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II - M.Sc., Mathematics

Measure and Integration.

UNIT-I - Measure on the Real Line.

Chapter-I : Lebesgue outer Measure

Definition: The Lebesgue outer Measure (or) outer Measure.

The Lebesgue outer Measure or more briefly the outer measure, of a set is given by $m^*(A) = \inf \Sigma l(I_n)$. where the infimum is taken over all finite or Countable Collections of intervals $[I_n]$ Such that $A \subseteq \cup I_n$.

Theorem-1:

(i) $m^*(A) \geq 0$

(ii) $m^*(\phi) = 0$

(iii) $m^*(A) \leq m^*(B)$ if $A \subseteq B$

(iv) $m^*([x]) = 0$ for any $x \in R$.

Proof: Since $m^*(A) = \inf \Sigma l(I_n)$

(i) Here infimum is taken over all finite or Countable.

$\therefore m^*(A) \geq 0$.

(ii) $\because l(\phi) = 0 \therefore m^*(\phi) = 0$

(iii) If $A \subseteq B$

Take measure, we get

$m^*(A) \leq m^*(B)$.

(iv) Since $x \in I_n = [x, x+(1/n)]$ for each n,

and $l(I_n) = 1/n$

$\therefore m^*([x]) = 0$ for any $x \in R$.

Theorem - 2:

The outer measure of an interval Equals its length.

Proof:

case I:

suppose that I is a closed interval, $[a,b]$, say.

Then for each $\epsilon > 0$, we have

$$m^*([a,b]) \leq m^*([a, b+\epsilon))$$
$$\leq b-a+\epsilon$$

$\therefore m^*(I) \leq b-a$. - - - - - - - - ①

Now To obtain the opposite inequality.

for each $\epsilon > 0$, I may be covered by a collection of intervals $[I_n]$ such that

$$m^*(I) \geq \sum_{n=1}^{\infty} l(I_n) - \epsilon, \text{ where } I_n = [a_n, b_n]$$

For each n, let $I_n' = (a_n - \epsilon/2^n, b_n)$

Then $\bigcup_{n=1}^{\infty} I_n' \supseteq I$

W.K.T. If A is a closed bounded set in R and $A \subseteq \bigcup_{\alpha \in I} G_\alpha$, where the sets G_α are open and I is some index set then $\exists$ a finite sub collection of the sets, say $[G_i, i=1, \dots n]$ whose union Contains A. (Heine - Borel Theorem]

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∴ a finite subcollection of the I_n', say $J_1, J_2, \ldots, J_N$, where $J_k = (c_k, d_k)$ covers I.
Then as we may suppose that no J_k is contained in any other, we have
suppose that $c_1 < c_2 < \ldots < c_N$.

$$d_N - c_1 = \sum_{k=1}^{N} (d_k - c_k) - \sum_{k=1}^{N-1} (d_k - c_{k+1})$$
$$< \sum_{k=1}^{N} l(J_k)$$

So we have
$$m^*(I) \geq \sum_{n=1}^{\infty} l(I_n) - \epsilon$$
$$\geq \sum_{n=1}^{\infty} l(I_n') - 2\epsilon$$
$$\geq \sum_{k=1}^{N} l(J_k) - 2\epsilon$$
$$\geq d_N - c_1 - 2\epsilon$$
$$> b - a - 2\epsilon$$
$$> l(I) - 2\epsilon \text{ ------ } ②$$

∴ from ① & ② ⇒ $m^*(I) = l(I)$.

Case (ii)

Suppose that $I = (a, b]$ and $a > -\infty$.
If $a = b$ then $m^*(\phi) = 0$.
If $a < b$, suppose that $0 < \epsilon < b - a$ and write $I' = [a+\epsilon, b]$, Then

$$m^*(I) \geq m^*(I')$$
$$= l(I) - \epsilon \text{ .... } ③$$

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But $I \subseteq I' = [a, b+\epsilon)$

So $m^*(I) \leq l(I')$

$= l(I) + \epsilon$ -------- (4)

Since (3) & (4) are true for all small ϵ,

$m^*(I) = l(I)$.

III[ly] we can prove $I = (a, b]$ and $I = [a, b)$

Case 3 :.

Suppose that I is an infinite interval.

Four types of interval occur.

Suppose that $I = (-\infty, a]$

For any $M > 0$, $\exists$ k such that the finite interval I_M. where $I_M = [k, k+M) \subseteq I$

So $m^*(I) > M$

and Hence $m^*(I) = \infty = l(I)$

III[ly] we can prove $[a, \infty)$, $(-\infty, a)$, (a, ∞).

$\therefore$ The outer measure of an interval Equals its length.

$\therefore$ $m^*(I) = l(I)$

Hence proved the theorem.

Theorem 3

For any sequence of sets $\{E_i\}$,

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} m^*(E_i).$$

Proof: For each i and for any $\epsilon > 0$,

$\exists$ a sequence of intervals $\{I_{i,j}, i = 1, 2, \ldots\}$

Such that $E_i \subseteq \bigcup_{j=1}^{\infty} I_{i,j}$ and

$$m^*(E_i) \ge \sum_{j=1}^{\infty} l(I_{i,j}) - \epsilon/2^i$$

Then, $\bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} I_i$

i.e. The sets $[I_{i,j}]$ ~~theorem~~ form a countable class covering $\bigcup_{i=1}^{\infty} E_i$, so

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i,j=1}^{\infty} l(I_{i,j})$$

$$\le \sum_{i=1}^{\infty} m^*(E_i) + \epsilon$$

where ϵ is arbitrary constant,

$$\therefore m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} m^*(E_i)$$

Hence proved the theorem.

Example:

Show that, for any set A and any $\epsilon > 0$, there is an open set O containing A and such that $m^*(O) \le m^*(A) + \epsilon$.

Solution:

Choose a sequence of intervals I_n such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and

$$\sum_{n=1}^{\infty} l(I_n) - \epsilon/2 \le m^*(A).$$

If $I_n = [a_n, b_n)$, let $I_n' = (a_n - \epsilon/2^{n+1}, b_n)$

so that $A \subseteq \bigcup_{n=1}^{\infty} I_n'$

Hence if $O = \bigcup_{n=1}^{\infty} I_n'$, O is an open set

and
$$m^*(O) \le \sum_{n=1}^{\infty} l(I_n')$$
$$= \sum_{n=1}^{\infty} l(I_n) + \epsilon/2$$
$$\le m^*(A) + \epsilon$$

$\therefore m^*(O) \le m^*(A) + \epsilon.$

Hence the solution.

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Measure and Integration

UNIT-I- continued

Chapter - 2 : Measurable sets

Definition: Lebesgue Measurable

The set E is Lebesgue Measurable or, more briefly, measurable if for each set A, we have,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap CE)$$

To prove E is measurable

we need only show, for each A, that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap CE).$$

Note: If $m^*(E) = 0$ then E is measurable.

Definition: σ-algebra

A class of subsets of an arbitrary space X is said to be a σ-algebra (or) σ-field, if X belongs to the class and the class is closed under the formation of countable unions and of complements.

Definition: Algebra

If in Definition of σ-algebra we consider only finite unions we obtain an algebra (or) a field.

Note:
We will denote by M the class of Lebesgue measurable set

Theorem: The class M is a σ-algebra.

Proof:
w.k.T, The set E is Measurable if for each set A, we have

$m^*(A) = m^*(A \cap E) + m^*(A \cap CE)$ ------ (1)

$\therefore R \in M$ and
The symmetric between E and CE implies that if $E \in M$ then $CE \in M$

$\therefore$ we have to shown that if $\{E_j\}$ is a Sequence of Sets in M then

$$E = \bigcup_{j=1}^{\infty} E_j \in M$$

Let A be an arbitrary Set, by (1), we have

$m^*(A) = m^*(A \cap E_1) + m^*(A \cap CE_1)$ [From (1)]
(E replaced by E_1)

and ~~$m^*(A)$~~ E replaced by E_2 and A by $(A \cap CE_1)$.
we have

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$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_2 \cap CE_1)$$
$$+ m^*(A \cap CE_1 \cap CE_2)$$

Continuing in this way we obtain, for $n \geq 2$

$$m^*(A) = m^*(A \cap E_1) + \sum_{i=2}^{n} m^*(A \cap E_i \cap \bigcap_{j<i} CE_j)$$
$$+ m^*(A \cap \bigcap_{j=1}^{n} CE_j)$$
$$= m^*(A \cap E_1) + \sum_{i=2}^{n} m^*(A \cap E_i \cap C \bigcup_{j<i} E_j)$$
$$+ m^*(A \cap C \bigcup_{j=1}^{n} E_j)$$
$$\geq m^*(A \cap E_1) + \sum_{i=2}^{n} m^*(A \cap E_i \cap C \bigcup_{j<i} E_j)$$
$$+ m^*(A \cap C \bigcup_{j=1}^{\infty} E_j).$$

$$\therefore m^*(A) \geq m^*(A \cap E_1) + \sum_{i=2}^{\infty} m^*(A \cap E_i \cap C \bigcup_{j<i} E_j)$$
$$+ m^*(A \cap C \bigcup_{j=1}^{\infty} E_j)$$
$$\geq m^*(A \cap \bigcup_{j=1}^{\infty} E_j) + m^*(A \cap C \bigcup_{j=1}^{\infty} E_j)$$
$$\geq m^*(A),$$

$$\therefore m^*(A) \geq m^*(A \cap \bigcup_{j=1}^{\infty} E_j) + m^*(A \cap C \bigcup_{j=1}^{\infty} E_j)$$

$\therefore \bigcup_{j=1}^{\infty} E_j$ is measurable

$\therefore$ The class $\mathcal{M}$ is a σ-algebra.

Hence proved

Example:

Show that if $F \in \mathcal{M}$ and $m^*(F \Delta G) = 0$, then G is measurable.

Solution:

W.K.T if $m^*(E) = 0$ then E is measurable.

$\therefore$ if $m^*(F \Delta G) = 0$, then $F \Delta G$ is measurable.

$\therefore$ The subset $F - G$ and $G - F$ are measurable.

$\therefore$ $F \cap G = F - (F - G)$ is measurable

So. $G = (F \cap G) \cup (G - F)$ is measurable.

$\therefore$ G is Measurable.

Theorem: If $\{E_i\}$ is any Sequence of disjoint measurable sets then

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i).$$

(ie) m^* is Countably additive on disjoint sets of $\mathcal{M}$.

Proof: W.K.T

$$m^*(A) \geq m^*\left(A \cap \bigcup_{j=1}^{\infty} E_j\right) + m^*\left(A \cap C \bigcup_{j=1}^{\infty} E_j\right) \geq m^*(A). \quad \longrightarrow (1)$$

Let $\{E_i\}$ are disjoint

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Take $A = \bigcup_{i=1}^{\infty} E_i$ in (1), we have

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(\bigcup_{i=1}^{\infty} E_i \cap \bigcup_{j=1}^{\infty} E_j\right) + m^*\left(\bigcup_{i=1}^{\infty} E_i \cap C\left(\bigcup_{j=1}^{\infty} E_j\right)\right)$$

$$\geq m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \text{ ---------- (2)}$$

W.K.T. for any sequence of sets $\{E_i\}$,

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$$

$\therefore$ (2) $\Rightarrow$

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i).$$

[$\because$ $\{E_i\}$ are disjoint]

Hence proved.

Theorem: Every interval is measurable

Proof:

We may suppose the interval to be of the form $[a, \infty)$, For any set A, we wish to show that

$$m^*(A) \geq m^*(A \cap (-\infty, a)) + m^*(A \cap [a, \infty)) \text{ --- (1)}$$

Let $A_1 = A \cap (-\infty, a)$ and $A_2 = A \cap [a, \infty)$.

Then for any $\epsilon > 0$, $\exists$ an intervals I_n, such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and

$m^*(A) \geq \sum_{n=1}^{\infty} l(I_n) - \epsilon$.

Let $I_n' = I_n \cap (-\infty, a)$ and

$I_n'' = I_n \cap [a, \infty)$ so that

$l(I_n') + l(I_n'') = l(I_n)$ Then

$A_1 \subseteq \bigcup_{n=1}^{\infty} I_n'$, $A_2 \subseteq \bigcup_{n=1}^{\infty} I_n''$

so. $m^*(A_1) + m^*(A_2) \leq \sum_{n=1}^{\infty} l(I_n') + \sum_{n=1}^{\infty} l(I_n'')$

$\leq \sum_{n=1}^{\infty} l(I_n)$

$\leq m^*(A) + \epsilon$.

Here ϵ is arbitrary,

$\therefore m^*(A) \geq m^*(A_1) + m^*(A_2)$

(1) $\Rightarrow m^*(A) \geq m^*(A \cap (-\infty, a)) + m^*(A \cap [a, \infty))$

$\therefore$ Every interval is measurable.

Measure and Integration.

Unit-1 - chapter-2 - Measurable set (continued)

Definition: Borel sets

We denote by $\mathcal{B}$ the σ-algebra generated by the class of intervals of the form $[a, b)$: its members are called the Borel sets of R.

Definition:

For any sequence of sets $\{E_i\}$

$$\lim\sup E_i = \bigcap_{n=1}^{\infty} \bigcup_{i \geq n} E_i$$

$$\lim\inf E_i = \bigcup_{n=1}^{\infty} \bigcap_{i \geq n} E_i$$

Note:

1. $\lim\inf E_i \subseteq \lim\sup E_i$
2. If $E_1 \subseteq E_2 \subseteq \ldots$ then $\lim E_i = \bigcup_{i=1}^{\infty} E_i$

   and if $E_1 \supseteq E_2 \supseteq \ldots$ Then $\lim E_i = \bigcap_{i=1}^{\infty} E_i$.

Theorem:

Let A be a class of subsets of a space X. Then there exists a smallest σ-algebra S containing A. We say that S is the σ-algebra generated by A.

Proof:-

Let $[S_\alpha]$ be any collection of σ-algebras of subsets of X. Then ~~by the~~ From the definition of σ-algebra, we have $\bigcap_\alpha S_\alpha$ is a σ-algebra.

But there exists a σ-algebra containing A. namely the class of all subsets of X.

So taking the intersection of the σ-algebras containing A, we get a σ-algebra, necessarily the smallest, containing A.

$\therefore$ $\exists$ a smallest σ-algebra S containing A.

Theorem:

(i) $B \subseteq M$, This is every Borel set is measurable.

(ii) B is the σ-algebra generated by each of the following classes:
The open intervals, The open sets, The G_δ-sets, the F_σ-sets.

Proof :

(i). W.K.T

The class M is a σ-algebra and Every interval is measurable.

Here $\mathcal{B} \subseteq M$, ------ (1)

∴ M - σ-algebra

i.e. M is measurable.

(1) ⇒ $\mathcal{B}$ is measurable sets.

⇒ ~~Bor~~. Every Borel set is measurable.

(ii) Let $\mathcal{B}_1$ be the σ-algebra generated by the open intervals.

Every open interval,

Since it is the union of a sequence of intervals of the form [a, b), is a Borel set.

So $\mathcal{B}_1 \subseteq \mathcal{B}$.

But every interval [a, b) is the intersection of a sequence of open intervals and so $\mathcal{B} \subseteq \mathcal{B}_1$.

So $\mathcal{B} = \mathcal{B}_1$.

Since every open set is the union of a sequence of open intervals.

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$\therefore$ $\mathcal{B}$ is the σ-algebra generated by each of open sets.

Since G_δ-sets and F_σ-sets are formed from open sets using only Countable intersections and Complements.

$\therefore$ $\mathcal{B}$ is the σ-algebra generated by each of the G_δ-sets and the F_σ-sets.

Example :

For any set A there exists a measurable set E containing A and such that $m^*(A) = m(E)$.

Solution:

W.K.T. for any set A and $\epsilon > 0$, $\exists$ an open set O containing A and such that

$$m^*(O) \leq m^*(A) + \epsilon \text{ .... } (1)$$

Let $\epsilon = 1/n$ and write O_n for corresponding open set. Then

The G_δ-set $E = \bigcap_{n=1}^{\infty} O_n$

Since for every n, $m(E) \leq m(O_n) \leq m^*(A) + \frac{1}{n}$.

$\therefore m^*(A) = m(E)$

Example.:

(i) show that Every non-empty open set has positive measure.

(ii) The rationals Q are enumerated as $q_1, q_2, \dots$, and the set G is defined by $G = \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)$

Prove that, for any closed set F, $m(G \Delta F) > 0$.

Solution:

(i) W.K.T., Each non empty open set G in R is the union of disjoint open intervals, atmost countable in number and The outer measure of an interval Equals its length.

∴ Every non-empty open set has positive measure.

(ii) If $m(G-F) > 0$, there is nothing to prove,

If $m(G-F) = 0$, then since G-F is open we have $G \subseteq F$.

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But G Contains Q whose closure in R,

so $F = R$ and $m(F) = \infty$.

But $m(G) \leq 2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

So $m(F - G) = \infty$

$\therefore m(G \Delta F) > 0$.

Example: show that $\exists$ uncountable set of zero measure

Solution:

we have to show that the cantor set P, is measurable and $m(P) = 0$

From the construction the sets P_n are measurable for each n, so

$$P = \bigcap_{n=1}^{\infty} P_n \text{ is measurable.}$$

Also $P^* = [0,1] - P$

$$= \bigcup_{n=1}^{\infty} \bigcup_{r=1}^{2^{n-1}} I_{n,r}$$, a union of disjoint sets.

so $m(P^*) = 0 \Rightarrow m(P) = 0$

$\therefore \exists$ an uncountable sets of zero measure.

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Theorem:

Let $\{E_i\}$ be a sequence of measurable sets. Then

(i) if $E_1 \subseteq E_2 \subseteq \ldots\ldots$, we have

$$m(\lim E_i) = \lim m(E_i)$$

(ii) if $E_1 \supseteq E_2 \supseteq \ldots\ldots$ and $m(E_i) < \infty$ for each i, Then we have $m(\lim E_i) = \lim m(E_i)$.

Proof:

(i) Write $F_1 = E_1$, $F_i = E_i - E_{i-1}$ for $i > 1$,

Then $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$ and

the sets F_i are measurable and disjoint.

So $m(\lim E_n) = m\left(\bigcup_{i=1}^{\infty} E_i\right)$

$$= \sum_{i=1}^{\infty} m(F_i)$$

$$= \lim \sum_{i=1}^{\infty} m(F_i)$$

$$= \lim m\left(\bigcup_{i=1}^{\infty} F_i\right)$$

$$= \lim m(E_n)$$

$$\therefore m(\lim E_n) = \lim m(E_n)$$

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(ii) we have $E_1 - E_1 \subseteq E_1 - E_2 \subseteq E_1 - E_3 \subseteq \cdots$

So by (i)

$$m(\lim (E_1 - E_i)) = \lim (m(E_1 - E_i))$$

$$= m(E_1) - \lim m(E_i).$$

But $\lim (E_1 - E_i) = \bigcup_{i=1}^{\infty} (E_1 - E_i)$

$$= E_1 - \bigcap_{i=1}^{\infty} E_i$$

$$= E_1 - \lim E_i \quad \text{------} \ (1)$$

Taking measure on both sides, we get

$$m(\lim (E_1 - E_i)) = m(E_1 - \lim E_i)$$

$$\therefore m(\lim E_i) = \lim m(E_i) \quad [\because \text{From } (1)]$$

Measure and Integration

Unit-I - continued

Chapter-3 - Regularity.

Theorem:

The following statements regarding the set E are Equivalent.

(i) E is measurable.

(ii) $\forall \epsilon > 0$, $\exists O$, an open set, $O \supseteq E$ such that $m^*(O - E) \leq \epsilon$.

(iii) $\exists G$, a G_δ-set, $G \supseteq E$ such that $m^*(G - E) = 0$

(ii)* $\forall \epsilon > 0$, $\exists F$, a closed set, $F \subseteq E$ such that $m^*(E - F) \leq \epsilon$.

(iii)* $\exists F$, a F_σ-set, $F \subseteq E$ such that $m^*(E - F) = 0$

Proof:

(i) $\Rightarrow$ (ii)

Suppose that $m(E) < \infty$, there is an open set $O \supseteq E$ such that

$m(O) \leq m(E) + \epsilon$

so $m(O - E) = m(O) - m(E) \leq \epsilon$

If $m(E) = \infty$, write $R = \bigcup_{n=1}^{\infty} I_n$.

a, union of disjoint finite intervals.

Then if $E_n = E \cap I_n$, we have

$m(E_n) < \infty$

So there is an open set $O_n \supseteq E_n$ such that $m(O_n - E_n) \leq \epsilon/2^n$

Write $O = \bigcup_{n=1}^{\infty} O_n$, an open set,

Then $O - E = \bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n$

$$\subseteq \bigcup_{n=1}^{\infty} (O_n - E_n).$$

So $m^*(O-E) \leq \sum_{n=1}^{\infty} m^*(O_n - E_n) \leq \epsilon$

$\therefore m^*(O-E) \leq \epsilon$.

(ii) $\Rightarrow$ (iii)

For each n, let O_n be an open set $O_n \supseteq E$, $m^*(O_n - E) < \frac{1}{n}$. Then if $G = \bigcap_{n=1}^{\infty} O_n$. G is a G_δ-set, $E \subseteq G$ and $m^*(G-E) \leq m^*(O_n - E) < \frac{1}{n}$ for each n

$\therefore m^*(G-E) = 0$

(iii) $\Rightarrow$ (i)

$E = G - (G-E)$

The set G is measurable, $G-E$ is measurable.

So E is measurable.

(i) ⇒ (ii)*

CE is measurable, $\exists$ an open set O

∴ There exists an open set

such that $O \supseteq CE$

and $m(O - CE) \leq \epsilon$

But $O - CE = E - CO$.

So taking $F = CO$.

∴ $\forall \epsilon > 0, \exists F$, a closed set $F \subseteq E$

such that $m^*(E - F) \leq \epsilon$.

(ii)* ⇒ (iii)*

For each n, let F_n be a closed set

$F_n \subseteq E$ and $m^*(E - F_n) < \frac{1}{n}$. Then

if $F = \bigcup_{n=1}^{\infty} F_n$. F is an F_σ-set, $F \subseteq E$,

and for each n, $m^*(E - F) \leq m^*(E - F_n)$
$< \frac{1}{n}$.

∴ $m^*(E - F) = 0$.

(iii)* ⇒ (i)

Since $E = F \cup (E - F)$

F is measurable and $E - F$ is measurable

∴ E is measurable.

Theorem:

If $m^*(E) < \infty$ then E is measurable.

$\Leftrightarrow$ $\forall \epsilon > 0$, $\exists$ disjoint finite intervals $I_1, \ldots I_n$ such that $m^*(E \Delta \bigcup_{i=1}^{n} I_i) < \epsilon$.

we may stipulate that the intervals I_i be open, closed (or) half-open.

Proof:

Suppose that E is measurable.

Then by last theorem,

$\forall \epsilon > 0$, $\exists$ an open set O containing E with $m(O - E) < \epsilon$. As $m(E)$ is finite.

so is $m(O)$.

$\therefore \exists$ an open set O is the union of disjoint open intervals I_i, $i = 1, 2, \ldots$ and $\exists n$, such that, $\sum_{i=n+1}^{\infty} l(I_i) < \epsilon$, write $U = \bigcup_{i=1}^{n} I_i$

Then $E \Delta U = (E - U) \cup (U - E)$

$\subseteq (O - U) \cup (O - E)$

$\therefore m^*(E \Delta U) < 2\epsilon$.

we first obtain the open intervals $I_1, I_2 \ldots I_n$ and for each i, choose a half-open interval $J_i \subset I_i$ such that

$m(I_i - J_i) < \epsilon/n$.

Then ~~m(E~~

$$m\left(E \Delta \bigcup_{i=1}^{n} J_i\right) \leq m\left(E \Delta \bigcup_{i=1}^{n} I_i\right) + m\left(\bigcup_{i=1}^{n} I_i \Delta \bigcup_{i=1}^{n} J_i\right) < 2\epsilon.$$

$\therefore m^*\left(E \Delta \bigcup_{i=1}^{n} I_i\right) < \epsilon.$

We prove the converse,

W.K.T. $\forall \epsilon > 0$, $\exists$, O open, $O \supseteq E$,

Such that $m^*(O) \leq m^*(E) + \epsilon$. ---------- ①

If we can show that $m^*(O-E)$ can be made arbitrarily small, then E is measurable.

by last theorem, write $J = \bigcup_{i=1}^{n} I_i$ and

$U = O \cap J$. Then

$m^*(O \Delta E) \leq m^*(O \Delta U) + m^*(U \Delta E) \rightarrow$ ②

Since $U \subseteq J$, we have $U - E \subseteq J - E$

~~U-E ⊆ J-E~~

and since $E \subseteq O$, we have $E - U = E - J$.

So, $U \Delta E \subseteq J \Delta E$ and $m^*(U \Delta E) < \epsilon$.

But $E \subseteq U \cup (U \Delta E)$

So $m^*(E) < m(U) + \epsilon$.

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(2b)

① $\Rightarrow m(O \Delta U) = m(O-U)$

$$= m(O) - m(U)$$

$$\leq m^*(O) - m(U) + 2\epsilon$$

$$< 2\epsilon$$

② $\Rightarrow m^*(O \Delta E) < 3\epsilon$.

$\therefore$ E is measurable.

Measure and Integration

Unit - 1 - Continued

Chapter - 4 - Measurable Functions

Note:

Let a is real

$a + \infty = \infty$ (a real (or) $a = \infty$)

$a \cdot \infty = \infty$ ($a > 0$)

$a \cdot \infty = -\infty$ ($a < 0$), $\infty \cdot \infty = \infty$

$0 \cdot \infty = 0$

Definition: Lebesque - Measurable function (Measurable function)

Let f be an extended real - valued function defined on the measurable set E. Then f is a Lebesque - measurable function or a measurable function. if, for each $\alpha \in R$, the set $[x : f(x) > \alpha]$ is measurable.

Theorem: The following statements are Equivalent.

(i) f is a measurable function,

(ii) $\forall \alpha$, $[x : f(x) \geq \alpha]$ is measurable.

(iii) $\forall \alpha$. $[x : f(x) < \alpha]$ is measurable.

(iv) $\forall \alpha$, $[x : f(x) \leq \alpha]$ is measurable.

Proof:

(i) ⇒ (ii)

Let f be measurable. Then

$$[x : f(x) \geq \alpha] = \bigcap_{n=1}^{\infty} [x : f(x) > \alpha - \frac{1}{n}]$$ is measurable

$\therefore [x : f(x) \geq \alpha]$ is measurable.

So (i) ⇒ (ii)

(ii) ⇒ (iii)

Let $[x : f(x) \geq \alpha]$ be measurable. Then

$[x : f(x) < \alpha] = C[x : f(x) \geq \alpha]$ is measurable

$\therefore [x : f(x) < \alpha]$ is measurable.

So (ii) ⇒ (iii)

(iii) ⇒ (iv)

Let $[x : f(x) < \alpha]$ be measurable. Then

$$[x : f(x) \leq \alpha] = \bigcap_{n=1}^{\infty} [x : f(x) < \alpha + \frac{1}{n}]$$ is measurable

$\therefore [x : f(x) \leq \alpha]$ is measurable.

So (iii) ⇒ (iv)

(iv) ⇒ (i)

Let $[x : f(x) \leq \alpha]$ be measurable. Then

$\therefore [x : f(x) > \alpha] = C[x : f(x) \leq \alpha]$ is measurable.

$\therefore [x : f(x) > \alpha]$ is measurable.

So (iv) ⇒ (i)

Example:

Show that if f is measurable, then $[x : f(x) = \alpha]$ is measurable for each extended real number α.

Solution:

For finite α,

$[x : f(x) = \alpha] = [x : f(x) \geq \alpha] \cap [x : f(x) \leq \alpha]$ is measurable.

For $\alpha = \infty$.

$[x : f(x) = \infty] = \bigcap_{n=1}^{\infty} [x : f(x) > n]$, is measurable.

H^{ly} $\alpha = -\infty$ $[x : f(x) = -\infty]$ is measurable.

$\therefore$ if f is measurable then $[x : f(x) = \alpha]$ is measurable for each extended real number α.

Note:

1. The constant functions are measurable.
   [$\because$ $[x : f(x) > \alpha]$, where f is constant]
2. The characteristic function χ_A of the set A is measurable iff A is measurable
   [$[x : \chi_A(x) > \alpha] = A, R$ (or) ϕ, where χ_A is the characteristic function.]

(3). Continuous functions are measurable.

[$\because$ $[x : f(x) > \alpha]$ is open, where f is continuous]

Theorem:

Let c be any real number and let f and g be real-valued measurable functions defined on the same measurable set E. Then $f+c$, cf, $f+g$, $f-g$ and fg are also measurable.

Proof: Let f and g be real valued measurable functions.

$\therefore$ $[x : f(x) > \alpha]$ and $[x : g(x) > \alpha]$ are measurable.

For each α,

$[x : f(x) + c > \alpha] = [x : f(x) > \alpha - c]$ is measurable.

$\therefore$ $f+c$ is measurable set.

If $c > 0$, $[x : cf(x) > \alpha] = [x : f(x) > c^{-1}\alpha]$ is measurable.

$\therefore$ cf is measurable.

$||^{ly}$ If $c < 0$, cf is measurable.

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To prove $f+g$ is measurable.

Let $x \in A = [x : f(x) + g(x) > \alpha]$ only if $f(x) > \alpha - g(x)$.

(i.e., if $\exists$ a rational r_i such that

$f(x) > r_i > \alpha - g(x)$,

i.e., $f(x) > \alpha - r_i$, But $g(x) > \alpha - r_i$

and so,

$x \in [x : f(x) > r_i] \cap [x : g(x) > \alpha - r_i]$

i.e., $\bigcup_{r=1}^{\infty} ([x : f(x) > r_i] \cap [x : g(x) > \alpha - r_i])$

is measurable

i.e., $f+g$ is measurable.

To prove $f-g$ is measurable

$f-g = f + (-g)$ is measurable

$\therefore f-g$ is measurable.

Finally: $fg = \frac{1}{4}((f+g)^2 + (f-g)^2)$

so it is sufficient to show that f^2 is measurable, whenever f is measurable.

If $\alpha < 0$, $[x : f^2(x) > \alpha] = R$
is measurable.

If $\alpha \geq 0$, $[x : f^2(x) > \alpha]$
$= [x : f(x) > \sqrt{\alpha}] \cap [x : f(x) < -\sqrt{\alpha}]$
is measurable.

$\therefore fg$ is measurable.

Theorem:
Let $\{f_n\}$ be a sequence of measurable functions defined on the same measurable set then.

(i) $\sup_{1 \leq i \leq n} f_i$ is measurable for each n,

(ii) $\inf_{1 \leq i \leq n} f_i$ is measurable for each n.

(iii) $\sup f_n$ is measurable.

(iv) $\inf f_n$ is measurable.

(v) $\lim \sup f_n$ is measurable.

(vi) $\lim \inf f_n$ is measurable.

Proof: Let $\{f_n\}$ be a sequence of measurable functions, Then.

(i) Since $[x : \sup_{1 \le i \le n} f_i(x) > \alpha]$

$$= \bigcup_{i=1}^{\infty} [x : f_i(x) > \alpha]$$

is measurable.

we have $\sup_{1 \le i \le n} f_i$ is measurable.

(ii) $\inf_{1 \le i \le n} f_i = -\sup_{1 \le i \le n} (-f_i)$ is measurable.

$\therefore \inf_{1 \le i \le n} f_i$ is measurable.

(iii) $[x : \sup f_n(x) > \alpha] = \bigcup_{n=1}^{\infty} [x : f_n(x) > \alpha]$

is measurable.

$\therefore \sup f_n$ is measurable.

(iv) $\inf f_n = -\sup(-f_n)$ is measurable.

$\therefore \inf f_n$ is measurable.

(v) $\limsup f_n = \inf \left(\sup_{i \ge n} f_i \right)$ is measurable.

$\therefore \limsup f_n$ is measurable.

(vi) $\liminf f_n = -\limsup(-f_n)$ is measurable.

$\therefore \liminf f_n$ is measurable.

Measure and Integration

chapter-4 : measurable functions (continued)

Definition: Borel measurable (Borel function)

The function f is Borel measurable or a Borel function if $\forall \alpha$, $[x : f(x) > \alpha]$ is a Borel set.

Definition: Essential Supremum [ess sup f]

Let f be a measurable function, Then $\inf [\alpha : f \leq \alpha \text{ a.e}]$ is called the essential supremum of f, denoted by ess sup f.

Definition: Essential Infimum : [ess inf f].

Let f be a measurable function, Then $\sup [\alpha : f \geq \alpha, \text{a.e}]$ is called The essential infimum of f. denoted by ess inf f.

Theorem:

Let f be a measurable function and let $f = g$ a.e. Then g is measurable.

Proof:

$$[x : f(x) > \alpha] \,\Delta\, [x : g(x) > \alpha]$$
$$\subseteq [x : f(x) \neq g(x)]$$

W.K.T. if $m^*(E) = 0$, Then E is measurable.

and if $F \in M$, and $M^*(F \Delta G) = 0$ Then g is measurable.

we have g is measurable.

Results:

1. Let $\{f_i\}$ be a sequence of measurable functions converging a.e.

2. If f is a measurable functions, ~~Then~~. then so are $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$.

3. The set of points on which a sequence of measurable functions $\{f_n\}$ Converges, is measurable.

Example:

Show that $f \leq \text{ess sup} f$, a.e.

Solution:

If $\text{ess sup} f = \infty$,

$\therefore$ The result is true.

i.e. $f \leq \text{ess sup} f$, a.e.

Suppose that $\text{ess sup} f = -\infty$,

Then $\forall n \in \mathbb{Z}$, $f \leq n$ a.e.

W.K.T. $\inf [\alpha : f \leq \alpha \text{ a.e}]$ is called the essential supremum, if f is measurable.

So $f = -\infty$ a.e.

Suppose that ess sup f is finite.

Write $E_n = [x : f(x) > \frac{1}{n} + \text{ess sup} f]$

and $E = [x : f(x) > \text{ess sup} f]$,

So $E = \bigcup_{n=1}^{\infty} E_n$.

But $m(E_n) = 0 \Rightarrow m(E) = 0$.

$\therefore$ E is measurable.

$\therefore f \leq \text{ess sup} f$ .a.e.

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Example:

Show that for any measurable functions f and g ess sup$(f+g) \leq$ ess sup f + ess sup g. and give and example of strict inequality.

Solution:

W.K.T. $f \leq$ ess sup f, a.e.

$\therefore f+g \leq$ ess sup f + ess sup g - a.e.

$\therefore$ ess sup$(f+g) \leq$ ess sup f + ess sup g —(1)

For inequality take,

$$f = \chi_{[-1,0)} - \chi_{[0,1]} \quad \text{and} \quad g = -f$$

Then L.H.S of (1) is 0

$\therefore$ R.H.S. of is 0.

Definition: Essentially bounded.

If f is a measurable function and ess sup $|f| < \infty$ then f is said to be essentially bounded.

Example.

$$\text{Ess sup} f = -\text{ess inf}(-f)$$

Solution.

$$\begin{aligned}\text{Ess sup} f &= \inf[\alpha : f \leq \alpha \text{ a.e}] \\ &= \inf[\alpha : -f \geq -\alpha \text{ a.e}] \\ &= -\sup[-\alpha : -f \geq -\alpha \text{ a.e}] \\ &= -\text{ess inf}(-f).\end{aligned}$$

Example: Let f be a measurable function and B a Borel set, then $f^{-1}(B)$ is a measurable set.

Solution:

we have $f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i)$

and $f^{-1}(CA) = C f^{-1}(A)$.

So the class of sets whose inverse images under f are measurable forms a σ-algebra. But this class contains the intervals.

So it must contain all Borel sets.

$\therefore f^{-1}(B)$ is a measurable set.

Assignment work.

1. The outer measure of an interval equals its length.

2. (a) For any sequence of sets $\{E_i\}$,
$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i).$$
(b) Show that for any set A and any $\epsilon > 0$, $\exists$ an open set O containing A and such that
$$m^*(O) \leq m^*(A) + \epsilon.$$

3. The class M is a σ-algebra.

4. (a) Every interval is measurable.
(b) (i) $B \subseteq M$, this is every Borel set is measurable.
(ii) B is the σ-algebra generated by each of the following classes: The open intervals, the open sets, the G_δ-sets, the F_σ-sets

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5. Let $\{E_i\}$ be a sequence of measurable set. Then.

(i) if $E_1 \subseteq E_2 \subseteq \dots$ we have $m(\lim E_i) = \lim m(E_i)$

(ii) if $E_1 \supseteq E_2 \supseteq \dots$ and $m(E_i) < \infty$ for each i, then we have $m(\lim E_i) = \lim m(E_i)$.

6. If $m^*(E) < \infty$ Then E is measurable.
$\Leftrightarrow$ $\forall\ \epsilon > 0$, $\exists$ disjoint finite intervals $I_1 \dots I_n$ such that $m^*(E \Delta \bigcup_{i=1}^{n} I_i) < \epsilon$.
we may stipulate that the intervals I_i, be open, closed, or half open.

7. Page No (21) → Regularity → Theorem.

8. Page No (27) → Measurable function → Theorem.

9. Page No (39) → Theorem.

10. Page No (32) → Theorem.

MEASURE AND INTEGRATION

UNIT-II – Integration of Functions of a Real Variable.

Chapter-1: Integration of non-negative Functions.

Definition: Simple function.

A non-negative finite valued function $\phi(x)$, taking only a finite number of different values, is called a simple function. If $a_1, a_2, \dots a_n$ are the distinct values taken by ϕ and $A_i = [x : \phi(x) = a_i]$, then clearly $\phi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$.

∴ If ϕ is a measurable function. Then the set A_i are measurable.

Definition: Integral of ϕ.

Let ϕ be a measurable simple function.

Then $\int \phi \, dx = \sum_{i=1}^{n} a_i \, m(A_i)$

Where $a_i, A_i, i = 1, 2, \dots n$ is called the integral of ϕ.

Example:

Let the sets A_i be defined as above,

Then $A_i \cap A_j = \phi$, $i \neq j$ and

$$\bigcup_{i=1}^{n} A_i = R.$$

Definition:

For any non-negative measurable function f, the integral of f, $\int f dx$, is given by $\int f dx = \sup \int \varphi dx$, where the supremum is taken over all measurable simple functions φ, $\phi \leq f$.

Definition:

For any measurable set E, and any non-negative measurable function f,

$\int_E f dx = \int f \chi_E dx$ is the integral of f over E.

Here, If the set E is an interval, such as $[a, b]$, then in te place of $\int_E f dx$, we write $\int_a^b f dx$, if $a > b$.

$$\therefore \int_a^b f dx = -\int_b^a f dx.$$

The integral defined above will be referred as the Lebesgue Integral.

Theorem:

If ϕ is a measurable simple function, then in the notation of $\phi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$. prove that

(i) $\int_E \phi\, dx = \sum_{i=1}^{n} a_i\, m(A_i \cap E)$, for any measurable set,

(ii) $\int_{A \cup B} \phi\, dx = \int_A \phi\, dx + \int_B \phi\, dx$, for any disjoint measurable sets A and B.

(iii) $\int a\phi\, dx = a \int \phi\, dx$, if $a > 0$.

Proof: (i)
W.K.T $\int \phi\, dx = \sum_{i=1}^{n} a_i m(A_i)$ and

$$\int_E f\, dx = \int f \chi_E\, dx$$

$\therefore$ we have $\int_E \phi\, dx = \sum_{i=1}^{n} a_i m(A_i \cap E)$,

for any measurable set E.

(ii) $\int_A \varphi\, dx + \int_B \varphi\, dx$

$$= \sum_{i=1}^{n} a_i m(A \cap A_i) + \sum_{i=1}^{n} a_i m(B \cap A_i)$$

$$= \sum_{i=1}^{n} a_i m((A \cup B) \cap A_i)$$

$$= \int_{A \cup B} \varphi\, dx.$$

(iii) As φ takes the values a_i

$a\varphi$ takes the distinct values $a a_i$,

So that $\int a\varphi\, dx = \sum_{i=1}^{n} a a_i m(A_i)$

$$= a \int \varphi\, dx.$$

$\therefore$ we have $\int a\varphi\, dx = a \int \varphi\, dx$.

Example:

Show that if f is non-negative measurable function, Then $f = 0$ a.e. $\Longleftrightarrow \int f\, dx = 0$.

Solution:

If $f = 0$ a.e. and φ is measurable simple function, $\varphi \leq f$, Then clearly $\int \varphi dx = 0$.

W.K.T. $\int f dx = \sup \int \varphi dx$

we have, $\int f dx = 0$.

Conversely, if $\int f dx = 0$ and

$E_n = [x : f(x) \geq \frac{1}{n}]$

Then $\int f dx \geq \int n^{-1} \chi_E dx = m(E_n)$

So $m(E_n) = 0$

But $[x : f(x) > 0] = \bigcup_{n=1}^{\infty} E_n$

So $f = 0$ a.e.

Theorem:

Let f and g be non-negative measurable functions, then

(i) If $f \leq g$, Then $\int f dx \leq \int g dx$

(ii) If A is a measurable set and $f \leq g$ on A, then $\int_A f dx \leq \int_A g dx$.

(iii) If $a \geq 0$, then $\int af dx = a \int f dx$.

(iv) If A and B are measurable sets and $A \supseteq B$ then $\int_A f dx \geq \int_B f dx$.

Proof:

(i) W.K.T $\int f dx = \sup \int \phi dx$ and

$\int_E f dx = \int f \chi_E dx$,

where supremum is taken over all measurable simple functions ϕ, $\phi \leq f$.

$\therefore$ we have If $f \leq g$ then $\int f dx \leq \int g dx$.

(ii) From (i), if A is measurable set and $f \leq g$ on A, then $\int_A f dx \leq \int_A g dx$.

(iii). Let $a \geq 0$, To prove $\int af dx = a \int f dx$

If $a = 0$, $\Rightarrow \int af dx = a \int f dx$ is true.

If $a > 0$, ϕ is a measurable Simple function with $\phi \leq af \iff \phi = a\psi$,

where ψ is simple, $\psi \leq f$, and

Then $\int \phi dx = a \int \psi dx$.

so $\int af dx = \sup \int \phi dx = a \sup \int \psi dx$

$= a \int f dx$.

$\therefore \int af dx = a \int f dx$.

(iv) We note that $\chi_A f \geq \chi_B f$ and from (i), we have If A and B are measurable sets and $A \supseteq B$ then $\int_A f dx \geq \int_B f dx$

Measure and Integration

unit - II - chapter - I (Continued)

Theorem: (Fatou's Lemma)

Let $\{f_n, n=1,2,\dots\}$ be a sequence of non-negative measurable functions. Then

$\liminf \int f_n dx \geq \int \liminf f_n dx$.

Proof:

Let $f = \liminf f_n$.

Then f is a non-negative measurable function.

W.K.T. $\int f dx = \sup \int \phi dx$, for each measurable simple function ϕ with $\phi \leq f$, we have

$\int \phi dx \leq \liminf \int f_n dx$. -------- (1)

case (i): Let $\int \phi dx = \infty$.

W.K.T $\int \phi dx = \sum_{i=1}^{n} a_i m(A_i)$, for some measurable set A, we have

$m(A) = \infty$ and $\phi > a > 0$ on A.

Write $g_k(x) = \inf_{j \geq k} f_j(x)$ and

$A_n = [x : g_k(x) > a, \text{ all } k \geq n]$,

a measurable set. Then

$A_n \subseteq A_{n+1}$, each n.

But, for each x, $\{g_k(x)\}$ is monotone increasing and

$\lim_{k\to\infty} g_k(x) = f(x) \geq \phi(x)$.

So $A \subseteq \bigcup_{n=1}^{\infty} A_n$.

Hence $\lim m(A_n) = \infty$,

But, for each n, $\int f_n dx \geq \int g_n dx > a\, m(A_n)$.

So $\liminf \int f_n dx = \infty$ and (1) holds.

$\therefore$ (ie) $\int \phi dx \leq \liminf \int f_n dx$.

Case (ii)

Let $\int \phi dx < \infty$.

Write $B = [x : \phi(x) > 0]$. Then $m(B) < \infty$.

Let M be the largest value of ϕ,

and if $0 < \epsilon < 1$.

Write $B_n = [x : g_k(x) > (1-\epsilon)\phi(x), k \geq n]$.

where $g_k(x) = \inf_{j \geq k} f_j(k)$,

Then the sets B_n are measurable,

$B_n \subseteq B_{n+1}$ for each n,

and $\bigcup_{n=1}^{\infty} B_n \supseteq B$

So $\{B - B_n\}$ is decreasing sequence of sets,

$\bigcap_{n=1}^{\infty} (B - B_n) = \phi$.

As $m(B) < \infty$, $\exists N$ such that

$m(B - B_n) < \epsilon \;\; \forall n \geq N.$

So if $n \geq N$,

$$\int g_n dx \geq \int_{B_n} g_n dx$$

$$\geq (1-\epsilon)\int_{B_n} \phi dx$$

$$= (1-\epsilon)\left(\int_B \phi dx - \int_{B-B_n} \phi dx\right)$$

$$\geq (1-\epsilon)\int \phi dx - \int_{B-B_n} \phi dx$$

$$\geq \int \phi dx - \epsilon \int \phi dx - \epsilon M.$$

Since ϵ is arbitrary, we have

$\liminf \int g_n dx \geq \int \phi dx.$

and since $f_n \geq g_n$.

$\therefore \liminf \int g_n dx \geq \int \liminf f_n dx.$

Hence proved.

Theorem: (Lebesgue's Monotone Convergence Theorem)

Let $\{f_n, n = 1, 2, \ldots\}$ be a sequence of non-negative measurable functions such that $\{f_n(x)\}$ is monotone increasing for each x. Let $f = \lim f_n$

Then $\int f dx = \lim \int f_n dx$.

Proof:

Fatou's lemma gives

$\int f dx = \int \liminf f_n dx \leq \liminf \int f_n dx$. ------ (1)

But $f \geq f_n$

W.K.T If $f \leq g$, then $\int f dx \leq \int g dx$.,

we have $\int f dx \geq \int f_n dx$. and hence.

$\int f dx \geq \limsup \int f_n dx$. ------- (2)

From (1) & (2), we have

$\int f dx = \lim \int f_n dx$.

Hence proved

Theorem:

Let f be a non-negative measurable function. Then $\exists$ a sequence $\{\phi_n\}$ of measurable simple functions such that, for each x, $\phi_n(x) \uparrow f(x)$.

Proof:

By construction,

Write, for each n,

$$E_{nk} = [x : (k-1)/2^n < f(x) \leq k/2^n],$$
$$k = 1, 2, \ldots n2^n$$

and $F_n = [x : f(x) > n]$.

Put $$\phi_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{E_{nk}} + n\chi_{F_n}.$$

Then the functions ϕ_n are measurable simple functions.

Also since the dissection of the range of f given ϕ_{n+1} is a refinement of that giving ϕ_n, it is easily seen that

$$\phi_{n+1}(x) \geq \phi_n(x) \text{ for each } x.$$

If $f(x)$ is finite,

$x \in CF_n$ $\forall$ large n, and then

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$|f(x) - \varphi_n(x)| \leq 2^{-n}$

So $\varphi_n(x) \uparrow f(x)$.

If $f(x) = \infty$ then $x \in \bigcap_{n=1}^{\infty} F_n$

So $\varphi_n(x) = n \ \forall n$. and hence

$\varphi_n(x) \uparrow f(x)$.

Hence proved.

Corollary:

$$\lim \int \varphi_n \, d\mu = \int f \, d\mu.$$

where φ_n and f are as in above theorem.

Measure and Integration

unit - II - chapter - I (Continued)

Theorem:- Let f and g be non-negative measurable functions. Then

$$\int f\,dx + \int g\,dx = \int (f+g)\,dx.$$

Proof:

consider $\int f\,dx + \int g\,dx = \int (f+g)\,dx$ ---- (1)

for measurable simple functions ϕ and ψ.

Let the values of ϕ be $a_1, a_2, \ldots a_n$ taken on sets $A_1, A_2, \ldots A_n$.

and let the values of ψ be $b_1, b_2, \ldots b_n$ taken on the sets $B_1, B_2, \ldots B_n$.

Then the simple function $\phi+\psi$ has the value $a_i + b_j$ on the measurable set $A_i \cap B_j$,

W.K.T. $\int_E \phi\,dx = \sum_{i=1}^{n} a_i\, m(A_i \cap E)$, for any measurable set E.

we have

$$\int_{A_i \cap B_j} (\phi+\psi)\,dx = \int_{A_i \cap B_j} \phi\,dx + \int_{A_i \cap B_j} \psi\,dx$$

⟶ (2)

But the union of the nm disjoint sets $A_i \cap B_j$ is R, so (2) gives

$$\int(\phi+\psi)\,dx = \int\phi\,dx + \int\psi\,dx \text{ ------- } (3)$$

Let f and g be any non-negative measurable functions.

Let $\{\varphi_n\}$, $\{\psi_n\}$ be sequence of measurable functions $\varphi_n \uparrow f$, $\psi_n \uparrow g$.

Then $\varphi_n + \psi_n \uparrow f+g$.

$\therefore$ (3) $\Rightarrow$ $\int(\varphi_n + \psi_n)\,dx = \int\varphi_n\,dx + \int\psi_n\,dx$.

So as $n \to \infty$, we have

$$\int(f+g)\,dx = \int f\,dx + \int g\,dx.$$

Hence proved.

Theorem:

Let $\{f_n\}$ be a sequence of non-negative measurable functions. Then

$$\int \sum_{n=1}^{\infty} f_n\,dx = \sum_{n=1}^{\infty} \int f_n\,dx.$$

Proof:

W.K.T $\int(f+g)\,dx = \int f\,dx + \int g\,dx$ ------ (1)

(1) applies to a sum of n functions.

So if $S_n = \sum_{i=1}^{n} f_i$, then

$$\int S_n \, dx = \sum_{i=1}^{n} \int f_i \, dx$$

But $S_n \uparrow f = \sum_{i=1}^{\infty} f_i$

$$\therefore \int \sum_{i=1}^{\infty} f_i \, dx = \sum_{i=1}^{\infty} \int f_i \, dx.$$

i.e., $\int \sum_{n=1}^{\infty} f_n \, dx = \sum_{n=1}^{\infty} \int f_n \, dx.$

Hence proved.

Example: Show that $\int_1^\infty \frac{dx}{x} = \infty$.

Solution: x^{-1} is a continuous function for $x > 0$ and so is measurable.

It is positive, so the integral is defined.

Also $\int_1^\infty x^{-1} dx > \int_1^n x^{-1} dx$.

But $x^{-1} > k^{-1}$ on $[k-1, k)$

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So $\int_1^n x^{-1}\,dx > \sum_{k=2}^{n} \int_1^n k^{-1} \chi_{[k-1,k)}\,dx$

$> \sum_{k=2}^{n} k^{-1} \to \infty$ as $n \to \infty$.

$\therefore \int_1^\infty x^{-1}\,dx > \int_1^n x^{-1}\,dx \to \infty$ as $n \to \infty$.

$\therefore \int_1^\infty x^{-1}\,dx \to \infty$

i.e. $\int_1^\infty x^{-1}\,dx = \infty$.

Example: $f(x)$, $0 \le x \le 1$ is defined by $f(x) = 0$ for x rational, if x is irrational, $f(x) = n$, where n is the number of zeros immediately after the decimal point, in the representation of x on the decimal scale. Show that f is measurable and find $\int_0^1 f\,dx$.

Solution.

For $x \in (0,1]$

Let $g(x) = n$ if $10^{-(n+1)} \le x < 10^{-n}$, $n = 0, 1, \ldots$

and $g(1) = 0$.

Then $f \leq g$, i.e. $f = g$ a.e.,

So f is measurable

$$\therefore \int_0^1 f\,dx = \int_0^1 g\,dx.$$

But $$\int_0^1 g\,dx = \sum_{n=0}^{\infty} n\left(\frac{1}{10^n} - \frac{1}{10^{n+1}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{9n}{10^{n+1}} = \frac{1}{9}$$

$$\therefore \int_0^1 f\,dx = \frac{1}{9}.$$

Chapter - II - The General Integral.

Definition: If $f(x)$ is any real function,

$f^+(x) = \max(f(x), 0)$, $f^-(x) = \max(-f(x), 0)$,

are said to be the positive and negative parts of f, respectively.

Definition: If f is a measurable function and $\int f^+ dx < \infty$, $\int f^- dx < \infty$, we say that f is integrable, and its integral is given by

$$\int f\,dx = \int f^+ dx - \int f^- dx.$$

clearly, a measurable function f is integrable $\Leftrightarrow$ $|f|$ is integrable and then

$$\int |f| \, dx = \int f^+ dx + \int f^- dx.$$

Definition: If E is a measurable set, f is a measurable function, and $\chi_E f$ is integrable we say that f is integrable over E, and its integral is given by

$$\int_E f \, dx = \int f \chi_E \, dx.$$

Definition: If f is a measurable function such that at least one of $\int f^+ dx$, $\int f^- dx$ is finite then $\int f \, dx = \int f^+ dx - \int f^- dx$.

Theorem: (i) $f = f^+ - f^-$; $|f| = f^+ + f^-$, $f^+, f^- \geq 0$

(ii) f is measurable $\Leftrightarrow$ f^+ and f^- are measurable.

Proof: (i) W.K.T. $f^+(x) = \max(f(x), 0)$, $f^-(x) = \max(-f, 0)$

$$\therefore f = f^+ - f^- \; ; \; |f| = f^+ + f^- \; ; \; f^+, f^- \geq 0.$$

(ii) W.K.T If f is a measurable function, then $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$ are measurable.

$\therefore f$ is measurable $\Leftrightarrow$ f^+ and f^- are both measurable.

Measure and Integration

chapter - II - The General Integral (Continued).

Theorem:

Let f and g be integrable functions

(i) af is integrable and $\int af\,dx = a\int f\,dx$.

(ii) $f+g$ is integrable and $\int (f+g)\,dx = \int f\,dx + \int g\,dx$.

(iii) If $f=0$ a.e. then $\int f\,dx = 0$.

(iv) If $f \le g$ a.e then $\int f\,dx \le \int g\,dx$.

(v) If A and B are disjoint measurable sets

then $\int_A f\,dx + \int_B f\,dx = \int_{A\cup B} f\,dx$.

Proof:

(i) suppose that $a \geq 0$

Then $(af)^+ = af^+$, $(af)^- = af^-$

so $\int (af)^+ dx < \infty$ and $\int (af)^- dx < \infty$

so af is integrable

and $\int af\,dx = \int af^+ dx - \int af^- dx = a\int f\,dx$.

Suppose that $a = -1$

Then $(-f)^+ = f^-$, $(-f^-) = f^+$

so $-f$ is integrable

and $\int (-f)\,dx = \int f^- dx - \int f^+ dx = -\int f\,dx$.

But for $a < 0$, $af = -|a|$

So $\int af\,dx = -\int |a| f\,dx$

$= -|a| \int f\,dx$

$= a\int f\,dx.$

i.e. $\int (af)\,dx = a\int f\,dx.$

(ii) $(f+g)^+ \leq f^+ + g^+$, $(f+g)^- \leq f^- + g^-$.

So $f+g$ is integrable.

Also $(f+g)^+ - (f+g)^- = f+g = f^+ + g^+ - f^- - g^-$

So $(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+ \longrightarrow (1)$

W.K.T. $\int (f+g)\,dx = \int f\,dx + \int g\,dx$

$(1) \Rightarrow \int \left[(f+g)^+ + f^- + g^-\right] dx = \int \left[(f+g)^- + f^+ + g^+\right] dx.$

i.e. $\int \left((f+g)^+ - (f+g)^-\right) dx = \int \left(f^+ + g^+ - f^- - g^-\right) dx.$

i.e. $\int (f+g)\,dx = \int (f^+ - f^-)\,dx + \int (g^+ - g^-)\,dx$

i.e. $\int (f+g)\,dx = \int f\,dx + \int g\,dx.$

(iii) Let $f = 0$ a.e.

$\therefore$ $f^+ = 0$ a.e and $f^- = 0$ a.e.

W.K.T. if f is a non-negative measurable function, then $f = 0$ a.e $\Longleftrightarrow \int f\,dx = 0$.

$\therefore$ $\int f^+ dx = 0$ and $\int f^- dx = 0$.

$\therefore$ $\int f\,dx = \int (f^+ - f^-)\,dx = 0$

$\Rightarrow \int f\,dx = 0$.

(iv) Let $f \leq g$ a.e

and $g = f + (g - f)$

So $\int g\,dx = \int f\,dx + \int (g-f)^+ dx - \int (g-f)^- dx$.

But $(g-f)^- = 0$ a.e. and from (iii), we have,

$$\int f\,dx \leq \int g\,dx.$$

(v) W.K.T. $\chi_{A \cup B} = \chi_A + \chi_B$ and

$$\int_E f\,dx = \int f \chi_E\,dx,$$ ~~we have~~

(ii) $\Rightarrow$ $(f+g)$ is integrable, and $\int (f+g)\,dx = \int f\,dx + \int g\,dx$.

we have, $\int_{A \cup B} f\,dx = \int_A f\,dx + \int_B f\,dx$.

Example:

Show that if f and g are measurable, $|f| \leq |g|$ a.e and g is integrable, then f is integrable.

Solution:

Let f and g are measurable.

Suppose that $|f| \leq |g|$ ----- (1)

W.K.T. $\int |f| dx = \int f^+ dx + \int f^- dx$

and $\int f dx = \int f^+ dx - \int f^- dx$.

Then (1) $\Rightarrow$ $f^+ \leq |g|$ and $f^- \leq |g|$

$\int f^+ dx \leq \int |g| dx$ and $\int f^- dx \leq \int |g| dx$

$\therefore$ f^+ and f^- are integrable

$\therefore$ f is integrable.

Example:

Show that if f is an integrable function, then $|\int f dx| \leq \int |f| dx$.

Solution:

W.K.T. $|f| - f \geq 0$ and $|f| + f \geq 0$

So $\int |f| dx \geq \int f dx$ and $\int |f| dx \geq -\int f dx$.

Hence $\int |f|\,dx \geq |\int f\,dx|$

Necessary Condition for equality:

If $\int f\,dx \geq 0$, then $\int |f|\,dx = \int f\,dx$,

i.e, $\int (|f| - f)\,dx = 0$

W.K.T. If f is a non-negative measurable function, then $f = 0$ a.e

$\Leftrightarrow \int f\,dx = 0$

we have $|f| = f$ a.e.

If $\int f\,dx < 0$ then $\int |f|\,dx = \int (-f)\,dx$,

i.e) $\int (|f| + f)\,dx = 0$

So $|f| = -f$ a.e.

Hence $f \geq 0$ a.e (or) $f \leq 0$ a.e.

is a necessary condition. and clearly this is also a sufficient condition.

$\therefore |\int f\,dx| \leq \int |f|\,dx$.

Example:

If f is measurable and g is measurable and α, β are real numbers such that $\alpha \le f \le \beta$ a.e., then $\exists\ \gamma$, $\alpha \le \gamma \le \beta$ such that $\int f|g|\,dx = \gamma \int |g|\,dx$.

Solution:

Write $|fg| \le (|\alpha| + |\beta|)|g|$ a.e.,

~~So~~ W.K.T. If f and g are measurable, $|f| \le |g|$ a.e and g is integrable, then f is integrable.

we have fg is ~~ing~~ integrable.

Also $\alpha|g| \le f|g| \le \beta|g|$ a.e.

so $\alpha \int |g|\,dx \le \int f|g|\,dx \le \beta \int |g|\,dx$.

If $\int |g|\,dx = 0$, then $g = 0$ a.e then the result is true.

If $\int |g|\,dx \ne 0$, take $\gamma = \left(\int f|g|\,dx\right) \cdot \left(\int |g|\,dx\right)^{-1}$

$\therefore \int f|g|\,dx = \gamma \int |g|\,dx$ is true.

Example:

Show that if f is integrable then f is finite - valued a.e.

Solution:

If $|f| = \infty$ on a set E with $m(E) > 0$,

then $\int |f| \, dx > n \, m(E) \;\; \forall n$,

giving a contradication.

$\therefore$ if f is integrable $\Rightarrow$ f is finite - valued.

Example: If f is measurable, $m(E) < \infty$ and $A \leq f \leq B$ on E, then

$$A \, m(E) \leq \int_E f \, dx \leq B \, m(E$$

Solution:

Let $A\chi_E$, $B\chi_E$ are integrable

W.K.T. If $f \leq g$ a.e then $\int f dx \leq \int g dx$.

$\therefore$ we have $A \leq f \leq B$ on E

i.e $A \, m(E) \leq \int_E f \, dx \leq B \, m(E)$.

Hence the result.

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Measure and Integration.

Chapter - II - The General Integral (Continued)

Theorem:

(Lebesgue's Dominated Convergence Theorem)

Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$, where g is integrable, and let $\lim f_n = f$ a.e. Then f is integrable and $\lim \int f_n dx = \int f dx$.

Proof:

Since for each n, $|f_n| \leq g$,

we have $|f| \leq g$ a.e. -------- (1)

W.K.T. If f and g are measurable, $|f| \leq |g|$ a.e and g is integrable, Then f is integrable.

$\therefore$ (1), we have f_n and f are integrable.

Also $\{g + f_n\}$ is a sequence of non-negative measurable functions,

so by Fatou's Lemma,

$$\liminf \int (g + f_n)\, dx \geq \int \liminf (g + f_n)\, dx.$$

$\Rightarrow \int g\,dx + \liminf \int f_n\,dx \geq \int g\,dx + \liminf \int f_n\,dx$

$\geq \int g\,dx + \int f\,dx.$

But $\int g\,dx$ is finite

$\Rightarrow \liminf \int f_n\,dx \geq \int f\,dx.$ ----- (2)

Again $\{g - f_n\}$ is also a sequence of non-negative measurable functions, so

$\liminf \int (g - f_n)\,dx \geq \int \liminf (g - f_n)\,dx.$

So $\int g\,dx - \limsup \int f_n\,dx \geq \int g\,dx - \int f\,dx.$

So $-\limsup \int f_n\,dx \geq -\int f\,dx$

$\Rightarrow \limsup \int f_n\,dx \leq \int f\,dx.$

$\Rightarrow \limsup \int f_n\,dx \leq \int f\,dx \leq \liminf \int f_n\,dx.$

[From (2)]

$\Rightarrow \int f\,dx = \lim \int f_n\,dx.$

Hence proved.

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Note: Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$, where g is integrable, and let $\lim f_n = f$ a.e. Then f is integrable and $\lim \int f_n dx = \int f dx$. Show that $\lim \int |f_n - f| dx = 0$.

Solution: Write $|f_n - f| \leq 2g$ for each n, from above theorem, $\lim \int |f_n - f| dx = 0$.

Theorem: Let $\{f_n\}$ be a sequence of integrable functions such that $\sum_{n=1}^{\infty} \int |f_n| dx < \infty$

Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges a.e. its sum $f(x)$ is integrable and

$$\int f dx = \sum_{n=1}^{\infty} \int f_n dx.$$

Proof:

W.K.T. If f and g are measurable, $|f| \leq |g|$ a.e. and g is integrable then f is integrable

Let $\sum_{n=1}^{\infty}\int |f_n|\,dx < \infty$ ----→ (1)

Let $\varphi(x) = \sum_{n=1}^{\infty} |f_n|$, $\int \varphi\,dx < \infty$,

and since if f is integrable then f is finite-valued a.e.

$\therefore$ φ is finite-valued.

$\Rightarrow \sum_{n=1}^{\infty} f_n(x)$ is absolutely convergent a.e.

and $|f| \leq \varphi$.

So f is integrable.

Write $g_n(x) = \sum_{i=1}^{\infty} f_i(x)$

Then $|g_n(x)| \leq \varphi(x)$ and $g_n(x) = f(x)$ a.e.

By Lebesgue's Dominated Convergence Theorem,

$\lim \int g_n\,dx = \int f\,dx$.

$\therefore \int f\,dx = \sum_{n=1}^{\infty} \int f_n\,dx$.

Hence proved.

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Note: ① For each $\xi \in [a,b]$, $-\infty \le a < b < \infty$, let f_ξ be a measurable function, $|f_\xi(x)| \le g(x)$, where g is an integrable function, and let $\lim_{\xi \to \xi_0} f_\xi(x) = f(x)$ a.e. where $\xi_0 \in [a,b]$. Then f is integrable and $\lim_{\xi \to \xi_0} \int f_\xi \, dx = \int f \, dx$. -----> ①

[Lebesque's Dominated Convergence Theorem deals with a sequence of functions $\{f_n\}$. state and prove a "continuous parameter" version of the theorem].

Proof:

Let $\{\xi_n\}$ be any sequence in $[a,b]$, $\lim \xi_n = \xi_0$. Then the sequence $\{f_{\xi_n}\}$ Satisfies the conditions of Lebesque's Dominated Convergence theorem.

$\therefore$ f is integrable.

i.e, $\lim_{\xi \to \xi_0} \int f_\xi \, dx = \int f \, dx$.

Suppose that ① does not hold

Then $\exists \delta > 0$, and a sequence $\{\beta_n\}$

with $\lim \beta_n = \xi_0$, such that

$\forall n, \left| \int f_{\beta_n} dx - \int f dx \right| > \delta$

i.e., we get a contradiction.

$$\therefore \lim_{\xi \to \xi_0} \int f_{\xi} dx = \int f dx.$$

Hence proved.

Note: ②

(i) If f is integrable then

$$\int f dx = \lim_{a \to \infty} \lim_{b \to -\infty} \int_a^b f dx = \lim_{b \to -\infty} \lim_{a \to \infty} \int_a^b f dx.$$

(ii) If f is integrable on $[a, b]$ and $0 < \epsilon < b - a$, then

$$\int_a^b f dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b f dx.$$

Solution:

W.K.T. $\int_b^a f dx = \int_{-\infty}^a \chi_{[b,\infty)} f dx.$

From note ① $\Rightarrow$

$$\lim_{b \to -\infty} \int_{-\infty}^a \chi_{[b,\infty)} f dx = \int_{-\infty}^a f dx.$$

from (i) $\Rightarrow$ f is integrable. Then

$$\int f\,dx = \lim_{a\to\infty} \lim_{b\to-\infty} \int_b^a f\,dx = \lim_{b\to-\infty} \lim_{a\to\infty} \int_b^a f\,dx.$$

|||^{ly} (ii) is proved.

(iii) $\int_a^b f\,dx = \lim_{\epsilon\to 0} \int_{a+\epsilon}^{b} f\,dx.$

Theorem: If f is continuous on the finite interval $[a,b]$, then f is integrable, and $F(x) = \int_a^x f(t)\,dt$ $(a<x<b)$ is differentiable function such that $F'(x) = f(x)$.

Proof: As f is continuous, it is measurable and $|f|$ is bounded. So f is integrable on $[a,b]$.

If $a<x<b$, we have $x+h \in (a,b)$ $\forall$ small h, and $F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt.$

The continuity of f, we have

$$\int_x^{x+h} f(t)\,dt = hf(\xi),\ \xi = x+\theta h,\ 0\le\theta\le 1.$$

so, suppose $h \neq 0$, deviding by h and let $h\to 0$,

we get $F'(x) = f(x)$.

Hence proved.

Measure and Integration.

Chapter - II - problems.

Example - 1.

Show that if $\alpha > 1$,

$$\int_0^1 \frac{x \sin x}{1+(nx)^\alpha}\, dx = o(n^{-1}) \text{ as } n \to \infty.$$

Solution:

We wish to show that $\lim_{n\to\infty} \int_0^1 \frac{nx \sin x}{1+(nx)^\alpha}\, dx = 0$.

Clearly $\lim_{n\to\infty} \frac{nx \sin x}{1+(nx)^\alpha} = 0$,

Let the sequence $f_n(x) = \frac{nx \sin x}{1+(nx)^\alpha}$, $n = 1, 2, \ldots$

We consider $h(x) = 1 + (nx)^\alpha - nx^{3/2}$.

So $h(0) = 1$, $h(1) = 1 + n^\alpha - n$ for $1 < \alpha \le 3/2$, h has no stationary point in $[0,1]$, $\forall$ large n; for $\alpha > 3/2$ it has a stationary point at which its value is easily seen to approach 1, for large n.

It follows that for large n, $h(x) > 0$ on $[0,1]$ and so $\left| \frac{nx \sin x}{1+(nx)^\alpha} \right| \le \frac{1}{\sqrt{x}}$.

and hence $\lim_{n\to\infty} \int_0^1 \frac{nx \sin x}{1+(nx)^\alpha}\, dx = 0$.

$\therefore \int_0^1 \frac{x \sin x}{1+(nx)^\alpha}\, dx = o(n^{-1})$ as $n \to \infty$.

Example 2:

show that $\lim \int_0^\infty \frac{dx}{(1+x/n)^n x^{1/n}} = 1$.

Solution.

For $n>1$, $x>0$

$$\left(1+x/n\right)^n = 1 + x + \frac{n(n-1)}{n^2}\frac{x^2}{2} + \cdots > \frac{x^2}{4}.$$

So we define $g(x) = 4/x^2$ $(x \geq 1)$,

$g(x) = x^{-1/2}$ $(0<x<1)$

We have $(1+x/n)^{-n} x^{-1/n} < g(x)$, $(n>1, x>0)$.

But g is integrable over $(0,\infty)$

So $\lim \int_0^\infty (1+x/n)^{-n} x^{-1/n} dx$

$$= \int_0^\infty e^{-x} dx$$

$$= 1$$

$\therefore \lim \int_0^\infty \frac{dx}{(1+x/n)^n x^{1/n}} = 1.$

Example-3.

show that $\lim_{n\to\infty} \int_0^\infty \frac{n^2 x e^{-n^2 x^2}}{1+x^2} dx = 0$,

for $a>0$, but not for $a=0$.

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Solution:

If $a > 0$, substitute $u = nx$

$$\int_a^\infty f_n(x)\,dx = \int_{na}^\infty \frac{ue^{-u^2}}{1+u^2/n^2}\,du$$

$$= \int_a^\infty \chi_{(na,\infty)} \frac{ue^{-u^2}}{1+u^2/n^2}\,du.$$

is integrable function.

But, as $a > 0$,

$$\lim_{n\to\infty} \chi_{(na,\infty)} (1+u^2/n^2)^{-1} u e^{-u^2} = 0.$$

By Lebesque's Dominated Convergence Theorem, $\lim \int f_n dx = \int f dx$,

we have $\lim_{n\to\infty} \int_a^\infty \frac{n^2 x e^{-n^2x^2}}{1+x^2}\,dx = 0.$ —— ①

If $a = 0$, then

$$\int_0^\infty f_n(x)\,dx = \int_0^\infty ue^{-u^2}(1+u^2/n^2)^{-1}\,du$$

$$\to \int_0^\infty ue^{-u^2}\,du = 1/2.$$

$\therefore$ If $a = 0$, then result ① is not true.

Example – 4:

Let f be a non-negative integrable function on $[0,1]$. Then $\exists$ a measurable function $\phi(x)$ such that ϕf is integrable on $[0,1]$ and $\phi(0+) = \infty$

Solution:

W.K.T $\lim_{a \to 0} \int_0^a f\,dx = 0$

So $\forall n$, $\exists\, x_n \; (0 < x_n < 1)$ such that

$\int_0^{x_n} f\,dx < n^{-3}$, and we may suppose

that $x_n \downarrow 0$ as $n \to \infty$.

We Define $\phi(x) = \sum_{k=2}^{\infty} (k-1)\, \chi_{(x_k, x_{k-1})}$

So $\phi(0+) = \infty$.

But $\int_{x_k}^{x_{k-1}} \phi f\,dx = \int_{x_k}^{x_{k-1}} (k-1) f\,dx < (k-1)^{-2}$.

So $\int_0^1 \phi f\,dx \leq \sum_{n=1}^{\infty} 1/n^2 < \infty$.

$\therefore \int_0^1 \phi f\,dx < \infty$

$\therefore$ ϕf is integrable on $[0,1]$ and $\phi(0+) = \infty$.

Chapter - (III) - Integration of series.

Example - 1

Show that $\int_0^1 \frac{x^{1/3}}{1-x} \log\frac{1}{x}\, dx = 9 \sum_{n=1}^{\infty} \frac{1}{(3n+1)^2}$

Solution:

Write $\frac{x^{1/3}}{1-x} \log\frac{1}{x} = x^{1/3} \log\frac{1}{x} \sum_{n=0}^{\infty} x^n \; (0<x<1)$

and W.K.T $\int \sum_{n=1}^{\infty} f_n\, dx = \sum_{n=1}^{\infty} \int f_n\, dx$, we have.

$$\int_0^1 \frac{x^{1/3}}{1-x} \log\frac{1}{x}\, dx = \sum_{n=0}^{\infty} \int_0^1 x^{n+1/3} \log\frac{1}{x}\, dx$$

$$= \sum_{n=0}^{\infty} \frac{9}{(3n+4)^2}$$

$$\left[\because \int x^n \log x\, dx = \frac{x^{n+1}}{n+1} \log x - \frac{x^{n+1}}{(n+1)^2} \right]$$

$$\therefore \int_0^1 \frac{x^{1/3}}{1-x} \log\frac{1}{x}\, dx = \sum_{n=1}^{\infty} \frac{9}{(3n+1)^2}.$$

Hence the solution.

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Example - 2:

Show that $\int_0^\infty \frac{\sin t}{e^t - x}\, dt = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2+1}, \quad -1 \le x \le 1.$

Solution:

The integrand $= \lim_{N\to\infty} \sum_{n=0}^{N} x^n \sin t\, e^{-(n+1)t}$.

But for $t > 0$,

$$\left| \sum_{n=0}^{N} x^n \sin t\, e^{-(n+1)t} \right| \le t e^{-t} \frac{1 - x^{N+1} e^{-(N+1)t}}{1 - x e^{-t}}$$

$$\le \frac{2t}{e^t - x},$$

is an integrable function.

$$\therefore \int_0^\infty \frac{\sin t}{e^t - x}\, dt = \sum_{n=0}^{\infty} x^n \int_0^\infty e^{-(n+1)t} \sin t\, dt$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{1 + (n+1)^2}$$

we have

$$\int_0^\infty \frac{\sin t}{e^t - x}\, dt = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2+1}, \quad -1 \le x \le 1.$$

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Example - 3

show that $\int_0^1 \sin x \log x \, dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!(2n)}$

Solution:

$$\sin x \log x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \log x.$$

$$= \sum_{n=0}^{\infty} f_n(x), \text{ say,}$$

But $\int_0^1 |f_n(x)| \, dx = (-1)^{n+1} \int_0^1 f_n(x) \, dx$

$$= \frac{(-1)^{n+1}}{(2n+2)(2n+2)!}$$

$$\therefore \int_0^1 |f_n(x)| \, dx = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+2)(2n+2)!}$$

Hence $\int_0^1 \sin x \log x \, dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)(2n)!}$

Measure and Integration.

chapter-4 - Riemam and Lebesque Integrals

Riemann Integrable

We consider The Riemann Integral of a bounded function f over a finite interval [a, b].

Let $a = \xi_0 < \xi_1 < \xi_2 < \dots < \xi_n = b$

be a partition. D, of [a, b].

Write $S_D = \sum_{i=1}^{n} M_i (\xi_i - \xi_{i-1})$

where M_i = Sup f in $[\xi_{i-1}, \xi_i]$, $i = 1, \dots n$.

$|||^{ly}$ on replacing M_i by m_i equal to inf f over the Corresponding interval,

we obtain $s_D = \sum_{i=1}^{n} m_i (\xi_i - \xi_{i-1})$.

Then f is said to be Riemann Integrable over [a, b] if given $\epsilon > 0$ $\exists$ D such that $S_D - s_D < \epsilon$.

In this case we have inf S_D = sup s_D, where infimum and Supremum are taken over all partitions D of [a, b], and we Write The common value as $R\int_a^b f dx$.

Definition:

If for each a and b, f is bounded and Riemann integrable on [a,b] and

$$\lim_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b f\,dx \quad \text{exists,}$$

then f is said to be Riemann integrable on $(-\infty, \infty)$ and the integral is written

$$R\int_{-\infty}^{\infty} f\,dx.$$

Theorem: If f is Riemann Integrable and bounded over the finite interval [a,b], then f is integrable and $R\int_a^b f\,dx = \int_a^b f\,dx$.

Proof:

Let $\{D_n\}$ be a sequence of partitions such that, for each n, $S_{D_n} - s_{D_n} < 1/n$,

It is easily seen that

$$S_{D_n} = \int_a^b u_n\,dx \quad \text{and} \quad s_{D_n} = \int_a^b l_n\,dx.$$

where u_n and l_n are step functions $u_n \geq f \geq l_n$.

Indeed we may, for Example, define $u_n = M_i$ on (ξ_{i-1}, ξ_i) and a partition point let u_n be the average of the values M_i corresponding to the intervals ending at that point.

write $U = \inf_n u_n$ and $L = \sup_n l_n$.

Now $[x : U(x) - L(x) > 0]$

$$= \bigcup_{k=1}^{\infty} [x : U(x) - L(x) > 1/k].$$

But if $U - L > 1/k$, then $u_n - l_n > 1/k$ for each n.

So if $m[x : U(x) - L(x) > 1/k] = a$ then

$\int (u_n - l_n) dx > a/k$, and so $a/k < 1/k$ for each n.

So $a = 0$.

Hence $U - L \le 1/k$ a.e. for each k,

So $U = L$ a.e.

But u_n, l_n and hence U, L are measurable.

Also $L \le f \le U$, so f is measurable and being bounded is integrable.

clearly $\int_a^b l_n dx \le \int_a^b f dx \le \int_a^b u_n dx$.

and let $n \to \infty$, we got.

$$R\int_a^b f dx = \int_a^b f dx.$$

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Note:
Converse does not hold.

Consider for example the function f on $[0,1]$.

$$f(x) = \begin{cases} 0 & , x \text{ rational} \\ 1 & , x \text{ irrational.} \end{cases}$$

Then f is measurable, indeed $f = 1$ a.e.

So $\int_0^1 f\,dx = 1$. But each $S_D = 1$ and each $s_D = 0$.

So f is not Rieman Integrable.

Theorem: Let f be a bounded function defined on the finite interval $[a,b]$, then f is Riemann Integrable over $[a,b]$ $\Longleftrightarrow$ if is continuous a.e.

Proof:
Suppose that f is Riemann Integrable over $[a,b]$.
and $U(x) = f(x) = L(x)$.
Where x is not a partition point of any D_n,
the D_n being sequence of partitions.
Then f is continuous at x, for otherwise there would exists $\epsilon > 0$ and a sequence $\{x_n\}$, $\lim x_k = x$ such that
for each k, $|f(x_k) - f(x)| > \epsilon$.
But $U(x) \geq L(x) + \epsilon$.

Now the set of all partition points of the D_n is countable and so has a measure zero, and the set $[x : U(x) \neq L(x)]$ has measure zero. (from above theorem)

So f is continuous a.e.

Conversely, suppose that f continuous a.e.

Choose a sequence $[D_n]$ of partitions of $[a,b]$ such that for each n, D_{n+1} contains the partition points of D_n and such that the length of the largest interval of $D_n \to 0$ as $n \to \infty$.

Then if u_n, l_n are the corresponding step functions. for each n. we have $u_{n+1} \leq u_n$ and $l_{n+1} \geq l_n$

Write $U = \lim u_n$ and $L = \lim l_n$.

Now suppose that f is continuous at x.

Then given $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$\sup f - \inf f < \epsilon$, where the sup and inf are taken over $(x-\delta, x+\delta)$.

$\forall n$ sufficiently large, an interval of D_n containing x will lie in $(x-\delta, x+\delta)$.

and so $u_n(x) - l_n(x) < \epsilon$.

But ϵ is arbitrary so $U(x) = L(x)$.

So $U = L$ a.e.

By Lebesgue's Dominated Convergence theorem,

$\lim \int u_n dx = \int U dx = \int L dx = \lim \int l_n dx$.

and so f is Riemann integrable.

Hence proved.

Theorem:

Let f be bounded and let f and $|f|$ be Riemann integrable on $(-\infty, \infty)$. Then f is integrable and $\int_{-\infty}^{\infty} f dx = R\int_{-\infty}^{\infty} f dx$.

Proof:

W.K.T. If f is Riemann integrable and bounded over the finite interval $[a,b]$, then f is integrable and $R\int_a^b f dx = \int_a^b f dx$,

we have $\int_a^b |f| dx = R\int_a^b |f| dx \leq R\int_{-\infty}^{\infty} |f| dx$,

$\forall$ a and b. So f is integrable.

$\Rightarrow \int_a^b f dx = R\int_a^b f dx$.

As $a \to -\infty$, $b \to \infty$, we have

$\int_{-\infty}^{\infty} f dx = R\int_{-\infty}^{\infty} f dx$.

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Measure and Integration

chapter - 4 - (continued)

Theorem: Let f be bounded and measurable on a finite interval $[a,b]$ and let $\epsilon > 0$. Then $\exists$ (i) a step function h such that

$$\int_a^b |f-h|\,dx < \epsilon.$$

(ii) continuous function g such that g vanishes outside a finite interval and

$$\int_a^b |f-g|\,dx < \epsilon.$$

Proof:

(i) As $f = f^+ - f^-$

we may assume throughout that $f \geq 0$.

Now $\int_a^b f\,dx = \sup \int_a^b \phi\,dx$, where $\phi \leq f$.

ϕ simple and measurable.

So we may assume that f is a simple measurable function, with $f = 0$ outside $[a,b]$.

So $f = \sum_{i=1}^{n} a_i \chi_{E_i}$ with $\bigcup_{i=1}^{n} E_i = [a,b]$.

$\longrightarrow$ (1)

Let $\epsilon' = \epsilon/nM$, where M = sup f on $[a,b]$, and M may obviously be supposed positive.

For each of the measurable sets E_i, $\exists$ an open intervals $I_1, I_2, \ldots I_k$ such that if $G = \bigcup_{r=1}^{k} I_r$, then $m(E_i \Delta G) < \epsilon'$.

But χ_G is a step function such that

$$\int |\chi_{E_i} - \chi_G| \, dx = m(E_i \Delta G) < \epsilon'.$$

Construct such step function, h_i, say, for each E_i, in (1), we have

$$\int_a^b |f - \sum_{i=1}^{n} a_i h_i| \, dx < \sum_{i=1}^{n} a_i \epsilon'$$

$$\leq nM\epsilon'$$

$$= \epsilon.$$

But $\sum_{i=1}^{n} a_i h_i$ is a step function.

(ii) From (i),

$\exists$ a step function h vanishing outside a finite interval such that

$$\int_a^b |f-h| \, dx < \epsilon/2.$$

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The proof is completed by constructing a continuous function g such that, $\int |h-g|\,dx < \epsilon/2$ and such that

$g(x) = 0$ whenever $h(x) = 0$.

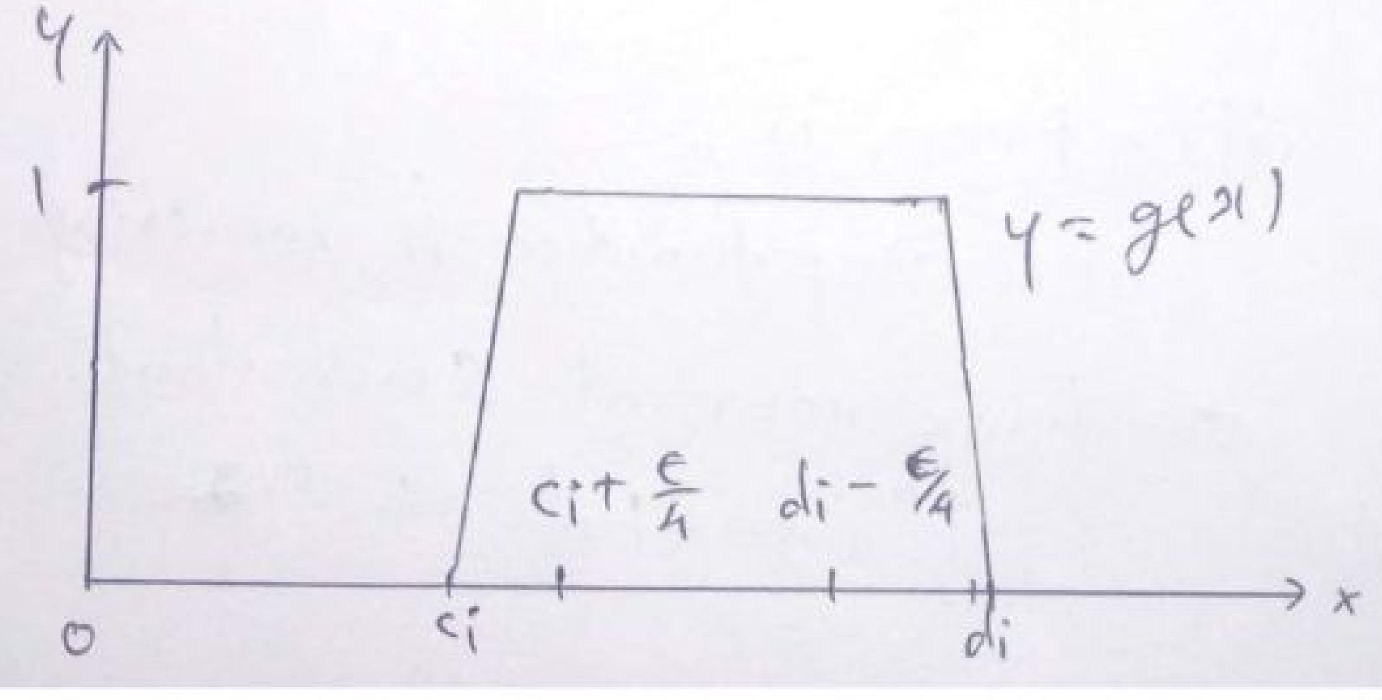
Let $h = \sum_{i=1}^{n} a_i \chi_{E_i}$, where E_i is the finite interval (c_i, d_i), $i = 1, \dots n$.

As it is sufficient to show that each χ_{E_i}, and we may suppose that $\epsilon < 2(d_i - c_i)$ and define g by:

$g = 1$ on $(c_i + \epsilon/4, d_i - \epsilon/4)$, $g = 0$

on $C(c_i, d_i)$.

Extend g by linearity to $(c_i, c_i + \epsilon/4)$ and $(d_i - \epsilon/4, d_i)$



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From above, Fig, to get a continuous function,

clearly $\int |\chi_{E_i} - g| \, dx < \epsilon/2$,

$\therefore \int_a^b |f-g| \, dx < \epsilon$.

<u>Example</u>: Let f be a bounded measurable function defined on the finite interval (a,b). Show that $\lim_{\beta \to \infty} \int_a^b f(x) \sin \beta x \, dx = 0$.

<u>Solution</u>:-

From above theorem,

$\forall \epsilon > 0, \exists h = \sum_{i=1}^{n} \xi_i \chi_{(a_i, b_i)}$, say,

with $\int_a^b |f-h| \, dx < \epsilon$.

Then $\left| \int_a^b f \sin \beta x \, dx \right| \leq \int_a^b |(f-h) \sin \beta x| \, dx$

$+ \left| \int_a^b h \sin \beta x \, dx \right|$

$< \epsilon + \left| \int_a^b h \sin \beta x \, dx \right|$.

Now $\left| \int_a^b \chi_{(a_i, b_i)} \sin \beta x \, dx \right|$

$= \left| 1/\beta \int_{\beta a_i}^{\beta b_i} \sin y \, dy \right|$

$\leq 2/\beta$

$< \epsilon/nM$ for $\beta > \beta_0$, say.

where $M = \max[\xi_i, i = 1, \ldots n]$.

So $\left| \int_a^b f \sin \beta x \, dx \right| < 2\epsilon$, for $\beta > \beta_0$.

$\therefore \lim_{\beta \to \infty} \int_a^b f(x) \sin \beta x \, dx = 0$.

Example:

Show that $f \in L(a+h, b+h]$ and $f_h(x) \equiv f(x+h)$, then $f_h \in L(a,b)$ and $\int_{a+h}^{b+h} f \, dx = \int_a^b f_h \, dx$.

Solution:

clearly $(f_h)^+ = (f^+)_h$, $(f_h)^- = (f^-)_h$.

So it is sufficient to prove the result for $f \geq 0$.

(i) $\exists$ a sequence of measurable simple functions $\{\varphi_n\}$ such that $\varphi_n \leq f$ and

$\int \varphi_n \, dx \uparrow \int f \, dx$.

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But then $(\varphi_n)_h \uparrow f_h$, and so by monotone convergence, we have

$$\int_{a+h}^{b+h} f\,dx = \lim \int_{a+h}^{b+h} \varphi_n\,dx$$

$$= \lim \int_a^b (\varphi_n)_h\,dx$$

$$= \int_a^b f_h\,dx.$$

$$\therefore \int_{a+h}^{b+h} f\,dx = \int_a^b f_h\,dx.$$

Assignment – II

1. State and prove "Fatou's Lemma".
2. State and prove Monotone Convergence Theorem.
3. State and prove Lebesgue's Dominated Convergence Theorem.

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4. If f is Riemann integrable and bounded over the finite interval [a, b], then f is integrable and $R\int_a^b f\,dx = \int_a^b f\,dx$.

5. Let f be a bounded function defined on the finite interval [a, b], then f is Riemann integrable over [a, b]. $\Leftrightarrow$ it is continuous a.e.

6. If f is continuous on the finite interval [a,b] then f is integrable, and $F(x) = \int_a^x f(t)\,dt$ $(a<x<b)$ is a differentiable function such that $F'(x) = f(x)$.

7. Let f be a non-negative measurable function. Then $\exists$ a sequence $\{\phi_n\}$ of measurable simple functions such that, for each x,

$\phi_n(x) \uparrow f(x)$.

8. Let $\{f_n\}$ be a sequence of non-negative measurable functions. Then.

$$\int \sum_{n=1}^{\infty} f_n\,dx = \sum_{n=1}^{\infty} \int f_n\,dx.$$

Measure and Integration

Unit - III - Abstract Measure Space

Chapter - I Measure and outer Measures

Definition: Ring:

A class of sets R, of some fixed space is called a ring if whenever $E \in R$ and $F \in R$, then $E \cup F$ and $E - F$ belong to R.

Ex:

The class of finite unions of intervals of the form $[a, b)$ forms a ring.

Definition: σ- Ring

A ring is called a σ- ring if it is closed under the formation of countable unions.

Ex:

Every algebra is a ring and every σ-algebra a σ- ring but that the Converse is not true.

Definition: Measure:

A set function μ defined on a ring R is a measure if

(i) μ is non-negative

(ii) $\mu(\phi) = 0$

(iii) for any sequence $\{A_n\}$ of disjoint set ~~of~~ of R such that $\bigcup_{n=1}^{\infty} A_n \in R$, we have $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

If R is a ring, the condition $\bigcup_{n=1}^{\infty} A_n \in R$ is clearly redundant.

Definition: Complete:

A measure μ on R is complete if whenever $E \in R$, $F \subseteq E$ and $\mu(E) = 0$ then $F \in R$.

Definition : σ-finite.

A measure μ on R is a σ-finite, if for every set $E \in R$, we have $E = \bigcup_{n=1}^{\infty} E_n$ for some sequence $\{E_n\}$ such that $E_n \in R$ and $\mu(E_n) < \infty$ for each n.

Ex: Lebesgue measure m defined on M, the class of measurable sets of R, is a σ-finite and complete.

Definition : Outer Measure :

If R is a ring, a set function μ^* defined on the class $H(R)$ is an outer measure if

(i) μ^* is non-negative

(ii) if $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$

(iii) $\mu^*(\phi) = 0$

(iv) for any sequence $\{A_n\}$ of sets of $H(R)$

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$$

(ie) μ^* is countably subadditive.

Example:

Show that if $A, B \in R$ and $A \subseteq B$ Then $\mu(A) \leq \mu(B)$.

Solution:

Let $B = A \cup (B-A)$ and as the measure μ is finitely additive.

$\therefore A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.

Theorem: $\exists$ a smallest ring and a smallest σ-ring containing a given class of subsets of a space: we refer to these as the generated ring and the generated σ-ring respectively.

Proof:

unit-I - pageno: (13) $\rightarrow$ Theorem.

Replace σ-algebra by σ-ring.

chapter - II - Completion of a Measure.

Theorem:

If μ is a measure on a σ-ring S, then the class $\overline{S}$ of sets of the form $E \Delta N$ for any set E, N such that $E \in S$ while N is contained in some set in S of zero measure. is a σ-ring. and the set function $\overline{\mu}$ defined by $\overline{\mu}(E \Delta N) = \mu(E)$ is a complete measure on $\overline{S}$.

Proof:

It is convenient to have two different descriptions of the sets of $\overline{S}$ so we prove the set theoretic identity.

$$E \Delta N = (E - M) \cup (M \cap (E \Delta N)) \text{ ------ } (1)$$

For any sets, E, M, N such that $M \supseteq N$.

Let $x \in E \Delta N$, then if $x \in M$, we have $x \in M \cap (E \Delta N)$.

While if $x \in CM$ we have $x \in CN$

so $x \in E-M \Rightarrow x \in E-M$.

To get the opposite inclusion in (1),

Suppose that $x \in R.H.S$,

If $x \in M \cap (E \Delta N)$, then

$x \in E \Delta N$; if $x \in E-M$,

we have $x \in E-N \subseteq E \Delta N$.

Let $D \in \bar{S}$, $D = E \Delta N$, with $N \subseteq M \in S$

where $\mu(M) = 0$ then by (1),

$D = F \cup A$, where $F \cap A = \phi$ and $F \in S$.

and $A \subseteq M \in S$ with $\mu(M) = 0$.

and since for F, A disjoint we have

$F \cup A = F \Delta A$ the two characterizations

of the sets of $\bar{S}$ are Equivalent.

Now if $D_i \in \bar{S}$, $i = 1, 2, \ldots$ on writing

$D_i = F_i \cap A_i$, we see that

$\bigcup_{i=1}^{\infty} D_i \in \overline{S}$. If $D_1 = E_1 \Delta N_1$ and $D_2 = E_2 \Delta N_2$

belong to S, we have,

$D_1 \Delta D_2 = (E_1 \Delta E_2) \Delta (N_1 \Delta N_2)$.

So $D_1 \Delta D_2 \in \overline{S}$, and so

$D_1 - D_2 = (D_1 \cup D_2) \Delta D_2 \in \overline{S}$

So $\overline{S}$ is a σ-ring.

Also $D_1 \Delta D_2 = \phi$ only if $E_1 \Delta E_2 = N_1 \Delta N_2$.

So if $E_1 \Delta N_1 = E_2 \Delta N_2$, we have

$\mu(E_1 \Delta E_2) = 0$ and hence $\mu(E_1) = \mu(E_2)$.

So $\overline{\mu}$ is a unambiguously defined.

Also $\overline{\mu}$ is a measure.

For clearly $\overline{\mu}(\phi) = 0$.

and if $\{D_i\}$ is a sequence of disjoint sets of $\overline{S}$. $D_i = F_i \cup A_i$ say.

So that $F_i \cap A_j = \phi \ \forall$ i and j, then

$\bar{\mu}(\cup D_i) = \bar{\mu}(\cup F_i \cup A_i)$

$= \bar{\mu}(\cup F_i \Delta \cup A_i)$

$= \mu(\cup F_i) = \sum \mu(F_i)$

$= \sum \bar{\mu}(F_i \cup A_i)$

$= \sum \bar{\mu}(D_i)$.

So $\bar{\mu}$ is countably additive.

Finally μ is Complete. ~~for let $D \subset D_0$~~

for let $D \subset D_0 \in \bar{S}$, where $\bar{\mu}(D_0) = 0$.

so $D_0 = E_0 \Delta N_0$, where $N_0 \subseteq M_0$,

$E_0, M_0 \in S$, $\mu(E_0) = \mu(M_0) = 0$.

and so $D_0 \subseteq M_0' = E_0 \cup M_0 \in S$.

and $\mu(M_0') = 0$ Then $D = E \Delta N$ with

$E = \phi$, $N = D \subseteq E_0 \cup M_0$ and so $D \in \bar{S}$.

$\therefore$ Hence proved.

Measure and Integration

Unit - III - chapter - II - Completion of a Measure.

Example:

Show that the extension $\bar{\mu}$ of above (page No: 5) Theorem is unique in the sense that if μ' is a complete measure on a ring $S' \supseteq S$ and $\mu' = \mu$ on S then $\mu' = \bar{\mu}$ on $\bar{S}$.

Solution:

Since μ' is complete it is easily seen that $S' \supseteq \bar{S}$.

For $D \in \bar{S}$, we have $D = F \cup A$;

F, A disjoint sets with $F \in S$,

$A \subseteq M \in S$ with $\mu(M) = 0$.

So, $\mu'(D) = \mu'(F) + \mu'(A)$

$= \mu(F)$

$= \bar{\mu}(D)$

We call $\bar{\mu}$ on $\bar{S}$ the completion of μ and S.

Hence the result.

Theorem:

The completion of a σ-finite measure is σ-finite

Proof:

Let $D \in \bar{S}$

As in the above theorem (Page No: 5)

$D = F \cup A$ where $F \in S$ and $\bar{\mu}(A) = 0$.

So $F = \bigcup_{i=1}^{\infty} F_i$, where $\mu(F_i) < \infty$,

and hence $D = \bigcup_{i=1}^{\infty} F_i$ is a countable union of sets of finite $\bar{\mu}$-Measure.

i.e., The completion of a σ-finite measure is σ-finite.

Hence proved.

Chapter - III - Measure Space.

Definition: Measurable space

A pair $[[X, S]]$, where S is a σ algebra of subsets of a space X, is called a measurable space.

The set of S are called measurable sets.

Definition: Measure space

A triple $[[X, S, \mu]]$ is called a measure space if $[[X, S]]$ is a measurable space and μ is a measure on S.

Example:

1. $[[R, M, m]]$ and $[[R, B, m]]$ are measure space, where B denotes the Borel sets and m is restricted to B. In the latter case m is called Borel measure on the real line.

2. Let $[[X, S]]$ be a measurable space and let $Y \in S$. Then if $S' = [B \cap Y : B \in S]$, we have that $[[Y, S']]$ is a measurable space.

Theorem:

Let $\{E_i\}$ be a sequence of measurable sets. we have

(i) if $E_1 \subseteq E_2 \subseteq \cdots$ then $\mu\left(\bigcup_{i=1}^{\infty} E_n\right) = \lim \mu(E_n)$.

(ii) if $E_1 \supseteq E_2 \supseteq \cdots$ and $\mu(E_1) < \infty$ then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim \mu(E_n)$$

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Proof:

unit - (I) - page No: (19) → see Theorem.

Definition: Measurable.

Let f be an extended real-valued function defined on X. Then f is said to be measurable if $\forall \alpha, [x : f(x) > \alpha] \in S$.

Definition:

If a property holds except on a measurable set E such that $\mu(E) = 0$, we say that it holds almost every where with respect to μ, written a.e.(μ).

Example: The limit of a pointwise convergent sequence of measurable function is measurable.

Example:

Let $f = g$ a.e.(μ). where μ is a complete measure. show that if f is measurable. so g is measurable.

Solution: Let f is measurable.

Write $E = [x : g(x) > \alpha]$,

$E_1 = [x : f(x) > \alpha]$, $E_2 = [x : f(x) \neq g(x)]$

Then E_1 and E_2 are measurable and, as μ is complete,

So $E \cap E_2$ is complete.

So $E = (E_1 - E_2) \cup (E \cap E_2)$ is measurable.

$\therefore$ g is also measurable.

Theorem: The measurability of f is equivalent to

(i) $\forall \alpha$, $[f(x) \geq \alpha] \in S$

(ii) $\forall \alpha$, $[x : f(x) < \alpha] \in S$

(iii) $\forall \alpha$, $[x : f(x) \leq \alpha] \in S$.

Proof:

Unit-(I) – page no: (27) → see Theorem.

Theorem:

If c is a real number and f, g measurable functions, then, $f+c$, cf, $f+g$, $g-f$ and fg are also measurable.

Proof:

Unit-(I) - Page no: (30) → see Theorem.

Theorem: If f_i is measurable, $i = 1, 2, \ldots n$ then $\sup_{1 \le i \le n} f_i$, $\inf_{1 \le i \le n} f_i$, $\sup f_n$, $\inf f_n$, $\limsup f_n$ and $\liminf f_n$ are also measurable.

Proof:

Unit-(I) — Page No: (32) - See Theorem.

Measure and Integration

unit-III - chapter-III

Integration with Respect To a Measure.

Definition:

A measurable simple function ϕ is one taking a finite number of non-negative value, each on a measurable set, so if $a_1 \ldots a_n$ are the distinct values of ϕ, we have $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$, where $A_i = [x : \phi(x) = a_i]$. Then the integral of ϕ with respect to μ is given by $\int \phi \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$

Definition:

Let f be measurable, $f : X \to [0, \infty)$.

Then the integral of f is

$$\int f \, d\mu = \sup[\int \phi \, d\mu : \phi \leq f, \phi \text{ a measurable simple function}].$$

Definition:

Let $E \in S$, and let f be a measurable function $f: E \to [0, \infty]$. Then the integral of f over E is

$$\int_E f \, d\mu = \int f \chi_E \, d\mu.$$

Theorem: (Fatou's Lemma:)

Let $\{f_n\}$ be a sequence of measurable function, $f_n: X \to [0, \infty]$. Then $\liminf \int f_n \, d\mu \geq \int \liminf f_n \, d\mu$.

Proof: Unit - (II) → page No (7) → see Theorem.

Theorem:

(Lebesgue's Monotone Convergence Theorem)

Let $\{f_n\}$ be a sequence of measurable functions. $f_n: X \to [0, \infty]$, such that $f_n(x) \uparrow$ for each x. and let $f = \lim f_n$. Then $\int f \, d\mu = \lim \int f_n \, d\mu$.

Proof:

Unit (II) → Page No (10) → See Theorem.

Theorem:

Let f be a measurable function $f: X \to [0, \infty]$. Then $\exists$ a sequence $[\varphi_n]$ of measurable simple functions such that, for each x, $\varphi_n(x) \uparrow f(x)$.

Proof:

Unit (II) → Page No (11) → see Theorem.

Theorem:

Let $\{f_n\}$ be a sequence of measurable functions. $f_n : X \to [0, \infty]$, then

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu.$$

Proof:

Unit (II) → Page No (14) → See Theorem.

Theorem:

Let $[[X, S, \mu]]$ be a measure space and f a non-negative measurable function. Then $\varphi(E) = \int_E f \, d\mu$ is a measure on the measurable space $[[X, S]]$. If in addition.

$\int f d\mu < \infty$ then $\forall \epsilon > 0$, $\exists \delta > 0$, Such that, if $A \in S$ and $\mu(A) < \delta$, then $\phi(A) < \epsilon$.

Proof:

The function ϕ is Countably additive since, if $\{E_n\}$ is a Sequence of disjoint sets of S,

$$\phi\left(\bigcup_{n=1}^{\infty} E_n\right) = \int \chi_{\cup E_n} f \, d\mu$$

$$= \sum_{n=1}^{\infty} \int \chi_{E_n} f \, d\mu.$$

$\therefore \phi$ is a measure on $[[X, S]]$.

Write $f_n = \min(f, n)$.

Then f_n is measurable, $f_n \uparrow f$ and

$\lim \int f_n d\mu = \int f d\mu$. (by monotone theorem)

So if $\int f d\mu < \infty$, then $\forall \epsilon > 0$, $\exists N$,

Such that $\int f d\mu < \int f_N d\mu + \epsilon/2$.

If $A \in S$ and $\mu(A) < \epsilon/2N$,

we have $\int_A f_N d\mu < \epsilon/2$.

So take $\delta = \epsilon/2N$ to get

$$\int_A f d\mu = \int_A (f - f_N) d\mu + \int_A f_N d\mu$$

$$\leq \int (f - f_N) d\mu + \epsilon/2.$$

$$< \epsilon.$$

i.e., $\int_A f d\mu < \epsilon$.

Hence proved.

Definition:

If f is measurable and both $\int f^+ d\mu$ and $\int f^- d\mu$ are finite. Then f is said to be integrable, and the integrable of f is $\int f^+ d\mu - \int f^- d\mu$. So

f is integrable $\Longleftrightarrow |f|$ is integrable.

Theorem:

Let f and g be integrable functions and let a and b be constants. Then $af + bg$ is integrable and

$$\int (af + bg)\, d\mu = a\int f\, d\mu + b\int g\, d\mu .$$

If $f = g$ a.e then $\int f\, d\mu = \int g\, d\mu$.

Proof:

Unit - (II) - page No - (19) → see theorem

Theorem:

Let f be integrable then $\left|\int f\, d\mu\right| \leq \int |f|\, d\mu$ with equality $\Leftarrow\Rightarrow$ $f = 0$ a.e (or) $f \leq 0$ a.e.

Proof:

Unit - (II) - page No (22) → see Theorem.

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Theorem (Lebesgue's Dominated Convergence Theorem)

Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$. where g is an integrable function. and $\lim f_n = f$ a.e. Then f is integrable, $\lim \int f_n d\mu = \int f d\mu$. and $\lim \int |f_n - f| d\mu = 0$.

Proof: unit (II) → page No (28) → see Theorem.
and. page No (28) → see Note:

Theorem: Let $\{f_n\}$ be a sequence of integrable functions such that $\sum_{n=1}^{\infty} \int |f_n| d\mu < \infty$.
Then $\sum_{n=1}^{\infty} f_n$ converges a.e. its sum, f is integrable, and $\int f d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$.

Proof:

unit-(II) → page No (28) → see Theorem.

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UNIT- IV - CONVERGENCE

Chapter - I - Convergence in Measure

Definition:

Let $\{f_n\}$ be a sequence of Measurable functions and f is a measurable function. Then f_n tends to f in Measure if for every positive ϵ,
$\lim \mu [x : |f_n(x) - f(x)| > \epsilon] = 0$

Definition:

A sequence of functions is said to be fundamental with respect to a particular kind of convergence if it forms a Cauchy sequence in that sence. Thus a sequence $\{f_n\}$ is fundamental in measure if for any $\epsilon > 0$, $\lim_{m,n \to \infty} \mu [x : |f_n(x) - f_m(x)| > \epsilon] = 0$

Theorem:

If a sequence of Measurable functions converges in Measure, then the limit furunction is unique a.e.

Proof:

Let $f_n \to f$ in measure and

$f_n \to g$ in measure.

Since $|f-g| \leq |f-f_n| + |g-f_n|$.

we have for any $\epsilon > 0$,

$$[x : |f(x) - g(x)| > 2\epsilon]$$
$$\subseteq [x : |f(x) - f_n(x)| > \epsilon] \cup [x : |g(x) - f_n(x)| > \epsilon].$$

But the measure of the set on the right-hand side tends to zero as $n \to \infty$.

∴ So $f = g$ a.e

∴ The limit function is unique.

Theorem: If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure, Then ∃ a measurable function f such that $f_n \to f$ in measure

Proof:

For every integer k we can find n_k such that for $n, m \geq n_k$,

$$\mu\left[x : |f_n(x) - f_m(x)| \geq \frac{1}{2^k}\right] < \frac{1}{2^k},$$

and we may assume that for each k, $n_{k+1} > n_k$.

Let $E_k = \left[x : |f_{n_k}(x) - f_{n_{k+1}}(x)| \geq \frac{1}{2^k}\right]$.

Then if $x \notin \bigcup_{k=m}^{\infty} E_k$, we have for $r > s \geq m$,

$$|f_{n_r}(x) - f_{n_s}(x)|$$

$$\leq \sum_{i=s+1}^{r} |f_{n_i}(x) - f_{n_{i-1}}(x)|$$

$$< \sum_{i=s+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^s} \text{ --------- (1)}$$

So $\{f_{n_k}(x)\}$ is a cauchy sequence for each $x \notin \limsup E_k = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k$.

But $\forall m$,

$$\mu(\limsup E_k) \leq \mu\left(\bigcup_{k=m}^{\infty} E_k\right)$$

$$\leq \sum_{k=m}^{\infty} 1/2^k$$

$$= \frac{1}{2^{m-1}}.$$

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So $\{f_{n_k}\}$ Converges a.e. to some measurable function. Also from (1), we have that $\{f_{n_k}\}$ is uniformly fundamental in $C\bigcup_{k=m}^{\infty} E_k$, for each m,

So $f_{n_k} \to f$ uniformly on $C\bigcup_{k=m}^{\infty} E_k$.

and hence, for every positive ϵ,

$$\mu[x : |f_{n_k}(x) - f(x)| > \epsilon/2] \to 0 \text{ as } k \to \infty \longrightarrow (2)$$

But $[x : |f_n(x) - f(x)| > \epsilon]$

$$\subseteq [x : |f_n(x) - f_{n_k}(x)| > \epsilon/2] \cup [x : |f(x) - f_{n_k}(x)| > \tfrac{\epsilon}{2}]$$

If n and n_k are sufficiently large, The measure of the first set on the right is arbitrarly small, as $\{f_n\}$ is fundamental in measure. But the second set has been shown to have arbitrarly small measure by (2),

$\exists$ a measurable function f such that $f_n \to f$ in measure.

Corollary:

Let $f_n \to f$ in measure where f and each f_n are measurable functions. Then there exists a subsequence $\{n_i\}$ such that $f_{n_i} \to f$ a.e.

Proof:

clearly $\{f_n\}$ is fundamental in measure, So from of the theorem we can find a subsequence $\{f_{n_i}\}$ and a measurable function g such that $f_{n_i} \to g$ a.e.

and in measure.

But $f_{n_i} \to f$ in measure.

W.K.T. If a sequence of measurable functions converges in measure, then the limit function is unique a.e.

we have $f = g$ a.e.

$\therefore \exists$ a subsequence $\{n_i\}$ such that $f_{n_i} \to f$ a.e.

Chapter - I - Convergence in Measure.

Theorem:-

Let $\{f_n\}$ be a sequence of non-negative measurable functions and let f be a measurable function such that $f_n \to f$ in measure. Then,

$\int f d\mu \leq \liminf \int f_n d\mu$.

Proof:

Suppose that $\int f d\mu < \infty$ and that $\int f d\mu > \liminf \int f_n d\mu$.

Then $\exists$ $\delta > 0$ and a sequence $\{n_i\}$ such that, for each i, $\int f_{n_i} d\mu < \int f d\mu - \delta$.

But $f_{n_i} \to f$ in measure.

So we can find a subsequence $\{n_i'\}$ of $\{n_i\}$ such that $f_{n_i'} \to f$ a.e.

But by Fatou's lemma,

$\int f d\mu \leq \liminf \int f_{n_i'} d\mu \leq \int f d\mu - \delta$,

giving a contradiction.

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Now suppose that $\int f d\mu = \infty$ and that $\liminf \int f_n d\mu < \infty$,

Then $\exists K > 0$ and a subsequence $\{f_{n_i}\}$ such that, for each i, $\int f_{n_i} d\mu < K$.

But again we find a subsequence $\{n_i'\}$ of $\{n_i\}$ such that $f_{n_i'} \to f$ a.e.

But by Fatou's lemma, $\liminf \int f_{n_i'} d\mu = \infty$, giving a contradiction.

So $\liminf \int f_n d\mu = \infty$.

$\therefore \int f d\mu \leq \liminf \int f_n d\mu$.

<u>Theorem</u>:

Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| < g$, an integrable function, and let $f_n \to f$ in measure, where f is measurable. Then f is integrable, $\lim \int f_n d\mu = \int f d\mu$ and $\lim \int |f_n - f| d\mu = 0$.

<u>Proof</u>:

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W.K.T. let $f_n \to f$ in measure where f and each f_n are measurable functions. Then $\exists$ a subsequence $\{n_i\}$ such that $f_{n_i} \to f$ a.e.

So we have $|f| \leq g$ and so $f \in L^1(\mu)$.

Also, for each n, $g + f_n \geq 0$, and $g + f_n \to g + f$ in measure. follows immediately from the fact that $f_n \to f$ in measure.

Then, W.K.T. $\int f d\mu \leq \liminf \int f_n d\mu$.

we have $\int g d\mu + \int f d\mu \leq \liminf \int (g + f_n) d\mu$.

So $\int f d\mu \leq \liminf \int f_n d\mu$.

we have, $|||^{ly}$, $g - f_n \geq 0$ and

$g - f_n \to g - g$ in measure.

So $\int g d\mu - \int f d\mu \leq \liminf \int (g - f_n) d\mu$

Hence $\int f d\mu \geq \limsup \int f_n d\mu$

$\geq \liminf \int f_n d\mu$

$\geq \int f d\mu$.

$\therefore \lim \int f_n d\mu = \int f d\mu. \longrightarrow (1)$

Also, it is clear from the definition of convergence in measure. That

$|f_n - f| \to 0$ in measure.

But $|f_n - f| \leq 2g$,

$\therefore \int |f_n - f| \, d\mu = 0$ (from (1))

Definition:-

Convergence in $L^p(\mu)$ is often described as convergence in the mean of order p, That is: $f_n \to f$ in the mean of order $p (p > 0)$ if $\lim \|f_n - f\|_p = 0$.
If $p = 1$, f_n is said to converge to f in the mean.

Theorem:

If $f_n \to f$ in the mean of order p ($p > 0$), then $f_n \to f$ in measure.

Proof:

Suppose not.

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Then $\exists$ $\epsilon > 0$, $\delta > 0$ such that

$\mu[x : |f_n - f| > \epsilon] > \delta$ for infinitely many n,

But then $\|f_n - f\|_p > \epsilon \delta^{1/p}$ for infinitely many n, giving a contradiction.

$\therefore$ $f_n \to f$ in measure.

Chapter - II - Almost uniform convergence.

Definition:

Let $\{f_n\}$ be a sequence of measurable functions and let f be a measurable function. Then we say that $f_n \to f$ almost uniformly, and write $f_n \to f$ a.u. if for any $\epsilon > 0$ $\exists$ a set E with $\mu(E) < \epsilon$ and such that on CE, $f_n \to f$ uniformly.

UNIT - IV - chapter II

Theorem:

Uniform convergence a.e implies almost uniform convergence.

proof:

Let $\{f_n\}$ be a sequence of measurable functions and let f be a measurable function.

Then we say that $f_n \to f$ almost uniformly, and write $f_n \to f$ a.u. if, for any $\epsilon > 0$, $\exists$ a set E with $\mu(E) < \epsilon$ and such that on CE, $f_n \to f$ uniformly.

$\therefore$ Uniform convergence a.e $\Rightarrow$ almost uniform convergence.

Theorem:

If $f_n \to f$ a.u. then $f_n \to f$ in measure.

Proof:

If $f_n \not\to f$ in measure.

∃ a positive number ϵ and δ
such that $\mu[x : |f_n(x) - f(x)| > \epsilon] > \delta$.
for infinitely many n.
But since ∃ exists a set E,
with $\mu(E) < \delta$, such that $f_n \to f$ uniformly
on CE,
we get a contradiction.
∴ If $f_n \to f$ a.u. then $f_n \to f$ in measure.

Theorem:

If $f_n \to f$ a.u. then $f_n \to f$ a.e.

Proof:

For each integer m, we can find a set E_m with $\mu(E_m) < \frac{1}{m}$, and on CE_m, $f_n \to f$ uniformly.

Then if $x \in \bigcup_{m=1}^{\infty} CE_m$, we have $x \in CE_N$,

so $\lim f_n(x) = f(x)$.

But $C\bigcup_{m=1}^{\infty} CE_m = \bigcap_{m=1}^{\infty} E_m$, a set of measure zero.

∴ $f_n \to f$ a.e.

Theorem:-

Let $f_n \to f$ a.e. If (i) $\mu(X) < \infty$

(ii) for each n, $|f_n| \leq g$, an integrable function, then we have $f_n \to f$ a.u.

Proof:-

write $E_{k,n} = \bigcap_{m=n}^{\infty} [x : |f_m(x) - f(x)| < \frac{1}{k}]$

It is sufficient to prove that for each k,

$\lim_{n\to\infty} \mu(CE_{k,n}) = 0$,

for then if $\epsilon > 0$, $\mu(CE_{k,n_k}) < \epsilon/2^k$

for an appropriate n_k.

So if $E = \bigcap_{k=1}^{\infty} E_{k,n_k}$, we have $\mu(CE) < \epsilon$

and on E, $|f_m - f| < \frac{1}{k}$ for $m \geq n_k$.

so $f_m \to f$ a.u.

Clearly $[x : \lim f_m(x) = f(x)] \subset \bigcup_{n=1}^{\infty} E_{k,n}$,

for each k, so the Complementary set,

$\bigcap_{n=1}^{\infty} CE_{k,n}$, has measure zero.

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So it is sufficient to prove that

$\mu(cE_{k,n}) < \infty$ for some n, and for each k.

$\therefore$ (i) $\mu(X) < \infty$ is proved.

for (ii), we have $|f_m - f| \leq 2g$.

$$\text{so } cE_{k,n} = \bigcup_{m=n}^{\infty} \left[x : |f_m(x) - f(x)| \geq \frac{1}{k}\right]$$

$$\subseteq \left[x : g(x) \geq \frac{1}{2k}\right].$$

But g is integrable

(ii) a set of finite measure,

for each n, $|f_n| \leq g$ an integrable function.

Then we have $f_n \to f$ a.u.

Example:

If $\{f_n\}$ be a sequence of measurable functions such that $\lim f_n = f$ uniformly, where $f \in L^p(\mu)$, and $\mu(X) < \infty$.

Then $f_n \to f$ in the mean of order p $(p > 0)$.

Solution :.

We have $|f_n|^p \leq 2^p(|f|^p+1)$ $\forall$ large n.

and so $f_n \in L^p(\mu)$.

But for any $\epsilon > 0$ and for all large n,

$|f_n - f| < \epsilon$ and so

$$\int |f_n - f|^p d\mu < \epsilon^p \mu(X).$$

$\therefore$ $f_n \to f$ in the mean of order p $(p>0)$.

Example :

Let g be a bounded function, measurable on $[0,1]$ and let $\{f_n\}$ be a sequence of functions integrable on $[0,1]$. show that if

(i) $\left\{ \int_0^1 |f_n| dx \right\}$ is bounded.

(ii) $\forall$ $\epsilon > 0$, $\exists$ $\eta > 0$ such that for each measurable subset H of $[0,1]$ with $m(H) < \eta$, we have $\left| \int_H f_n dx \right| < \epsilon$ $\forall n$,

(iii) $\lim \int_0^u f_n dx = 0$ for each $u \in [0,1]$,

Then $\lim \int_0^1 g f_n dx = 0$.

Solution:

By (iii) we have $\lim \int_I f_n = 0$ for any interval I.

Let A is measurable set and $\eta > 0$.

$\exists$ intervals $I_1, I_2, \ldots I_N$ such that

$m\left(A \Delta \bigcup_{k=1}^{N} I_k\right) < \eta$.

Then by (ii), $\left| \int_A f_n dx - \int_E f_n dx \right| < 2\epsilon, \forall n$.

Where $E = \bigcup_{k=1}^{N} I_k$.

But $\lim \int_E f_n dx = 0$,

So $\lim \int_A f_n dx = \lim \int \chi_A f_n dx = 0$.

So $\lim \int \phi f_n dx = 0$.

where ϕ is any measurable simple function.

As we may consider g^+ and g^- separately we may suppose that $g \geq 0$.

Then we can find a sequence $\{\varphi_n\}$ of measurable simple functions,

$\varphi_n \uparrow g$ on $[0,1]$.

So $\forall \epsilon > 0$, $\exists H$ with $m(H) < \eta$,

and k such that $|\varphi_k - g| < \epsilon$ on $[0,1] - H$.

Now, $$\int_0^1 g f_n dx = \int_0^1 \varphi_k f_n dx + \int_H (g - \varphi_k) f_n dx + \int_{[0,1]-H} (g - \varphi_k) f_n dx.$$

But $$\left| \int_{[0,1]-H} (g - \varphi_k) f_n dx \right| < \epsilon \int_0^1 |f_n| dx \leq M\epsilon, \forall n.$$

Now choose n so that $\left| \int_0^1 \varphi_k f_n dx \right| < \epsilon$.

If $g \leq N$ on $[0,1]$, we have

$$\left| \int_H (g - \varphi_k) f_n dx \right| \leq N \int_H |f_n| dx \leq 2N\epsilon,$$

by (ii), $\lim \int_0^1 g f_n dx = 0$.

Hence proved.

unit - IV

chapter - III - Signed Measures and the Hahn Decomposition

Definition: Signed Measure:

A set function ν defined on a measurable space $[[X, S]]$ is said to be a signed measure, if the values of ν are extended real numbers and

(i) ν takes at most one of the values $\infty, -\infty$,

(ii) $\nu(\phi) = 0$,

(iii) $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_i)$, if $E_i \cap E_j = \phi$ for $i \neq j$, where if the left-hand side is infinite, the series on the right-hand side has sum ∞ or $-\infty$ as the case may be.

Clearly, Every Measure is a signed measure.

Definition: Positive set :-

A is a positive set with respect to the signed measure ν on $[[X, S]]$, if $A \in S$ and $\nu(E) \geq 0$ for each measurable subset E of A.

Ex:
If A is a positive set with respect to ν and if, for $E \in S$, $\mu(E) = \nu(E \cap A)$, Then μ is a measure

Definition : Negative set

A is a negative set with respect to ν if it is a positive set with respect to $-\nu$.

Definition: Null set

A is a null set with respect to ν, or a ν-null set, if it is both a positive set with respect to ν and a negative set with respect to ν.

Equivalently, A is a ν- null set if $A \in S$ and $\nu(E) = 0$ $\forall$ $E \in S$, $E \subseteq A$.

Ex: If A is a positive set with respect to ν, then every measurable subset of A is a positive set,

The same holds for negative sets and null sets.

Example:

Show that if $\phi(E) = \int_E f\,d\mu$, where $\int f\,d\mu$ is defined, then ϕ is a signed measure.

Solution:

we have $\int f^+ d\mu < \infty$ or $\int f^- d\mu < \infty$.

Let $\{E_i\}$ be a sequence of disjoint sets of S and for $E \in S$

write $\phi^+(E) = \int_E f^+ d\mu$,

$$\phi^-(E) = \int_E f^- d\mu,$$

$\therefore$ ϕ^+ and ϕ^- are measures.

Then

$$\phi\left(\bigcup_{i=1}^{\infty} E_i\right) = \phi^+\left(\bigcup_{i=1}^{\infty} E_i\right) - \phi^-\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$= \sum_{i=1}^{\infty} \phi^+(E_i) - \sum_{i=1}^{\infty} \phi^-(E_i)$$

$$= \sum_{i=1}^{\infty} \phi(E_i)$$

$\therefore$ ϕ is a signed measure.

Theorem :

A countable union of sets positive with respect to a signed measure ν is a positive set.

Proof: Let $\{A_n\}$ be a sequence of positive sets. Then we have

$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$, where the sets

$B_n \in S$, $B_n \subseteq A_n$ and $B_n \cap B_m = \phi$ if $n \neq m$.

Now let $E \subseteq \bigcup_{n=1}^{\infty} A_n$.

Then $E = \bigcup_{n=1}^{\infty} (E \cap B_n)$.

So $\nu(E) = \sum_{n=1}^{\infty} \nu(E \cap B_n) \geq 0$,

as $E \cap B_n$ is a positive set for each n,

So $\bigcup_{n=1}^{\infty} A_n$ is a positive set.

Corollary:

A countable union of negative or of null sets is respectively, a negative or a null set.

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Theorem: Let v be a Signed measure on $[[X, S]]$. Let $E \in S$ and $v(E) > 0$. Then there exists A, a set positive with respect to v, such that $A \subseteq E$ and $v(A) > 0$.

Proof:

If E contains no set of negative v-measure, then E is a positive set and $A = E$

$\therefore v(A) > 0$, gives the result.

Otherwise there exists $n \in N$ such that there exists $B \in S$, $B \subseteq E$ and $v(B) < -1/n$.

Let n_1 be the smallest such integer and E_1 a corresponding measurable subset of E with $v(E_1) < -1/n_1$

Let n_i be the smallest positive integer such that there is a measurable subset E_i of

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$E - \bigcup_{i=1}^{k-1} E_i$ with $v(E_i) < -1/n_k$.

From the construction, $n_1 \le n_2 \le \cdots$ and we have a corresponding sequence $\{E_i\}$ of disjoint subsets of E.

If the process stops at n_M say, and $C = E - \bigcup_{i=1}^{M} E_i$. Then

C is a positive set, and $v(C) > 0$, for $v(C) = 0$ would imply that

$v(E) = \sum_{i=1}^{M} v(E_i) < 0$.

So C is the desired set.

If the process does not stop.

put $A = E - \bigcup_{k=1}^{\infty} E_k$.

We wish to show that A is positive set.

$v(E) = v(A) + v\left(\bigcup_{k=1}^{\infty} E_k\right)$ ------- (1)

But v cannot take both the values $\infty, -\infty$, $v(E) > 0$, $v\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} v(E_k) < 0$.

So the second term on the right-hand side of ① is finite.

So $\sum_{k=1}^{\infty} \nu(E_k) > -\infty$.

Hence $\sum_{k=1}^{\infty} 1/n_k < \infty$ and

in particular, $\lim_{k \to \infty} n_k = \infty$ and

$n_k > 1$ for $k > k_0$, say.

So let $B \in S$, $B \subseteq A$ and $k > k_0$.

Then $B \subseteq E - \bigcup_{i=1}^{k} E_i$,

So $\nu(B) \geq -\frac{1}{n_k - 1}$ ⋯⋯→ ②

By definition of n_k.

But ② holds for all $k > k_0$, so letting $k \to \infty$ we have $\nu(B) \geq 0$ and so A is a positive set. As before

$\nu(A) = 0$ would imply $\nu(E) < 0$,

so $\nu(A) > 0$

Hence proved.

Theorem: Hahn decomposition

Let ν be a signed measure on $[[X, S]]$. Then there exists a positive set A and negative set B such that $A \cup B = X$, $A \cap B = \phi$. The pair A, B is said to be a Hahn decomposition of X with respect to ν. It is unique to the extent that if A_1, B_1 and A_2, B_2 are Hahn decompositions of X with respect to ν. Then $A_1 \Delta A_2$ is a ν-nullset.

Proof:

We may suppose that $\nu < \infty$ on S, for otherwise we consider $-\nu$,

$\Rightarrow$ The result is true for ν.

Let $\lambda = \sup[\nu(C) : C$ a positive set$]$

so $\lambda \geq 0$.

Let $\{A_i\}$ be a sequence of positive set such that $\lambda = \lim \nu(A_i)$.

W.K.T A countable union of sets positive with respect to a signed measure ν is a positive set.

$\therefore A = \bigcup_{i=1}^{\infty} A_i$ is a positive set and from the definition of λ, $\lambda \geq \nu(A)$.

But $A - A_i \subseteq A$ and hence is a positive set, so for each i,

$\nu(A) = \nu(A_i) + \nu(A - A_i) \geq \nu(A_i)$.

So $\nu(A) \geq \lim \nu(A_i) = \lambda$

and hence $\nu(A) = \lambda$.

i.e. The value of λ is achieved on a positive set.

Write $B = CA$.

Then if B contains a set D of positive ν-measure,

we have $0 < \nu(D) < \infty$.

$\therefore$ D contains a positive set E such that $0 < \nu(E) < \infty$.

But then $\nu(A \cup E) = \nu(A) + \nu(E) > \lambda$

Contradicting the definition of λ,

So $\nu(D) \leq 0$. and B is a negative set.

and A, B form a Hahn Decomposition.

For the last part note that

$A_1 - A_2 = A_1 \cap B_2$

and hence is a positive set. and negative set. and so a null set.

$||^{ly}$ $A_2 - A_1$ is a null set.

and so $A_1 \Delta A_2$ is a null set.

Hence proved the theorem.

Unit - IV

Chapter - IV - The Jordan Decomposition

Definition: Let v_1, v_2 be measures on $[[X,S]]$. Then v_1 and v_2 are said to be mutually singular, If for some $A \in S$, $v_2(A) = v_1(CA) = 0$; and we than write $v_1 \perp v_2$.

Example:

Let μ be a measure and let the measures v_1, v_2 be given by $v_1(E) = \mu(A \cap E)$, $v_2(E) = \mu(B \cap E)$, where $\mu(A \cap B) = 0$ and $E, A, B \in S$. show that $v_1 \perp v_2$.

Solution:

$$v_1(B) = \mu(A \cap B) = 0$$

$$v_2(CB) = \mu(\phi) = 0.$$

$$\therefore v_1 \perp v_2$$

Theorem: The Jordan Decomposition

Let ν be a signed measure on $[[X, S]]$. Then there exists measures ν^+ and ν^- on $[[X, S]]$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$. The measures ν^+ and ν^- are uniquely defined by ν, and $\nu = \nu^+ - \nu^-$ is said to be the Jordan decomposition of ν.

Proof:-

Let A, B be a Hahn decomposition of X with respect to ν, and define ν^+ and ν^- by

$$\nu^+(E) = \nu(E \cap A), \quad \nu^-(E) = -\nu(E \cap B). \longrightarrow (1)$$

for $E \in S$. Then ν^+ and ν^- are measures and $\nu^+(B) = \nu^-(A) = 0$

So $\nu^+ \perp \nu^-$

Also, for $E \in S$

$$\nu(E) = \nu(E \cap A) + \nu(E \cap B)$$
$$= \nu^+(E) - \nu^-(E)$$

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So $\nu = \nu^+ - \nu^-$
and the proof will be complete when we show that the decomposition is unique.

Let $\nu = \nu_1 - \nu_2$ be any decomposition of ν into mutually singular measures.
Then we have $X = A \cup B$,
where $B = CA$ and $\nu_1(B) = \nu_2(A) = 0$.
Let $D \subseteq A$, then
$\nu(D) = \nu_1(D) - \nu_2(D) = \nu_1(D) \geq 0$,

So A is a positive set with respect to ν.

$|||^{ly}$ B is negative set.
For each $E \in S$ we have
$\nu_1(E) = \nu_1(E \cap A) = \nu(E \cap A)$ and
$\nu_2(E) = -\nu(E \cap B)$,

So every such decomposition of ν is obtained from a Hahn decomposition of X.

So it is enough to show that if A, B and A', B' are two Hahn decompositions

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then the measures obtained as in (1) are the same, we have

$$\nu(A \cup A') = \nu(A \cap A') + \nu(A \Delta A')$$

$$= \nu(A \cap A')$$

From a Hahn decompositions theorem.

For each $E \in S$, as $A \cup A'$ is a positive set, we have

$$\nu(E \cap (A \cap A')) \leq \nu(E \cap A)$$

$$\leq \nu(E \cap (A \cup A'))$$

and

$$\nu(E \cap (A \cap A')) \leq \nu(E \cap A')$$

$$\leq \nu(E \cap (A \cup A')).$$

But the first and last terms in each of these inequalities are the same so

$$\nu(E \cap A) = \nu(E \cap A')$$

and $\nu^+(E) = \nu(E \cap A)$ is unique.

But then $\nu^- = \nu^+ - \nu$ is also uniqe.

$\therefore$ ν^+ and ν^- are unique.

$\therefore$ Hence proved

Example:

Let $[[X, S, \mu]]$ be a measure space and let $\int f\,d\mu$ exist. Define v by $v(E) = \int_E f\,d\mu$, for $E \in S$. Find a Hahn decomposition with respect to v and the Jordan decomposition of v.

Solution:

W.K.T if $\phi(E) = \int_E f\,d\mu$ where $\int f\,d\mu$ is defined, then ϕ is a signed measure.

$\therefore v(E) = \int_E f\,d\mu$

$\Rightarrow v$ is sighned measure

Let $A = \{x : f(x) \geq 0\}$.

$B = \{x : f(x) < 0\}$.

Then A, B form a Hahn decomposition, while v^+, v^- given by

$v^+(E) = \int_E f^+\,d\mu$, $v^-(E) = \int_E f^-\,d\mu$.

form the Jordan decomposition.

Hence the result.

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Definition: The total variation of a signed measure ν is $|\nu| = \nu^+ + \nu^-$, where $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of ν.

Clearly $|\nu|$ is a measure on $[[X,S]]$, and for each $E \in S$, $|\nu(E)| \le |\nu|(E)$.

Definition: A signed measure ν on $[[X,S]]$ is σ-finite if $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n \in S$ and, for each n, $|\nu(X_n)| < \infty$.

Example:

show that the signed measure ν is finite or σ-finite respectively $\Longleftrightarrow$ $|\nu|$ is, or $\Longleftrightarrow$ both ν^+ and ν^- are.

Solution:

Suppose $|\nu(E)| < \infty$. Then as ν^+ and ν^- are not both infinite we have $\nu^+(E) < \infty$ and $\nu^-(E) < \infty$ and hence $|\nu|(E) < \infty$.

obviously ν is finite if $|\nu|$ is finite.

The corresponding results on σ-finiteness are an immediate consequence

UNIT- V

MEASURE AND INTEGRATION IN A PRODUCT SPACE

chapter - I - MEASURABILITY IN A PRODUCT SPACE

Definition:

If X and Y are sets, their Cartesian product $X \times Y$ is the set of ordered pairs $[(x, y) : x \in X, y \in Y]$. If X and Y are spaces, $X \times Y$ is the product space.

Definition:

A set in $X \times Y$ is a rectangle if it may be written $A \times B$ for $A \subseteq X$, $B \subseteq Y$.

Definition:

A measurable rectangle in $X \times Y$ is any set which may be written as $A \times B$ for $A \in S$, $B \in T$.

Definition: The class of elementary sets E consists of those sets which may be written as the union of a finite number of disjoint measurable rectangles.

Theorem:

E is an algebra.

Proof:

clearly E is closed under finite disjoint unions.

It is closed also under finite intersections. for let

$$P = \bigcup_{i=1}^{n} U_i \in E, \quad Q = \bigcup_{j=1}^{m} V_j \in E,$$

where U_i, V_j are measurable rectangles,

$\cdot U_i \cap U_k = \phi$ for $i \neq k$.,

$V_j \cap V_s = \phi$ for $j \neq s$, Then

$$P \cap Q = \bigcup_{\substack{1 \le i \le n \\ 1 \le j \le m}} (U_i \cap V_j) \in E$$

as the intersection of two measurable rectangles is a measurable rectangle.

If $A \times B$ is a measurable rectangle,

$$C(A \times B) = (CA \times Y) \cup (A \times CB) \in E.$$

③

So if $P = \bigcup_{i=1}^{n} U_i \in E$

$$CP = \bigcap_{i=1}^{n} CU_i \in E$$

Also E is closed under finite unions for if $P \in E$, $Q \in E$, then

$P \cup Q = (P - Q) \cup Q$, a disjoint union, belongs to E.

Since clearly $X \times Y \in E$,

∴ E is an algebra.

Definition:

$S \times T$ denotes the σ-algebra generated by the class of measurable rectangles. Also $[[X \times Y, S \times T]]$ is the product of the measurable space $[[X, S]]$ and $[[Y, T]]$

Ex:

$S \times T = S(E)$,

The σ-algebra generated by E.

Definition:

A class M_0 of subsets of a space is a monotone class if for any increasing or decreasing sequence of sets $\{E_n\}$ of M_0, $\lim E_n \in M_0$.

Theorem: If Y is any class of subsets of X, there exists a smallest monotone class, denoted by $M_0(Y)$ containing Y.

Proof:

obivously, $P(Y)$ is a monotone class, Also the intersection of monotone classes is a monotone class.

So the intersection of all the monotone classes which contain Y provides the required monotone class $M_0(Y)$.

Hence proved.

unit- 5

Theorem:

If A is an algebra, $S(A) = M_0(A)$ that is the σ-algebra generated by A is the smallest monotone class containing A.

Proof: For brevity write M_0 in place of $M_0(A)$.

Since every σ-algebra is a monotone class, $S(A) \supseteq M_0$.

To prove the opposite inclusion it is sufficient to show that M_0 is an algebra, since each countable union can be written as the limit of finite unions.

Let $M_0 = [A : cA \in M_0]$;

Then it easily seen that M_0 is a monotone class and that $A \subseteq M_0$.

So $M_0 \subseteq M_0$ i.e, M_0 is closed under the taking of complements.

(6)

We wish to show that the same is true for finite unions. For each $F \in M_0$.

let $k(F) = [E : E \in M_0, E \cup F \in M_0]$.

Then it is sufficient to show that $M_0 \in k(F)$ for each $F \in M_0$.

Now $k(F)$ is a monotone class.

Since, for example,

if $E_n \in k(F)$, $E_n \subseteq E_{n+1}$, $n = 1, 2, \ldots$

we have $\bigcup_{n=1}^{\infty} E_n \in M_0$ and

$$\left(\bigcup_{n=1}^{\infty} E_n\right) \cup F = \lim (E_n \cup F) \in M_0.$$

So $\lim E_n \in k(F)$.

Also, if $G \in A$, $k(G)$ contains A.

Since $M_0 \supseteq A$, so $k(G) = M_0$.

So for any $H \in M_0$, $H \in k(G)$.

so $A \subseteq k(H)$ for each $H \in M_0$ and as $k(H)$ is a monotone class $M_0 = k(H)$ as required.

Corollary: $S \times T = M_0(E)$

Definition:

If $E \subseteq X \times Y$, we define the x-section of E to be the set $E_x = [y : (x, y) \in E]$. and the y-section of E to be the set $E^y = [x : (x, y) \in E]$.

Example

Show that if $\{A_i\}$ is a monotone sequence of sets, then $\lim (A_i)^y = (\lim A_i)^y$ and $\lim (A_i)_x = (\lim A_i)_x$.

Solution:

This follows from the fact that

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^y = \bigcup_{i=1}^{\infty} A_i^y \text{ and}$$

$$\left(\bigcap_{i=1}^{\infty} A_i\right)^y = \bigcap_{i=1}^{\infty} A_i^y$$

$\parallel^{ly}$ for y-sections.

Theorem:

If $E \in S \times T$, Then for each $x \in X$ and $y \in X$, $E_x \in T$ and $E^y \in S$.

Proof:

Let $\Omega = [E : E \in S \times T, E_x \in T \text{ each } x \in X]$.

If $A \in S$ and $B \in T$, Then

$(A \times B)_x = B$ or ϕ according as

$x \in A$ or $x \in CA$.

So Ω contains the measurable rectangles. If $E \in \Omega$ then

Since $(CE)_x = [y : (x,y) \in CE]$

$= C[y : (x,y) \in E]$

$= CE_x$.

we have $CE \in \Omega$. Also if $E_n \in \Omega$, $n = 1, 2, \ldots$ we have for each $x \in X$.

$$\left(\bigcup_{n=1}^{\infty} E_n\right)_x = \bigcup_{n=1}^{\infty} (E_n).$$

So Ω is a σ-algebra and hence $\Omega \in S \times T$.

III^{ly} $E^y \in S$ for each $y \in Y$.

Definition: Let f be a function defined on $X \times Y$. Then, given $x \in X$, the x-section of f is the function on Y: $f_x(y) = f(x,y)$, and given $y \in X$, the y-section of f is the function on X: $f^y(x) = f(x,y)$.

Theorem: Let f be an $S \times T$ measurable function on $X \times Y$: then for each $x \in X$ and $y \in Y$, f_x is a T-measurable function and f^y is an S-measurable function.

Proof: Let $E = [(x,y) : f(x,y) > \alpha]$.

Then for a fixed $x \in X$, $E_x = [y : f_y(y) > \alpha]$ belongs to $\mathcal{T}$ for each α,

$\therefore f_x$ is T-measurable.

III^{ly} f_y is measurable

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Definition: Let f be a function defined on $X \times Y$. Then, given $x \in X$, the *x-section* of f is the function on Y: $f_x(y) = f(x, y)$, and given $y \in X$, the y-Section of f is the function on X: $f^y(x) = f(x, y)$.

Theorem: Let f be an $S \times T$ measurable function on $X \times Y$: then for each $x \in X$ and $y \in Y$, f_x is a T-measurable function and f^y is an S-measurable function.

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$\therefore f_x$ is T-measurable.

|||ly f_y is measurable

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The product Measure and Fubini's Theorem.

Theorem:

Let $[[X, S, \mu]]$ and $[[Y, T, \nu]]$ be a σ-finite measure space. For $V \in S \times T$ write $\phi(x) = \nu(V_x)$, $\psi(y) = \mu(V^y)$, for each $x \in X$, $y \in Y$. Then ϕ is S-measurable, ψ is T-measurable, and

$$\int_X \phi \, d\mu = \int_Y \psi \, d\nu$$

Proof:

We suppose first that result holds if μ and ν are finite measures. We may write $X = \bigcup_{n=1}^{\infty} X_n$, $Y = \bigcup_{m=1}^{\infty} Y_m$, decomposing X and Y into disjoint sequences of sets of finite measure, and the result is therefore assumed true for each rectangle $X_n \times Y_m$, where we are considering μ and ν restricted to the measurable subsets of X_n and Y_m respectively.

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Let $V \in S \times T$ and write $V_{n,m} = V \cap (X_n \times Y_m)$;

Then for each x; $V_x = \bigcup_{n,m} (V_{n,m})_x$.

By hypothesis $\nu((V_{n,m})_x)$ is a measurable function of x on X_n for each m,

So $\sum_{m=1}^{\infty} \nu((V_{n,m})_x)$ is measurable on X_n

Hence $\phi(x) = \nu(V_x) = \sum_{m,n=1}^{\infty} \nu((V_{n,m})_x)$

is measurable with respect to S.

$|||^{ly}$ $\psi(y) = \sum_{m,n=1}^{\infty} \mu(V_{n,m}^{y})$ is measurable

with respect to T., we have

$$\int \phi \, d\mu = \sum_{n=1}^{\infty} \int_{X_n} \phi \, d\mu$$

$$= \sum_{n=1}^{\infty} \int_{X_n} \sum_{m=1}^{\infty} \nu((V_{n,m})_x) d\mu$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_{X_n} \nu((V_{n,m})_x) d\mu$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_{Y_m} \mu(V_{n,m}^{y}) \, d\nu.$$

$$= \int \psi \, d\nu.$$

So we suppose that μ and ν are finite measures on S and T respectively, and write Ω for the class of sets $V \in S \times T$ for which

$$\int_X \varphi \, d\mu = \int_Y \psi \, d\nu \text{ holds.}$$

Then Ω contains every measurable rectangle $A \times B$. Since

$$\nu((A \times B)_x) = \chi_A(x) \nu(B),$$

$$\mu((A \times B)) = \chi_B(y) \mu(A),$$

so $\int_X \varphi \, d\mu = \int_Y \psi \, d\nu$ holds.

i.e., Ω contains that algebra E,

If $V_1 \subseteq V_2 \subseteq \ldots$, where $V_i \in \Omega$ for each i, and if $V = \bigcup_{i=1}^{\infty} V_i$, Then $V \in \Omega$.

For, write $\varphi_i(x) = \nu((V_i)_x)$,

$$\psi_i(y) = \mu(V_i^y).$$

These are measurable functions

$$\therefore \varphi_i(x) \uparrow \varphi(x) = \nu(V_x)$$

$$\psi_i(y) \uparrow \psi(y) = \mu(V^y).$$

(13)

So φ and ψ are measurable and

$$\int \varphi \, d\mu = \lim \int \varphi_i \, d\mu$$
$$= \lim \int \psi_i \, d\nu$$
$$= \int \psi \, d\nu.$$

If $V_1 \supseteq V_2 \supseteq \ldots$, where $V_i \in \Omega$ for each i,

and if $V = \bigcap_{i=1}^{\infty} V_i$, we obtain similarly

sequences $\varphi_i \downarrow \varphi$, $\psi_i \downarrow \psi$.

Since μ and ν are finite we may

since $\varphi_i \leq \nu(Y)$, a function integrable

over X and $\psi_i \leq \mu(X)$.

which is integrable over Y.

$\therefore \int_X \varphi \, d\mu = \int_Y \psi \, d\nu$ holds for φ and ψ.

So Ω is monotone class contained in $S \times T$

and containing E and. Hence proved the

Theorem.

Corollary: with the notation and the σ-finiteness condition of the theorem we have $\int_X d\mu(x) \int_Y \chi_V(x,y)\, d\nu(y)$

$= \int_Y d\nu(y) \int_X \chi_V(x,y)\, d\mu(x)$.

Proof: We have $\phi(x) = \nu(V_x)$

$= \int_Y \chi_V(x,y)\, d\nu(y)$.

and ||ly for $\psi(y) = \nu(V_y)$

$= \int_X \chi_V(x,y)\, d\nu(x)$.

Hence from above theorem, we have

$\int_X d\mu(x) \int_Y \chi_V(x,y)\, d\nu(y)$

$= \int_Y d\nu(y) \int_X \chi_V(x,y)\, d\mu(x)$.

Definition: Let $[[X,S,\mu]]$ and $[[Y,T,\nu]]$ be a σ-finite measure space. Then the product measure $\mu \times \nu$ on $S \times T$ is given by

$(\mu \times \nu)(V) = \int_X \nu(V_x)\, d\mu = \int_Y \mu(V^y)\, d\nu$.

for each $V \in S \times T$.

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Example: show that if μ and ν are σ-finite measures, then $\mu \times \nu$ as given in above definition is the only measure on $S \times T$ giving to each measurable rectangle. $A \times B$ the measure $\mu(A)\mu(B)$.

Solution.

The required measure must have value $\sum_{i=1}^{n} \mu(A_i)\nu(B_i)$ on the elementary set which decomposes into disjoint measurable rectangles as $\bigcup_{n=1}^{n} (A_i \times B_i)$.

Now $\mu \times \nu$ clearly takes the correct value on measurable rectangles.

$\therefore$ it is a measure on E.

So it takes the correct value on the set of E and indeed in clearly a σ-finite measure on the σ-algebra. E.

But the extension from E to $S(E) = S \times T$ is then unique.

Hence proved the example.

Theorem: Let f be a non-negative $S \times T$ measurable function and let

$$\varphi(x) = \int_Y f_x \, d\nu, \quad \psi(y) = \int_X f^y \, d\mu$$

for each $x \in X$, $y \in Y$. Then φ is S-measurable, ψ is T-measurable and

$$\int_X \varphi \, d\mu = \int_{X \times Y} f \, d(\mu \times \nu) = \int_Y \psi \, d\nu.$$

Proof:

Let f is the characteristic function of an $S \times T$ measurable set, and hence for a measurable simple functions.

In general case, let $\{f_n\}$ be a sequence of measurable simple functions such that $f_n \uparrow f$.

$\therefore \varphi_n(x) = \int (f_n)_x \, d\nu$ is S-measurable

and $\int_X \varphi_n \, d\mu = \int_{X \times Y} f_n \, d(\mu \times \nu) \longrightarrow (1)$

As $n \to \infty$, $(f_n)_x \uparrow f_x$.

so $\varphi_n \uparrow \varphi$ and so φ is measurable.

$\therefore \int_X \varphi_n \, d\mu = \int_{X\times Y} f_n \, d(\mu\times\nu)$ is true.

$|||^{ly}$ ψ is T-measurable and

$$\int_X \varphi \, d\mu = \int_{X\times Y} f \, d(\mu\times\nu) = \int_Y \psi \, d\nu.$$

is true.

Theorem:

Let f be an $S\times T$ - measurable function and let $\varphi^*(x) = \int_Y |f|_x \, d\nu$, $\psi(y) = \int_X |f|^y d\mu$ for each $x \in X$, $y \in Y$, then the conditions $\varphi^* \in L^1(\mu)$, $\varphi^* \in L^1(\nu)$, $f \in L^1(\mu\times\nu)$ are equivalent.

Proof:

we apply the last theorem page no. 16 to $|f|$, and gives the result.

Theorem: (Fubini's Theorem)

If $f \in L^1(\mu \times \nu)$ Then $f_x \in L^1(\nu)$ for almost all $x \in X$, $f^y \in L^1(\mu)$ for almost all $y \in Y$. The functions ϕ and ψ defined as ~~in~~ $\phi(x) = \int_Y f_x d\nu$, $\psi(y) = \int_X f^y d\mu$, for each $x \in X$, $y \in Y$: Then

$$\int_X \phi d\mu = \int_{X \times Y} f d(\mu \times \nu) = \int_Y \psi d\nu \text{ holds.}$$

~~is true~~.

Proof:

From the measurable functions f^+, f^-, we obtain the functions ϕ_1, ϕ_2 as ϕ was obtained from $f \in L^1(\mu \times \nu)$.

Since $f^+, f^- \in L^1(\mu \times \nu)$.

$$\therefore \int_X \phi d\mu = \int_{X \times Y} f d(\mu \times \nu) = \int_Y \psi d\nu,$$

$\Rightarrow \phi_1, \phi_2 \in L^1(\mu)$.

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So $\forall x$, both $\varphi_1(x)$ and $\varphi_2(x)$ are finite, and for such x,

Since $f_x = f_x^+ - f_x^-$, we have $f_x \in L^1(\nu)$ and $\varphi(x) = \varphi_1(x) - \varphi_2(x)$.

Hence φ is integrable.

Also $\int_X \varphi \, d\mu = \int_{X\times Y} f \, d(\mu\times\nu)$

holds for φ_1 and f^+ and for φ_2 and f^-.

$|||^{ly}$ $\int_Y \psi \, d\nu = \int_{X\times Y} f \, d(\mu\times\nu)$ about f^y and ψ are proved.

Hence

$$\int_X \varphi \, d\mu = \int_{X\times Y} f \, d(\mu\times\nu) = \int_Y \psi \, d\nu.$$

Hence proved.

Example 4: The condition $V \in \mathcal{S} \times \mathcal{T}$ is necessary in Theorem 6.

Solution: For X and Y take the set of ordinals $[\alpha: \alpha \leq \omega_1]$ were ω_1 is the first uncountable ordinal ([11], p. 69). For $\mathcal{S}$ and $\mathcal{T}$ take the σ-algebra generated by the countable subsets. Let μ and ν be zero for countable sets and 1 for uncountable measurable sets. Let $V = [(x, y): x < y]$. Then if $x = \omega_1$, the x-section $V_x = \phi$; otherwise V_x is uncountable but $\mathbf{C}V_x$ is countable and so V_x is measurable . If $y = \omega_1$,

$V^y = [x: x < \omega_1]$, a measurable set, and for $y < \omega_1$, V^y is countable and so measurable. But $\int_X d\mu \int_Y \chi_V \, dv = \int v(V_x)\, d\mu = \mu[x: x < \omega_1] = 1$, whereas $\int_Y dv \int \chi_V \, d\mu = \int \mu (V^y)\, dv$. So (10.2) does not hold for this V.

The next example shows how Theorem 6, p. 172, breaks down if μ and v are not both σ-finite. The same example shows that σ-finiteness is essential in Theorems 7, 8 and 9.

Example 5: Let $X = Y = [0, 1]$, $\mathcal{S} = \mathcal{T} = \mathcal{B}$. Take $\mu = m$ on the Borel subsets of $[0, 1]$, and for v take the counting measure on $[0, 1]$, that is, $v(E) = \text{Card } E$. Take $V = [(x, y): x = y, (x, y) \in X \times Y]$. Then V is $\mathcal{S} \times \mathcal{T}$-measurable, for if n is any positive integer put $I_j = [(j-1)/n, j/n]$, $j = 1, ..., n$ and $V_n = (I_1 \times I_1) \cup .... \cup (I_n \times I_n)$. So V_n is measurable, and so therefore is $V = \bigcap_{n=1}^{\infty} V_n$. (A diagram may assist.) However

$$\int_Y dv \int_X \chi_V \, d\mu = 0 \quad \text{but} \quad \int_X d\mu \int_Y \chi_V \, dv = 1.$$

Example 6: The condition $f \in L^1 (\mu \times v)$ in Theorem 9, is necessary if the order of integration is to be interchangeable.

Solution: Take X , Y, $\mathcal{S}$, $\mathcal{T}$ as in the last example and let $\mu = v = m$, restricted to $[0, 1]$. Let $0 < \alpha_1 < ... < \alpha_n < ... < 1$, $\lim \alpha_n = 1$. For each n choose a continuous function g_n such that $[t: g_n(t) \neq 0] \subseteq (\alpha_n, \alpha_{n+1})$ and also $\int_0^1 g_n \, dt = 1$. Let $f(x, y) = \sum_{n=1}^{\infty} g_n(y)(g_n(x) - g_{n+1}(x))$. For each (x, y) only one term in this series can be non-zero, so f is well defined. Also f is meaurable, indeed f is continuous except at $(1, 1)$. But

$$\int_0^1 f(x, y)\, dy = \int_0^1 \sum_{n=1}^{\infty} g_n(y)(g_n(x) - g_{n+1}(x))\, dx =$$

$$= g_n(y)\left(\int_{\alpha_n}^{\alpha_{n+1}} g_n \, dx - \int_{\alpha_{n+1}}^{\alpha_{n+2}} g_{n+1} \, dx \right) = 0$$

for each y. However

$$\int_0^1 f(x, y)\, dy = \sum_{n=1}^{\infty} (g_n(x) - g_{n+1}(x)) \int_0^1 g_n \, dy = g_1(x),$$

so $\int_0^1 dx \int_0^1 f(x, y)\, dy = 1$ and the iterated integrals are therefore unequal. However, Fubini's theorem is not contradicted since f is not integrable. For, writing $I_i = (\alpha_i, \alpha_{i+1})$, we have

$$\int |f(x, y)|\, dx\, dy = \sum_{i,j=1}^{\infty} \int_{I_i \times I_j} |\sum_{n=1}^{\infty} g_n(y)(g_n(x) - g_{n+1}(x))|\, dx\, dy$$

$$= \sum_{i,j=1}^{\infty} \int_{I_i \times I_j} |g_j(y)(g_j(x) - g_{j+1}(x)|\, dx\, dy$$

$$= \sum_{j=1}^{\infty} \int_{I_j \times I_j} + \int_{I_{j+1} \times I_j} |g_j(y)(g_j(x) - g_{j+1}(x))|\, dx\, dy = \infty.$$

In some product spaces, for example in the plane, the 'natural' measure to consider is not $\mu \times \nu$ but its completion, obtained in Theorem 8, p. 91. Results equivalent to Theorem 9 can be obtained in this case but details of the statements are no longer the same; for example, the functions ϕ and ψ of Theorem 6 onwards need not now be measurable but only equal a.e. to measurable functions. For details of an approach using complete measures see, for example, [12], Chapter 12.