & induction hypothesis we have $\bar{U}_p \subset U_q$.

By normality of X, we can find an open set $U_r \in X \ni$:

$$\overline{U}_p \subset U_r \ \& \ \overline{U}_r \subset U_q.$$

We P.T ⊛ holds for every pair of elts. of P_{n+1}.

If both elts. lies in P_n, then ⊛ holds [By induc]

If one of them is r & other is s of P_n, then

either $s \leq p \Rightarrow \overline{U}_s \subset \overline{U}_p \subset U_r$

or $s \geq q \Rightarrow \overline{U}_r \subset U_q \subset U_s$

Thus for every pair of elts of P_{n+1} ⊛ holds.

By induction, we've U_p defined $\forall\, p \in P$.

Arranging elts. of P in an infinite seq.:-

$$P = \{1, 0, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, \dots\}$$

By defining U_0 & U_1 we can also define

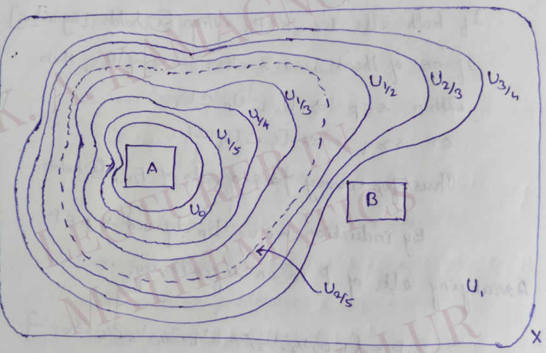
$U_{1/2} \Rightarrow \overline{U}_0 \subset U_{1/2}$ & $\overline{U}_{1/2} \subset U_1$

Then $U_0 \subset U_{1/3} \subset U_{1/2}$

$U_{1/2} \subset U_{2/3} \subset U_1$

... ...

At 9th step we've $U_{1/3} \subset U_{2/5} \subset U_{1/2}$ and so on.



Step 2:- Extending the defn. of U_p of all to all rational nos. p in $\mathbb{R}$ by

$$U_p = \phi \text{ if } p < 0$$
$$U_p = X \text{ if } p > 1$$

It's true $\forall$ rational nos. p & q,

$$p < q \Rightarrow \bar{U}_p \subset U_q.$$

Step 3 :- Given $x \in X$, let the set $Q(x) = \{p \mid x \in U_p\}$ rational nos.

Since no x is in U_p for $p < 0 \Rightarrow Q(x) \ni^{ns}$ no no. < 0.

Since every x is in U_p for $p > 1 \Rightarrow Q(x) \ni^{ns}$ every no. > 1.

$\therefore Q(x)$ is bounded below & its g.l.b is a pt. of $[0,1]$.

Define $f(x) = \inf Q(x) = \inf\{p \mid x \in U_p\}$

Step 4 :- T.P f is cont.

If $x \in A \Rightarrow x \in U_p \ \forall p \geq 0$

$\therefore Q(x) =$ set of all nonnegative rationals

& $f(x) = \inf Q(x) = 0$

$|||^{ly}$, if $x \in B$, then $x \in U_p$ for no $p \leq 1$

$\therefore Q(x)$ consists of all rational nos. >1 & $f(x) = 1$.

First, w.p.t (1) $x \in \bar{U}_r \Rightarrow f(x) \leq r$

(2) $x \notin U_r \Rightarrow f(x) \geq r$

T.P (1) :- If $x \in \bar{U}_r \Rightarrow x \in U_s \ \forall \ s > r$

$\therefore$ $\mathbb{Q}(x)$ $\supset^{ns}$ all rational nos. $> r$

So, by defn. $f(x) = \inf \mathbb{Q}(x) \leq r$.

T.P (2) :- If $x \notin U_r \Rightarrow x \notin U_s$ for any $s < r$.

$\therefore$ $\mathbb{Q}(x)$ $\supset^{ns}$ no rational nos. $< r$

So, $f(x) = \inf \mathbb{Q}(x) \geq r$.

Now, T.P continuity of f

Given a pt. $x_0 \in X$ & (c,d) in $\mathbb{R}$ containing the pt. $f(x_0)$.

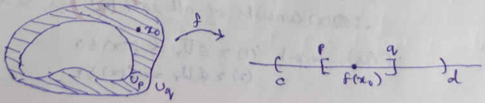
We've to find a nbhd. U of x_0 $\ni$: $f(U) \subset (c,d)$

Choose, rational nos. p & q $\ni$:

$$c < p < f(x_0) < q < d.$$

We show that the open set

$U = U_q - \bar{U}_p$ is the nbhd. of x_0.



First, we've $x_0 \in U$

i.e., (2) $\Rightarrow x_0 \in U_q$ for $f(x_0) < q$ &

(1) $\Rightarrow x_0 \notin \bar{U}_p$ for $f(x_0) > p$

Second, w.s.t $f(U) \subset (c, d)$

Let $x \in U$.

Then $x \in U_q \subset \bar{U}_q \Rightarrow f(x) \leq q$ [By (1)]

And $x \notin \bar{U}_p \Rightarrow x \notin U_p$ & $f(x) \geq p$ [By (2)]

Thus $f(x) \in [p, q] \subset (c, d)$.

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Defn:- If A & B are two subsets of the topo sp. X, and if there is a continuous fn. $f : X \to [0,1] \ni : f(A) = \{0\}$ & $f(B) = \{1\}$, then A & B can be separated by a continuous fn.

Defn:- A space X is completely regular if one-pt sets are closed in X & if for each pt. x_0 & each closed set A not containing x_0, there is a continuous fn. $f : X \to [0,1]$ such that $f(x_0) = 1$ & $f(A) = \{0\}$.

§ Thm 2:- A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular.

Pf:- Let X be completely regular.

Let Y be a subspace of X.

Let $x_0 \in Y$ and let A be a closed set of Y disjoint from x_0.

Now, $A = \bar{A} \cap Y$, where $\bar{A}$ is closure of A in X.

$\therefore x_0 \notin \bar{A}$

~~$\therefore Y$ is comp. reg~~.

Since X is completely regular,

Choose a continuous fn. $f : X \to [0,1] \ni: f(x_0) = 1$ & $f(\bar{A}) = \{0\}$.

The restriction of f to Y is the continuous fn. on Y.

$\therefore$ Y is comp. reg.

Let $X = \prod X_\alpha$ be a product of completely regular spaces.

Let $b = (b_\alpha)$ be a pt. of X &

let A be a closed set of X disjoint from b.

Choose a basis elt. $\prod U_\alpha \ni b$ but $\prod U_\alpha \not\supset A$.

Then $U_\alpha = X_\alpha$ except for finitely many α, say $\alpha = \alpha_1, \ldots, \alpha_n$.

Given $i = 1, \ldots, n$ choose a continuous fn.

$f_i : X_{\alpha_i} \to [0,1] \ni : f_i(b_{\alpha_i}) = 1$ &

$f_i(X - U_{\alpha_i}) = \{0\}$

Let $\phi_i(x) = f_i(\pi_{\alpha_i}(x))$, then $\phi_i : X \xrightarrow{\text{into}} \mathbb{R}$ is continuous, and ϕ_i vanishes outside $\pi_{\alpha_i}^{-1}(U_{\alpha_i})$.

The product $f(x) = \phi_1(x) \cdot \phi_2(x) \cdots \phi_n(x)$ is the cont. fn. on X.

i.e., $f(x) = 1$ at b & vanishes outside $\prod U_\alpha$.

Note:

1) If J is uncountable, the product space $\mathbb{R}^J$ is not normal.

2) If X is a well-ordered set. ~~The set~~ Given $\alpha \in X$, the set $S_\alpha = \{x \mid x \in X \ \& \ x < \alpha\}$ is called the section of X by α.

3) $\exists$ a well-ordered set A having a largest elt Ω, $\ni$: the section S_Ω of A by Ω is uncountable.

4) The product space $S_\Omega \times \bar{S}_\Omega$ is not normal.

5) The spaces $\mathbb{R}_l^2$ & $S_\Omega \times \bar{S}_\Omega$ are completely regular but not normal.

6) A is a G_δ set in X if A is the $\cap^{tion}$ of a countable collection of open sets of X.

7) Strong form of the Urysohn Lemma:- Let X be a normal space. There is a cont. fn. $f: X \to [0,1] \ni: f(x) = 0$ for $x \in A$ & $f(x) = 1$ for $x \in B$ and $0 < f(x) < 1$ otherwise, iff A & B are disjoint closed G_δ sets in X.

8) A space X is said to be perfectly normal if X is normal & if every closed set in X is a G_δ set in X.

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Thm:3:- Every topological group is completely regular.

Pf:. Let V_0 be a nbhd. of the identity elt e, in the topo. gp. of G.

In general, choose V_n as nbhd of $e \ni: V_n \cdot V_n \subset V_{n-1}$

Consider the set of all rationals p of the form $k/2^n$ with k & n integers.

$\forall\ p \in [0,1]$ define an open set $U(p)$ inductively as follows:-

$U(1) = V_0$ & $U(1/2) = V_1$

Given n, if $U(k/2^n)$ is defined for $0 < k/2^n \leq 1$,

define $U(1/2^{n+1}) = V_{n+1}$,

$U((2k+1)/2^{n+1}) = V_{n+1} \cdot U(k/2^n)$ for $0 < k < 2^n$.

For $p \leq 0$, let $U(p) = \phi$ &

for $p \geq 1$, let $U(p) = G$.

S.T $V_n \cdot U(k/2^n) \subset U((k+1)/2^n)$ $\forall\ k\ \&\ n$.

By Urysohn Lemma, we've that the given topological group is completely regular.

The Urysohn Metrization Theorem

Thm 1 :- (Urysohn Metrization Thm) Every regular sp. X with a countable basis is metrizable.

Pf Let X be a regular sp.

Let Y be a metrizable space.

W.S.T. X is metrizable by imbedding X in Y.

i.e., T.P X is homeomorphic with a subspace of Y.

Step 1 :- T.P :- $\exists$ a countable collection of continuous functions $f_n : X \to [0,1]$ having the property that given any pt. x_0 of X & any nbhd U of x_0, $\exists$ an index n $\ni$: f_n is +ve at x_0 & vanishes outside U.

Let $\{B_n\}$ be a countable basis for X.

For each pair n, m indices, $\bar{B}_n \subset B_m$

By Urysohn Lemma, choose a cont. fn.

$g_{n,m} : X \to [0,1] \ni: g_{n,m}(\bar{B}_n) = \{1\}$ &

$g_{n,m}(X - B_m) = \{0\}$

Given x_0 & given a nbhd U of x_0, we can choose a basis elt. $B_m \ni x_0$ & $B_m \subset U$.

By regularity, choose $B_n \Rightarrow x_0 \in B_n$ & $\overline{B}_n \subset B_m$.

Then the ~~pair n, m~~ function $g_{n,m}$ is defined for the pair n, m & it is positive at x_0 & vanishes outside U.

Since, collection $\{g_{n,m}\}$ is indexed with a subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$, it's countable.

$\therefore$ $\{g_{n,m}\}$ can be reindexed ~~to $\{f_n\}$~~ the collection

$\Rightarrow$ ~~$\{f_n\}$~~ collection $\{f_n\}$ is countable.

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Step 2:- Given a fn. f_n of step 1, take $\mathbb{R}^\omega$ in the product topology & define $F: X \to \mathbb{R}^\omega$ by

$$F(x) = (f_1(x), f_2(x), \ldots).$$

W.p.t. F is an imbedding.

First, F is continuous because $\mathbb{R}^\omega$ has the prod topo. and each f_n is continuous.

Second, F is injective, since given $x \neq y$,

w.k.t $\exists$ an index $n \ni: f_n(x) > 0$ & $f_n(y) = 0$

$\therefore F(x) \neq F(y)$.

Finally, T.P : F is a homeomorphism of X into its image, the sub sp. $Z = F(X)$ of $\mathbb{R}^\omega$.

w.k.t. F defines a cont. bijection of X with Z

w.s.t. $\forall$ open set $U \in X$, the set $F(U)$ is open in Z.

Let $z_0 \in F(U)$.

We shall find an open set W of $Z \ni:$

$$z_0 \in W \subset F(U)$$

Let $x_0 \in U \ni: F(x_0) = z_0$.

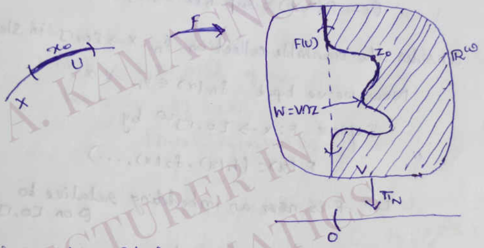
Choose an index N for which

$f_N(x_0) > 0$ & $f_N(X - U) = \{0\}$

Take $(0, +\infty)$ in $\mathbb{R}$ & let V be the open set $V = \pi_N^{-1}((0, +\infty))$ of $\mathbb{R}^\omega$.

Let $W = V \cap Z$

Then W is open in Z [By defn. of subsp. topo.]



T.P: $z_0 \in W \subset F(U)$

First, $z_0 \in W$, since

$$\pi_N(z_0) = \pi_N(F(x_0)) = f_N(x_0) > 0$$

Second, $W \subset F(U)$

If $z \in W$, then $z = F(x)$ for some $x \in X$ & $\pi_N(z) \in (0, \infty)$

Since, $\pi_N(z) = \pi_N(F(x)) = f_N(x)$ & f_N vanishes outside U

$\therefore x \in U$.

$\Rightarrow z = F(x) \in F(U)$

Thus F is an imbedding of X in $\mathbb{R}^\omega$.

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Step 3: Here we imbed X in metric space $(\mathbb{R}^{\omega}, \bar{\rho})$

i.e., imbed X in the subspace $[0,1]^{\omega}$ on which

$$\bar{\rho} = \rho(x,y) = \sup\{|x_i - y_i|\}.$$

Consider the countable collection $f_n : X \to [0,1]$ in Step 1

Now, we've t.p.t $f_n(x) \le 1/n \ \forall x$.

Define $F: X \to [0,1]^{\omega}$ by

$$F(x) = (f_1(x), f_2(x), \ldots)$$

T.P F is ~~now~~ an imbedding relative to ρ on $[0,1]^{\omega}$

By step 2, F is injective.

W.k.t. F: open set of X into open set of subsp. $Z = F(X)$ of the product topology on $[0,1]^{\omega}$.

Assume $f_n(x) \le 1/n$.

Let $x_0 \in X$ & let $\epsilon > 0$.

T.P continuity, we've to find a nbhd U of $x_0 \ni$:

$$x \in U \Rightarrow \rho(F(x), F(x_0)) < \epsilon.$$

Choose N large enough that $1/N \le \epsilon/2$.

$\forall\ n = 1, \dots, N$ by continuity of f_n choose a nbhd U_n of $x_0 \ni$:

$$|f_n(x) - f_n(x_0)| \le \epsilon/2 \text{ for } x \in U_n.$$

Let $U = U_1 \cap \dots \cap U_N$

W.s.t. U is the nbhd of x_0.

Let $x \in U$. If $n \le N$,

$$|f_n(x) - f_n(x_0)| \le \epsilon/2$$

If $n > N$,

$$|f_n(x) - f_n(x_0)| < 1/N \le \epsilon/2$$

$[\because f_n : X \text{ into } [0, 1/n]]$

$\therefore\ \forall\ x \in U,\ \rho(F(x), F(x_0)) \le \epsilon/2 < \epsilon.$

Hence the theorem.

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Thm 22:- (Imbedding thm):- Let X be a space in which one-pt sets are closed. Suppose that $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of cont. fns. $f_\alpha : X \to \mathbb{R}$ satisfying the requirement that for each pt. x_0 of X & each nbhd. U of x_0, there is an index $\alpha \ni$: f_α is +ve at x_0 & vanishes outside U.

Then the function $F: X \to \mathbb{R}^J$ defined by $F(x) = (f_\alpha(x))_{\alpha \in J}$ is an imbedding of X in $\mathbb{R}^J$. If f_α maps X into $[0,1]$ for each α, then F imbeds X in $[0,1]^J$.

Pf:-

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Cor: 3 :- A space X is completely regular iff it is homeomorphic to a subspace of $[0,1]^J$ for some J.

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The Tietze Extension Theorem

Thm 1: (Tietze extension thm.):- Let X be a normal space, let A be a closed subspace of X.

a) Any conti map of A into the closed interval $[a, b]$ of $\mathbb{R}$ may be extended to a continuous map of all of X into $[a, b]$.

b) Any cont. map of A into $\mathbb{R}$ may be extended to a a cont. map of all of X into $\mathbb{R}$.

Pf:-

Step 1:- To construct a function g defined on all of X.

Let $f : A \to [-r, r]$

w.p.t $\exists$ a cont. fn. $g : X \to \mathbb{R} \ni$:

$$|g(x)| \le \frac{1}{3} r \quad \forall x \in X$$

$$|g(a) - f(a)| \le \frac{2}{3} r \quad \forall a \in A.$$

Function g is constructed as follows:-

Divide $[-r, r]$ into three equal intervals of length $\frac{2}{3} r$.

$I_1 = [-r, -\frac{1}{3}r]$, $I_2 = [-\frac{1}{3}r, \frac{1}{3}r]$, $I_3 = [\frac{1}{3}r, r]$

Let B & C be subsets

$B = f^{-1}(I_1)$ & $C = f^{-1}(I_3)$ of A.

Since f is cont., B & C are closed disjoint subsets of A.

$\therefore$ They are closed in X.

By Urysohn Lemma, $\exists$ cont. fn. $g : X \to [-\frac{1}{3}r, \frac{1}{3}r]$

$\ni : g(x) = -\frac{1}{3}r \ \forall x \in B$ &

$g(x) = \frac{1}{3}r \ \forall x \in C$

$\Rightarrow |g(x)| \leq \frac{1}{3}r \ \forall x$.

W.P.T $\forall a \in A$, $|g(a) - f(a)| \leq \frac{2}{3}r$.

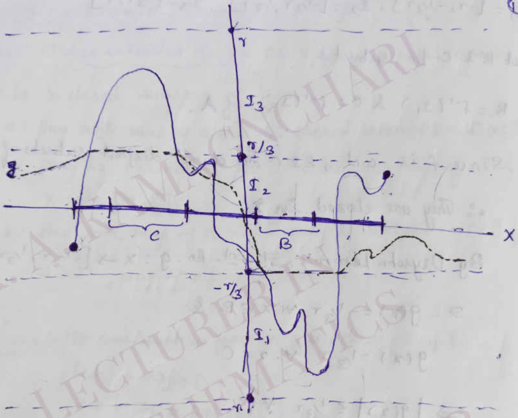
$\therefore$ There are three cases.

(i) If $a \in B \Rightarrow$ both $f(a)$ & $g(a) \in I_1$

(ii) If $a \in C \Rightarrow$ both $f(a)$ & $g(a) \in I_3$

(iii) If $a \notin B \Rightarrow f(a)$ & $g(a) \in I_2$

In each case $|g(a) - f(a)| \leq \frac{2}{3}r$.



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Step 2 : T.P. (a) of Tietze thm. :

Let $f : X \to [-1, 1]$ be a cont. map.

Then f satisfies the hypothesis of Step 1, with $r = 1$.

$\therefore$ $\exists$ a cont. real-valued fn. g, defined on all X $\ni$:

$|g(x)| \le 1/3$ for $x \in X$

$|f(a) - g(a)| \le 2/3$ for $a \in A$.

Now, $\underline{f - g_1} : A \xrightarrow{\text{into}} [-2/3, 2/3]$ $(f-g_1)(a)$

By applying step 1, let $r = 2/3$.

We obtain a real valued fn $\underline{\underline{g_2}}$ defined on all of $X \ni$;

$|g_2(x)| \le 1/3(2/3)$ for $x \in X$

$|f(a) - g_1(a) - g_2(a)| \le (2/3)^2$ for $a \in A$.

Then by applying step 1 to the fn $f - g_1 - g_2$. And so on.

In general, the real-valued fns. $g_1, \ldots, g_n$ defined on all of $X \ni$;

$|f(a) - g_1(a) - \cdots - g_n(a)| \le (2/3)^n$ for $a \in A$.

Applying step 1 to $f - g_1 - \cdots - g_n$, with $r = (2/3)^n$.

we obtain a real valued fn. g_{n+1} defined on all of $X \ni$:

$|g_{n+1}(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^n$ for $x \in X$.

$|f(a) - g_1(a) - \cdots - g_{n+1}(a)| \leq \left(\frac{2}{3}\right)^{n+1}$ for $a \in A$.

By induction, the fns. g_n are defined for all n.

Now define, $g(x) = \sum_{n=1}^{\infty} g_n(x) \ \forall x \in X$.

By comparison with geometric series $\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$

$g(x)$ converges.

T.S.T g is continuous, w.s.t the seq. s_n converges to g uniformly.

By Weierstrass M-test, if $k > n$, then

$$|s_k(x) - s_n(x)| = \left|\sum_{i=n+1}^{k} g_i(x)\right|$$

$$\leq \frac{1}{3}\sum_{i=n+1}^{k}\left(\frac{2}{3}\right)^{i-1}$$

$$< \frac{1}{3}\sum_{i=n+1}^{\infty}\left(\frac{2}{3}\right)^{i-1} = \left(\frac{2}{3}\right)^n$$

Let n be fixed & $k \to \infty$, then

$|g(x) - S_n(x)| \le (2/3)^n \quad \forall x \in X$

$\therefore$ S_n converges to g uniformly.

w.s.t, $g(a) = f(a)$ for $a \in A$.

Let $S_n(x) = \sum_{i=1}^{n} g_i(x)$, the nth partial sum of series.

Then $g(x)$ is the limit of the infinite seq $S_n(x)$

Since,

$|f(a) - \sum_{i=1}^{n} g_i(a)| = |f(a) - S_n(a)| \le (2/3)^n \; \forall a \in A$.

$\Rightarrow S_n(a) \to f(a) \; \forall a \in A$.

$\therefore f(a) = g(a)$ for $a \in A$.

Finally, w.s.t. $g : X \xrightarrow{\text{into}} [-1, 1]$.

Since the series $(1/3) \sum (2/3)^n \to 1$. ~~the proof is essential~~.

If $g : X \xrightarrow{\text{into}} \mathbb{R}$ then $r \circ g$, where $r : \mathbb{R} \to [-1, 1]$ is the map

$r(y) = y$ if $|y| \le 1$.
$r(y) = y/|y|$ if $|y| \ge 1$. is an extension of $f : X \xrightarrow{\text{into}} [-1, 1]$.

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Step 3 : T.P (b) of Tietze thm., here $f : A \xrightarrow{\text{into}} \mathbb{R}$.

Since $(-1,1)$ is ~~homeomo~~ homeomorphic to $\mathbb{R}$,

let ~~f : be~~ $f : A \xrightarrow{\text{into}} (-1,1)$ be continuous.

Now, extend f to a continuous map

$g : X \to [-1, 1]$.

Given g, let D be a subset of X

defined by, $D = g^{-1}(\{-1\}) \cup g^{-1}(\{1\})$.

Since g is continuous $\Rightarrow$ D is a closed subset of X

$\because$ $g(A) = f(A) \subset (-1, 1)$

$\Rightarrow$ the set A is disjoint from D.

By Urysohn lemma, $\exists$ cont. fn.

$\phi : X \to [0,1] \ni \phi(D) = \{0\}$ & $\phi(A) = \{1\}$.

Define $h(x) = \phi(x)\, g(x)$.

Then h is continuous

Also, h is an extension of f, since for $a \in A$,

$h(a) = \phi(a) g(a) = 1 \cdot g(a) = f(a)$.

Finally T.P:- $h : X \xrightarrow{\text{into}} (-1, 1)$

If $x \in D \Rightarrow h(x) = 0 \cdot g(x) = 0$

If $x \notin D \Rightarrow |g(x)| < 1$

~~$\Rightarrow |h(x)| \leq 1 \cdot |g(x)|$~~

$|g(x)|$

$|h(x)| \leq 1 \cdot g(x) < 1$

$\therefore |h(x)| \leq 1$