CORE COURSE IX

CLASSICAL DYNAMICS

Objectives

1. To give a detailed knowledge of the mechanical system of particles.
2. To study the applications of Lagrange's and Hamilton's equations .

UNIT I

Introductory concepts: The mechanical system - Generalised Coordinates - constraints - virtual work - Energy and momentum.

UNIT II

Lagrange's equation: Derivation and examples - Integrals of the Motion - Small oscillations.

UNIT III

Special Applications of Lagrange's Equations: Rayleigh's dissipation function - impulsive motion - Gyroscopic systems - velocity dependent potentials.

UNIT IV

Hamilton's equations: Hamilton's principle - Hamilton's equations - Other variational principles - phase space.

UNIT V

Hamilton - Jacobi Theory: Hamilton's Principal Function – The Hamilton - Jacobi equation - Separability.

TEXT BOOKS.

1. Donald T. Greenwood,Classical Dynamics, PHI Pvt. Ltd., New Delhi-1985.

UNIT – I	Chapter 1: Sections 1.1 to 1.5
UNIT – II	Chapter 2: Sections 2.1 to 2.4
UNIT – III	Chapter 3 : Sections 3.1 to 3.4
UNIT – IV	Chapter 4: Sections 4.1 to 4.4
UNIT – V	Chapter 5: Sections 5.1 to 5.3

REFERENCES.

1. H. Goldstein, Classical Mechanics, (2nd Edition), Narosa Publishing House, New Delhi.
2. Narayan Chandra Rana & Promod Sharad Chandra Joag, Classical Mechanics, Tata McGrawHill, 1991.

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Classical Dynamics

UNIT - I

Introduction

Basic Concepts

1. **Particles**

A particle can be thought of as a scap of matter with no size but with a definitie position.

The mathematical model of particle is a point.

The most of particles in space is, therefore, the most of the point in space.

2. **Rigid body**

A rigid body is made up of particles which never undergoes any change of size or shape.

3. **Mass**

The mass of the particle is the amount of matter contained in the particle.

4. **Force**

A force is one which has

(i) A point of application

(ii) A direction and

(iii) A magnitude

$\therefore$ Force is a vector quantity and is usually denoted by $\bar{F}$

In general the forces that act on a body may be classified as

(i) Contact force (ii) body force.

Forces that are transmitted to the body by a direct mechanical pull or push are contact forces.

The contact forces generally are applied only at the boundary surface of the body.

The body forces are associated with action at a distance and applied throughout the body.

Relative to a basis frame of reference, a particle of mass 'm' subject to a force 'F' moves in accordance with

the equation $\bar{F} = k m \bar{a}$.
where $\bar{a}$ is the acceleration of particle and
k is a intial positive constant.
we make $k = 1$, by a special choice of the unit of force and thus we get $F = m\bar{a}$.

5. <u>Frame of reference</u>

Keeping sun as the origin and assuming that it is rotating with respect to the fixed star, such frame of reference the astronomical frame of reference. Later it is called newtonian frame of reference or inertial frame of reference.

If a frame of reference is called newtonian or inertial then the newton's law of motion, namely $\bar{F} = m\bar{a}$ holds good.

6. <u>Generalised Co-ordinates</u>

The Configuration of given system can be expressed by using various sets of co-ordinates. If the system consists of N particles, then the configuration of the system is specified by 3N Cartesian Co-ordinates written as $x_1, x_2, \ldots, x_{3N}$

Let us now consider two sets of co-ordinates describes the same system. The process of obtaining one set from the other is known as co-ordinate transformation.

<u>Example (1)</u>

Consider a particle which moves on a fixed circle path of radius a. The polar angle θ made with the axis ox, by the line p joining the centre of circle of path and the position of the particle specifying the configuration

Let us take $q_1 = \theta$ and $q_2 = a$.

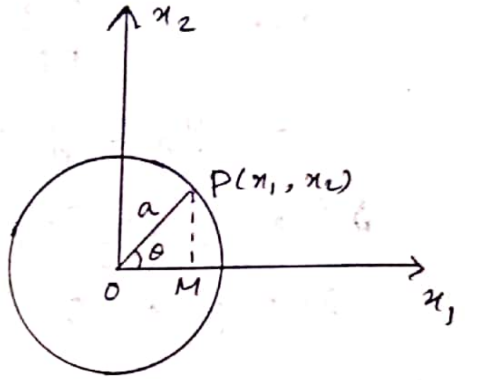
Then the transformation equations are

$x_1 = q_2 \cos q_1$ and $x_2 = q_2 \sin q_1$.

x_1 and x_2 are connected by the equations of constraints $(x_1^2 + x_2^2)^{1/2} = a$. (2)

Now, the jacobian of the transformation is

$$J\left(\frac{x_1, x_2}{q_1, q_2}\right) = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} \end{vmatrix}$$



$$= \begin{vmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{vmatrix}$$

$$= -q_2 \sin^2 q_1 - q_2 \cos^2 q_1$$

$$= -q_2.$$

∴ Transformation is only possible if $q_2 \neq 0$ and the transformation equations are

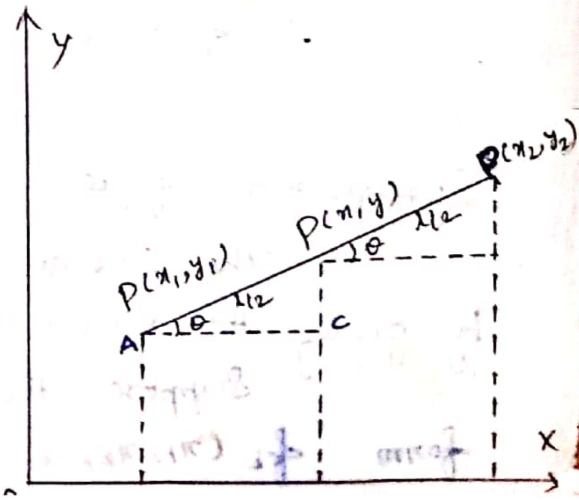
$$q_1 = \tan^{-1}\left(\frac{x_2}{x_1}\right), \quad q_2 = (x_1^2 + x_2^2)^{1/2} \quad \text{where } 0 \le q_1 \le 2\pi, \; 0 < q_2 \le \infty$$

This transformation equation apply all points on the finite plane except at the origin.

Example (2)

Particles A and B are connected by rigid rod of length 'l'. The configuration of the system is given by the cartesian co-ordinates (x_1, y_1, x_2, y_2) or by the generalised co-ordinates (x, y, θ). Write the transformation equation giving the cartesian co-ordinates in terms of generalised co-ordinates. Define a fourth generalised co-ordinates $q_4 = l$ and evaluate the Jacobian $\frac{\partial(x_1, y_1, x_2, y_2)}{\partial(x, y, \theta, q_4)}$.

Solve the generalised co-ordinates interms of cartesian co-ordinates.



From the figure

$$\cos\theta = \frac{AC}{AP} = \frac{x - x_1}{l/2}$$

$$\Rightarrow x_1 = x - \frac{l}{2}\cos\theta \quad \text{-----} \; (1)$$

$$\sin\theta = \frac{PC}{AP} = \frac{y - y_1}{l/2}$$

$$\Rightarrow y_1 = y - \frac{l}{2}\sin\theta \quad \text{----} \; (2)$$

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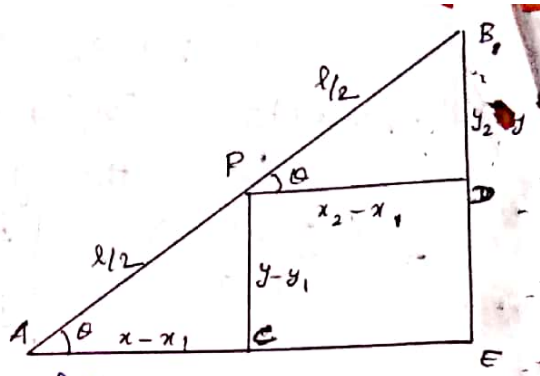
$$\cos\theta = \frac{PD}{PB} = \frac{x_2 - x}{l/2}$$

$$\Rightarrow x_2 = x + \frac{l}{2}\cos\theta \quad \text{------} (3)$$

$$\sin\theta = \frac{BD}{PB} = \frac{y_2 - y}{l/2}$$

$$\Rightarrow y_2 = y + \frac{l}{2}\sin\theta \quad \text{-----} (4)$$



$$\frac{\partial(x_1, y_1, x_2, y_2)}{\partial(x, y, \theta, q_4)} = \begin{vmatrix} \frac{\partial x_1}{\partial x} & \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_1}{\partial q_4} \\ \frac{\partial y_1}{\partial x} & \frac{\partial y_1}{\partial y} & \frac{\partial y_1}{\partial \theta} & \frac{\partial y_1}{\partial q_4} \\ \frac{\partial x_2}{\partial x} & \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial \theta} & \frac{\partial x_2}{\partial q_4} \\ \frac{\partial y_2}{\partial x} & \frac{\partial y_2}{\partial y} & \frac{\partial y_2}{\partial \theta} & \frac{\partial y_2}{\partial q_4} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & \frac{l}{2}\sin\theta & -\frac{1}{2}\cos\theta \\ 0 & 1 & -\frac{l}{2}\cos\theta & -\frac{1}{2}\sin\theta \\ 1 & 0 & -\frac{l}{2}\sin\theta & \frac{1}{2}\cos\theta \\ 0 & 1 & \frac{l}{2}\cos\theta & \frac{1}{2}\sin\theta \end{vmatrix}$$

$$= -l.$$

The generalised co-ordinates in terms of the co-ordinates are

$$(1)+(3) \Rightarrow x = \frac{x_1}{2} + \frac{x_2}{2}.$$

and $(2)+(4) \Rightarrow y = \frac{y_1}{2} + \frac{y_2}{2}.$

$$\therefore \theta = \tan^{-1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

$$\therefore q_4 = l = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2\right]^{1/2}.$$

Configuration force

The configuration of the system of n particles is specified by giving the values of $3N$ cartesian co-ordinates $x_1, x_2, \dots, x_{3N}$.

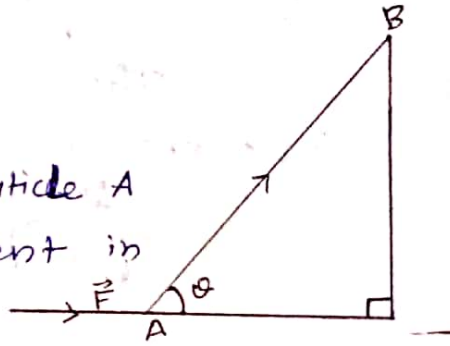
Suppose there are l- equations of the constraints of the form $f_i(x_1, x_2, \dots, x_{3N}, t) = \alpha_i$, $i = 3N-(l-1), 3N-(l-2), \dots, 3N$

Let $q_1, q_2, \ldots, q_n$ be the n generalised Co-ordinates given (3)
by the transformation equation

$$x_j = x_j(q_1, q_2, \ldots, q_n, t), \quad j = 1, 2, \ldots, 3N.$$

The system is specified by a set of n values of generalised Co-ordinates. There n numbers can be thought of as the Co-ordinates of the single point in an n-dimensional space and this n-dimensional space is configuration space.

If n generalised Co-ordinates be choosen so that they are independent, $3N - l = n$, the dimension of the configuration space.



Virtual Work

Work

Let F be the force acting on a particle A and giving it an infinite small displacement in the direction $\overline{AB}$.

Then we say that some work been done by the force. If we take δw as the work done by the force $\bar{F}$.

Then $\delta w = F \cdot \cos\theta \cdot \delta_s$, where δ_s is the displacement of the particle and θ is the angle between $\bar{F}$ and $\overline{AB}$.

Let us consider the system of N particles whose Configuration is given by $3N$ Cartesian Co-ordinates $x_1, x_2, \ldots, x_{3N}$.

Suppose the Co-ordinates move an infinitesimal displacement $\delta x_1, \delta x_2, \ldots, \delta x_{3N}$ which are virtual or imaginary.

Such a displace is called a "virtual displacement".

Suppose the force components $F_1, F_2, \ldots, F_{3N}$ are applied to the corresponding Co-ordinates in the +ve sense that the virtual of there Co-ordinates $(F_1, F_2, \ldots, F_{3N})$ are

$$\delta w = \sum_{i=1}^{3N} F_i \delta x_i = \sum_{i=1}^{3N} \bar{F}_i \cdot \delta_{xi}$$

Suppose the system is subject to k-holonomic Constraints given by

$$\phi_i(x_1, x_2, \dots, x_{3N}, t) = 0, \quad i = 1, 2, \dots, k.$$

The total differential is

$$d\phi_i = \sum_{j=1}^{3N} \frac{\partial \phi_i}{\partial x_j} dx_j + \frac{\partial \phi_i}{\partial t} dt = 0, \quad i = 1, 2, \dots, k$$

In a virtual displacement of the time t is held fixed.

$$\therefore \sum_{j=1}^{3N} \frac{\partial \phi_i}{\partial x_j} \delta x_j = 0$$

Mly If the system has l non-holonomic constraints of the form

$$\sum_{i=1}^{3N} a_{ji} dx_i + a_{jt} dt = 0, \quad j = 1, 2, \dots, l.$$

Then any virtual displacement conforming to these Constraint must have the δx's by the l equations

$$\sum_{i=1}^{3N} a_{ji} \delta x_i = 0, \quad j = 1, 2, \dots, l.$$

Workless Constraints

The workless Constraints can be defined as a bilateral Constraints for which the virtual work of the Corresponding constrained forces is zero, for any virtual displacement consist with the Constraints.

The virtual work of the Constraint force is

$$\delta w_c = \sum_{i=1}^{N} \bar{R}_i \cdot \delta \bar{r}_i$$

For a system having only workless constraints we have $\delta w_c = 0$.

$$\text{ie,} \quad \sum_{i=1}^{N} \bar{R}_i \cdot \delta \bar{r}_i = 0.$$

Principle of virtual work

The necessary and sufficient condition for the static equilibrium of an intially motionless scleronomic system which is subject to the workless Constraints, is that the virtual workdon

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by the applied force in moving through an arbitrary virtual displacement satisfying the constraints is zero. (4)

Proof

Necessary part

Let us consider a scleronomic system of N particles is ~~at~~ static equilibrium. Then for each particle we have $\bar{F}_i + \bar{R}_i = 0$, where $\bar{F}_i$ is the applied force and $\bar{R}_i$ is the constraint force at the i^{th} particle.

For any arbitrary virtual displacement $\delta \bar{r}_i$, consistent with the constraints, $\sum_{i=1}^{N} (\bar{F}_i + \bar{R}_i) \delta \bar{r}_i = 0$.

$$\Rightarrow \sum_{i=1}^{N} \bar{F}_i \cdot \delta \bar{r}_i + \sum_{i=1}^{N} \bar{R}_i \cdot \delta \bar{r}_i = 0 \quad \text{----} (1)$$

If we assume that all the constraints are workless, then $\sum_{i=1}^{N} \bar{R}_i \cdot \delta \bar{r}_i = 0$

$$(1) \Rightarrow \sum_{i=1}^{N} \bar{F}_i \delta \bar{r}_i = 0$$

ie, The system is static equilibrium the virtual workdone by the applied force is zero.

Sufficient part

Suppose that the virtual workdone by the applied force in any infinitsimal displacement is zero.

Let us assume that the system is not in equilibrium. Then, by newton's law of motion, one or more particles of the system will start to move in the direction of resultant forces acting on it.

Hence, the virtual workdone by the forces,

$$\delta w > 0.$$

ie, $\sum_{i=1}^{N} (\bar{F}_i + \bar{R}_i) \delta r_i > 0$

$$\Rightarrow \sum_{i=1}^{N} \bar{F}_i \cdot \delta \bar{r}_i + \sum_{i=1}^{N} \bar{R}_i \cdot \delta \bar{r}_i > 0 \quad \text{----} (2)$$

But the Constraints are workless $\sum_{i=1}^{N} \bar{R}_i \cdot \delta \bar{r}_i = 0$

② $\Rightarrow \sum_{i=1}^{N} \bar{F}_i \cdot \delta \bar{r}_i > 0.$

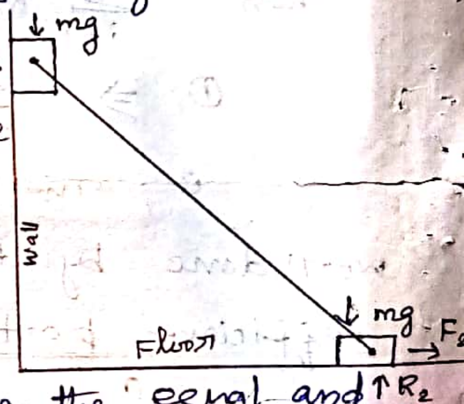
This contradicts to the given condition, may be the virtual workdone by the applied force is zero.

Hence, the system must be in equilibrium.

Example for workless constraints

Application of the principle of virtual work.

1. Two friction less blocks of equal mass 'm' are connected by a massless rigid rod. The system is constrained to move in the vertical plane as shown in the following figure. It is required to solve for the force F_2 acting on the lower block.



Solution

The external constrained forces are the reactions R_1 and R_2 of the wall and the floor respectively, which are acting perpendicular to the surfaces of contact.

The internal constrained forces are the equal and opposite compulsive forces in the rod.

Since, there are all workless constraints, the total virtual work of these constrained forces is zero.

The applied forces acting on the system are the gravitational force 'mg' acting vertically downwards on the blocks and the external force F_2 along the floor.

Let x_1, x_2 be the distances measure down the wall and along the floor respectively.

By the principle of virtual work, the condition for static equilibrium is

$mg\,\delta x_1 + F_2\,\delta x_2 = 0$ ------①

But δx_1 and δx_2 are related by the displacement components along the rod at the ends must be equal.

$$\therefore \sin\theta \cdot \delta x_1 = \cos\theta \cdot \delta x_2 \quad \text{------} \ (2)$$

$$\Rightarrow \delta x_2 = \tan\theta \, \delta x_1 .$$

$$(1) \Rightarrow mg\delta x_1 + F_2 \tan\theta \cdot \delta x_1 = 0 .$$

$$\delta x_1 (mg + F_2 \tan\theta) = 0 \Rightarrow mg = -F_2 \tan\theta .$$

$$\therefore F_2 = \frac{-mg}{\tan\theta} \Rightarrow F_2 = -mg\cot\theta .$$

Hence the force acting required to keep the initially motionless system in equilibrium is $-mg\cot\theta$ along the force.

<u>Note</u>

This is an example of scleronomic system of workless constraints.

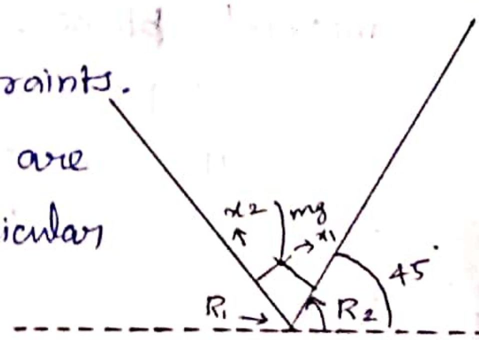
<u>Example</u>

A system consists of a cube of mass 'm' which is resting in static equilibrium at a corner formed by two frictionless, mutually perpendicular planes, as shown in the figure. Assume that any motion is restricted in the vertical plane.

<u>Solution</u>

This is example of unilateral constraints.

Here the external constrained forces are the reaction R_1, R_2 of the planes, perpendicular to the planes as shown in the fig.



The only applied force acting on the system is the gravitational force mg, acting vertically downwards on the plane.

Let x_1, x_2 be the distances measured along the two planes the unilateral constrained equations are

$$x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0 .$$

The components of mg along x_1 and x_2 are resp..

$F_1 = F_2 = -mg\cos 45° = -mg.\frac{1}{\sqrt{2}}$.

The virtual workdone by the applied force is

$\delta w = F_1\,\delta x_1 + F_2\,\delta x_2 \Rightarrow \delta w = \frac{-mg}{\sqrt{2}}(\delta x_1 + \delta x_2)$.

ie, the virtual work $\delta w \leq 0$, for any virtual displacement consistent by the unilateral constraints.

- D' Alembert's principle.

Let us consider a system of N particles.

Let $\bar{F_i}$ be the applied force and $\bar{R_i}$ be the constrained force acting at the i^{th} particle.

Then the equation of motion for each particle can be written as $\bar{F_i} + \bar{R_i} = M_i\ddot{\bar{r}}_i$, $i = 1,2,\ldots,N$

$\Rightarrow \bar{F_i} + \bar{R_i} - M_i\ddot{\bar{r}}_i = 0$ ------ ①.

The term $-M_i\ddot{\bar{r}}_i$ is known as the inertial (or) the reserved effective force acting on the i^{th} particle.

$\ddot{\bar{r}}_i$ is the acceleration of the i^{th} particle relative to an inertial plane.

∴ $\bar{F_i}$ and $\bar{R_i}$ are called the real or actual forces

Thus it can be stated that "the sum of all the force, real and inertial, acting on each particle of the system is zero".

This is known as "D' Alembert's principle".

It can also be stated as the reserved effective forces and the real forces together give statical equilibrium

∴ We can obtained the principle of virtual work on this forces system including the inertial forces.

The total virtual workdone by the forces in an arbitrary displacement $\delta \bar{r}_i$ is

$$\delta w = \sum_{i=1}^{N} (\bar{F}_i + \bar{R}_i - M_i \ddot{\bar{r}}_i)\, \delta \bar{r}_i = 0.$$

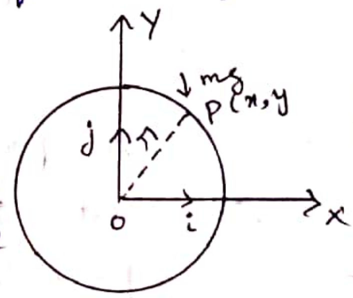
If $\bar{R}_i$'s are workless constraints then $\sum_{i=1}^{N} \bar{R}_i \cdot \delta \bar{r}_i = 0$.

$\therefore$ we get $\delta w = \sum_{i=1}^{N} (\bar{F}_i - M_i \ddot{\bar{r}}_i) \cdot \delta \bar{r}_i = 0.$

This equation is the Lagrange form of D'Alembert's principle and is one of the most important equation of classical dynamics.

Problems

1. A particle of mass 'm' can slide without friction on a fixed circular wire of radius 'r' which lies in a vertical plane. Using D'Alembert's principle and the equation of constraints, show that $y\ddot{x} = x\ddot{y} + gx$.



Proof

Let 'O' be the centre of the circular wire, ox be the horizontal line through 'O' in the plane of wire and oy be the vertical line through 'O'.

Let $p(x,y)$ be the cartesian co-ordinates of the position of the particle and $\vec{i}$ & $\vec{j}$ be the unit vectors along ox and oy respectively.

Now the applied force acting on a particle is a gravitational force 'mg', acting vertically downwards.

$\therefore \quad F = -mgj$

Also, $\bar{r} = xi + yj$ and $\ddot{\bar{r}} = \ddot{x}i + \ddot{y}j$

A virtual displacement consistent with the instantaneous constrained is $\delta \bar{r} = \delta x i + \delta y j$

$\therefore$ By the Lagrange's form of D'Alembert's principle,

$$\sum_{i=1}^{N} (\bar{F}_i - M_i \ddot{\bar{r}}_i) \cdot \delta \bar{r}_i = 0$$

we get, $[-Mgj - m(\ddot{x}i + \ddot{y}j)] \cdot (\delta x i + \delta y j) = 0.$

$\Rightarrow -m\ddot{x}\,\delta x - (mg + m\ddot{y})\,\delta y = 0.$

$$\Rightarrow \ddot{x}\,\delta x + (g+\ddot{y})\,\delta y = 0 \quad \text{-----} \; (1)$$

Now the particle slice over the circular wire with the frictio

The Constrained equation is

$$x^2 + y^2 = r^2$$

$$\therefore 2x\,\delta x + 2y\,\delta y = 0. \Rightarrow 2[x\,\delta x + y\,\delta y] = 0$$

$$\Rightarrow x\,\delta x + y\,\delta y = 0 \quad \text{------} \; (2)$$

From equations (1) and (2)

we have,

$$x\ddot{x}\,\delta x + x(g+\ddot{y})\,\delta y = 0$$

$$x\ddot{x}\,\delta x + \ddot{x}y\,\delta y = 0.$$

ie, $x(g+\ddot{y}) - \ddot{x}y\,\delta y = 0$

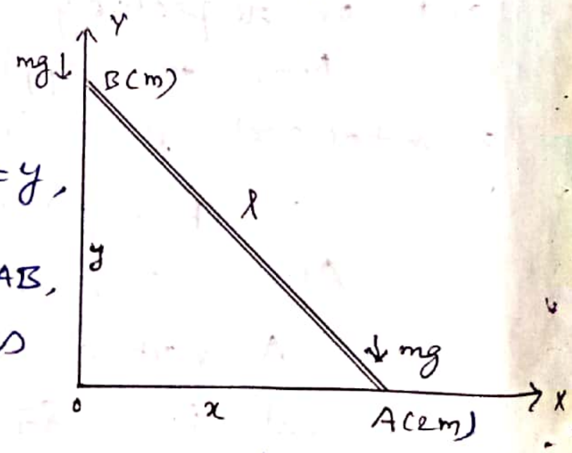
$$x(g+\ddot{y}) - \ddot{x}y = 0 \Rightarrow y\ddot{x} = \ddot{y}x + gx.$$

Hence the result.

2. A particle 'A' of mass '2m' and 'B' of mass 'm' are connected by a massless rod of length 'l'.

particle 'A' is constrained to move along the horizontal X_axis while particle 'B' can move only on the vertical axis.

What is the equation of Constraints relating x & y? Use D'-Alembert's principle to obtained the equation of motion $2y\ddot{x} + gx + x\ddot{y} = 0$.



Proof

Let 'O' be the origin, $OA = x$, $OB = y$, and $AB = l$.

From the right angled triangle OAB, we get the equations of Constraints as

$$x^2 + y^2 = l^2.$$

If we take i and j as the unit vectors along OX and OY respectively, the applied forces acting on the particles A and B are

$$\bar{F}_1 = -2mg\,j \quad \text{and} \quad F_2 = -mg\,j.$$

Also the acceleration of the two particles are

$$\ddot{\bar{r}}_1 = \ddot{x}\,i \quad \text{and} \quad \ddot{\bar{r}}_2 = \ddot{y}\,j$$

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Virtual displacements consistent with the Constraints are
$\delta\bar{r}_1 = \delta x i$ and $\delta\bar{r}_2 = -\delta y j$.

$\therefore$ The Lagrange's form of D'-Alembert's principle

$$\sum_{i=1}^{N} (\bar{F}_i - M_i \ddot{\bar{r}}_i)\,\delta\bar{r}_i = 0.$$

ie,
$$\sum_{i=1}^{2} (\bar{F}_i - M_i \ddot{\bar{r}}_i)\,\delta\bar{r}_i = 0.$$

$$\Rightarrow \left[(\bar{F}_1 - m_1 \ddot{\bar{r}}_1)\cdot\delta\bar{r}_1\right] + \left[(\bar{F}_2 - m_2\ddot{\bar{r}}_2)\cdot\delta\bar{r}_2\right] = 0.$$

$$\Rightarrow \left[(-2mgj - 2m\ddot{x}i)\cdot\delta x i\right] + \left[(-mgj - m\ddot{y}j)\cdot(-\delta y j)\right] = 0$$

$$\Rightarrow \left[-2m\ddot{x}\,\delta x\right] + (mg + m\ddot{y})\,\delta y = 0$$

$$\Rightarrow -2\ddot{x}\,\delta x + (g + \ddot{y})\,\delta y = 0 \quad \text{-----} \ (1)$$

Now the Constraint equation is

$$x^2 + y^2 = l^2$$

$$2x\,\delta x + 2y\,\delta y = 0 \Rightarrow x\,\delta x + y\,\delta y = 0$$

$$\Rightarrow \delta y = \frac{-x}{y}\,\delta x \quad \text{----} \ (2)$$

Substuting equation (2) in (1)

$$-2\ddot{x}\,\delta x + (g + \ddot{y})\left(\frac{-x}{y}\,\delta x\right) = 0$$

$$\Rightarrow 2\ddot{x}\,\delta x + (g + \ddot{y})\left(\frac{x}{y}\right)\cdot\delta x = 0$$

$$\Rightarrow \left[2\ddot{x} + (g + \ddot{y})\frac{x}{y}\right]\delta x = 0$$

$$\Rightarrow 2\ddot{x}y + \ddot{y}x + gx = 0$$

Hence the result.

Generalised forces

Consider a system of N particles acted upon by the forces with components $F_1, F_2, \ldots, F_{3N}$. where the configuration of the system is given by cartesian Co-ordinate $x_1, x_2, \ldots, x_{3N}$.

Then the virtual work of there force in a virtual displacement δx is given by

$$\delta w = \sum_{i=1}^{3N} F_i \, \delta x_i \quad ----(1)$$

Suppose that the 3N co-ordinates $x_1, x_2, \ldots, x_{3N}$ are related to n-generalised co-ordinates $q_1, q_2, \ldots, q_n$ by the transformation equation

$$x_i = x_i(q_1, q_2, \ldots, q_n, t), \quad i = 1, 2, \ldots, 3N.$$

Hence $$\delta x_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \delta q_j \quad (i = 1, 2, \ldots, 3N) \quad ---(2)$$

Here, $\delta t = 0$. as we consider the virtual displacement.

Substuting equation (2) in (1)

$$\delta w = \sum_{i=1}^{3N} F_i \left(\sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \delta q_j \right)$$

$$= \sum_{i=1}^{3N} \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} F_i \, \delta q_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{3N} \frac{\partial x_i}{\partial q_j} F_i \right) \delta q_j$$

$$\Rightarrow \delta w = \sum_{j=1}^{n} Q_j \, \delta q_j, \quad \text{where } Q_j = \sum_{i=1}^{3N} \frac{\partial x_i}{\partial q_j} \cdot F_i, \quad j = 1, 2, \ldots n.$$

This is called the "generalised force".

If the generalised co-ordinates are independent

$$\delta w = 0 \quad \Rightarrow \sum_{j=1}^{n} Q_j \, \delta q_j = 0.$$

$$\Rightarrow Q_j = 0 \quad \forall \, j = 1, 2, \ldots n.$$

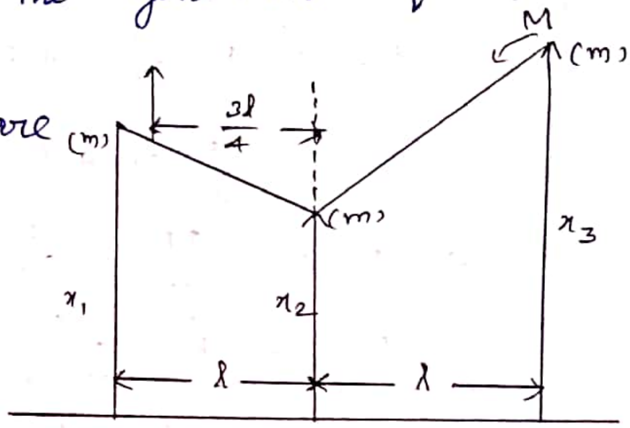
$\therefore$ If the configuration of an initially motionless holonomic system having the workless, fixed constraints is except interms of independent generalised co-ordinates.

Then the necessary and sufficient condition for static equilibrium is that all the generalised forces due to the applied forces be zero.

Problems

1. Three particles are connected by two rigid rods having a pointed between them to form the system as shown in the fig. A vertical force 'F' and a moment 'M' are applied as

shown the configuration of the system is given by ordinary Co-ordinates (x_1, x_2, x_3) (or) by the generalised Co-ordinates (8)
(q_1, q_2, q_3). $x_1 = q_1 + q_2 + \frac{1}{2} q_3$, $x_2 = q_1 - q_3$,
$x_3 = q_1 - q_2 + \frac{1}{2} q_3$ Find the generalised forces.



<u>Solution</u>

The transformation equations are

$$x_1 = q_1 + q_2 + \frac{1}{2} q_3$$

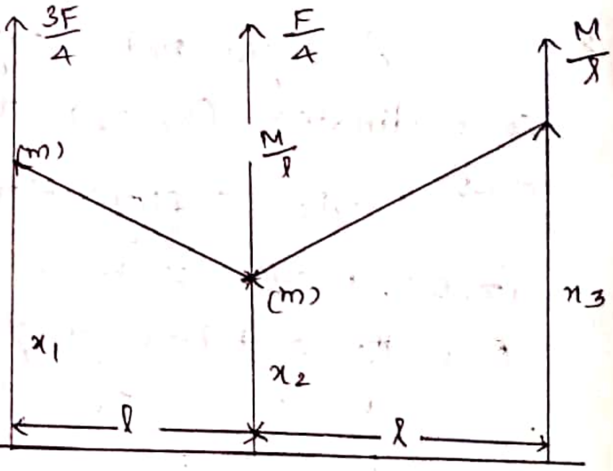
$$x_2 = q_1 - q_3$$

$$x_3 = q_1 - q_2 + \frac{1}{2} q_3.$$

$\therefore$ By Jacobian

$$J\left(\frac{x_1, x_2, x_3}{q_1, q_2, q_3}\right) = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \frac{\partial x_1}{\partial q_3} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_2}{\partial q_2} \\ \frac{\partial x_3}{\partial q_1} & \frac{\partial x_3}{\partial q_2} & \frac{\partial x_3}{\partial q_3} \end{vmatrix} = \begin{vmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 0 & -1 \\ 1 & -1 & \frac{1}{2} \end{vmatrix}$$

$$= -3 \neq 0$$



$\Rightarrow$ q's are independent.

The force F can be replaced by the forces $\frac{3F}{4}$ at x_1 and $\frac{F}{4}$ at x_2.

Also the momentum 'F' can be replaced by equal and opposite force of magnitude '$\frac{M}{l}$' acting on the direction of $-x_2$ & x_3.

Thus, we have the system of forces, say F_1 at x_1, F_2 at x_2 and F_3 at x_3.

where $F_1 = \frac{3F}{4}$, $F_2 = \frac{F}{4} - \frac{M}{l}$ and $F_3 = \frac{M}{l}$.

Now, the generalised force

$$Q_j = \sum_{j=1}^{3N} F_i \frac{\partial x_i}{\partial q_j}$$

$$\therefore Q_1 = F_1 \cdot \frac{\partial x_1}{\partial q_1} + F_2 \cdot \frac{\partial x_2}{\partial q_1} + F_3 \cdot \frac{\partial x_3}{\partial q_1}$$

$$= \frac{3F}{4}(1) + \left(\frac{F}{4} - \frac{M}{l}\right)(1) + \frac{M}{l}(1)$$

$$\Rightarrow Q_1 = F.$$

$$Q_2 = F_1 \frac{\partial x_1}{\partial q_2} + F_2 \cdot \frac{\partial x_2}{\partial q_2} + F_3 \cdot \frac{\partial x_3}{\partial q_2}$$

$$= \frac{3F}{4}(1) + \left(\frac{F}{4} - \frac{M}{l}\right)(0) + \frac{M}{l}(-1)$$

$$\Rightarrow Q_2 = \frac{3F}{4} - \frac{M}{l}$$

$$Q_3 = F_1 \cdot \frac{\partial x_1}{\partial q_3} + F_2 \cdot \frac{\partial x_2}{\partial q_3} + F_3 \cdot \frac{\partial x_3}{\partial q_3}$$

$$= \frac{3F}{4}\left(\frac{1}{2}\right) + \left(\frac{F}{4} - \frac{M}{l}\right)(-1) + \frac{M}{l}\left(\frac{1}{2}\right)$$

$$\Rightarrow Q_3 = \frac{F}{8} + \frac{3M}{2l}.$$

$\therefore$ The generalised forces are

F, $\frac{3F}{4} - \frac{M}{l}$ and $\frac{F}{8} + \frac{3M}{2l}$.

2. A rigid rod length 'l' undergoes small motion in which the co-ordinates (x_1, x_2) represent the vertical displacements of the ends. The configuration is also given by the generalised co-ordinates (z, θ). where z is a vertical displacement of the centre and θ is the rotation angle.

What are the transformation equation?

For the given applied forces of the ends, Evaluate the generalised forces Q_z + Q_θ.

Solution

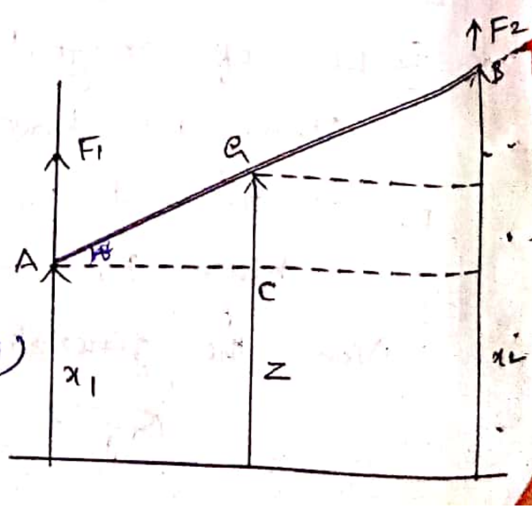
From the Δ^{le} ACG

$$\tan\theta = \frac{z - x_1}{l/2} \Rightarrow z - x_1 = \frac{l}{2}\tan\theta.$$

$$\Rightarrow x_1 = z - \frac{l}{2}\tan\theta$$

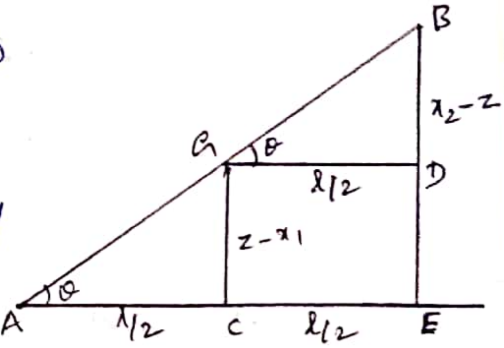
$$\simeq z - \frac{l\theta}{2} \quad (\theta \text{ is very small})$$

$\quad\quad$ ----- (1)



IIIly $\quad \frac{x_2 - z}{l/2} = \tan\theta \Rightarrow x_2 = z + \frac{l}{2}\tan\theta$

$$\simeq z + \frac{l}{2}\theta \quad \text{---- (2)}$$



From equations (1) and (2) are required transformation equations.

Now, the generalised forces are given by

$$Q_j = \sum_{i=1}^{3N} F_i \frac{\partial x_i}{\partial q_j}$$

$$Q_1 = F_1 \cdot \frac{\partial x_1}{\partial q_1} + F_2 \cdot \frac{\partial x_2}{\partial q_1} \Rightarrow Q_z = F_1 \cdot \frac{\partial x_1}{\partial z} + F_2 \cdot \frac{\partial x_2}{\partial z}$$

$$Q_2 = F_1 \cdot \frac{\partial x_1}{\partial q_2} + F_2 \cdot \frac{\partial x_2}{\partial q_2} \Rightarrow Q_\theta = F_1 \cdot \frac{\partial x_1}{\partial \theta} + F_2 \cdot \frac{\partial x_2}{\partial \theta}.$$

Hence $Q_z = F_1 + F_2$ and $Q_\theta = \frac{l}{2}(F_2 - F_1)$

Energy and momentum

Potential energy

Consider a system of N particles. Let A be the standard configuration and B be any other configuration.

Let us take the system from B to A and the workdone by all the forces acting on the system be denoted by 'W', during the process.

If the workdone is independent of the way, then the system is said to be conservative and the workdone W is called the potential energy of the system at the configuration B and is denoted by V.

Let us consider a single particle whose position is given, by the cartesian co-ordinates (x, y, z).

Suppose the total force F acting on the particles has components $F_x = \frac{-\partial V}{\partial x}$, $F_y = \frac{-\partial V}{\partial y}$, $F_z = \frac{-\partial V}{\partial z}$

where V is the single value scalar function is called the potential energy function, then F is called a conservative force.

Now, the workdone by $\bar{F}$ is an infinitesimal displacement $d\bar{r}$ is

$$dw = \bar{F} \cdot d\bar{r}$$
$$= (F_x i + F_y j + F_z k) \cdot (dx i + dy j + dz k)$$
$$= F_x dx + F_y dy + F_z dz.$$
$$= -\frac{\partial V}{\partial x} dx - \frac{\partial V}{\partial y} dy - \frac{\partial V}{\partial z} dz$$
$$\Rightarrow dw = -dV(x, y, z)$$

$\therefore$ The workdone by the vector F as the particle moves over certain path between a and b is

$$W = \int_A^B \bar{F} \cdot d\bar{r} = -\int_A^B dV.$$
$$= -(V_B - V_A) \Rightarrow W = V_A - V_B.$$

WKT, The potential energy is a function of position only.
$\therefore$ The workdone on the particle depends upon the initial and final positions, but is independent of any path connecting these points.

Note

When A and B coincide $W = 0$.

ie, The workdone in moving around any closed path is zero.

ie, $\oint \bar{F} \cdot d\bar{r} = 0$, for any conservative force F.

Book work

Thm (The principle of work and kinetic energy)

The increase in the kinetic energy of an particle, as it moves from one arbitrary point to another, is equal to the workdone by the forces acting on a particle during the given interval.

Proof

Let a particle of mass 'm' move with velocity v relative to an inertial frame of reference.

Then the kinetic energy T of the particle is defined as (b)

$$T = \frac{1}{2} m v^2.$$

Now, the workdone by F as a particle moves over certain path between the points A and B is

$$W = \int_A^B \bar{F} \cdot d\bar{r} \quad , \text{where} \quad \bar{F} = m\ddot{\bar{r}}$$

$$= \int_A^B m\ddot{\bar{r}}\, d\bar{r} \Rightarrow W = m\int_A^B \ddot{\bar{r}}\, d\bar{r} \quad \text{-----} (1)$$

Now, Consider $\frac{d}{dt}(\dot{\bar{r}}.\dot{\bar{r}})dt = \left[\frac{d\dot{\bar{r}}}{dt}\cdot\dot{\bar{r}} + \dot{\bar{r}}\frac{d\dot{\bar{r}}}{dt}\right]dt$

$$= [2\dot{\bar{r}}.\ddot{\bar{r}}]\,dt = 2\left[\frac{d\bar{r}}{dt}.\ddot{\bar{r}}\right]dt$$

$$\Rightarrow \frac{d}{dt}(\dot{\bar{r}}.\dot{\bar{r}})dt = 2\left[\frac{d\bar{r}}{dt}.\ddot{\bar{r}}\right]dt \quad \text{----} (2)$$

$$\Rightarrow \frac{d}{dt}(\dot{\bar{r}}.\dot{\bar{r}})dt = 2(\ddot{\bar{r}}.d\bar{r}) \quad \text{---}$$

$$\Rightarrow \frac{1}{2}\left[\frac{d}{dt}(\dot{\bar{r}}.\dot{\bar{r}})\right]dt = \ddot{\bar{r}}.d\bar{r} \quad \text{-----} (3)$$

Substuting equation (3) in (1)

$$W = m\int_A^B \frac{1}{2}\left[\frac{d}{dt}(\dot{\bar{r}}.\dot{\bar{r}})\right]dt = \frac{m}{2}\int_A^B d(v^2)$$

$$= \frac{1}{2}m(v^2)_A^B = \frac{1}{2}m(v_B^2 - v_A^2)$$

$$= \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 \Rightarrow W = T_B - T_A.$$

$\Rightarrow$ The increase in kinetic energy is equal to the workdone by the forces.

Conservation of energy

If a particle is acted upon by conservative forces only, then we have, workdone by the forces is $W = V_A - V_B$.

Also, we have $W = T_B - T_A$.

Hence $T_B - T_A = V_A - V_B$.

$$T_B + V_B = T_A + V_A = E \quad (\text{say})$$

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Since the points A and B are arbitrary, we conclude that E is a constant and this E is called total mechanical energy.

Thus, the sum of kinetic and potential energy is a constant which is known as the principle of conservation of energy.

Note

Let us now consider a system of N particles, whose configuration is specified by 3N cartesian co-ordinates $x_1, x_2, \ldots, x_{3N}$.

If the only forces which do work on the system during its motion are given by

$$F_i = -\frac{\partial V}{\partial x_i} \quad \text{-----} \quad (1)$$

where the potential energy $V(x_1, x_2, \ldots, x_{3N})$ is a single value function of position only, then the total energy E is connected.

Suppose a configuration is also specified by a generalised co-ordinates $q_1, q_2, \ldots, q_n$ by a transformation equation

$$x_i = x_i(q_1, q_2, \ldots, q_n), \quad i = 1, 2, \ldots, 3N.$$

Then the generalised force

$$Q_j = \sum_{i=1}^{3N} F_i \frac{\partial x_i}{\partial q_j}, \quad j = 1, 2, \ldots, n \quad \text{----} \quad (2)$$

Substuting equation (1) in (2)

$$Q_j = \sum_{i=1}^{3N} \left(\frac{-\partial V}{\partial x_i}\right)\left(\frac{\partial x_i}{\partial q_j}\right)$$

ie, $$Q_j = \frac{-\partial V}{\partial q_j}, \quad j = 1, 2, \ldots, n.$$

where the potential energy V is now expressed as the function of generalised co-ordinates.

Each generalised force Q_j may be considered to be component of a generalised force $\overline{Q}$, in an n-dimensional configuration.

If no other generalised force work on the system, then we write

$$W = \int_A^B \overline{Q} \cdot d\overline{q} = -\int_A^B dV = V_A - V_B$$

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Where the points A and B are now considered as end (11)
points of the path in q-space.

Thus, in this case also, W is independent of the path and the total energy is preserved.

Equilibrium and stability

Let us now show that an equilibrium configuration of a conservative holonomic system with workless fixed constraints must occur at a position where the potential energy has a stationary value.

Consider, a system of n particles whose configuration is specified by $3N$ cartesian co-ordinates $x_1, x_2, \ldots, x_{3N}$.

Let the applied forces be conservative and we obtained from the potential energy function $V(x_1, x_2, \ldots, x_{3N})$.

Then the virtual work δW of these forces in a virtual displacement δx is given by

$$\delta W = \sum_{i=1}^{3N} F_i \,\delta x_i = \sum_{i=1}^{3N} -\frac{\partial V}{\partial x_i}\,\delta x_i$$

ie, $\delta W = -\delta V$.

By the principle of virtual work a necessary and sufficient condition for the static equilibrium of the system is

$$\delta W = 0$$

ie, $\delta V = 0$.

For every virtual displacement consistent with the constraints.

Let V is expressed as the function of generalised co-ordinates $q_1, q_2, \ldots, q_n$.

Then we get, $\delta V = \sum_{j=1}^{n} \frac{\partial V}{\partial q_i}\,\delta q_j = 0$.

$\therefore$ For a holonomic system in the independent q's.

we get, $\frac{\partial V}{\partial q_j} = 0$, $j = 1, 2, \ldots, n$.

Let $\Delta V = V - V_0$ is the change in the potential energys. From its value at equilibrium

(1) If $\Delta V > 0$ for every virtual displacement having atleast one of the δq's non-zero, then the reference position is one of

the minimum potential energy corresponding to the stable equilibrium.

(2) If $\Delta V < 0$ for any one virtual displacement, then the equilibrium is unstable.

(3) If $\Delta V = 0$ for some virtual displacement the equilibrium is called a neutral stability. It is also considered as a form of instability.

Kinetic energy of system

Konig's thm

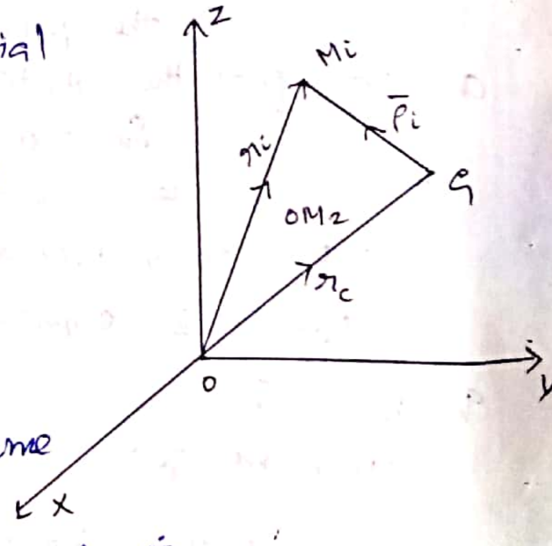
Statement

The total kinetic energy of the system is equal to the sum of

(1) the kinetic energy due to a particle having a mass equal to the total mass of a system and moving with a velocity of the centre of the mass. and

(2) the kinetic energy due to the motion of the system relative to its centre of the mass.

Proof



Let 'O' be the origin in an inertial frame of reference.

Consider a system of N particles and Let $\bar{r}_i$ be the position of i^{th} particle relative to 'O'.

Then the kinetic energy T of the system with respect to the inertial frame is given by

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\bar{r}}_i^2 \text{, where } \dot{\bar{r}}_i^2 = \dot{\bar{r}}_1 \cdot \dot{\bar{r}}_1$$

Let G be a position of the centre of the mass and let its position vector with respect to 'O' be R_p.

Let ρ_i be the position of the ith particle with respect to G.

Then, we have $\vec{r}_i = \vec{r}_c + \vec{\rho}_i$

i.e, $\dot{\vec{r}}_i = \dot{\vec{r}}_c + \dot{\vec{\rho}}_i$

$\therefore$ The total kinetic energy

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} = \sum_{i=1}^{N} \frac{1}{2} m_i (\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i)$$

$$= \frac{1}{2} \sum_{i=1}^{N} m_i \left[(\dot{\vec{r}}_c + \dot{\vec{\rho}}_i) \cdot (\dot{\vec{r}}_c + \dot{\vec{\rho}}_i) \right]$$

$$= \frac{1}{2} \sum_{i=1}^{N} m_i \left[\dot{\vec{r}}_c^{\,2} + 2 \dot{\vec{r}}_c \dot{\vec{\rho}}_i + \dot{\vec{\rho}}_i^{\,2} \right]$$

$$\Rightarrow T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{r}}_c^{\,2} + \sum_{i=1}^{N} m_i \dot{\vec{r}}_c \dot{\vec{\rho}}_i + \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{\rho}}_i^{\,2} \quad \text{-----} (1)$$

Since ρ_i is measure from the centre of the mass of system.

$$\sum_{i=1}^{N} m_i \dot{\vec{r}}_c \dot{\vec{\rho}}_i = 0 \quad \text{-----} (2)$$

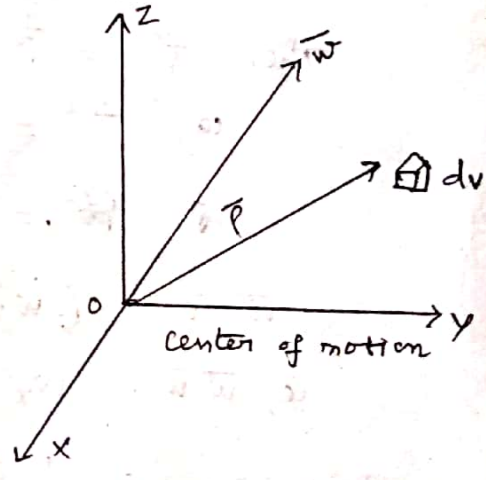
Substuting equation (2) in (1)

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{r}}_c^{\,2} + \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{\rho}}_i^{\,2} \quad \text{-----} (3)$$

Let us now consider a rigid body is general motion. Let dv be a small volume having density ρ'.

Each element of body will be translating and rotating.

Hence, Considering each element as a particle of very small or infinitesimal mass.



we have,

$$\sum_{i=1}^{N} m_i = \int_V \rho' \, dv.$$

$$(3) \Rightarrow \quad T = \frac{1}{2} m \dot{\vec{r}}_c^{\,2} + \frac{1}{2} \int_V \rho' \dot{\vec{\rho}}^{\,2} dv \quad \text{-----} (4), \quad \text{where } m = \sum_{i=1}^{N} m_i$$

Here, $\frac{1}{2} m \dot{\vec{r}}_c^{\,2}$ is called the translational kinetic energy and $\frac{1}{2} \int_V \rho' \cdot \dot{\vec{\rho}}^{\,2} dv$ is called the rotational kinetic energy.

Thm

To prove that $T_{rot} = \frac{1}{2} w^T I w$ (or) $T_{rot} = \frac{1}{2} I w^2$.

Proof Suppose that the body is rotated in the angular velocity w.

WKT, $\dot{\bar{\rho}} = \bar{w} \times \bar{\rho}$

Now, $(\dot{\bar{\rho}})^2 = \dot{\bar{\rho}} \cdot \dot{\bar{\rho}} = (\bar{w} \times \bar{\rho}) \cdot \dot{\bar{\rho}} \Rightarrow (\dot{\bar{\rho}})^2 = \bar{w} \cdot (\bar{\rho} \times \dot{\bar{\rho}})$

Now, rotational Kinetic energy $= T_{rot}$

$$= \frac{1}{2} \int_V \rho' \cdot \dot{\bar{\rho}}^2 dv = \frac{1}{2} \bar{w} \int_V \rho' (\bar{\rho} \times \dot{\bar{\rho}}) \, dv.$$

$$= \frac{1}{2} \bar{w} \cdot \int_V \rho' [\bar{\rho} \times (\bar{w} \times \bar{\rho})] \, dv$$

$$= \frac{1}{2} \bar{w} \int_V \rho' [(\bar{\rho} \cdot \bar{\rho}) \bar{w} - (\bar{\rho} \cdot \bar{w}) \bar{\rho}] \, dv.$$

$$T_{rot} = \frac{1}{2} \bar{w} \int_V \rho' [\bar{\rho}^2 \bar{w} - (\bar{\rho} \cdot \bar{w}) \bar{\rho}] \, dv \quad \text{-----} \; (1)$$

If $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along the axis of cartesian Co-ordinate system, rotating with the body having the origin at the centre of mass.

$$\bar{\rho} = x\vec{i} + y\vec{j} + z\vec{k} \quad \text{and} \quad \bar{w} = w_x \vec{i} + w_y \vec{j} + w_z \vec{k}$$

Then $\bar{\rho} \cdot \bar{w} = (x\vec{i} + y\vec{j} + z\vec{k}) \cdot (w_x \vec{i} + w_y \vec{j} + w_z \vec{k}) = xw_x + yw_y + zw_z$.

ie, $\bar{\rho}^2 = \bar{\rho} \cdot \bar{\rho} = (x\vec{i} + y\vec{j} + z\vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k})$

$\bar{\rho}^2 = x^2 + y^2 + z^2$. and

$$\bar{w} \cdot \bar{w} = (w_x \vec{i} + w_y \vec{j} + w_z \vec{k}) \cdot (w_x \vec{i} + w_y \vec{j} + w_z \vec{k})$$

ie, $\bar{w} \cdot \bar{w} = w_x^2 + w_y^2 + w_z^2$

$$(1) \Rightarrow T_{rot} = \frac{1}{2} \bar{w} \int_V \rho' [(x^2 + y^2 + z^2) \cdot \bar{w} - (xw_x + yw_y + zw_z) \cdot \bar{\rho}] \, dv$$

$$= \frac{1}{2} \int_V \rho' [(x^2 + y^2 + z^2)(\bar{w} \cdot \bar{w}) - (xw_x + yw_y + zw_z) \cdot (\bar{w} \cdot \bar{\rho})] \, dv.$$

$$= \frac{1}{2} \int_V \rho' [(x^2 + y^2 + z^2)(w_x^2 + w_y^2 + w_z^2) - (xw_x + yw_y + zw_z)(xw_x + yw_y + zw_z)] \, dv$$

$$= \frac{1}{2} \int_V \rho' [(x^2 + y^2 + z^2)(w_x^2 + w_y^2 + w_z^2) - (xw_x + yw_y + zw_z)^2] \, dv$$

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i.e., $T_{rot} = \frac{1}{2}\int_V \rho' \left[(y^2+z^2)\omega_x^2 + (z^2+x^2)\omega_y^2 + (x^2+y^2)\omega_z^2 \right.$
$\left. - 2xy\omega_x\omega_y - 2yz\omega_y\omega_z - 2zx\omega_z\omega_x \right] dv$ ----- (2)

Now, the moment of inertia are

$$I_{xx} = \int_V \rho'(y^2+z^2)\,dv$$
$$I_{yy} = \int_V \rho'(z^2+x^2)\,dv$$
$$I_{zz} = \int_V \rho'(x^2+y^2)\,dv$$

and the product of inertia are

$$I_{xy} = I_{yx} = -\int_V \rho' xy\,dv$$
$$I_{yz} = I_{zy} = -\int_V \rho' yz\,dv$$
$$I_{zx} = I_{xz} = -\int_V \rho' zx\,dv.$$

(2) $\Rightarrow$ $T_{rot} = \frac{1}{2}I_{xx}\omega_x^2 + \frac{1}{2}I_{yy}\omega_y^2 + \frac{1}{2}I_{zz}\omega_z^2 + I_{xy}\omega_x\omega_y$
$+ I_{yz}\omega_y\omega_z + I_{zx}\omega_z\omega_x$ ---- (3)

$\therefore T_{rot} = \frac{1}{2}\sum_i\sum_j I_{ij}\,\omega_i\,\omega_j$ ------ (4)

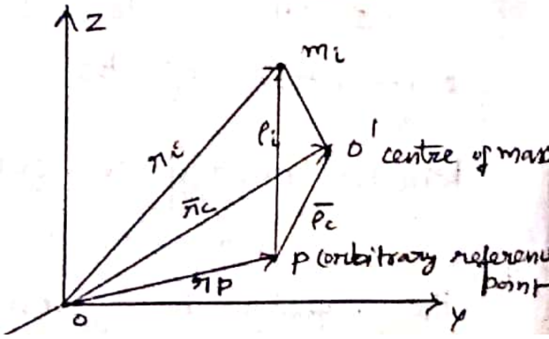
Using the matrix notation the above equ (3) can be written as

$$T_{rot} = \frac{1}{2}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}^T \begin{bmatrix} I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} \quad \text{----- (5)}$$

$T_{rot} = \frac{1}{2}\omega^T I \omega$ (OR) $T_{rot} = \frac{1}{2}I\omega^2$ ------ (6)

Hence the result.

Kinetic energy in terms of the motion with respect to an ordinary reference point



Let us consider a system of N particles with position vectors $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_N$ with respect to the origin 'o' of a fixed inertial frame reference.

Let P be an arbitrary reference point.

Let $OP = \bar{r}_p$.

From the figure $\bar{r}_i = \bar{r}_p + \bar{\rho}_i \Rightarrow \dot{\bar{r}}_i = \dot{\bar{r}}_p + \dot{\bar{\rho}}_i$

$$\dot{\bar{r}}_i \cdot \dot{\bar{r}}_i = (\dot{\bar{r}}_p + \dot{\bar{\rho}}_i).(\dot{\bar{r}}_p + \dot{\bar{\rho}}_i)$$

$$(\dot{\bar{r}}_i)^2 = \dot{\bar{r}}_i \cdot \dot{\bar{r}}_i = \dot{\bar{r}}_p^{\,2} + 2\,\dot{\bar{r}}_p \cdot \dot{\bar{\rho}}_i + \dot{\bar{\rho}}_i^{\,2}$$

Now, the kinetic energy of the system

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\bar{r}}_i \cdot \dot{\bar{r}}_i)$$

$$= \frac{1}{2}\sum_{i=1}^{N} m_i\left[\dot{\bar{r}}_p^{\,2} + 2\dot{\bar{r}}_p \cdot \dot{\bar{\rho}}_i + \dot{\bar{\rho}}_i^{\,2}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\bar{r}}_p^{\,2} + \frac{1}{2}\sum_{i=1}^{N} m_i\, 2\dot{\bar{r}}_p \cdot \dot{\bar{\rho}}_i + \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\bar{\rho}}_i^{\,2}$$

ie, $T = \frac{1}{2} m(\dot{\bar{r}}_p)^2 + \dot{\bar{r}}_p \sum_{i=1}^{N} m_i \dot{\bar{\rho}}_i + \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\bar{\rho}}_i^{\,2})$ ------ ①

But the position vector O with respect to P is $\bar{\rho}_c$.

$$\bar{\rho}_c = \frac{1}{m}\sum_{i=1}^{N} m_i \bar{\rho}_i \Rightarrow m\bar{\rho}_c = \sum_{i=1}^{N} m_i \bar{\rho}_i$$

$$\Rightarrow m\dot{\bar{\rho}}_c = \sum_{i=1}^{N} m_i \dot{\bar{\rho}}_i \quad \text{------ ②}$$

Substuting equation ② in ①

$$T = \frac{1}{2} m(\dot{\bar{r}}_p)^2 + \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\bar{\rho}}_i)^2 + \dot{\bar{r}}_p \cdot m\dot{\bar{\rho}}_c \quad \text{------ ③}$$

Thus the kinetic energy of the system consists of three parts, namely,

(1) The kinetic energy due to a particle having a mass m and moving with the reference point P.

(2) The kinetic energy of the system due to its motion relative to P and

(3) The scalar product of the velocity of the P and the linear momentum of the system relative to P.

Kinetic energy in generalised Co-ordinates

Consider a system of N particles. Let the configuration of the system be given by $3N$ cartesian co-ordinates $x_1, x_2, \ldots, x_{3N}$

relative to an inertial frame. (14)

Let m_i be the mass of the i^{th} particle and let $m_i = m_{i+1} = m_{i+2}$ for $i = 1, 2, \dots, N$.

Let the total Kinetic of the system in cartesian co-ordinate

$$T = \frac{1}{2} \sum_{i=1}^{3N} m_i (\dot{x}_i)^2 \quad \text{------} (1)$$

Let the transformation equations relating the cartesian co-ordinates to the generalised co-ordinates $q_1, q_2, \dots, q_n$ be

$$x_i = x_i (q_1, q_2, \dots, q_n, t), \quad i = 1, 2, \dots, 3N.$$

Let us assume that these functions are twice differential with respect to q's and t.

Then $$\dot{x}_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \dot{q}_j + \frac{\partial x_i}{\partial t} \quad \text{------} (2)$$

$\Rightarrow$ $\dot{x}_i$'s is a function in q's, $\dot{q}$'s and t and it is linear in $\dot{q}$.

$$\therefore T = \frac{1}{2} \sum_{i=1}^{3N} m_i \left(\sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \dot{q}_j + \frac{\partial x_i}{\partial t} \right)^2$$

We can write $T = T_2 + T_1 + T_0$.

where (1) $T_2 = \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{n} l_{kj} \dot{q}_k \dot{q}_j$ with $l_{kj} = l_{jk} = \sum_{i=1}^{3N} m_i \frac{\partial x_i}{\partial q_k} \frac{\partial x_i}{\partial q_j}$

(2) $T_1 = \sum_{k=1}^{n} a_k \dot{q}_k$ with $a_k = \sum_{i=1}^{3N} m_i \frac{\partial x_i}{\partial q_k} \cdot \frac{\partial x_i}{\partial t}$.

and (3) $T_0 = \frac{1}{2} \sum_{i=1}^{3N} m_i \left(\frac{\partial x_i}{\partial t} \right)^2$

Thus $T = 0$ only if the system is motionless otherwise $T > 0$.

Note

(1) For a system in which any moving constraints or moving reference frames are held fixed.

we have $\frac{\partial x_i}{\partial t} = 0$.

In this case the total Kinetic energy $T = T_2$.

(2) T_1 and T_2 are non-zero only for the case of holonomic system and for as scleronomic system T is a homogenous quadratic function of the $\dot{q}$'s.

Angular momentum

The linear momentum of the particle of mass 'm' is defined as

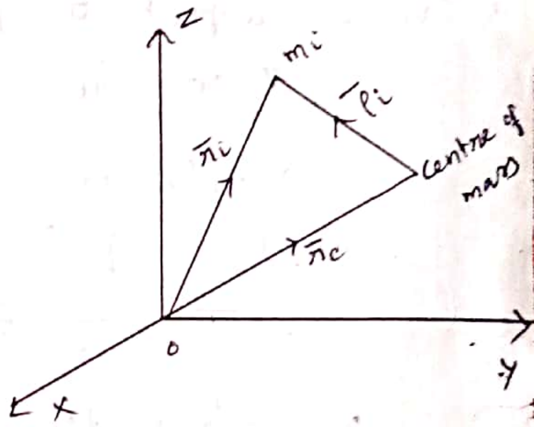
$$p = m \cdot \frac{d\bar{r}}{dt} \Rightarrow p = m\dot{\bar{r}}$$

The vector $\bar{r}$ is the position vector of the particle with respect to an origin 'O'.

Let us consider a system of N particles of mass $m_1, m_2, \ldots, m_N$ and position vectors $\bar{r}_1, \bar{r}_2 \cdots \bar{r}_N$ respectively with respect to a fixed point O.

Then the total angular momentum of the system about O is defined as

$$\bar{H} = \sum_{i=1}^{N} \bar{r}_i \times m\dot{\bar{r}}_i \quad \text{----} (1)$$



If $\bar{r}_c$ is the position of the centre of mass with respect to O and ρ_i is the position of the particle of mass m_i with reference to centre of mass.

then $\bar{r}_i = \bar{r}_c + \bar{\rho}_i \Rightarrow \dot{\bar{r}}_i = \dot{\bar{r}}_c + \dot{\bar{\rho}}_i$

Hence,

$$\bar{H} = \sum_{i=1}^{N} (\bar{r}_c + \bar{\rho}_i) \times m(\dot{\bar{r}}_c + \dot{\bar{\rho}}_i)$$

$$= \sum_{i=1}^{N} \bar{r}_c \times m_i \dot{\bar{r}}_c + \sum_{i=1}^{N} \bar{r}_c \times m_i \dot{\bar{\rho}}_i + \sum_{i=1}^{N} \bar{\rho}_i \times m_i \dot{\bar{r}}_c + \sum_{i=1}^{N} \bar{\rho}_i \times m_i \dot{\bar{\rho}}_i$$

$$\Rightarrow \bar{H} = \bar{r}_c \times m\dot{\bar{r}}_c + \bar{r}_c \times \sum_{i=1}^{N} m_i \dot{\bar{\rho}}_i + \sum_{i=1}^{N} m_i \bar{\rho}_i \times \dot{\bar{r}}_c + \sum_{i=1}^{N} \bar{\rho}_i \times m_i \dot{\bar{\rho}}_i$$

Here $\sum_{i=1}^{N} m_i \bar{\rho}_i = 0 \Rightarrow \sum_{i=1}^{N} m_i \dot{\bar{\rho}}_i = 0.$

$$\therefore \bar{H} = \bar{r}_c \times m\dot{\bar{r}}_c + \sum_{i=1}^{N} \bar{\rho}_i \times m_i \dot{\bar{\rho}}_i \quad \text{-----} (2)$$

Thus the angular momentum of a system of particles of total mass m about a fixed point O is equal to the sum of

(i) angular momentum about O of single particle of mass m moving with the centre of mass and

(ii) angular momentum of the system about the centre of mass.

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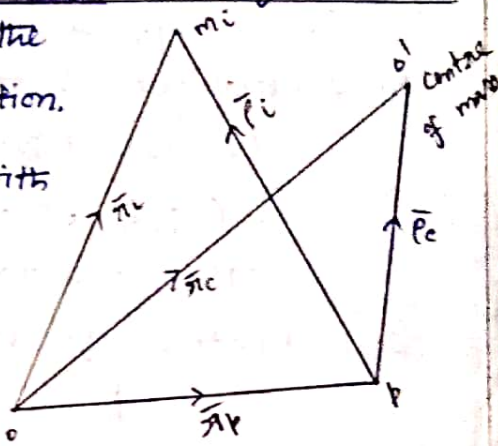
Angular momentum in the case of rigid body arbitrary motion (15)

10m

If we apply the above result to the case of the rigid body in arbitrary motion. We find the total angular momentum with respect to a fixed point o is

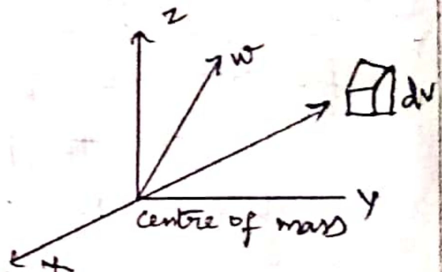


$$\bar{H} = \bar{r}_c \times m\dot{\bar{r}}_c + \bar{H}_\rho$$

$$\left[\because \bar{H} = \bar{r}_c \times m\dot{\bar{r}}_c + \sum_{i=1}^{N} \bar{\rho}_i \times m_i \dot{\bar{\rho}}_i\right]$$

$= \cancel{\bar{r}_c \times m\dot{\bar{r}}_c \times \dots}$

Where (i) $\bar{H}_\rho = \int_V \rho' \bar{\rho} \times (\bar{w} \times \bar{\rho})\, dv.$



(ii) ρ' is the density of the element volume dv.

(iii) w is the angular velocity of the body.

If P is an arbitrary point with position vector $\bar{r}_p$, then the angular momentum about P is

$$\bar{H}_p = \sum_{i=1}^{N} \bar{\rho}_i \times m_i \dot{\bar{\rho}}_i ,$$ where ρ_i is the position vector of the i^{th} particle with respect to P.

But $\bar{\rho}_i + \bar{r}_p = \bar{r}_i$

$\Rightarrow \bar{\rho}_i = \bar{r}_i - \bar{r}_p$, where $\bar{r}_p = \bar{r}_c - \bar{\rho}_c$

$= \bar{r}_i - (\bar{r}_c - \bar{\rho}_c) \Rightarrow \bar{\rho}_i = \bar{r}_i - \bar{r}_c + \bar{\rho}_c$.

$$\therefore \bar{H}_p = \sum_{i=1}^{N} (\bar{r}_i - \bar{r}_c + \bar{\rho}_c) \times m_i (\dot{\bar{r}}_i - \dot{\bar{r}}_c + \dot{\bar{\rho}}_c)$$

$$\Rightarrow \bar{H}_p = \sum_{i=1}^{N} (\bar{r}_i \times m_i \dot{\bar{r}}_i - \bar{r}_i \times m_i \dot{\bar{r}}_c + \bar{r}_i \times m_i \dot{\bar{\rho}}_c)$$

$$- \sum_{i=1}^{N} (\bar{r}_c \times m_i \dot{\bar{r}}_i - \bar{r}_c \times m_i \dot{\bar{r}}_c + \bar{r}_c \times m_i \dot{\bar{\rho}}_c) + \sum_{i=1}^{N} (\bar{\rho}_c \times m \dot{\bar{r}}_i - \bar{\rho}_c \times m_i \dot{\bar{r}}_i + \bar{\rho}_c \times m_i \dot{\bar{\rho}}_c) \quad \text{------} (1)$$

But $\bar{r}_c = \frac{1}{m} \sum_{i=1}^{N} m_i \bar{r}_i \Rightarrow \sum_{i=1}^{N} m_i \bar{r}_i = m\bar{r}_c$ & $\sum_{i=1}^{N} m_i \dot{\bar{r}}_i = m\dot{\bar{r}}_c$

$$(1) \Rightarrow \bar{H}_p = \sum_{i=1}^{N} \bar{r}_i \times m_i \dot{\bar{r}}_i - m\bar{r}_c \times \dot{\bar{r}}_c + m\bar{r}_c \times \dot{\bar{\rho}}_c - \bar{r}_c \times m\dot{\bar{r}}_c$$
$$+ \bar{r}_c \times m\dot{\bar{r}}_c - \bar{r}_c \times m\dot{\bar{\rho}}_c + \bar{\rho}_c \times m\dot{\bar{r}}_c - \bar{\rho}_c \times m\dot{\bar{r}}_c + \bar{\rho}_c \times m\dot{\bar{\rho}}_c$$

ie, $\bar{H}_p = \sum_{i=1}^{N} \bar{r}_i \times m_i \dot{\bar{r}}_i - \bar{r}_c \times m\dot{\bar{r}}_c + \bar{\rho}_c \times m\dot{\bar{\rho}}_c$

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Generalised momentum

Suppose configuration of the system is described by n - generalised co-ordinates $q_1, q_2, \dots, q_n$.

Let us define the lagrange's function $L(q,\dot{q},t)$ as

$$L(q,\dot{q},t) = T - V \quad \text{-----} ①$$

The generalised momentum p_i associated with the generalised co-ordinates q_i is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \text{------} ②$$

Since L is atmost quadratic in $\dot{q}$'s, p_i is a linear function of $\dot{q}$'s.

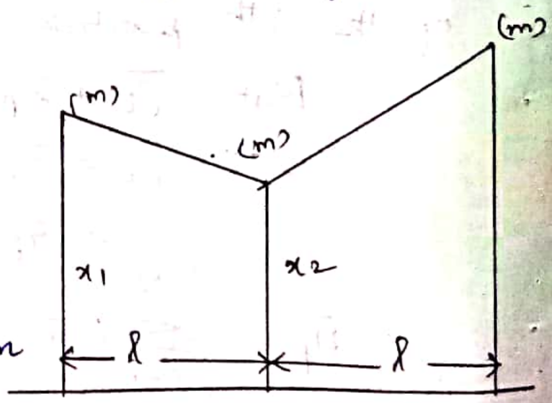
If the potential energy is of the form $V(q,t)$.

Then $p_i = \dfrac{\partial L}{\partial \dot{q}_i} = \dfrac{\partial T}{\partial \dot{q}_i}$ $\left[\because \dfrac{\partial V}{\partial \dot{q}_i} = 0 \Rightarrow p_i = \dfrac{\partial T}{\partial \dot{q}_i}\right.$

Example

Three particles are connected by two rigid road having a pointed between them, to form the system as shown the following figure. Find the expression for the kinetic energy and generalised momentum.

Solution



$$x_1 = q_1 + q_2 + \frac{1}{2} q_3$$

$$x_2 = q_1 - q_3$$

$$x_3 = q_1 - q_2 + \frac{1}{2} q_3$$

Now the configuration of the system is given by the ordinary co-ordinary x_1, x_2, x_3 and a transformation equation to the generalised co-ordinates are

$$x_1 = q_1 + q_2 + \frac{1}{2} q_3$$

$$x_2 = q_1 - q_3$$

$$x_3 = q_1 - q_2 + \frac{1}{2} q_3.$$

$\therefore$ The total kinetic energy

$$T = \frac{1}{2} m \left[(\dot{x}_1)^2 + (\dot{x}_2)^2 + (\dot{x}_3)^2\right]$$

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$$\therefore T = \frac{1}{2} m \left[(\dot{q}_1 + \dot{q}_2 + \tfrac{1}{2}\dot{q}_3)^2 + (\dot{q}_1 - \dot{q}_3)^2 + (\dot{q}_1 - \dot{q}_2 + \tfrac{1}{2}\dot{q}_3)^2 \right]$$

$$\Rightarrow T = \frac{1}{2} m \left[3\dot{q}_1^2 + 2\dot{q}_2^2 + \frac{3}{2}\dot{q}_3^2 \right]$$

$\because (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

$\because (a-b+c)^2 = a^2+b^2+c^2-2ab-2bc-2ca$

$$\therefore \frac{\partial T}{\partial \dot{q}_1} = \frac{1}{2} m (6\dot{q}_1) \Rightarrow \frac{\partial T}{\partial \dot{q}_1} = 3m\dot{q}_1$$

$$\frac{\partial T}{\partial \dot{q}_2} = \frac{1}{2} m (4\dot{q}_2) \Rightarrow \frac{\partial T}{\partial \dot{q}_2} = 2m\dot{q}_2$$

$$\frac{\partial T}{\partial \dot{q}_3} = \frac{1}{2} m (3\dot{q}_3) \Rightarrow \frac{\partial T}{\partial \dot{q}_3} = \frac{3}{2} m\dot{q}_3$$

$\therefore$ The generalised momentum are

$p_1 = 3m\dot{q}_1$; $p_2 = 2m\dot{q}_2$ and $p_3 = \frac{3}{2} m\dot{q}_3$.

UNIT-II

LAGRANGE'S EQUATION

Derivation of lagrange's equations for holonomic system

Let us consider a system of N particles.

Let the configuration of the system be specified by 3N Cartesian co-ordinates $x_1, x_2, \dots, x_{3N}$.

Let $q_1, q_2, \dots, q_n$ be the n generalised co-ordinates, describing the system and let the transformation equation be

$$x_i = x_i(q_1, q_2, \dots, q_n, t) \qquad (i = 1, 2, \dots, 3N)$$

Hence
$$\frac{dx_i}{dt} = \sum_{i=1}^{3N} \frac{\partial x_i}{\partial q_j} dq_j + \frac{\partial x_i}{\partial t}, \quad j = 1, 2, \dots, n$$

$$\therefore \dot{x}_i = \sum_{i=1}^{3N} \frac{\partial x_i}{\partial q_j} \dot{q}_j + \frac{\partial x_i}{\partial t}, \quad j = 1, 2, \dots, n \text{ -----} (1)$$

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j} \text{ for } i = 1, 2, \dots, 3N \text{ \& } j = 1, 2, 3, \dots, n \text{ -----} (2)$$

This result is known as the cancellation of dots.

Again from eqn (1)

$$\frac{\partial \dot{x}_i}{\partial q_K} = \sum_{i=1}^{3N} \frac{\partial^2 x_i}{\partial q_K \partial q_j} \dot{q}_j + \frac{\partial^2 x_i}{\partial q_K \partial t}$$

$$\Rightarrow \frac{dx_i}{dt} = \sum_{i=1}^{3N} \frac{\partial x_i}{\partial q_j} \dot{q}_j + \frac{\partial x_i}{\partial t}$$

Again from eqn (1)

$$\frac{\partial}{\partial q_K}\left(\frac{dx_i}{dt}\right) = \sum_{i=1}^{3N} \frac{\partial^2 x_i}{\partial q_K \partial q_j} \dot{q}_j + \frac{\partial^2 x_i}{\partial q_K \partial t}$$

Since the order of differentiation is admissible.

We have, $$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_K}\right) = \sum_{i=1}^{3N} \frac{\partial^2 x_i}{\partial q_K \partial q_j} \dot{q}_j + \frac{\partial^2 x_i}{\partial q_K \partial t}$$

$$\Rightarrow \frac{\partial \dot{x}_i}{\partial q_K} = \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_K}\right) \text{ -----} (3)$$

Now the generalised momentum P_j can be written as

$$P_j = \frac{\partial T}{\partial \dot{q}_j}$$

$$\Rightarrow \quad = \frac{\partial}{\partial \dot{q}_j}\left[\frac{1}{2}\sum_{i=1}^{3N} m_i (\dot{x}_i)^2\right]$$

$$= \frac{1}{2}\sum_{i=1}^{3N} m_i \frac{\partial}{\partial \dot{q}_j}(\dot{x}_i)^2$$

$$= \frac{1}{2}\sum_{i=1}^{3N} m_i \, 2\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j}$$

$$= \sum_{i=1}^{3N} m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j}$$

$$= \sum_{i=1}^{3N} m_i \frac{\partial x_i}{\partial q_j} \dot{x}_i \qquad [\because \text{Using equ } 2]$$

Here, $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left[\sum_{i=1}^{3N} m_i \dot{x}_i \frac{\partial x_i}{\partial q_j}\right]$$

$$= \sum_{i=1}^{3N} m_i \left[\frac{d}{dt}\left(\dot{x}_i \frac{\partial x_i}{\partial q_j}\right)\right]$$

$$= \sum_{i=1}^{3N} m_i \left[\ddot{x}_i \frac{\partial x_i}{\partial q_j} + \dot{x}_i \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right)\right]$$

$$= \sum_{i=1}^{3N} m_i \left[\ddot{x}_i \frac{\partial x_i}{\partial q_j} + \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j}\right] \qquad \left[\because \text{Using equ } 3\right]$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \sum_{i=1}^{3N} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} + \sum_{i=1}^{3N} m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j} \quad \text{-----} \; (4)$$

But $$\frac{\partial T}{\partial q_j} = \frac{\partial}{\partial q_j}\left[\frac{1}{2}\sum_{i=1}^{3N} m_i (\dot{x}_i)^2\right]$$

$$= \frac{1}{2}\sum_{i=1}^{3N} m_i \frac{\partial}{\partial q_j}(\dot{x}_i)^2$$

$$= \frac{1}{2}\sum_{i=1}^{3N} m_i \, 2\dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j}$$

$$\frac{\partial T}{\partial q_j} = \sum_{i=1}^{3N} m_i x_i \frac{\partial \dot{x}_i}{\partial q_j} \quad \text{-----} \; (5)$$

Substuting equ (5) in (4)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \sum_{i=1}^{3N} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} + \frac{\partial T}{\partial q_j}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \sum_{i=1}^{3N} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} \quad \text{-----} (6) \quad (3)$$

Generalised force is given by

$$Q_j = \sum_{i=1}^{3N} F_i \frac{\partial x_i}{\partial q_j} \quad \text{-----} (7)$$

D'Alembert's principle is given by

$$\sum_{i=1}^{3N} (F_i - m_i \ddot{x}_i) \delta x_i = 0 \quad \text{-----} (8)$$

$$\sum_{i=1}^{3N} (F_i - m_i \ddot{x}_i) \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \delta q_j = 0$$

$$\sum_{i=1}^{3N} \sum_{j=1}^{n} (F_i - m_i \ddot{x}_i) \frac{\partial x_i}{\partial q_j} \delta q_j = 0$$

$$\sum_{j=1}^{n} \left[\sum_{i=1}^{3N} F_i \frac{\partial x_i}{\partial q_j} - \sum_{i=1}^{3N} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} \right] \delta q_j = 0 \quad \text{----} (9)$$

Substuting equs (6) & (7) in (9)

$$\sum_{j=1}^{n} \left\{ Q_j - \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right] \right\} \delta q_j = 0 \quad \text{----} (10)$$

For the holonomic system $q_1, q_2, \dots, q_n$ are all independence and therefore the co-efficient of $\delta q_j = 0$

Hence, $Q_j - \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right] = 0 \quad , j = 1, 2, 3, \dots, n$

ie, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad , j = 1, 2, \dots, n. \quad \text{----} (11)$

These n-equations are known as lagrange's equations.

Let us now make the additional condition that all the generalised force are derived form a potential function $V(q, t)$ is given by

$$Q_j = \frac{-\partial V}{\partial q_j} \quad , j = 1, 2, \dots, n$$

$$(11) \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \frac{-\partial V}{\partial q_j} \quad , j = 1, 2, \dots, n$$

ie, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0 \quad , j = 1,2,\dots,n$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T - V) = 0 \quad , j=1,2,\dots,n \quad \text{-----} (12)$$

Now the lagrangian function $L(q, \dot{q}, t) = T - V$

So that $\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$

$$(12) \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = 0 \quad , j = 1,2,\dots,n \quad \text{-----} (13)$$

This is standard form of the lagrange's equation for a holonomic system.

Note

Suppose the generalised forces Q_j are given by

$$Q_j = \frac{-\partial V}{\partial q_j} + Q_j' \quad , j = 1,2,\dots,n.$$

Where Q_j' are not derived form a potential function.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad \text{for } j = 1,2,\dots,n$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \frac{-\partial V}{\partial q_j} + Q_j' \quad , j = 1,2,\dots,n$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j'$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T - V) = Q_j'$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j' \quad , j = 1,2,\dots,n \quad \text{where } L = T - V$$

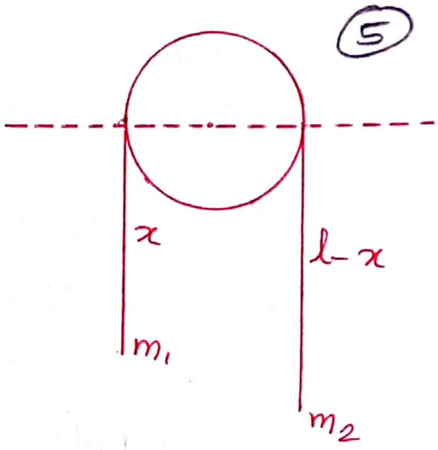
Problems

1. Two particles of masses m_1 and m_2 are connected by a light string of length l which passes over a smooth polley obtained the equation of motion.

Proof

That is only one independent co-ordinate x, the position of the other weight being determined by the constraints.

ie, The length of the string l.



$\therefore$ The total kinetic energy

$$T = \frac{1}{2} m_1 \dot{x}^2 - \frac{1}{2} m_2 (-\dot{x}^2)$$

$$\text{ie, } T = \frac{1}{2} (m_1 + m_2) \dot{x}^2 \quad \text{-----} \; (1)$$

$\therefore$ The potential energy

$$V = -m_1 g x - m_2 g (l - x)$$

$\therefore$ The lagrangian function $L = T - V$

$$\text{ie, } L = \frac{1}{2} (m_1 + m_2) \dot{x}^2 - \left[-m_1 g x - m_2 g (l - x)\right]$$

$$L = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + m_1 g x + m_2 g (l - x)$$

$$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2) \dot{x} \quad \text{and} \quad \frac{\partial L}{\partial x} = m_1 g - m_2 g$$

$$\frac{\partial L}{\partial x} = (m_1 - m_2) g$$

The lagrange's equation of motion is

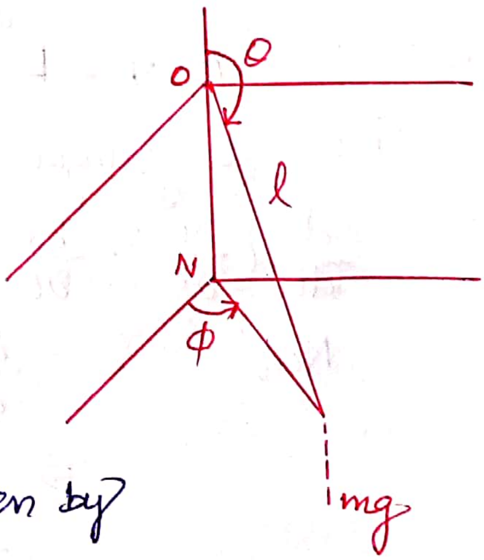
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\therefore \frac{d}{dt}\left[(m_1 + m_2) \dot{x}\right] - (m_1 - m_2) g = 0$$

$$\Rightarrow (m_1 + m_2) \ddot{x} - (m_1 - m_2) g = 0$$

(2) Find the differential equations of motion for a spherical pendulum of length l.

Solution



Let m be the mass of the particle suspended by the mass less wire of length l, from a fixed point 'o' form a spherical pendulum.

In the cartesian co-ordinate w.r.t the axis are given in figure are given by

$$x = l \sin\theta \cos\phi, \quad y = l \sin\theta \sin\phi$$

$$\& \quad z = -l \cos\theta.$$

Now, $\dot{x} = l(\cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi})$

$\dot{y} = l(\cos\theta\sin\phi\,\dot{\theta} + \sin\theta\cos\phi\,\dot{\phi})$

and $\dot{z} = l\sin\theta\,\dot{\theta}$

$\therefore$ The ~~total~~ kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$= \frac{1}{2}ml^2\Big[(\cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi})^2 + (\cos\theta\sin\phi\,\dot{\theta}+\sin\theta\cos\phi\,\dot{\phi})^2 + (\sin\theta\,\dot{\theta})^2\Big]$$

$$= \frac{1}{2}ml^2\Big[\cos^2\theta\cos^2\phi\,\dot{\theta}^2 + \sin^2\theta\sin^2\phi\,\dot{\phi}^2 - 2\cos\theta\cos\phi\,\dot{\theta}\sin\theta\sin\phi\,\dot{\phi} + \cos^2\theta\sin^2\phi\,\dot{\theta}^2 + \sin^2\theta\cos^2\phi\,\dot{\phi}^2 + 2\cos\theta\sin\phi\,\dot{\theta}\sin\theta\cos\phi\,\dot{\phi} + \sin^2\theta\,\dot{\theta}^2\Big]$$

$$= \frac{1}{2}ml^2\Big[\cos^2\theta\,\dot{\theta}^2(\cos^2\phi+\sin^2\phi) + \sin^2\theta\,\dot{\phi}^2(\sin^2\phi+\cos^2\phi) + \sin^2\theta\,\dot{\theta}^2\Big]$$

$$= \frac{1}{2}ml^2\Big[\cos^2\theta\,\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2 + \sin^2\theta\,\dot{\theta}^2\Big]$$

$$\therefore T = \frac{1}{2}ml^2\Big[\dot{\theta}^2(\cos^2\theta+\sin^2\theta) + \sin^2\theta\,\dot{\phi}^2\Big]$$

$$\Rightarrow T = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)$$

$\therefore$ Potential energy, $V = -mgl\,(ON)$

$$= -mgl\cos(180-\theta)$$

$$\Rightarrow V = mgl\cos\theta.$$

$\therefore$ The Lagrangian function $L = T - V$

ie, $L = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) - mgl\cos\theta$ ----- (1)

Lagrangian equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \;\text{-----}(2) \quad \& \quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0 \;\text{-----}(3)$$

Now, $\frac{\partial L}{\partial\dot{\theta}} = \frac{1}{2}ml^2(2\dot{\theta}+0) - 0$

$\Rightarrow \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}$ ----- (4)

and $\frac{\partial L}{\partial\theta} = \frac{1}{2}ml^2(0 + 2\sin\theta\cos\theta\,\dot{\phi}^2) - mgl(-\sin\theta)$

$\Rightarrow \frac{\partial L}{\partial\theta} = ml^2\sin\theta\cos\theta\,\dot{\phi}^2 + mgl\sin\theta$ ----- (5)

Substuting eqns 4 & 5 in 2

$$\frac{d}{dt}(ml^2\dot{\theta}) - [ml^2 \sin\theta\cos\theta\,\dot{\phi}^2 + mgl\sin\theta] = 0$$

$$ml^2\ddot{\theta} - ml^2\sin\theta\cos\theta\,\dot{\phi}^2 - mgl\sin\theta = 0$$

$$ml(l\ddot{\theta} - l\sin\theta\cos\theta\,\dot{\phi}^2 - g\sin\theta) = 0$$

$$\Rightarrow l\ddot{\theta} - l\sin\theta\cos\theta\,\dot{\phi}^2 - g\sin\theta = 0 \quad \text{-----} (6)$$

III'ly

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2}ml^2(0 + \sin^2\theta\, 2\dot{\phi})$$

$$\Rightarrow \frac{\partial L}{\partial \dot{\phi}} = ml^2\sin^2\theta\,\dot{\phi} \quad \text{---} (7) \qquad \text{and} \qquad \frac{\partial L}{\partial \phi} = 0 \quad \text{-----} (8)$$

Substuting eqns 7 & 8 in 3

$$\frac{d}{dt}(ml^2\sin^2\theta\,\dot{\phi}) - 0 = 0$$

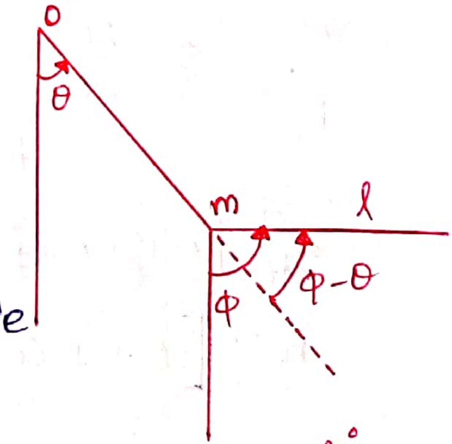
$$\therefore ml^2(\sin^2\theta\,\ddot{\phi} + 2\sin\theta\cos\theta\,\dot{\phi}) = 0$$

$$\Rightarrow \sin^2\theta\,\ddot{\phi} + 2\sin\theta\cos\theta\,\dot{\phi} = 0 \quad \text{-----} (9)$$

Equations 6 & 9 are the required differential equation of the motion of the system.

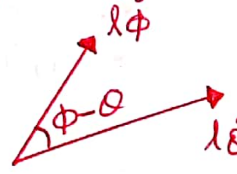
3 A double pendulum consists of two particles suspended by mass less rods as shown the following figure. Assume that all motion takes place the vertical plane. Find the differential equation of motion.

Solution



Let 'o' be the point of suspension.

Let the rod connecting to the upper particle 'o' make an angle θ with vertical and the rod connecting the lower particle to the upper particle make an angle ϕ with the vertical.



Now, the absolute velocity of lower particle

$$= l[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\phi-\theta)]^{1/2}$$

$\therefore$ The total kinetic energy is

$$T = \frac{1}{2}ml^2[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\phi-\theta)]$$

By triangle law
$\because$ Resultant velocity
$= (l\dot{\theta})^2 + (l\dot{\phi})^2$
$+ 2(l\dot{\theta})(l\dot{\phi})\cos(\phi-\theta)$

If we choose O as the reference level the potential energy

$$V = -mgl\cos\theta - (mgl\cos\theta + mgl\cos\phi) \quad (8)$$

ie, $V = -mgl\cos\theta - mgl\cos\theta - mgl\cos\phi$

$$= -2mgl\cos\theta - mgl\cos\phi$$

$$\Rightarrow V = -mgl(2\cos\theta + \cos\phi)$$

The lagrangian function L is given by

$$L = T - V$$

$$\therefore L = \frac{1}{2}ml^2[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\phi-\theta)] + mgl(2\cos\theta + \cos\phi) \text{ ---- } (1)$$

Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \text{ ----- } (2) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \text{ ----- } (3)$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}ml^2[2\dot{\theta} + 0 + 2\dot{\phi}\cos(\phi-\theta)(1)] + mgl(0+0)$$

$$= \frac{1}{2}ml^2[2\dot{\theta} + 2\dot{\phi}\cos(\phi-\theta)]$$

$$\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = ml^2[\dot{\theta} + \dot{\phi}\cos(\phi-\theta)] \text{ ----- } (4)$$

and $\frac{\partial L}{\partial \theta} = \frac{1}{2}ml^2[0 + 0 + 2\dot{\theta}\dot{\phi}(-\sin(\phi-\theta))(-1)]$

$$+ mgl(-2\sin\theta + 0)$$

$$\Rightarrow \frac{\partial L}{\partial \theta} = ml^2\dot{\theta}\dot{\phi}\sin(\phi-\theta) - 2mgl\sin\theta \text{ ----- } (5)$$

Substuting equs (4) & (5) in (2)

$$\frac{d}{dt}\left[ml^2(\dot{\theta} + \dot{\phi}\cos(\phi-\theta))\right] - \left[ml^2\dot{\theta}\dot{\phi}\sin(\phi-\theta) - 2mgl\sin\theta\right] = 0$$

$$ml^2[\ddot{\theta} + \ddot{\phi}\cos(\phi-\theta) + \dot{\phi}(-\sin(\phi-\theta))(\dot{\phi}-\dot{\theta})]$$

$$- ml^2\dot{\theta}\dot{\phi}\sin(\phi-\theta) + 2mgl\sin\theta = 0$$

$$\Rightarrow ml^2[\ddot{\theta} + \ddot{\phi}\cos(\phi-\theta) - \dot{\phi}\sin(\phi-\theta)(\dot{\phi}-\dot{\theta}) - \dot{\theta}\dot{\phi}\sin(\phi-\theta)]$$

$$+ 2mgl\sin\theta = 0 \text{ ---- } (6)$$

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2}ml^2[0 + 2\dot{\phi} + 2\dot{\theta}(1)\cos(\phi-\theta)] + 0$$

$$\therefore \frac{\partial L}{\partial \dot{\phi}} = ml^2[\dot{\phi} + \dot{\theta}\cos(\phi-\theta)] \text{ ----- } (7)$$

and $$\frac{\partial L}{\partial \phi} = \frac{1}{2}ml^2\left[0+0+2\dot{\theta}\dot{\phi}(-\sin(\phi-\theta))(1)\right]+mgl(0-\sin\phi)$$

$$= \frac{1}{2}ml^2\left[-2\dot{\theta}\dot{\phi}\sin(\phi-\theta)\right]-mgl\sin\phi$$

$$\Rightarrow \frac{\partial L}{\partial \phi} = -\left[ml^2\dot{\theta}\dot{\phi}\sin(\phi-\theta)+mgl\sin\phi\right] \text{ ----- (8)}$$

Substuting eqns (7) & (8) in (3)

$$\frac{d}{dt}\left[ml^2(\dot{\phi}+\dot{\theta}\cos(\phi-\theta))\right]+ml^2\dot{\theta}\dot{\phi}\sin(\phi-\theta)+mgl\sin\phi=0$$

$$\therefore ml^2\left[\ddot{\phi}+\ddot{\theta}\cos(\phi-\theta)+\dot{\theta}(-\sin(\phi-\theta))(\dot{\phi}-\dot{\theta})\right]$$
$$+ml^2\dot{\theta}\dot{\phi}\sin(\phi-\theta)+mgl\sin\phi=0$$

$$\Rightarrow ml^2\left[\ddot{\phi}+\ddot{\theta}\cos(\phi-\theta)-\dot{\theta}(\dot{\phi}-\dot{\theta})\sin(\phi-\theta)+\dot{\theta}\dot{\phi}\sin(\phi-\theta)\right]$$
$$+mgl\sin\phi=0 \text{ --- (9)}$$

Equations (6) & (9) are the required differential equations of motion of the system.

Lagrange's equations for non-holonomic system.

For a non-holonomic system it is not possible to find a set of independent generalised co-ordinates

Hence a non-holonomic system always requires more co-ordinates for that description.

Thus if there are m non-holonomic constrained equations

$$\sum_{j=1}^{n} a_{kj}\,dq_j + a_{kj}\,dt = 0 \quad , k=1,2,\ldots,m \text{ ----- (1)}$$

and then $$\sum_{j=1}^{n} a_{kj}\,\delta q_j = 0 \quad , k=1,2,\ldots,m \text{ ----- (2)}$$

Let all the generalised applied forces be derived from a potential function $V(q,t)$ as

$$Q_j = \frac{-\partial V}{\partial q_j} \quad , j=1,2,\ldots,n$$

Let us assume that the constraints to be workless so that the generalised constrained forces C_j satisfy the condition

$$\sum_{j=1}^{n} C_j\,\delta q_j = 0 \text{ ----- (3)}$$ for any virtual displacement consistent with the constraints.

Let us introduce λ_K known as lagrange's multiplier.

Multiply equ ② by λ_K.

$$\lambda_K \sum_{j=1}^{n} a_{Kj}\, \delta q_j = 0 \quad , \; K = 1,2,\dots,m$$

$$\Rightarrow \sum_{K=1}^{m} \lambda_K \sum_{j=1}^{n} a_{Kj}\, \delta q_j = 0$$

$$\Rightarrow \sum_{j=1}^{n} \left(\sum_{K=1}^{m} \lambda_K a_{Kj} \right) \delta q_j = 0 \quad \text{-----} \; ④$$

$$③ - ④ \Rightarrow \sum_{j=1}^{n} c_j\, \delta q_j - \sum_{j=1}^{n} \left(\sum_{K=1}^{m} \lambda_K a_{Kj} \right) \delta q_j = 0$$

$$\sum_{j=1}^{n} \left(c_j - \sum_{K=1}^{m} \lambda_K a_{Kj} \right) \delta q_j = 0 \quad \text{-----} \; ⑤$$

Choosing lagrange's multipliers λ's such that

$$c_j = \sum_{K=1}^{m} \lambda_K a_{Kj} \qquad \text{for } j = 1,2,\dots n$$

Then the coefficients of δq's in equ ⑤ are all zero and therefore equ ⑤ will be applied to any set of δq's.

∴ The lagrange's equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j' = c_j$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = \sum_{K=1}^{m} \lambda_K a_{Kj} \; , \; j = 1,2,\dots,n \quad \text{----} \; ⑥$$

This is the standard form of lagrange's equation for a non-holonomic system.

Problems

① A block of mass m_2 can slide on another block of mass m_1 which inturns slides on the horizontal surface. It is assume that all surfaces are functions. Using x_1 and x_2 as co-ordinates shown in the figure.

Obtain the differential equations of motion and Solve for acceleration of the two blocks as they move under the influence of gravity. Also find the force of interation between the blocks.

Solution

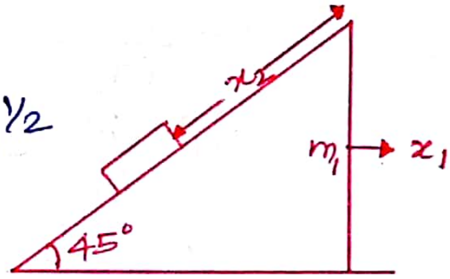


Figure (a)

(a) The absolute velocity of

$$m_2 = [\dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\dot{x}_2\cos(90^\circ+45^\circ)]^{1/2}$$

$$= [\dot{x}_1^2 + \dot{x}_2^2 - 2\dot{x}_1\dot{x}_2\sin 45^\circ]^{1/2}$$

$$= [\dot{x}_1^2 + \dot{x}_2^2 - 2\dot{x}_1\dot{x}_2\frac{1}{\sqrt{2}}]^{1/2}$$

$$\therefore\ m_2 = [\dot{x}_1^2 + \dot{x}_2^2 - \sqrt{2}\,\dot{x}_1\dot{x}_2]^{1/2}$$

The total kinetic energy

$$T = \frac{1}{2}m_1(\dot{x}_1^2) + \frac{1}{2}\left\{m_2\left[(\dot{x}_1^2 + \dot{x}_2^2 - \sqrt{2}\,\dot{x}_1\dot{x}_2)\right]^{1/2}\right\}^2$$

$$= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}\left[m_2\left[(\dot{x}_1^2 + \dot{x}_2^2 - \sqrt{2}\,\dot{x}_1\dot{x}_2)\right]\right]$$

ie, $$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{\sqrt{2}}m_2\dot{x}_1\dot{x}_2 \quad \text{---}(1)$$

Also the potential energy,

$$V = m_2 g x_2 \cos(90^\circ+45^\circ)$$

$$V = -\frac{1}{\sqrt{2}}m_2 g x_2$$

$\therefore$ The lagranjian function L is

$$L = T - V$$

ie, $$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{\sqrt{2}}m_2\dot{x}_1\dot{x}_2 + \frac{1}{\sqrt{2}}m_2 g x_2 \quad \text{---}(2)$$

WKT, The lagrange's of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \quad \text{-----}(3) \quad \& \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \quad \text{----}(4)$$

$$\frac{\partial L}{\partial \dot{x}_1} = \frac{1}{2}m_1 2\dot{x}_1 + \frac{1}{2}m_2 2\dot{x}_1 - \frac{1}{\sqrt{2}}m_2\dot{x}_2(1)$$

$$\therefore \frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1 + m_2\dot{x}_1 - \frac{1}{\sqrt{2}}m_2\dot{x}_2$$

ie, $\frac{\partial L}{\partial \dot{x}_1} = (m_1+m_2)\dot{x}_1 - \frac{1}{\sqrt{2}} m_2 \dot{x}_2$ ----- (5)

and $\frac{\partial L}{\partial x_1} = 0.$ ----- (6)

$$\frac{\partial L}{\partial \dot{x}_2} = 0 + \frac{1}{2} m_2 \, 2\dot{x}_2 - \frac{1}{\sqrt{2}} m_2 \dot{x}_1 (1) + 0$$

ie, $\frac{\partial L}{\partial \dot{x}_2} = m_2 \dot{x}_2 - \frac{1}{\sqrt{2}} m_2 \dot{x}_1$ ----- (7)

and $\frac{\partial L}{\partial x_2} = 0+0+0+\frac{1}{\sqrt{2}} m_2 g$

$\therefore \frac{\partial L}{\partial x_2} = \frac{1}{\sqrt{2}} m_2 g$ ----- (8)

Substuting equs (5) and (6) in (3)

$$\frac{d}{dt}(m_1+m_2)\dot{x}_1 - \frac{1}{\sqrt{2}} m_2 \dot{x}_2 - 0 = 0$$

ie, $(m_1+m_2)\ddot{x}_1 - \frac{1}{\sqrt{2}} m_2 \ddot{x}_2 = 0$ ----- (9)

Substuting equs (7) and (8) in (4)

$$\frac{d}{dt}\left[(m_2 \dot{x}_2) - \frac{1}{\sqrt{2}} m_2 \dot{x}_1\right] - \frac{1}{\sqrt{2}} m_2 g = 0$$

ie, $m_2 \ddot{x}_2 - \frac{1}{\sqrt{2}} m_2 \ddot{x}_1 - \frac{1}{\sqrt{2}} m_2 g = 0$ ----- (10)

$$(9) + \frac{1}{\sqrt{2}}(10) \Rightarrow m_1 \ddot{x}_1 + m_2 \ddot{x}_1 - \frac{1}{\sqrt{2}} m_2 \ddot{x}_2 + \frac{1}{\sqrt{2}} m_2 \ddot{x}_2 - \frac{1}{2} m_2 \ddot{x}_1 - \frac{1}{2} m_2 g = 0$$

$$\Rightarrow m_1 \ddot{x}_1 + m_2 \ddot{x}_1 - \frac{1}{2} m_2 \ddot{x}_1 - \frac{1}{2} m_2 g = 0$$

$$\Rightarrow m_1 \ddot{x}_1 + \frac{1}{2} m_2 \ddot{x}_1 = \frac{1}{2} m_2 g$$

$$\Rightarrow \left(m_1 + \frac{1}{2} m_2\right)\ddot{x}_1 = \frac{1}{2} m_2 g$$

$$\Rightarrow \left(\frac{2m_1+m_2}{2}\right)\ddot{x}_1 = \frac{1}{2} m_2 g$$

$$\Rightarrow (2m_1+m_2)\ddot{x}_1 = m_2 g$$

$$\Rightarrow \ddot{x}_1 = \frac{m_2 g}{2m_1+m_2}$$ ----- (11)

Substuting equ (11) in (9)

$$(m_1+m_2)\left(\frac{m_2 g}{2m_1+m_2}\right) - \frac{1}{\sqrt{2}} m_2 \ddot{x}_2 = 0$$

$$(m_1+m_2)\left(\frac{m_2 g}{2m_1+m_2}\right) = \frac{1}{\sqrt{2}} m_2 \ddot{x}_2$$

$$\therefore \ddot{x}_2 = \frac{\sqrt{2}(m_1+m_2) g}{2m_1+m_2}$$

(b) To find the force of interation between the two blocks.

Let us use lagrange's multiplier.

This interaction force is normal to the surface of contact and that may be considered as the generalised constrained force corresponding to the co-ordinate x_3 which is shown in the figure (b)

There are only two degrees of freedom. Since $x_3 = 0$.

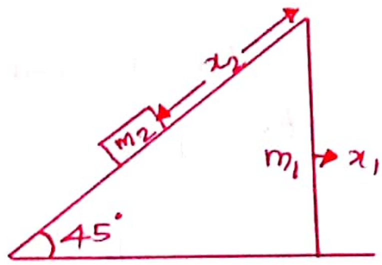
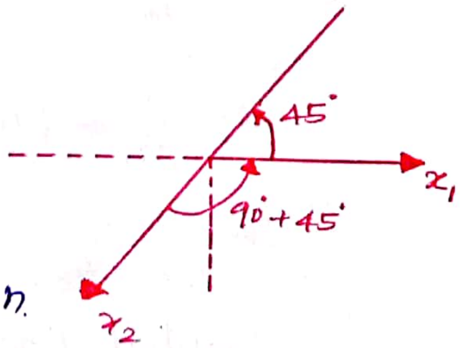


Fig (b)

Let us write this holonomic constrained equation in the form $\dot{x}_3 = 0$ is regular to an equation of non holonomic constraints.

Now non-holonomic constraint equation are of the form



$$\sum_{j=1}^{n} a_{kj}\dot{q}_j + a_{kj} = 0 \quad , \quad K = 1, 2, \dots, n.$$

Here $n = 3$, $m = 1$

$\therefore$ we have,

$$a_{11}\dot{q}_1 + a_{12}\dot{q}_2 + a_{13}\dot{q}_3 + a_{11} + a_{12} + a_{13} = 0$$

Comparing this with $\dot{x}_3 = 0$

we have, $a_{11} = 0$, $a_{12} = 0$ and $a_{13} = 1$

(14)

From $C_j = \sum_{k=1}^{m} \lambda_k a_{kj}$, $j = 1, 2, \ldots, n$.

$C_1 = \lambda_1 a_{11} = 0$, $C_2 = \lambda_1 a_{12} = 0$, $C_3 = \lambda_1 a_{13} = 0$

Writing the vertical and horizontal velocity components separately. We get

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \left[\left(\dot{x}_1 - \frac{\dot{x}_2 + \dot{x}_3}{\sqrt{2}} \right)^2 + \left(\frac{\dot{x}_3 - \dot{x}_2}{\sqrt{2}} \right)^2 \right]$$

$$T = \frac{1}{2}(m_1 + m_2) \dot{x}_1^2 + \frac{1}{2} m_2 \left[\dot{x}_2^2 + \dot{x}_3^2 - \sqrt{2}\, \dot{x}_1 (\dot{x}_2 + \dot{x}_3) \right] \quad \text{---- (1)}$$

and $V = \frac{-1}{\sqrt{2}} m_2 g (x_3 - x_2)$ ----- (2)

∴ The Lagrangian function L is

$$L = T - V$$

i.e, $$L = \frac{1}{2}(m_1 + m_2) \dot{x}_1^2 + \frac{1}{2} m_2 \left[\dot{x}_2^2 + \dot{x}_3^2 - \sqrt{2}\, \dot{x}_1 (\dot{x}_2 + \dot{x}_3) \right] + \frac{1}{\sqrt{2}} m_2 g (x_3 - x_2) \quad \text{--- (3)}$$

The Lagrangian equation of motion corresponding to x_1 is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1} \right) - \frac{\partial L}{\partial x_1} = \lambda_1 a_{11} \quad \text{----- (4)}$$

(3) $\Rightarrow$ $$\frac{\partial L}{\partial \dot{x}_1} = \frac{1}{2}(m_1 + m_2) 2\dot{x}_1 + \frac{1}{2} m_2 \left[0 + 0 - \sqrt{2} (\dot{x}_2 + \dot{x}_3)(1) \right] + 0$$

i.e, $$\frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2) \dot{x}_1 - \frac{1}{\sqrt{2}} m_2 (\dot{x}_2 + \dot{x}_3) \quad \text{---- (5)}$$

and $\frac{\partial L}{\partial x_1} = 0$ ----- (6)

Substuting eqns (5) and (6) in (4)

$$\frac{d}{dt}\left[(m_1 + m_2) \dot{x}_1 - \frac{1}{\sqrt{2}} m_2 (\dot{x}_2 + \dot{x}_3) \right] - 0 = 0$$

$$(m_1 + m_2) \ddot{x}_2 - \frac{1}{\sqrt{2}} m_2 (\ddot{x}_2 + \ddot{x}_3) = 0 \quad \text{---- (6)}$$

lll'y The other equations of motion are

$$-\frac{1}{2} m_2 \ddot{x}_1 + m_2 \ddot{x}_2 - \frac{1}{\sqrt{2}} m_2 g = 0$$

and $\quad \frac{-1}{2} m_2 \ddot{x}_1 + m_2 \ddot{x}_3 + \frac{1}{\sqrt{2}} m_2 g = \lambda_1$

Now, putting $\dot{x}_3 = 0$. we get

$$\frac{-1}{2} m_2 \ddot{x}_1 + \frac{1}{\sqrt{2}} m_2 g = \lambda_1 \quad \text{-----} (7)$$

ie, $\quad \frac{-1}{2} m_2 \ddot{x}_1 + m_2 \ddot{x}_2 = \frac{1}{2} m_2 g \quad \text{-----} (8)$

and $\quad (m_1 + m_2)\ddot{x}_1 - \frac{1}{\sqrt{2}} m_2 \ddot{x}_2 = 0 \quad \text{-----} (9)$

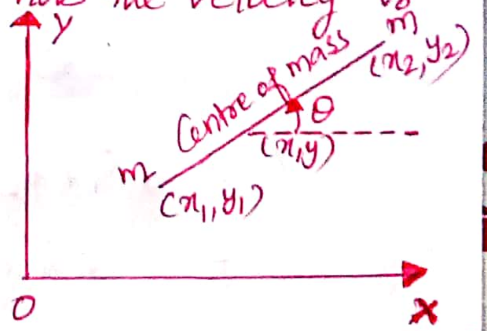
Solving these three equations

we get, $\quad \ddot{x}_1 = \frac{m_2 g}{2m_1 + m_2} \quad \& \quad \ddot{x}_2 = \frac{\sqrt{2}(m_1 + m_2) g}{2m_1 + m_2}$

and the Constraint force $\quad C_3 = \lambda_1 = \frac{\sqrt{2} m_1 m_2 g}{2m_1 + m_2}$

(2) Two particles are connected by a rigid massless rod of length l which rotates in a horizontal plane with a constant angular velocity ω edge supports at the two particles precent either particle from having a velocity component of the rod, But the particles can slide without friction in a direction perpendicular to the rod.

Find the generalised constraint forces, if the centre of mass initially at the origin and has the velocity V_0 in the positive direction

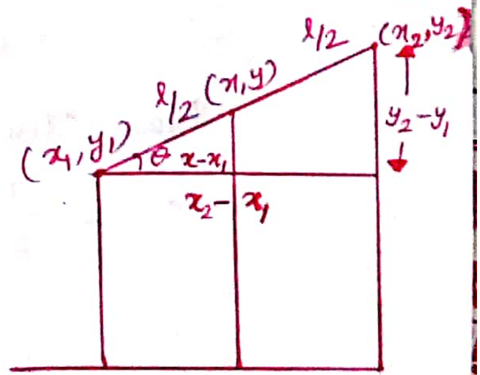


Proof

Let the cartesian Co-ordinates of the two particles be (x_1, y_1) & (x_2, y_2)

Let (x, y) be the Co-ordinates of the centre of mass.

Let θ be the angle made by the rod with OX as shown in the figure



So that $\theta = \omega t$

Then $(x_1 - x_2)^2 + (y_1 - y_2)^2 = l^2$ ----- ①

and $y_2 - y_1 = (x_2 - x_1)\tan\theta$

$\Rightarrow y_2 - y_1 = (x_2 - x_1)\tan\omega t$ ----- ②

Equations ① & ② are two equations of holonomic Constraints.

The non-holonomic constrained equations, restrict the velocity of the centre of the rod the direction perpendicular rod is

$$(\dot{x}_1 + \dot{x}_2)\cos\theta + (\dot{y}_1 + \dot{y}_2)\sin\theta = 0$$

$\Rightarrow (\dot{x}_1 + \dot{x}_2)\cos\omega t + (\dot{y}_1 + \dot{y}_2)\sin\omega t = 0$ ---- ③

As we choose (x, y) as the centre of mass, the transformation equations are

$$x_1 = x - \frac{l}{2}\cos\omega t \quad , \quad x_2 = x + \frac{l}{2}\cos\omega t$$

& $$y_1 = y - \frac{l}{2}\sin\omega t \quad , \quad y_2 = y + \frac{l}{2}\sin\omega t$$

Hence $\dot{x}_1 = \dot{x} + \frac{l}{2}\omega\sin\omega t \quad , \quad \dot{x}_2 = \dot{x} - \frac{l}{2}\omega\sin\omega t$

$$\Rightarrow \dot{x}_1 + \dot{x}_2 = 2\dot{x}$$

$|||^{ly}$

$$\dot{y}_1 = \dot{y} - \frac{l}{2}\omega\cos\omega t \quad , \quad \dot{y}_2 = y + \frac{l}{2}\omega\cos\omega t$$

$$\Rightarrow \dot{y}_1 + \dot{y}_2 = 2\dot{y}$$

③ $\Rightarrow (\dot{x}_1 + \dot{x}_2)\cos\omega t + (\dot{y}_1 + \dot{y}_2)\sin\omega t = 0$

$\Rightarrow 2\dot{x}\cos\omega t + 2\dot{y}\sin\omega t = 0$

$\Rightarrow \dot{x}\cos\omega t + \dot{y}\sin\omega t = 0$ ----- ④

Now, the total Kinetic energy

$$T = \frac{1}{2}m\left[\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2\right]$$

ie, $T = \frac{1}{2} m \left[(\dot{x} + \frac{1}{2} wl \sin wt)^2 + (\dot{x} - \frac{1}{2} lw \sin wt)^2 \right.$

$\left. + (\dot{y} - \frac{1}{2} lw \cos wt)^2 + (\dot{y} + \frac{1}{2} lw \cos wt)^2 \right]$

$= \frac{1}{2} m \left[2\dot{x}^2 + \frac{1}{2} l^2 w^2 \sin^2 wt + 2\dot{y}^2 + \frac{1}{2} l^2 w^2 \cos^2 wt \right]$

$= \frac{1}{2} m \left(2\dot{x}^2 + \frac{1}{2} l^2 w^2 (\sin^2 wt + \cos^2 wt) + 2\dot{y}^2 \right)$

$\therefore\ T = \frac{1}{2} m \left[2(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} l^2 w^2 \right]$

ie, $T = m(\dot{x}^2 + \dot{y}^2) + \frac{1}{4} m l^2 w^2$

For this system of potential energy is $V = 0$.

The lagrangian function is $L = T - V$

$\therefore\ L = m(\dot{x}^2 + \dot{y}^2) + \frac{1}{4} m l^2 w^2$ ---- (5)

$\frac{\partial L}{\partial \dot{x}} = 2m\dot{x} \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 2m\ddot{x}$

and $\frac{\partial L}{\partial x} = 0$

$|||^{ly}$ $\frac{\partial L}{\partial \dot{y}} = 2m\dot{y} \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) = 2m\ddot{y}$.

and $\frac{\partial L}{\partial y} = 0$

Now, the standard form lagrangian equation for non-holonomic system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \sum_{K=1}^{m} \lambda_K a_{Kj}, \quad j = 1, 2, \ldots, m.$$

Here $K = 1$ and $j = 2$

$\therefore$ The equation are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \lambda_1 a_{11}$$

$\Rightarrow\ 2m\ddot{x} = \lambda_1 a_{11}$

and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \lambda_1 a_{12}$

$$\Rightarrow \quad 2m\ddot{y} = \lambda_1 a_{12}$$

From equation (4)

We have, $a_{11} = \cos wt$ & $a_{12} = \sin wt$

$\therefore$ Equations of motion are

$$2m\ddot{x} = \lambda_1 \cos wt \quad \& \quad 2m\ddot{y} = \lambda_1 \sin wt \text{ -----(5)}$$

(4) $\Rightarrow$ $\dot{x}\cos wt + \dot{y}\sin wt = 0$

$$\dot{x}\left(\frac{2m\ddot{x}}{\lambda_1}\right) + \dot{y}\left(\frac{2m\ddot{y}}{\lambda_1}\right) = 0$$

$$\frac{2m\dot{x}\ddot{x} + 2m\dot{y}\ddot{y}}{\lambda_1} = 0$$

ie, $2m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) = 0$

$$\Rightarrow 2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} = 0$$

$$\Rightarrow \frac{d}{dt}(\dot{x}^2 + \dot{y}^2) = 0$$

$$\Rightarrow \dot{x}^2 + \dot{y}^2 = v_0$$

Where v_0 is the initial velocity of centre of mass. Since the direction of motion is always perpendicular to the rod

$$\dot{x} = v_0 \cos wt \quad \& \quad \dot{y} = v_0 \sin wt$$

$$\Rightarrow x = \frac{-v_0}{w}\sin wt \quad \& \quad y = \frac{-v_0}{w}\cos wt$$

with the initial Conditions $x=0, y=0$ when $t=0$

Integrating again and using the initial Condition $x=0$, $y=0$ when $t=0$.

We get, $x = \frac{v_0}{w}(\cos wt - 1)$ & $y = \frac{v_0}{w}\sin wt$

$\therefore \quad \dot{x} = -v_0 \sin wt \quad \& \quad \dot{y} = v_0 \cos wt$

Now, $\ddot{x} = -v_0 w \cos wt \quad \& \quad \ddot{y} = -v_0 w \sin wt$

$(6) \Rightarrow 2m\ddot{x} = \lambda_1 \cos wt$

$-2mv_0 w \cos wt = \lambda_1 \cos wt$

$\lambda_1 = -2mv_0 w$

The generalised Constrained forces are

$C_1 = \lambda_1 a_{11} = -2mv_0 w \cos wt$

$\& \quad C_2 = \lambda_1 a_{12} = -2mv_0 w \sin wt$

INTEGRALS OF MOTION

If the Configuration of a holonomic system is Specified by n generalised Co-ordinates, then the equations of motion Consists of n-second order differential equations with time of the independent variables.

Solutions of these n-second order differential equations Contain $2n$- Constant of integration.

The $2n$-Constant can be evaluated from $2n$-initial Condition

The general Solution can be expressed in the form

$$f_l(q, \dot{q}, t) = \alpha_l \quad , \quad l = 1, 2, \dots, 2n \quad \text{----} (1)$$

where α's are arbitrary Constant

$\therefore$ These $2n$- functions are called the integrals or Constants of motion.

These $2n$-equations can be solved for q's and $\dot{q}$'s interms of α and t.

Thus we can write $q_j = q_j(\alpha_1, \alpha_2, \dots, \alpha_{2n}, t)$

and $\dot{q}_j = \dot{q}_j(\alpha_1, \alpha_2, \dots, \alpha_{2n}, t)$, $j = 1, 2, \dots, n$

Such that equ (1) is satisfied for all n.

IGNORABLE CO-ORDINATES

Let the configuration of a holonomic system be described by n-generalised co-ordinates $q_1, q_2, \dots, q_n$.

Suppose that the lagrangian function L of the system contain all the n at some of the q's say $q_1, q_2, \dots, q_k$ are missing in L then these k-coordinates namely $q_1, q_2, \dots, q_k$ are called ignorable co-ordinates or a cyclic co-ordinates.

Theorem

The generalised momentum conjucate to a cyclic (ignorable) co-ordinates is concerned.

Proof

Consider a holonomic system with n-generalised co-ordinates $q_1, q_2, \dots, q_n$.

Suppose $q_1, q_2, \dots, q_k$ are ignorable co-ordinates.

The lagrangian function $L(q, \dot{q}, t)$ does not contain there co-ordinates $q_1, q_2, \dots, q_k$.

$$\therefore \frac{\partial L}{\partial q_j} = 0 \quad , \quad j = 1, 2, \dots, k \quad \text{-----} (1)$$

Now lagrange's equation of motion for a holonomic system are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad , \quad j = 1, 2, \dots, n \quad \text{----} (2)$$

Substuting eqn (1) in (2)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \quad , \quad j = 1, 2, \dots, k$$

$$\frac{\partial L}{\partial \dot{q}_j} = \beta_j \quad j = 1, 2, \dots, k$$

$\therefore$ Generalised momentum for ignorable Co-ordinates are $p_j = \frac{\partial L}{\partial \dot{q}_j} = \beta_j$ is a Constant

$\Rightarrow p_j$ is preserved.

$\Rightarrow$ The generalised momentum corresponding to each ignorable Co-ordinates is a constant, Otherwise it is an integrals of motion.

Example

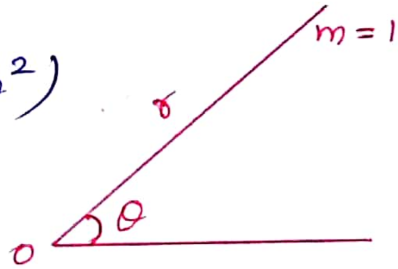
KEPLER PROBLEM

A particle of unit mass moves an attraction to a fixed point 'o' by inverse square gravitation force. Using polar Co-ordinates. Find the equation of motion.

Solution.

The kinetic energy, $T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$

and the potential energy, $V = \frac{-\mu}{r}$



$\therefore$ The Lagrangian function L is

$$L = T - V$$

ie, $L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu}{r}$ ----- (1)

$\Rightarrow L$ is does not contain θ

$\Rightarrow \theta$ is an ignorable Co-ordinates

Now, The θ-equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad ----- (2)$$

(1) $\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}(0 + r^2 2\dot{\theta}) \Rightarrow \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta}$ and $\frac{\partial L}{\partial \theta} = 0$

(2) $\Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0$

$\Rightarrow r^2\dot{\theta} = \beta$ a constant and is equal to the angular momentum of the particle about 'o'

$\therefore$ Thus one integral of motion has been obtain immediately.

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ROUTHAIAN FUNCTION

Let the configuration of the holonomic system be described by n-generalised co-ordinates $q_1, q_2, \dots, q_n$.

Suppose $q_1, q_2, \dots, q_K$ are ignorable co-ordinates.

Then the lagrangian function L is a function of $q_{K+1}, q_{K+2}, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t$.

$$\therefore \frac{\partial L}{\partial q_j} = 0 \quad , \quad j = 1, 2, \dots, K.$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \quad , \quad j = 1, 2, \dots, K$$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_j} = \beta_j \quad , \quad j = 1, 2, \dots, K \quad (\text{a Constant})$$

Let us define a function $R = L - \sum_{j=1}^{K} \beta_j \dot{q}_j$ ----- (1)

This function is called the "Routhian function".

Let us discuss the Routhian procedure to eliminate the ignorable Co-ordinates from the equation of motion.

We Now define $\quad R = L - \sum_{j=1}^{K} \beta_j \dot{q}_j$ ----- (2)

$\Rightarrow$ R is the function of $(q_{K+1}, q_{K+2}, \dots, q_n, \dot{q}_{K+1}, \dot{q}_{K+2}, \dots, \dot{q}_n, \beta_1, \beta_2, \dots, \beta_K, t)$. where $\beta_j = \frac{\partial L}{\partial \dot{q}_j}$, $j = 1, 2, \dots, K$

Let us make an arbitrary variation of all the variable in the Routhian function.

$$\delta R = \sum_{j=K+1}^{n} \frac{\partial R}{\partial q_j}\delta q_j + \sum_{j=K+1}^{n} \frac{\partial R}{\partial \dot{q}_j}\delta \dot{q}_j + \sum_{j=1}^{K} \frac{\partial R}{\partial \beta_j}\delta \beta_j + \frac{\partial R}{\partial t}\delta t \quad \text{-----} (3)$$

Also $\quad R = L - \sum_{j=1}^{K} \beta_j \dot{q}_j$

$$\therefore \delta\left(L - \sum_{j=1}^{K} \beta_j \dot{q}_j\right) = \sum_{j=K+1}^{n} \frac{\partial L}{\partial q_j}\delta q_j + \sum_{j=K+1}^{n} \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j + \frac{\partial L}{\partial t}\delta t - \sum_{j=1}^{K} \beta_j \delta \dot{q}_j - \sum_{j=1}^{K} \dot{q}_j \delta \beta_j \quad \text{-----} (4)$$

$$\text{ie, } \delta\left(L - \sum_{j=1}^{K} \beta_j \dot{q}_j\right) = \sum_{j=K+1}^{n} \frac{\partial L}{\partial q_j}\delta q_j + \sum_{j=K+1}^{n} \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j - \sum_{j=1}^{K} \dot{q}_j \delta \beta_j + \frac{\partial L}{\partial t}\delta t \quad \text{-----} (5)$$

$\left[\because \beta_j = \frac{\partial L}{\partial \dot{q}_j}, \; j = 1, 2, \dots, K\right.$

Thus assuming that the varied Co-ordinates equations ③ and ⑤ are independent.

We get, (i) $\frac{\partial L}{\partial q_j} = \frac{\partial R}{\partial q_j}$ and $\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial R}{\partial \dot{q}_j}$, $j = k+1, \dots, n$

(ii) $\dot{q}_j = \frac{-\partial R}{\partial \beta_j}$, $j = 1, 2, \dots, k$

(iii) $\frac{\partial L}{\partial t} = \frac{\partial R}{\partial t}$

Substuting these in lagrange equation of the form

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_j}\right) - \frac{\partial R}{\partial q_j} = 0 \quad , j = k+1, k+2, \dots, n \quad \text{-----} ⑥$$

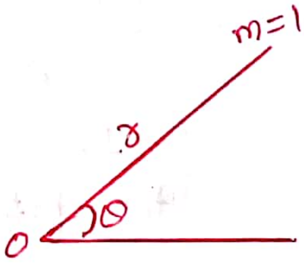
These we have obtained $n-k$ second order differential equations in the $n-k$ non-ignorable co-ordinates.

The ignorable Co-ordinates have been eliminatted from the equations of motion and thus reducing the number of degrees of freedom to $(n-k)$.

PROBLEM

Obtain the Routhian function and the equation of motion for the Kepler problem.

Solution



The lagrangian function is $L = T - V$

ie, $L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{M}{r}$ ----- ①

Here θ is an ignorable Co-ordinates

and $\frac{\partial L}{\partial \dot{\theta}} = \beta$.

① $\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}(0 + r^2 2\dot{\theta}^2) + 0$

$\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta}$

$\beta = r^2\dot{\theta}$

$\Rightarrow \dot{\theta} = \frac{\beta}{r^2}$ ----- ②

Now, The Routhian function $R = L - \beta\dot{\theta}$ ----- ③

Substuting eqns ① and ② in ③

ie, $R = \frac{1}{2}\left(\dot{r}^2 + r^2\frac{\beta^2}{r^4}\right) + \frac{M}{r} - \beta\frac{\beta}{r^2}$

$= \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{\beta^2}{r^2} + \frac{M}{r} - \frac{\beta^2}{r^2}$

$R = \frac{1}{2}\dot{r}^2 - \frac{1}{2}\frac{\beta^2}{r^2} + \frac{M}{r}$

This gives Routhian function for Kepler problem.

$\therefore$ The equation of motion is

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) - \frac{\partial R}{\partial r} = 0 \quad \text{-----} \quad (4)$$

Here, $\frac{\partial R}{\partial r} = \frac{\beta^2}{r^3} - \frac{M}{r^2}$ and $\frac{\partial R}{\partial \dot{r}} = \dot{r} \Rightarrow \frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) = \ddot{r}$

$(4) \Rightarrow \quad \ddot{r} - \frac{\beta^2}{r^3} + \frac{M}{r^2} = 0$

CONSERVATIVE SYSTEM

When the forces acting on a system are such ~~that~~ the workdone by them, in the passage of the system from the Configuration to the Standard Configuration is independent of the way in which the passage is carried out, then the System is said to be "Conservative".

$\therefore$ A Conservative force field in such that

(1) The generalised force components are obtaine from the potential energy function by $Q_j = \frac{-\partial V}{\partial q_j}$

(2) $W = \int_A^B \bar{Q}\cdot\delta\bar{q} = \sum_{j=1}^{n}\int_{A_j}^{B_j} Q_j\, dq_j$ is independent of the path between the given end points in Q spaces.

DEFINITION

CONSERVATIVE SYSTEM

A system is said to be conservative if

(1) This standard form lagrange's equation applies to the system.

(2) The lagrangian function L is not an explicit function

(3) Any constrained equation can be expressed in the differential form

$$\sum_{j=1}^{n} a_{kj}\, dq_j = 0 \quad , \; k = 1, 2, \ldots, m$$

ie, All the co-efficients a_{kj} are equal to zero.

JACOBI INTEGRAL / ENERGY INTEGRAL

To ensure the existence of the energy's constant. We have to show that the three conditions are sufficient for a system to be conservative.

Proof

Consider a standard form non-holonomic system of lagrangian equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{ji} \quad , i = 1, 2, \ldots, n \quad \text{------} (1)$$

Where $L = (q, \dot{q})$

Let m-equations of constraint be

$$\sum_{i=1}^{n} a_{ij}\, dq_i = 0 \quad , j = 1, 2, \ldots, m \quad \text{------} (2)$$

$$\text{ie, } \sum_{i=1}^{n} a_{ij}\, \dot{q}_i = 0 \quad , j = 1, 2, \ldots, m$$

Any holonomic constrained function $\phi_j(q)$ cannot be an explicit function of time.

$$\text{Then } a_{ji} = \frac{\partial \phi_j}{\partial t} = 0 \quad \text{-----} (3)$$

Now, the lagrangian function L is a function of q's and $\dot{q}$'s.

$$\therefore \frac{dL}{dt} = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \frac{dq_i}{dt} + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt}$$

$$\text{ie, } \frac{dL}{dt} = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \dot{q}_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \quad \text{-----} (4)$$

$$(1) \Rightarrow \frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \sum_{j=1}^{m} \lambda_j a_{ji} \quad , i = 1, 2, \ldots, n. \text{ ---} (5)$$

Substuting equation ( ) in (4)

$$\therefore \frac{dL}{dt} = \sum_{i=1}^{n} \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \sum_{j=1}^{m} \lambda_j a_{ji} \right) \dot{q}_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i$$

ie, $$\frac{dL}{dt} = \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i - \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_j a_{ji} \dot{q}_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

$$\frac{dL}{dt} = \sum_{i=1}^{n}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right) \quad [\because \text{Using equ } ②$$

ie, $$\frac{dL}{dt} = \frac{d}{dt}\left[\sum_{i=1}^{n}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i\right]$$

By integrating on both sides

$$L = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = h \text{ (Constant)} \quad \text{------} ⑥$$

Which is known as the Jacobi integral

This lagrangian function can be written in the form

$L = (T_2 + T_1 + T_0) - V$. where T_0 and V are functions of q only w.r.t

$$\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i = 2T_2 + T_1.$$

From equ. ⑥ $\quad 2T_2 + T_1 - L = h.$

NATURAL SYSTEM (CONSERVATIVE)

A natural system is a conservative system which has the additional property.

(1). It is described by standard holonomic from lagrange's equation.

(2) The kinetic energy enpressed as homogenous quadratic of q's.

(3) $$T = T_2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j$$

where m_{ij} may be functions of q's but not time.

LIOUVILLE'S SYSTEM

A natural system having the kinetic energy and potential energy of the form

$\therefore\ 2T_2 + T_1 - [(T_2 + T_1 + T_0) - V] = h$

$\Rightarrow T_2 - T_0 + V = h.$

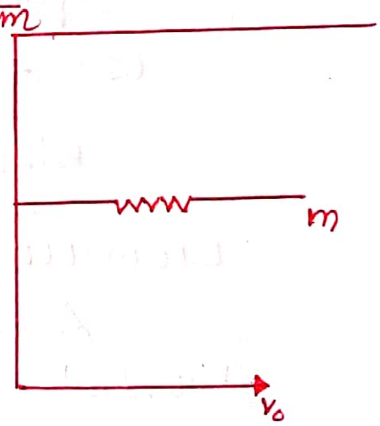
ie, $T' + V' = h$ a (Constant), where $T' = T_2$ & $V' = V - T_0$.

Thus the energy $T' + V'$ is Constant for any Conservative System.

EXAMPLE

A mass spring system is attached to a frame which is translating with uniform velocity V_0 as shown in the figure the unstreched spring length 't' and the elongation in x.

Find the Jacobi integral for this system



Solution

The Kinetic energy $T = \frac{1}{2} m (V_0 + \dot{x})^2$

$$T = \frac{1}{2} m (V_0^2 + \dot{x}^2 + 2V_0 \dot{x})$$

ie, $T = \frac{1}{2} m \dot{x}^2 + m V_0 \dot{x} + \frac{1}{2} m V_0^2$

$\Rightarrow T = T_2 + T_1 + T_0$

where $T_2 = \frac{1}{2} m \dot{x}^2$, $T_1 = m V_0 \dot{x}$ & $T_0 = \frac{1}{2} m V_0^2$

The potential energy, $V = \frac{1}{2} K x^2$, where K is a Stiffness of the spring.

Here T & V are not enplicit function of time.

The only generalised Co-ordinated force Q_x is derivables from V.

Thus the system under Consideration is the holonomic Conservative System.

Although that having frame does work is the system resulting in a change in total energy $T + V$.

$\therefore$ The Jacobi integral enists which is equal to $T_2 - T_0 + V$.

ie, $T_2 - T_0 + V = h$ (a constant)

$$\frac{1}{2}m\dot{x}^2 - \frac{1}{2}mv_0^2 + \frac{1}{2}kx^2 = h \quad (\text{Constant})$$

Which is a Jacobi integral. Also $T_0 = \frac{1}{2}mv_0^2$ (Constant;

NATURAL SYSTEM (Conservative)

A natural system is a Conservative system which has the additional property.

(1) It is described by standard holonomic form of lagrange's equation.

(2) The kinetic energy is expressed as homogeneous Quadratic of $\dot{q}$'s.

(3) $T = T_2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j$

where m_{ij} may be functions of q's but not time.

LIOUVILLE'S SYSTEM

A natural system having the kinetic energy and potential energy of the form.

$$T = \frac{1}{2}f\sum_{i=1}^{n} m_i(q_i)(\dot{q}_i)^2$$

and $V = \frac{1}{f}\sum_{i=1}^{n} v_i q_i$, where $f = \sum_{i=1}^{n} f_i(q_i) > 0$

is called the liouville's system.

It is an orthogonal system.

SMALL OSCILLATION

EQUATION OF MOTION

Let us consider a natural system whose configuration is specified by n-generalised co-ordinates $q_1, q_2, \ldots, q_n$.

Let the q's be measured from the position of equilibrium.

Let us make the reference value V_0 as 0.

So that potential energy can be written in the form

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V}{\partial q_i\,\partial q_j} q_i q_j + \ldots$$

Neglecting the higher powers.

we get, $V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V}{\partial q_i \partial q_j} q_i q_j$

Where the stiffness co-efficients are $k_{ij} = k_{ji} = \frac{\partial^2 V}{\partial q_i \partial q_j}$

Thus V is a homogenous quadratic equation of q's for small motion.

NEAR A POSITION OF EQUILIBRIUM

Suppose the system consists of N particles whose position, are given by 3N Cartesian Co-ordinates $x_1, x_2, \ldots, x_{3N}$.

Then the kinetic energy is of the form

$$T = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j, \text{ where } m_{ij} = m_{ji} = \sum_{i=1}^{3N} m_k \frac{\partial}{\partial q_i} \frac{\partial x_k}{\partial q_j}$$

Here T is a +ve definite quadratic function of $\dot{q}$.

The lagrangian function is $L = T - V$

$$\Rightarrow L = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} q_i q_j \quad \text{----} (1)$$

From the lagrangian equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \text{-----} (2)$$

$$(1) \Rightarrow \frac{\partial L}{\partial \dot{q}_i} = \frac{1}{2} \sum_{j=1}^{n} m_{ij} \dot{q}_j$$

ie, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{1}{2} \sum_{j=1}^{n} m_{ij} \ddot{q}_j$ ----(3) and $\frac{\partial L}{\partial q_i} = \frac{-1}{2} \sum_{j=1}^{n} k_{ij} q_j$ ---(4)

Substuting equations (3) & (4) in (2)

$$\frac{1}{2} \sum_{j=1}^{n} m_{ij} \ddot{q}_j + \frac{1}{2} \sum_{j=1}^{n} k_{ij} q_j = 0$$

In matrix form

$$\bar{m} \ddot{\bar{q}} + \bar{k} \bar{q} = 0$$

This equation of motion are linear second order ordinary differential equation m and k are constant symmetric $n \times n$ matrix.

NATURAL MODE

Let us consider a system whose differential equation of motion are given by

$$\sum_{j=1}^{n} m_{ij}\ddot{q}_j + \sum_{j=1}^{n} k_{ij} q_j = 0 \quad , \quad i = 1,2,\dots,k \quad \text{-----} (1)$$

Assume the solution of the form

$$q_j = A_j \, c \cos(wt + \theta) \quad , \quad j = 1,2,\dots,n \quad \text{-----} (2)$$

Where the amplitude of the oscillation in q_j is equal to the product of the constant A_j, c.

C - Scale factor for q's and A_j - relative magnitude.

Equation (2) diff. w.r.t

$$\dot{q}_j = -A_j c \sin(wt+\theta) . w$$

Again diff.

$$\ddot{q}_j = -A_j c \cos(wt+\theta) w^2 \quad \text{-----} (3)$$

Substuting equations (2) & (3) in (1)

$$\sum_{j=1}^{n} m_{ij}\left[-A_j c \cos(wt+\theta) w^2\right] + \sum_{j=1}^{n} k_{ij} A_j c \cos(wt+\theta) = 0$$

$$\sum_{j=1}^{n} (k_{ij} - m_{ij} w^2) A_j c \cos(wt+\theta) = 0$$

ie, $$\sum_{j=1}^{n} (k_{ij} - m_{ij} w^2) A_j = 0$$

Since A_j's are not all zero the determinant of the coefficients vanishes.

$$\begin{vmatrix} (k_{11} - m_{11} w^2) & (k_{12} - m_{12} w^2) & \cdots\cdots & (k_{1n} - m_{1n} w^2) \\ (k_{21} - m_{21} w^2) & (k_{22} - m_{22} w^2) & \cdots\cdots & (k_{2n} - m_{2n} w^2) \\ \vdots & & & \\ (k_{n1} - m_{n1} w^2) & (k_{n2} - m_{n2} w^2) & \cdots\cdots & (k_{nn} - m_{nn} w^2) \end{vmatrix} = 0$$

The evaluation of this determinant result in n^{th} degree algebraic equation in w^2 is called characteristic equation.

The n-roots w^2, where $k = 1,2,\dots,n$ are known as "Eigen values or Characteristic values".

Special applications of lagrange's equation

WKT, The standard form of lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad (i = 1, 2, \ldots, n) \quad \text{------} ①$$

This is applicable to the holonomic systems whose generalised forces are derivable from a potential function $V(q,t)$ by

$$Q_i = \frac{-\partial V}{\partial q_i}$$

If these applied forces are functions of $\dot{q}$'s and are not wholly derivable from the potential functions, there forces are frequently represented by Q_i'.

Equation ① becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i', \quad (i = 1, 2, \ldots, n) \quad \text{-----} ②$$

$$\left[\because Q_i = \frac{-\partial V}{\partial q_i} + Q_i'\right.$$

Sometimes Q_i' are of the form

$$Q_i' = -\sum_{j=1}^{n} c_{ij}(q,t)\,\dot{q}_j \quad \text{----} ③$$

Where c's are known as damping co-efficients and $[c_{ij}]$ is a real symmetric matrix. $(\because A = A^T)$

These generalised forces are dissipative in nature and result in a loss of energy whenever Q_i' is non-zero.

Bookwork

Define Rayleigh's dissipation function and show that it is equal to half the instantaneous rate of dissipation of total mechanical energy?

$Q_i' = -\sum_{j=1}^{n} c_{ij}(q,t)\,\dot{q}_j$, where c's are damping co-efficients forming a real symmetric matrix, the generalised forces are dispative in nature and result in a loss of energy, whenever Q_i' is non zero.

The dissipation function can be defined as

$$F(q,\dot{q},t) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\,\dot{q}_i\dot{q}_j \quad \text{------} ④$$

From equations ② and ③

we can write the equation of motion as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = 0 , \quad i = 1,2,\dots,n \quad \text{-----} (5)$$

Here, we have assumed that the generalised forces are not derived from the potential function V.

The frictional forces do work on the system is

$$\sum_{i=1}^{n} Q_i' \dot{q}_i = -\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\, \dot{q}_i \dot{q}_j$$

$$= -2F.$$

$\therefore$ the dissipation function $F = \frac{1}{2}$ instantaneous rate of dissipation of the total mechanical energy.

Since the rate of energy of dissipation must be positive are zero at all lines, it follows that F is +ve definite function or a +ve semi-definite function of the $\dot{q}$'s.

As the rate of energy dissipation is independent of the co-ordinates used to described the configuration, F is invarid with respect to the co-ordinate transformation.

<u>Note</u>

(1) If F and V are both positive definite, the total energy T+V decreases continuously due to damping except when $\dot{q} = 0$ or $T = 0$.

(2) Since the only equilibrium position is at $q = 0$, corresponding to a minimum potential energy $V = 0$, we see that the system approach the condition $T = V = 0$ as $t \to \infty$

This type of stability is called "assymptotic stability."

(3) If F is a +ve definite but V is not, we find that the asymptotic stability does not result.

(4) If V is a +ve definite but F is a +ve semi-definite the system may not be asymptotically stable because it may be possible to find a continuing motion for which energy dissipation is zero.

Book work

Define impulsive forces (or) implusive of momentum and find the principle of linear impluse and linear momentum also find the principle of angular impluse and angular momentum?

Solution

A force of large magnitude acting on a small duration is called an impulsive force and if F is the force, t is the time. Ft is called the impulse.

Consider a system having N particles at a distances $r_1, r_2, \ldots, r_N$ from O.

∴ the equation of motion is $\bar{F} = \dot{\bar{p}}$ ----- (1)

where F is the total external force and the total linear momentum

$$\bar{p} = \sum_{i=1}^{N} m_i \dot{\bar{r}}_i = m \dot{\bar{r}}_c,$$ where m is the total mass

and r_c is the position of the centre of mass.

Integrating (1) with respect to time between t_1 & t_2

we get, $$\int_{t_1}^{t_2} F\,dt = \int_{t_1}^{t_2} \dot{\bar{p}}\,dt = \bar{p}(t_2) - \bar{p}(t_1)$$

ie, $\hat{F} = p_2 - p_1$ ------ (2)

where $\hat{F}$ is the total impulsive of the external forces.

Thus the "principle of linear impulse and linear momentum" is as follows.

The change of the total linear momentum of a given system during a given time interval is equal to the total impulse of the external forces acting over the same period.

Consider the rotational motion is given by

$\bar{M} = \dot{\bar{H}}$, ---(3) where $\bar{M}$ is the momentum of the external forces and $\bar{H}$ is the total angular momentum.

Integrating eqn (3) with respect to time over the interval t_1 to t_2.

We have, $\int_{t_1}^{t_2} \bar{M}\, dt = \int_{t_1}^{t_2} \dot{\bar{H}}\, dt$.

ie, $\hat{M} = H_2 - H_1$, where $\hat{M}$ is the angular implusive.

Hence the "principle of angular impulse and angular momentum as follows

The change in the total angular momentum of a system during a given time interval is equal to the total angular impulse of the external forces acting over the same interval, provided the reference points of $\bar{M}$ and $\bar{H}$ are either fixed is an inertial frame or is taken at the centre of mass

Book work

Discuss the lagrangian approach to impulsive motion when the generalised co-ordinates are independent or non independent.

Solution

Consider the independent generalised co-ordinates $q_1, q_2, \ldots, q_n$. when Q_i are the applied forces equation of motion is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i, \; (i = 1, 2, \ldots n) \quad \text{------} (1)$$

If the impulsive forces are applied interval Δt and generalised momentum $p_i = \frac{\partial T}{\partial \dot{q}_i}$

$\therefore$ Equation (1) becomes

$$\frac{d}{dt}(p_i) - \frac{\partial T}{\partial q_i} = Q_i$$

Integrating over time interval Δt.

we have, $$\int_t^{t+\Delta t} \left(\dot{p}_i - \frac{\partial T}{\partial q_i}\right) dt = \int_t^{t+\Delta t} Q_i\, dt.$$

The term $-\frac{\partial T}{\partial q_i}$ in the integral can be neglected because it is finite in time t and $\lim_{\Delta t \to 0} \int_t^{t+\Delta t} \frac{\partial T}{\partial q_i}\, dt = 0$.

If $\hat{Q}_i = \int_t^{t+\Delta t} Q_i \, dt$, the above integral becomes $\Delta p_i = \hat{Q}_i$ (3)

$(i = 1, 2, \dots, n)$

But $\Delta p_i = \sum_{j=1}^{n} m_{ij} \Delta \dot{q}_j$, where $m_{ij}(q,t)$ are continuous.

$$\therefore \sum_{j=1}^{n} m_{ij}(q,t) \Delta \dot{q}_j = \hat{Q}_i, \; (i = 1, 2, \dots, n)$$

ie, $\Delta \dot{q} = m^{-1} \hat{Q}$ ----- (2)

[$\because$ The inertia matrix m is positive definite $\therefore$ Inverse exists

If q's are independent and if the constraints are workless, the constrained impulse will not contribute the $\hat{Q}$'s.

The co-ordinates are not independent and when the constraints are holonomic or non-holonomic. we expressed the constrained equation in the form

$$\sum_{i=1}^{n} a_{ji} \dot{q}_i + a_{jt} = 0, \; (j = 1, 2, \dots, n)$$

$\therefore$ Lagrange's equation takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + C_i, \quad i = 1, 2, \dots, n \quad \text{----- (3)}$$

Where c's are generalised constrained forces given by

$C_i = \sum_{j=1}^{m} \lambda_j a_{ji}$, where λ's are lagrangian multipliers and Q_i's are generalised forces associated with applied forces.

$\therefore$ Virtual work

$$\delta w = \sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - Q_i - C_i\right] \delta q_i \quad \text{--- (4)}$$

From equation (3)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - Q_i - C_i = 0$$

$\therefore$ (4) $\Rightarrow$ $\delta w = 0$

If the constraints are workless and δq's conform to instantaneous constraints.

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Then $\sum_{i=1}^{n} c_i \delta q_i = 0$

④ $\Rightarrow$ $$\sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} - Q_i \right] \delta q_i = 0 \quad \text{-----} ⑤$$

This equation is the generalisation of a lagrange's form of D' Alembert's principle.

If the implusive forces are applied to the system during the interval Δt, integrating eqn ⑤ with respect to time over the interval Δt.

we have, $$\sum_{i=1}^{n} [\Delta p_i - \hat{Q}_i] \delta q_i = 0 \quad \text{-----} ⑥$$

If u_i are virtual velocities corresponding δq_i

Then equation ⑥ becomes

$$\sum_{i=1}^{n} \left[\sum_{j=1}^{n} m_{ij} (\dot{q}_j - \dot{q}_{j0}) - \hat{Q}_i \right] u_i = 0 \quad \text{----} ⑦$$

where $\Delta q_j = \dot{q}_j - \dot{q}_{j0}$

If u's satisfy the instantaneous Constraints namely

$$\sum_{j=1}^{n} a_{ji} u_i = 0 \quad \text{------} ⑧$$

we can choose $n-m$ independent set of virtual velocity components which meet instantaneous constraints ~~coordinates~~ condition (8).

Here we obtained $n-m$ equations of the form ⑦ and m equations of constraints

$$\sum_{i=1}^{n} a_{ji} \dot{q}_i + a_{jt} = 0, \ (j = 1, 2, \ldots, m)$$

These n equations can be solved for nq's immediately after the impulses have been applied.

Note

We can also making use of the lagrangian multiplies method.

Impulsive constraints

An impulsive constraints is a suddently applied constraint which is represented by a discontinuous constrained equation.

The Constrained equations are

$$\sum_{i=1}^{n} a_{ji}\dot{q}_i + a_{jt} = 0 \quad , \; j = 1, 2, \ldots n$$

Here, we assume that a_{ji}, a_{jt} as Continuous

If one or more of a's are discontinuous at the give time t, it follows for a sudden appearence of the constrained or a sudden change in motion.

Constraint impulse

Constraint impulse are constraint forces of impulsive nature which may ~~arrise~~ ~~as~~ arise as a result of implusive constraints or of applied implu~~ses~~ $\hat{Q}_i$.

Energy consideration

Let us now show that the energy - because of sudden appearence of a fixed constraint is equal to the kinetic energy of the relative motion.

The sudden application of a Constraint or an implusive nearly changes the kinetic energy of a system because in general the q's are suddently changed.

Let us suppose that the kinetic energy of a system is a quadratic function in q's.

The change in kinetic energy due to Δt is

$$T - T_0 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_{i0}\dot{q}_{i0} \quad \text{-----} (1)$$

WKT, $\quad p_i = \sum_{j=1}^{m} m_{ij}\dot{q}_j$

we have, $\quad T - T_0 = \frac{1}{2}\sum_{i=1}^{n} p_i\dot{q}_i - \frac{1}{2}\sum_{i=1}^{n} p_{i0}\dot{q}_{i0} \quad \text{-----} (2)$

$$\sum_{i=1}^{n} p_i\dot{q}_{i0} = \sum_{i=1}^{n}\left[\sum_{j=1}^{n} m_{ij}\dot{q}_j\right]\dot{q}_{i0}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_{i0}\dot{q}_j = \sum_{j=1}^{n} p_{j0}\dot{q}_j$$

ie, $\quad \sum_{i=1}^{n} p_i\dot{q}_{i0} = \sum_{i=1}^{n} p_{i0}\dot{q}_i$

$$② \Rightarrow T - T_0 = \frac{1}{2}\left[\sum_{i=1}^{n} p_i \dot{q}_i + \sum_{i=1}^{n} p_i \dot{q}_{i0} - \sum_{i=1}^{n} p_{i0}\dot{q}_i - \sum_{i=1}^{n} p_{i0}\dot{q}_{i0}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left[(p_i - p_{i0})(\dot{q}_i + \dot{q}_{i0})\right]$$

$$= \frac{1}{2}\sum_{i=1}^{n} \Delta p_i \, (\dot{q}_i + \dot{q}_{i0})$$

$$= \frac{1}{2}\sum_{i=1}^{n} (\hat{Q}_i + \hat{C}_i)(\dot{q}_i + \dot{q}_{i0})$$

$$\therefore T - T_0 = \sum_{i=1}^{n} (\hat{Q}_i + \hat{C}_i)\left(\frac{\dot{q}_i + \dot{q}_{i0}}{2}\right) \quad \text{-----} ③$$

In the general case, where the a's are continuous at t and $a_{ji} = 0, \forall j$

then $\sum_{i=1}^{n} a_{ji}\dot{q}_i = 0$ both before and after impulse

$$\therefore \sum_{i=1}^{n} \hat{C}_i \dot{q}_i \Rightarrow \sum_{i=1}^{n}\left[\sum_{j=1}^{n} \lambda_j a_{ji}\right]\dot{q}_i$$

$$\therefore \sum_{i=1}^{n} \hat{C}_i \dot{q}_i = 0$$

|||ly $\sum_{i=1}^{n} \hat{C}_i \dot{q}_{i0} = 0$

$$③ \Rightarrow T - T_0 = \sum_{i=1}^{n}\left[\hat{Q}_i\left(\frac{\dot{q}_i + \dot{q}_{i0}}{2}\right)\right]$$

$$k = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}(\dot{q}_i - \dot{q}_{i0})(\dot{q}_j - \dot{q}_{j0})$$

Consider the system having impulsive constraint given by

$$\sum_{i=1}^{n} a_{ij}\dot{q}_i = 0$$

Assume $\hat{Q}_i = 0$, where a's are discontinuous function of time.

$\therefore$ The basic equation is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}(\dot{q}_j - \dot{q}_{j0})\dot{q}_i = 0$$

$\sum_{i=1}^{n} m_{ij}\dot{q}_j\dot{q}_i - \sum\sum m_{ij}\dot{q}_{j0}\dot{q}_{i0}$

$\dot{q}_j\dot{q}_i = \dot{q}_i\dot{q}_{j0} = \dot{q}_{i0}\dot{q}_j$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_{i0}\dot{q}_j - \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_{j0} = 0$$

$$\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_{i0}\dot{q}_j = \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_{j0}$$

$$\therefore K = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_{io}\dot{q}_{jo}$$

ie, $K = T - T_0$.

Thus the energy lost of because of sudden appearence of fixed Constraint is equal to the kinetic energy of relative motion.

$|||^{ly}$ we can show that the increasing in kinetic energy of a system due to sudden start of moving Constraint is equal to the kinetic energy of the relative motion.

Gyroscopic system

Let us consider an explicit form of the equation of motion,

$$\sum_{j=1}^{n} m_{ij}\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n}[jl, i]\dot{q}_j\dot{q}_i + \sum_{j=1}^{n}\gamma_{ij}\dot{q}_j$$

$$+ \sum_{j=1}^{n}\frac{\partial m_{ij}}{\partial t}\dot{q}_j + \frac{\partial q_i}{\partial t} - \frac{\partial T_0}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0, \quad i = 1, 2, \ldots, n$$

The term $\gamma_{ij}, \dot{q}_j$ is known as "gyroscopic term" and a system whose equation of motion contain gyroscopic term is known as "gyroscopic system".

To illustrate law routhian procedure can result in gyroscopic terms in the equation of motion

Let us consider a system in which q_1 is ignorable.

Suppose the original lagrangian function is of the form

$$L = \frac{1}{2}\sum\sum m_{ij}\dot{q}_i\dot{q}_j + \sum_{i=1}^{n} a_i\dot{q}_i + T_0 - V \quad \text{------} (1)$$

The generalised momentum p_1 is a Constant of motion

$$p_i = \sum_{j=1}^{n} m_{ij}\dot{q}_j + a_i$$

$$p_1 = \sum_{j=1}^{n} m_{1j}\dot{q}_j + a_1 \quad \text{------} (2)$$

$\Rightarrow p_1 = \beta_1$ ($\because$ It is a constant of motion)

$$\Rightarrow \beta_1 = \sum_{j=1}^{n} m_{1j}\dot{q}_j + a_1$$

$$\dot{q}_1 = \frac{1}{m_{1j}}\left[\beta_1 - a_1 - \sum_{j=2}^{n}(m_{1j}\dot{q}_j)\right] \quad \text{------} (3)$$

The routhian function in this case is

$$R = L - \beta_1\dot{q}_1 \quad \text{------} (4)$$

Substuting equation (1) in (4)

$$R = \frac{1}{2}\sum\sum m_{ij}\dot{q}_i\dot{q}_j + \sum_{i=1}^{n} a_i\dot{q}_i + T_0 - V - \beta_1\dot{q}_1$$

$$= \frac{1}{2}\left(m_{11}\dot{q}_1^2 + 2\sum_{j=2}^{n} m_{1j}\dot{q}_1\dot{q}_j\right) + \frac{1}{2}\sum_{i=2}^{n}\sum_{j=2}^{n} m_{ij}\dot{q}_i\dot{q}_j + a_1\dot{q}_1$$

$$+ \sum_{i=2}^{n} a_j\dot{q}_j + T_0 - V - \beta_1\dot{q}_i \quad \text{------} (5)$$

Substuting equation (3) in (5)

we have, $R = \left(\frac{\beta_1 - a_1}{m_{11}}\right)\sum_{j=2}^{n} m_{ij}\dot{q}_j + \sum_{j=2}^{n} a_j\dot{q}_j$

This equation show that the linear term in $\dot{q}$'s appears in the "routhian function".

Gyeoscopic stability

The response of non-ignored Co-ordinate be an "infinitesimal distance", from a reference equilibrium position at the origin of Configuration space.

If the response of the system remains infinitesimal.

ie, $|q_i| < \epsilon < 1$

Then the system is stable. Otherwise the system is unstable.

Find the necessary and sufficient condition for the stability of a gyeoscopic system with two degree of freedom.

Solution

Consider a gyeoscopic system with two degrees of freedom whose equation of motion are

(6)

$$m_{11}\ddot{q}_1 + m_{12}\ddot{q}_{12} + \gamma_{12}\dot{q}_2 + k_{11}q_1 + k_{12}q_2 = 0$$

$$m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + \gamma_{21}\dot{q}_2 + k_{21}q_1 + k_{22}q_2 = 0$$

$$\Rightarrow m_{12}\ddot{q}_1 + m_{22}\ddot{q}_2 - \gamma_{12}\dot{q}_2 + k_{12}q_1 + k_{22}q_2 = 0$$

Where m and k are symmetric and γ is skew symmetric

Assuming solution of the form,

$$q_j = A_j\, c\, e^{\lambda t}$$

The characteristic equation is

$$|\lambda^2 m - \lambda\gamma + k| = 0$$

Here $m = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$, $\gamma = \begin{bmatrix} 0 & \gamma_{12} \\ -\gamma_{12} & 0 \end{bmatrix}$, $k = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$

$\therefore$ The characteristic equation is

$$\begin{vmatrix} \lambda^2 m_{11} + k_{11} & \lambda^2 m_{12} + \lambda\gamma_{12} + k_{12} \\ \lambda^2 m_{21} - \lambda\gamma_{12} + k_{21} & \lambda^2 m_{22} + k_{22} \end{vmatrix} = 0$$

$$\Rightarrow \lambda^4[m_{11}m_{22} - m_{12}m_{21}] + \lambda^2[m_{11}k_{22} + m_{22}k_{11}] - \lambda^2(m_{21}k_{12})$$
$$+ \lambda^2\gamma_{12}^2 + \lambda^2 m_{12}k_{21} - \lambda(\gamma_{12}k_{21} + \gamma_{12}k_{12}) - \lambda^3(m_{21}\gamma_{12} - m_{12}\gamma_{12})$$
$$+ k_{11}k_{22} - k_{12}k_{21} = 0$$

$\because m_{ij} = m_{ji}$. we have,

$$\lambda^4[m_{11}m_{22} - m_{12}^2] + \lambda^2[m_{11}k_{22} + k_{11}m_{22} - 2k_{12}m_{12} + \gamma_{12}^2]$$
$$+ k_{11}k_{22} - k_{12}^2 = 0.$$

$[\because m_{ij} = m_{ji} \text{ \& } k_{ij} = k_{ji}]$

The system is stable if the ~~origin~~ eigen value λ^2 is negative real and distinct resulting non repeated imaginary roots.

WKT, the kinetic energy is positive definite for any real system.

$$\therefore m_{11}m_{22} - m_{12}^2 > 0$$

$$\Rightarrow k_{11}k_{22} - k_{12}^2 > 0 \quad \text{and}$$

$$[m_{11}k_{22} + k_{11}m_{22} - 2k_{12}m_{12} + \gamma_{12}^2] > \sqrt{(m_{11}m_{22} - m_{12})^2(k_{11}k_{22} - k_{12}^2)}$$

This is necessary and sufficient condition for the stability of the gyroscopic system with two degrees of freedom.

Velocity dependent potential

WKT, The basic form of the lagrangian equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \; , \; i = 1, 2, \ldots, n \quad \text{------} \; (1)$$

Q_i is a generalised force (component)

Suppose that Q's can be obtained from the velocity dependent potential function $V(q, \dot{q}, t)$.

Then,
$$Q_i = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_i}\right) - \frac{\partial V}{\partial q_i} \quad \text{------} \; (2)$$

From equations (1) & (2)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_i}\right) - \frac{\partial V}{\partial q_i}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_i}\right) = \frac{\partial}{\partial q_i}(T-V)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \; , \quad \text{where } L = T - V$$

Electro-magnetic forces

An example of a velocity dependent potential. Consider the electro-magnetic forces acting on a charged particle.

$\bar{F} = e(\bar{E} + \bar{V} \times \bar{B})$, where e - charge, $\bar{V}$ - velocity, $\bar{E} = -\nabla\phi - \frac{\partial \bar{A}}{\partial t}$

and $\bar{B} = \nabla \times \bar{A}$.

ϕ and $\bar{A}$ are functions of position and time

$$\bar{F} = e\left[-\nabla\phi - \frac{\partial \bar{A}}{\partial t} + \bar{V} \times (\nabla \times \bar{A})\right]$$

Let us assume the position of a particle is given by the cartesian co-ordinates (x, y, z)

If we designate the cartesian component of $\bar{A}$ by A_x, A_y, A_z.

$$\nabla \times \bar{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \vec{j}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \vec{k}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$\bar{V} \times (\nabla \times \bar{A}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ V_x & V_y & V_z \\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) & \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) & \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \end{vmatrix}$$

Let us consider the x-component of the term, $\bar{V} \times (\nabla \times \bar{A})$

ie, $$[\bar{V} \times (\nabla \times \bar{A})]_x = V_y\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right] - V_z\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]$$

Adding and substracting $V_x \frac{\partial A_x}{\partial x}$, then we get

$$[\bar{V} \times (\nabla \times \bar{A})]_x = \left[V_y \frac{\partial A_y}{\partial x} + V_z \frac{\partial A_z}{\partial x}\right] - V_y \frac{\partial A_x}{\partial y} - V_z \frac{\partial A_x}{\partial z} + V_x \frac{\partial A_x}{\partial x} - V_x \frac{\partial A_x}{\partial x}$$

$$\Rightarrow [\bar{V} \times (\nabla \times \bar{A})]_x = \left[V_y \frac{\partial A_y}{\partial x} + V_z \frac{\partial A_z}{\partial x}\right] + V_x \frac{\partial A_x}{\partial x} - V_y \frac{\partial A_x}{\partial y} - V_z \frac{\partial A_x}{\partial z} - V_x \frac{\partial A_x}{\partial x}.$$

But $\bar{A}$ is the function of position and time.

$$\frac{d}{dt}(A_x) = \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} + \frac{\partial A_x}{\partial t}$$

ie, $$\frac{d}{dt}(A_x) = \frac{\partial A_x}{\partial x} V_x + \frac{\partial A_x}{\partial y} V_y + \frac{\partial A_x}{\partial z} V_z + \frac{\partial A_x}{\partial t}$$

$$[\bar{V} \times (\nabla \times \bar{A})]_x = \frac{\partial}{\partial x}(\bar{V}.\bar{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t} \quad \text{-----}$$

$$\therefore \frac{\partial}{\partial x}(\bar{V}.\bar{A}) = \frac{\partial}{\partial x}[V_x A_x + V_y A_y + V_z A_z]$$

$$= V_x \frac{\partial A_x}{\partial x} + V_y \frac{\partial A_y}{\partial x} + V_z \frac{\partial A_z}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial x}(\bar{V}.\bar{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t} = V_x \frac{\partial A_x}{\partial x} + V_y \frac{\partial A_y}{\partial x} + V_z \frac{\partial A_z}{\partial x} - V_x \frac{\partial A_x}{\partial x} - V_y \frac{\partial A_y}{\partial y} - V_z \frac{\partial A_z}{\partial z} - \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial t}$$

$$\therefore \frac{\partial}{\partial x}(\bar{V}.\bar{A}) = V_y \frac{\partial A_y}{\partial x} + V_z \frac{\partial A_z}{\partial x} - V_y \frac{\partial A_y}{\partial y} - V_z \frac{\partial A_z}{\partial z}$$

ie, $$\frac{\partial}{\partial x}(\bar{V}.\bar{A}) = \bar{V} \times (\nabla \times \bar{A})$$

$\therefore$ The x-component of the electro-magnetic force

$$F_x = e\left[-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} + \frac{\partial}{\partial x}(\bar{V}.\bar{A}) - \frac{d}{dt}(A_x) + \frac{\partial A_x}{\partial t}\right]$$

$$\therefore \bar{F}_x = e\left[-\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial x}(\bar{v}.\bar{A}) - \frac{d}{dt}(A_x)\right]$$

Next we observe that

$$\frac{d}{dt}(A_x) = \frac{d}{dt}\left[\frac{\partial}{\partial x}(\bar{v}.\bar{A})\right]$$

If we take $U = e(\phi - \bar{v}.\bar{A})$

we obtain,

$$F_x = \frac{d}{dt}\left(\frac{\partial v}{\partial \dot{x}}\right) - \frac{\partial u}{\partial v}$$

Similar expressions occur for F_y, F_z.

Thus the electro magnetic forces on a particle are represented by the velocity dependent potential in U.

Note

The lagrangian function is $L = T - V$.

$$\Rightarrow L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - e(\phi - A_x\dot{x} - A_y\dot{y} - A_z\dot{z})$$

where m is a mass of the particle.

ie, $L = \frac{1}{2}mv^2 - e[\phi - (\bar{v}.\bar{A})]$

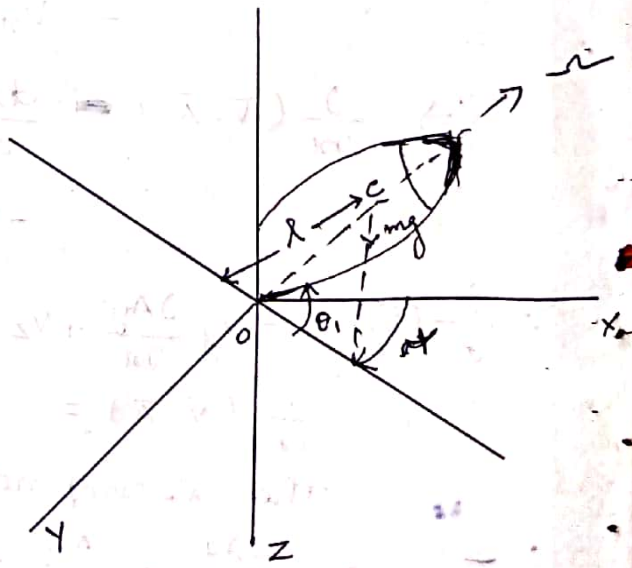
Thm

Consider the motion of a top whose configuration is expressed interms of Eulerian angles. Obtain the differential equations of motion using routhia procedure and show the nature of small motions near a reference condition of a steady precession.

Proof

Let the moment of inertia about the axis symmetry be I_a.

Let the transverse moment of inertia about the fixed point 'o' be I_t.



The axial component of the angular velocity ω is called the total spin Ω and is given by

$$\Omega = \dot{\phi} - \dot{\psi}\sin\theta.$$

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⑧

(4) $\Rightarrow I_t \ddot{\varphi} \cos^2\theta - I_t \dot{\theta}\dot{\varphi}\, 2\sin\theta\cos\theta - \beta_\varphi \cos\theta . \dot{\theta} = 0$ ----- (A)

∴ Lagrangian equation w.r.to θ is

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\theta}}\right) - \frac{\partial R}{\partial \theta} = 0 \quad \text{-----} \ (5)$$

From eqn (3)

$$\frac{\partial R}{\partial \dot{\theta}} = \frac{1}{2} I_t (2\dot{\theta}) \Rightarrow \frac{\partial R}{\partial \dot{\theta}} = I_t \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\theta}}\right) = I_t \ddot{\theta} \text{ and } \frac{\partial R}{\partial \theta} = \frac{1}{2} I_t \left\{\left[\dot{\varphi}^2 2\cos\theta(-\sin\theta)\right] - \beta_\varphi \dot{\varphi}\cos\theta - mgl\cos\theta\right\}$$

(5) $\Rightarrow I_t \ddot{\theta} + I_t \dot{\varphi}^2 \cos\theta \sin\theta + \beta_\varphi \dot{\varphi} \cos\theta + mgl\cos\theta = 0$ ----- (B)

The gyroscopic terms are

$-\beta_\varphi \cos\theta . \dot{\theta}$ in φ equation (of lagrangian) and

$\beta_\varphi \dot{\varphi} \cos\theta$ in θ equation (of lagrangian).

To determine the study precessia rate.

Let us set $\ddot{\theta} = 0$ and solve eqn (B) for $\dot{\varphi}$.

we obtain, $I_t \dot{\varphi}^2 \cos\theta \sin\theta + \beta_\varphi \dot{\varphi} \cos\theta + mgl\cos\theta = 0$ ----- (6)

This is a quadratic equation w.r.to $\dot{\varphi}$

$$\therefore \dot{\varphi} = \frac{-\beta_\varphi \cos\theta \pm \sqrt{\beta_\varphi^2 \cos^2\theta - 4 I_t \cos^2\theta \sin\theta\, mgl}}{2 I_t \cos\theta \sin\theta}.$$

$$\Rightarrow \dot{\varphi} = \frac{-\beta_\varphi \cos\theta \pm \cos\theta \sqrt{\beta_\varphi^2 - 4 I_t \sin\theta\, mgl}}{2 I_t \cos\theta \sin\theta}.$$

$$= \frac{-\beta_\varphi \pm \sqrt{\beta_\varphi^2 - 4 I_t \sin\theta\, mgl}}{2 I_t \sin\theta}.$$

ie, $$\dot{\varphi} = \frac{-\beta_\varphi}{2 I_t \sin\theta}\left[1 \pm \sqrt{1 - \left(\frac{4 I_t \sin\theta\, mgl}{\beta_\varphi^2}\right)}\right]$$

Now there are two possible steady precession rates provided that the square root is real.

ie, $\beta_\varphi^2 > 4 I_t\, mgl \sin\theta$.

The component of the angular velocity ω is the vector sum of the orthogonal components $\dot{\theta}$ and $\dot{\psi}\cos\theta$.

Hence, The total kinetic energy is

$$T = \frac{1}{2} I_a (\dot{\phi} - \dot{\psi}\sin\theta)^2 + \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2\cos^2\theta)$$

and the potential energy, $V = mgl\sin\theta$.

Thus, the Lagrangian function

$$L = T - V$$

$$L = \frac{1}{2} I_a (\dot{\phi} - \dot{\psi}\sin\theta)^2 + \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2\cos^2\theta) - mgl\sin\theta$$

ie, $L = \frac{1}{2} I_a (\dot{\phi}^2 + \dot{\psi}^2\sin^2\theta - 2\dot{\phi}\dot{\psi}\sin\theta) + \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2\cos^2\theta)$
$- mgl\sin\theta$ ---- (1)

Then, Ignore ϕ only

$$P_\phi = I_a(\dot{\phi} - \dot{\psi}\sin\theta) = \beta_\phi$$

$$\Rightarrow \beta_\phi = I_a(\dot{\phi} - \dot{\psi}\sin\theta) \Rightarrow \beta_\phi = I_a \Omega$$

$$\Rightarrow \dot{\phi} = \frac{\beta_\phi}{I_a} + \dot{\psi}\sin\theta \quad \text{-----} (2)$$

$$\Rightarrow R = L - \beta_\phi \dot{\phi}$$

$$= \frac{1}{2}\left(\frac{\beta_\phi^2}{I_a}\right) + \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2\cos^2\theta) - mgl\sin\theta - \frac{\beta_\phi^2}{I_a} - \beta_\phi\dot{\psi}\sin\theta$$

ie, $R = \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2\cos^2\theta) - \beta_\phi\dot{\psi}\sin\theta - mgl\sin\theta - \frac{\beta_\phi^2}{2I_a}$

We have omitted the constant terms $\left(\frac{-\beta_\phi^2}{I_a}\right)$.

$$\therefore R = \frac{1}{2} I_t (\dot{\theta}^2 + \dot{\psi}^2\cos^2\theta) - \beta_\phi\dot{\psi}\sin\theta - mgl\sin\theta \quad \text{----} (3)$$

The Lagrangian equation w.r.to ψ is

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\psi}}\right) - \frac{\partial R}{\partial \psi} = 0 \quad \text{----} (4)$$

From equation (3)

$$\frac{\partial R}{\partial \psi} = 0 \quad \text{and} \quad \frac{\partial R}{\partial \dot{\psi}} = \frac{1}{2} I_t (2\dot{\psi}\cos^2\theta) - \beta_\phi\sin\theta$$

$$\therefore \frac{\partial R}{\partial \dot{\psi}} = I_t\dot{\psi}\cos^2\theta - \beta_\phi\sin\theta.$$

ie, $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\psi}}\right) = I_t\ddot{\psi}\cos^2\theta + I_t\dot{\theta}\dot{\psi}2(-\sin\theta)\cos\theta - \beta_\phi\cos\theta.\dot{\theta}$

$$\therefore \Omega^2 > \frac{4 I_t mgl \sin\theta}{I_a^2} \qquad (\because \beta_\phi = I_a \Omega)$$

ie, the magnitude of the total spin must be large enough to provide gyroscopic stabilization.

For the common case of large spin

$$\Omega^2 >> \frac{4 I_t mgl \sin\theta}{I_a^2}$$

The slower precession rate is the one usually observed

$$\dot{\psi} = -mgl / \beta_\phi = \frac{-mgl}{I_a \Omega}$$

principle of least action

statement

The actual path of a conservative holonomic system in the configuration space is such that the action is stationary with respect to { valid paths } having the same end points in the configuration space.

Proof

Consider a conservative holonomic system. we assume that δq's are consistent with the instantaneous constraints.

we consider the most general variation is

$$I = \int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt$$

$$dI = \int_{t_0}^{t_1} \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i\right) dt + \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial t} \delta t - \left(\sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L\right)\right]$$

$$\frac{d}{dt}(\delta t)\, dt - \int_{t_0}^{t_1} \left[\sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right] \delta q_i\, dt \quad \text{----} (1)$$

If all the applied forces are derivable from a potential in $V(q)$,

we have, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$ is lagrangian equation

Hence the ~~the~~ third integral on the right side of (1) vanishes

If the valid paths have fixed end points in the configuration space.

We have, $\int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\,\delta q_i\right) dt = \left[\frac{\partial L}{\partial \dot{q}_i}\,\delta q_i\right]_{t_0}^{t_1} = 0$.

Hence the first integral in (1) vanishes.

Now suppose we continue valid paths whiles have an energy integral

$$\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\,\dot{q}_i - L = h, \text{ where } h \text{ is a constant.}$$

Also we assume that the variations are non_contemporaneous in which $\delta t \neq 0$.

Since $\frac{\partial L}{\partial t} = 0$ for a conservative system (1) reduces to.

$$\partial I = -\int_{t_0}^{t_1} h \frac{d}{dt}(\delta t) = -h(\delta t_1 - \delta t_0) \quad \text{------} (2)$$

Now action is defined as

$$A = \int_{t_0}^{t_1} \left\{\sum_i p_i \dot{q}_i\right\} dt \text{ (i.e.) } A = \int_{t_0}^{t_1} (L+h)\, dt$$

For the assumed path variation, we obtain,

$$\delta A = \delta\int_{t_0}^{t_1} (L+h)\, dt = \delta\int_{t_0}^{t_1} L\,dt + \delta\int_{t_0}^{t_1} h\,dt.$$

$$= \delta I + \delta h\int_{t_0}^{t_1} dt + h\,\delta\int_{t_0}^{t_1} \delta(dt)$$

$$= \delta I + \delta h\,(t_1 - t_0) + h\,(\delta t)_{t_0}^{t_1}$$

$$= \delta I + \delta h\,(t_1 - t_0) + h\,(\delta t_1 - \delta t_0)$$

$$\Rightarrow \delta A = -h(\delta t_1 - \delta t_0) + \delta h\,(t_1 - t_0) + h(\delta t_1 - \delta t_0) + h(\delta t_1 - \delta t_0) \quad \text{-----} (3)$$

If we restrict the valid path to there for which 'h' has the same values as the actual path

we have, $\delta h = 0$ this leads to $\delta A = 0$

Hence $\delta A = \delta\int_{t_0}^{t_1} \sum p_i \dot{q}_i\, dt = 0$.

This is called the principle of least action.

HAMILTON'S EQUATIONS

HAMILTON'S PRINCIPLE :-

BOOK WORK

To find the stationary values of the function $f(q_1, q_2, \ldots, q_n)$

SOLUTION

Consider the function $f(q_1, q_2, \ldots, q_n)$, assumed to be continuous through the second partial derivatives.

The first variation of f at the reference point q_0 is

$$\delta q = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\right) \delta q_i$$

, where δq's are the variations in the individual q's.

The necessary and sufficient condition that f has a stationary value at $\bar{q}_0$ is that $\delta f = 0$ at all geometrically positive δq's, where $\bar{q} = \bar{q}_0 + \bar{\delta q}$.

If δq's are independent and reversible then $\left(\frac{\partial f}{\partial q_i}\right)_0 = 0$, $i = 1, 2, \ldots, n$.

The second variation of the function f about the stationary point $\bar{q}_0$ is

$$\partial^2 f = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 f}{\partial q_i \partial q_j}\right)_0 \delta q_i \, \delta q_j \quad \text{at } q_0.$$

$$\therefore \partial^2 f = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (K_{ij}) \, \delta q_i \, \delta q_j, \quad \text{where } K_{ij} = \left(\frac{\partial^2 f}{\partial q_i \partial q_j}\right)_0$$

Form the elements of symmetric $n \times n$ matrix K.

If δq's are again independent and reversible, then the condition $\bar{q}_0$ to the local minimum if the matrix K must be positive definite.

Conversely, if K is negative definite, then the point $\bar{q}_0$ is the local maximum.

If K is indefinite, then q_0 is a saddle point.

Book WORK

To find the stationary value of the function $f(q_1, q_2, \ldots, q_n)$ subject to m constraints of the form $\phi_j(q_1, q_2, \ldots, q_n) = 0$

Proof:-

Consider the function $F(q_1, q_2, \ldots, q_n, \lambda_1, \lambda_2, \ldots, \lambda_m)$ is defined by

$$F = f + \sum_{j=1}^{m} \lambda_j \phi_j$$, where λ_j are called lagrangian multipliers.

Here, we consider n q's and m λ's has independent variables.

The necessary and sufficient condition for F to be stationary are

$$\left(\frac{\partial F}{\partial q_i}\right)_0 = 0, \quad i = 1, 2, \ldots, n \quad \& \quad \left(\frac{\partial F}{\partial \lambda_j}\right) = 0, \quad j = 1, 2, \ldots, n$$

$$\left(\frac{\partial f}{\partial q_i}\right)_0 + \sum_{j=1}^{m} \lambda_j \left(\frac{\partial \phi}{\partial q_i}\right)_0 = 0, \quad i = 1, 2, \ldots, n \quad \text{----} (1)$$

and $\phi_j(q_1, q_2, \ldots, q_n) = 0, \quad j = 1, 2, \ldots, m$ ---- (2)

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When the assume that the n equations in eqn ① are consistence $m\lambda$'s can be found. ②

If $C_{ij} = \left(\frac{\partial \phi_j}{\partial q_i}\right)$.

Then the $m \times n$ matrix $C = C_{ij}$ is of rank m.

Since eqn ② are all independent constraints that is the stationary values of $f(q_1, q_2, \dots, q_n)$ subject to m constraints $\phi_j(q_1, q_2, \dots, q_n)$ is solved by finding $m\lambda$'s and nq's from $(m+n)$ equations given by equations ① & ②.

BOOK WORK :-

To find the stationary values of the function $f = z$ subject to the constraints $\phi_1 = x^2 + y^2 + z^2 - 4 = 0$, $\phi_2 = xy - 1 = 0$.

ie, To find the higest and lowest points on the curve formed by the intersection of the sphere and the hyperbolic cylinder.

Solution

Given the function is $f = z$ -----①

Constraints are $\phi_1 = x^2 + y^2 + z^2 - 4 = 0$ ----②

and $\phi_2 = xy - 1 = 0$ ----③

The augmend function, $F = f + \sum_{i=1}^{2} \lambda_i \phi_i$

ie, $F = z + \lambda_1(x^2 + y^2 + z^2 - 4) + \lambda_2(xy - 1)$

The stationary values are given by

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0 \text{ and } \frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial \lambda_1} = 0, \quad \frac{\partial F}{\partial \lambda_2} = 0$$

ie, $2x\lambda_1 + y\lambda_2 = 0$ ----- (4)

$2y\lambda_1 + x\lambda_2 = 0$ ----- (5)

$1 + 2z\lambda_1 = 0$ ----- (6)

Eliminating λ_1 and λ_2 from equations (4) & (5)

$$\begin{vmatrix} 2x & y \\ 2y & x \end{vmatrix} = 0$$

$\Rightarrow 2x^2 - 2y^2 = 0$

$\Rightarrow 2(x^2 - y^2) = 0$

$\Rightarrow x^2 - y^2 = 0$

$\Rightarrow x^2 = y^2$

$\Rightarrow x = y$

From eqn (2)

$x^2 + x^2 + z^2 - 4 = 0$

$2x^2 + z^2 - 4 = 0$ ---- (7)

From eqn (3)

$x^2 - 1 = 0$

$x^2 = 1 \Rightarrow x = \pm 1$

When $x = \pm 1$

(7) $\Rightarrow z = \pm\sqrt{2}$.

$\therefore$ We have four points $(1, 1, \sqrt{2}), (1, 1, -\sqrt{2}), (-1, -1, \sqrt{2})$ and $(-1, -1, -\sqrt{2})$.

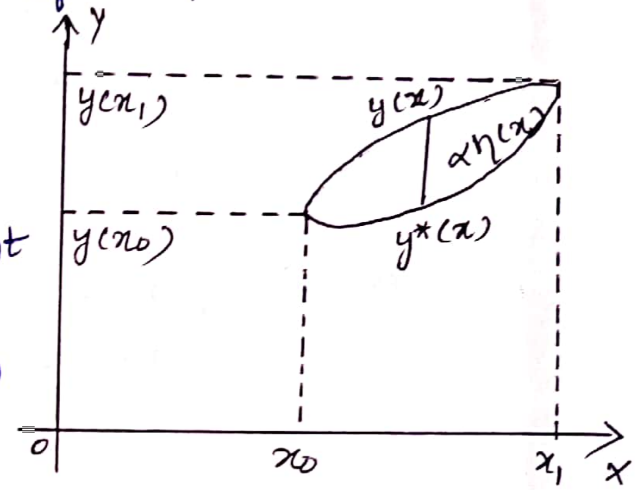
$(1, 1, \sqrt{2})$ & $(-1, -1, \sqrt{2})$ are the constraint maximum points of $f = z$.

$(1, 1, -\sqrt{2})$ & $(-1, -1, -\sqrt{2})$ are the constraint minimum points of $f = z$.

Where $\eta(x)$ is an arbitrary function having the required smoothness and α is a small parameter which does not depend upon x. 3

$\therefore$ y is an function of α & x

ie, $y(\alpha, x) = y^*(x) + \alpha\eta(x)$



Now, Let us make the addiontional assumption that

$\eta(x_0) = 0 = \eta(x_1)$

$\Rightarrow y(x_0)$ and $y(x_1)$ are fixed

Now, for a given $\eta(x)$,

I is a function of α only

$\therefore$ The necessary Condition that $y^*(x)$ result in a stationary value of I is that $\delta I = 0$.

ie, $\left(\frac{dI}{d\alpha}\right)_{\alpha \neq 0} \delta\alpha = 0$ an arbitrary $\eta(x)$ and non zero α.

Since x_0 and x_1 are not dependent on α

We can differentiable and integral sign.

$$\therefore \frac{dI}{d\alpha} = \int_{x_0}^{x_1} \left(\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial \alpha}\right) dx$$

$$\frac{dI}{d\alpha} = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\right] dx$$

ie, $$\int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\right] dx = 0 \quad \text{-----} (1)$$

$[\because y(x,\alpha) = y^* + \alpha\eta'(x)]$

Now Consider,

$$\int_{x_0}^{x_1} \frac{\partial f}{\partial y'}\eta'(x)dx = \left[\frac{\partial f}{\partial y'}\eta(x)\right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx$$

$[\because \int u\,dv = uv - \int v\,du]$

$$= 0 - \int_{x_0}^{x_1} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx$$

The lagrange's multipliers are

(6) $\Rightarrow 1 + 2z\lambda_1 = 0$

$$\lambda_1 = \frac{-1}{2z}$$

$$= \frac{-1}{2(\pm\sqrt{2})}$$

$$\lambda_1 = \mp \frac{1}{2\sqrt{2}}$$

(4) $\Rightarrow y\lambda_2 = -2x\lambda_1$

$$\lambda_2 = -2\lambda_1 \qquad [\because x = y$$

$$\lambda_2 = -2\left(\mp\frac{1}{2\sqrt{2}}\right)$$

$$\lambda_2 = \pm\frac{1}{\sqrt{2}}$$

BOOK WORK :-

Derive the Euler lagrange equation for a single dependent variable. (OR)

To find the stationary values of a definite integral $I = \int_{x_0}^{x_1} f[y(x), y'(x), x]\, dx$,

where $y'(x) = \frac{dy}{dx}$ and the elements x_0 and x_1 are fixed.

Proof :-

The given definite integral $I = \int_{x_0}^{x_1} f[y(x), y'(x), x]\, dx$,

Where $f(y, y', x)$ has two continuous derivatives in each of its elements.

Let $y(x) = y^*(x) + \delta y(x)$, where $\delta y(x)$ is a small variation in y.

In this convenient to represent the variation δy in the form $\delta y = \alpha\eta(x)$.

ie, $$\int_{x_0}^{x_1} \frac{\partial f}{\partial y'} \eta'(x)\,dx = -\int_{x_0}^{x_1} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\,dx \quad \text{-----} (2)$$

Substuting equ (2) in (1).

$$\int_{x_0}^{x_1} \frac{\partial f}{\partial y} \eta(x)\,dx - \int_{x_0}^{x_1} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\,dx = 0$$

ie, $$\int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\,dx = 0 \quad \text{-----} (3)$$

As $\eta(x)$ is arbitrary, the necessary condition for integral (3) to be zero.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad \text{-----} (4)$$

The sufficient condition is apparent from the fact that (4) implies that the integral in equ (2) vanishes, resulting the variation δI to be zero.

$$\therefore \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

is necessary and sufficient condition for the stationary values of integral I.

This is called the Euler Lagrange's equation.

Brochitochrome problem

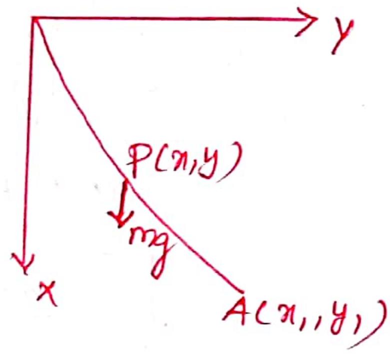
Statement

To find a curve $y(x)$ with the region 'o' and the point (x_1, y_1) such that the particle starting from restato and sliding down the curve without friction and the influence of uniform gravitational field, will reach the end of the curve of minimum time

Proof

Let us assume that the gravitational force is directed along the positive x axis as shown in figure.

By the Conservation of energy



$\frac{1}{2}mv^2 = mgx \Rightarrow v^2 = 2gx$

$\Rightarrow v = \sqrt{2gx}$ ----- (1)

Let ds be the infinitesimal length it is

$ds^2 = dx^2 + dy^2$

$ds^2 = dx^2\left[1 + \frac{dy^2}{dx^2}\right]$

$\frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2} \Rightarrow \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

$(ds)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right](dx)^2 \Rightarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2 (dx^2)}$

$\therefore \; ds = \sqrt{1 + y'^2}\, dx$ ----- (2)

The time required to each the point (x_1, y_1) is found

$t = \int_{x=0}^{x=x_1} \frac{ds}{v}$ $\quad \left[\because ds = vdt \Rightarrow \int dt = \int \frac{ds}{v}\right.$

$= \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2gx}}\, dx$ $\quad \left[\because \text{Using eqns (1) \& (2)}\right.$

This is of the form $I = \int_{x_0}^{x_1} f(y, y', x)\, dx.$

where $f(y, y', x) = \frac{\sqrt{1+y'^2}}{\sqrt{2gx}}$ ----- (3)

Euler Lagrangian is given by

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ ----- (4)

(3) $\Rightarrow \frac{\partial f}{\partial y} = 0$

(4) $\Rightarrow 0 - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$\Rightarrow -\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$\Rightarrow \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$\Rightarrow \frac{\partial f}{\partial y'} = \text{Constant}.$

$$\frac{\partial}{\partial y'}\sqrt{\frac{1+y'^2}{2gx}} = c \Rightarrow \frac{\partial}{\partial y'}\left[\frac{\sqrt{1+y'^2}}{\sqrt{2gx}}\right] = c$$ (5)

$$\frac{1}{\sqrt{2gx}}\frac{\partial}{\partial y'}\left[\sqrt{1+y'^2}\right] = c \Rightarrow \frac{1}{\sqrt{2gx}}\frac{1}{2\sqrt{1+y'^2}}(2y') = c$$

$$\therefore \frac{y'}{\sqrt{2gx}\sqrt{1+y'^2}} = c \Rightarrow y' = c\sqrt{2gx}\sqrt{1+y'^2}$$

Squaring on both sides

$$y'^2 = c^2(2gx)(1+y'^2) \Rightarrow y'^2 = 2gxc^2 + 2gxc^2y'^2$$

$$y'^2 - 2gxc^2y'^2 = 2gxc^2 \Rightarrow y'^2\left[1-c^2 2gx\right] = 2gxc^2$$

$$y'^2 = \frac{2gxc^2}{1-c^2 2gx} \Rightarrow y' = \frac{\sqrt{2gxc^2}}{\sqrt{1-2gxc^2}}$$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{2gxc^2}}{\sqrt{1-2gxc^2}} \text{ ----- } (5)$$

Put $x = a(1-\cos\theta)$, where $a = \frac{1}{4gc^2}$

$$\frac{dx}{d\theta} = a(-(-\sin\theta))$$

$$dx = a\sin\theta\, d\theta.$$

From eqn (5)

$$dy = \frac{\sqrt{2gxc^2}}{\sqrt{1-2gxc^2}}dx.$$

$$= \frac{\sqrt{2g\frac{1}{4gc^2}(1-\cos\theta)c^2}}{\sqrt{1-2g\frac{1}{4gc^2}(1-\cos\theta)c^2}}a\sin\theta\, d\theta$$

$$= \frac{\sqrt{\frac{1-\cos\theta}{2}}}{\sqrt{1-\frac{1}{2}(1-\cos\theta)}}a\sin\theta\, d\theta = \frac{\sqrt{\frac{1-\cos\theta}{2}}}{\sqrt{\frac{2-(1-\cos\theta)}{2}}}a\sin\theta\, d\theta$$

$$= \frac{\sqrt{1-\cos\theta}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2-1+\cos\theta}}a\sin\theta\, d\theta$$

$$\therefore dy = \frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}\, a\sin\theta\, d\theta \Rightarrow dy = \frac{\sqrt{1-\cos\theta}\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}\sqrt{1-\cos\theta}}\, a\sin\theta\, d\theta$$

ie, $$dy = \frac{1-\cos\theta}{\sqrt{1-\cos^2\theta}}\, a\sin\theta\, d\theta \Rightarrow dy = \frac{1-\cos\theta}{\sqrt{\sin^2\theta}}\, a\sin\theta\, d\theta$$

$$dy = \frac{1-\cos\theta}{\sin\theta}\, a\sin\theta\, d\theta \Rightarrow dy = a(1-\cos\theta)\, d\theta$$

Integrating on both sides,

$$\int dy = a\int(1-\cos\theta)\, d\theta \Rightarrow y = a(\theta - \sin\theta)$$

As the path starts from '0' the constant of integration is zero.

$x = a(1-\cos\theta)$, $y = a(\theta - \sin\theta)$ represents a Cycloid.

The parameter θ increases continuously as the particle proceeds along this path even through x may decreases during in the lateral p.

A constant 'a' can be choosen such that the path goes to the final point (x_1, y_1), where $x_1 > 0$. So along this curve the time of travel is minimum.

Geodesic problem

Statement

To find the shortest path between two points in the given space.

ie, To find the path of the minimum length between two given points on the two dimensional surface of a sphere of radius 'r'.

Solution

This is the problem for finding the shortest path between the two points on the surface.

Now, we consider with two points on the two dimensional surface of a sphere of radius 'r'

Here the use spherical Co-ordinates (θ, ϕ) as the variables. (6)

The infinitesimal ds is given by

$$ds^2 = dx^2 + dy^2$$

$$= (r\,d\theta)^2 + (r\sin\theta\, d\phi)^2 \Rightarrow ds^2 = r^2(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2$$

$$ds^2 = r^2(d\theta)^2 \cdot \left[1 + \sin^2\theta \frac{(d\phi)^2}{d\theta^2}\right] \Rightarrow ds^2 = r^2(d\theta)^2\left[1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right]$$

Square on both sides.

$$ds = r\sqrt{1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\, d\theta \quad \text{-----} (1)$$

Integrating between the limits θ_0 & θ_1

$$\int ds = r\int_{\theta_0}^{\theta_1} \sqrt{1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\, d\theta$$

$$\therefore s = r\int_{\theta_0}^{\theta_1} \sqrt{1 + \sin^2\theta\,\phi'^2}\, d\theta, \quad \text{where } \phi' = \frac{d\phi}{d\theta}.$$

This is of the form $I = \int_{x_0}^{x_1} f(y, y', x)\, dx$.

where $f = \sqrt{1 + \sin^2\theta\,\phi'^2}$ ------ (2)

The Euler Lagrangian equation is given by

$$\frac{\partial f}{\partial \phi} - \frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) = 0 \quad \text{-----} (3)$$

$$(2) \Rightarrow \frac{\partial f}{\partial \phi} = 0$$

$$(3) \Rightarrow 0 - \frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) = 0 \Rightarrow \frac{-d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) = 0$$

$$\therefore \frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) = 0 \Rightarrow \frac{\partial f}{\partial \phi'} = c \quad (\text{Constant})$$

$$\frac{\partial}{\partial \phi'}\left(\sqrt{1 + \sin^2\theta\,\phi'^2}\right) = c \Rightarrow \frac{1}{2\sqrt{1 + \sin^2\theta\,\phi'^2}}(2\phi'\sin^2\theta) = c$$

$$\phi'\sin^2\theta = c\sqrt{1 + \sin^2\theta\,\phi'^2}$$

Squaring on both sides

$$(\phi'\sin^2\theta)^2 = c^2(1 + \sin^2\theta\,\phi'^2) \Rightarrow \phi'^2\sin^4\theta = c^2 + c^2\phi'^2\sin^2\theta$$

ie, $\phi'^2\sin^4\theta - c^2\phi'^2\sin^2\theta = c^2 \Rightarrow \phi'^2\sin^2\theta\left[\sin^2\theta - c^2\right] = c^2$

$$\phi'^2 = \frac{c^2}{\sin^2\theta\left[\sin^2\theta - c^2\right]} \Rightarrow \phi' = \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}$$

i.e, $\frac{d\phi}{d\theta} = \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}} \Rightarrow d\phi = \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}\, d\theta$

$\therefore\ d\phi = \frac{c\sin\theta}{\sin^2\theta\sqrt{\sin^2\theta - c^2}}\, d\theta \Rightarrow d\phi = \frac{c\sin\theta}{\sin^3\theta\sqrt{1-\frac{c^2}{\sin^2\theta}}}\, d\theta$

i.e, $d\phi = \frac{c\,\mathrm{cosec}^2\theta}{\sqrt{1-c^2\,\mathrm{cosec}^2\theta}}\, d\theta \Rightarrow d\phi = \frac{c\,\mathrm{cosec}^2\theta}{\sqrt{1-c^2(1+\cot^2\theta)}}\, d\theta$

$= \frac{c\,\mathrm{cosec}^2\theta}{\sqrt{(1-c^2)-c^2\cot^2\theta}}\, d\theta \Rightarrow d\phi = \frac{c\,\mathrm{cosec}^2\theta}{c\sqrt{\frac{1-c^2}{c^2}-\cot^2\theta}}\, d\theta$

i.e, $d\phi = \frac{\mathrm{cosec}^2\theta}{\sqrt{\frac{1-c^2}{c^2}-\cot^2\theta}}\, d\theta$

Integrating on both sides.

$\int d\phi = \int \frac{-dx}{\sqrt{\frac{1-c^2}{c^2}-x^2}}$

$\left[\because \text{put } x = \cot\theta \quad \frac{dx}{d\theta} = -\mathrm{cosec}^2\theta \quad -dx = \mathrm{cosec}^2\theta\, d\theta\right.$

$\left[\because \int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) \text{ \& } \int\frac{-dx}{\sqrt{a^2-x^2}} = \cos^{-1}\left(\frac{x}{a}\right)\right.$

$\therefore\ \phi = \cos^{-1}\left[\frac{x}{\sqrt{\frac{1-c^2}{c^2}}}\right] + \phi_0$

$\phi = \cos^{-1}\left[\frac{x}{\frac{\sqrt{1-c^2}}{c}}\right] + \phi_0$

$\therefore\ \phi - \phi_0 = \cos^{-1}\left[\frac{cx}{\sqrt{1-c^2}}\right]$

i.e, $\phi - \phi_0 = \cos^{-1}\left(\frac{c\cot\theta}{\sqrt{1-c^2}}\right)$

$\Rightarrow \cos(\phi-\phi_0) = \frac{c\cot\theta}{\sqrt{1-c^2}}$ ----- (4)

Change into cartesian Co-ordinates

$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$ ---- (5)

From eqn (4)

$\cos\phi\cos\phi_0 + \sin\phi\sin\phi_0 = \frac{c}{\sqrt{1-c^2}}\cot\theta$

Multiplying by r on both sides

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$$r\cos\phi\cos\phi_0 + r\sin\phi\sin\phi_0 = \frac{c}{\sqrt{1-c^2}}\, r\,\frac{\cos\theta}{\sin\theta} \quad (7)$$

$$r\sin\theta\cos\phi\cos\phi_0 + r\sin\theta\sin\phi\sin\phi_0 = \frac{c}{\sqrt{1-c^2}}\, r\cos\theta \text{ ----- } (6)$$

Substituting equ (5) in (6)

$$x\cos\phi_0 + y\sin\phi_0 = \frac{c}{\sqrt{1-c^2}}\, z$$

$$\Rightarrow x\cos\phi_0 + y\sin\phi_0 - \frac{c}{\sqrt{1-c^2}}\, z = 0$$

This equation of plane through the origin.

This plane intersects of a sphere in a great circle which is geodesic of the problem.

A constant c and ϕ_0 are choosen such that the curve goes through the required two points.

BOOK WORK

A necessary and sufficient condition that

$$I = \int_{x_0}^{x_1} f(y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n', x)\,dx$$ has a stationary value. (OR)

Derive the Euler lagrange equation for $f(y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n', x)$ having n independent variables.

Proof

The problem is to find the functions $y_1(x), y_2(x), \dots, y_n(x)$ which lead to a stationary value of

$$I = \int_{x_0}^{x_1} f(y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n', x)\,dx$$

Where the values of each function $y_1(x)$ are specified at the fixed end points x_0 and x_1.

We also assume that $y_1(x)$ and the variations $\delta y_i(x)$ have two continuous derivatives.

Let the variations be of the form $\delta y_i(x) = \alpha\eta_i(x)$,

where $\eta_i(x_0) = \eta_i(x_1) = 0$

For any given set of η. I is a function of the parameter α.

$\therefore$ For a stationary value of I.

$$\delta I = 0$$

ie, $$\frac{dI}{d\alpha} = 0$$

$$\int_{x_0}^{x_1} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i} \frac{\partial y_i}{\partial \alpha} + \frac{\partial f}{\partial y_i'} \frac{\partial y_i'}{\partial \alpha} \right) dx = 0$$

$$\int_{x_0}^{x_1} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i} \eta_i(x) + \frac{\partial f}{\partial y_i'} \eta_i'(x) \right) dx = 0 \quad \text{-----} (1)$$

Consider, $$\int_{x_0}^{x_1} \left(\frac{\partial f}{\partial y_i'} \eta_i'(x) \right) dx = \int_{x_0}^{x_1} \frac{\partial f}{\partial y_i'} d(\eta_i(x))$$

$$= \left(\frac{\partial f}{\partial y_i'} \eta_i(x) \right)_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta_i(x) \frac{d}{dx} \left(\frac{\partial f}{\partial y_i'} \right) dx$$

$$= (0 - 0) - \int_0^\infty \eta_i(x) \frac{d}{dx} \left(\frac{\partial f}{\partial y_i'} \right) dx$$

$$\therefore \int_{x_0}^{x_1} \left(\frac{\partial f}{\partial y_i'} \eta_i(x) \right) dx = -\int_0^\infty \eta_i(x) \frac{d}{dx} \left(\frac{\partial f}{\partial y_i'} \right) dx \quad \text{-----} (2)$$

Substuting eqn (2) in (1)

$$\int_{x_0}^{x_1} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i} \eta_i(x) - \eta_i(x) \frac{d}{dx} \left(\frac{\partial f}{\partial y_i'} \right) \right) dx = 0$$

$$\int_{x_0}^{x_1} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial f}{\partial y_i'} \right) \right) \eta_i(x)\, dx = 0$$

We have chosen δy_i's and hence $\eta_i(x)$'s to be independent.

$$\therefore \frac{\partial f}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial f}{\partial y_i'} \right) = 0 \quad , \quad i = 1, 2, \ldots, n.$$

These n-equations are the necessary and sufficient condition that $\delta I = 0$

ie, These are the conditions for which I may have stationary values.

BOOK WORK

Obtain Hamilton's principle for a holonomic dynamical system (OR)

The actual path in the configuration space followed by a holonomic dynamical system during a fixed interval t_0 to t_1 is such that $I = \delta \int_{t_0}^{t_1} L\,dt$ is stationary with respect to the path variations, which vanish at the end points.

Proof

Consider a system of N particles whose configuration relative to an inertial frame is given by vectors $\bar{r}_1, \bar{r}_2, \dots, \bar{r}_N$.

Using lagrange's form of D'Alembert's principle

$$\sum_{i=1}^{N} (F_i - m_i \ddot{\bar{r}}_i)\, \delta \bar{r}_i = 0 \quad \text{-----} (1)$$

Where F_i is the applied force acting on the i^{th} particle.

We assume that the virtual displacement $\delta \bar{r}_i$ are reversible and consistent with the instantaneous constraints which are considered to be workless.

The variation in the kinetic energy is

$$\delta T = \delta \left(\frac{1}{2} \sum_{i=1}^{N} m_i \dot{\bar{r}}_i^{\,2} \right)$$

ie, $$\delta T = \frac{1}{2} \sum_{i=1}^{N} m_i\, 2\dot{\bar{r}}_i\, \delta \dot{\bar{r}}_i$$

$$\therefore \delta T = \sum_{i=1}^{N} m_i \dot{\bar{r}}_i\, \delta \dot{\bar{r}}_i \quad \text{-----} (2)$$

But, $$\frac{d}{dt}\left(\sum_{i=1}^{N} m_i \dot{\bar{r}}_i\, \delta \bar{r}_i \right) = \sum_{i=1}^{N} m_i \ddot{\bar{r}}_i\, \delta \bar{r}_i + \sum_{i=1}^{N} m_i \dot{\bar{r}}_i\, \delta \dot{\bar{r}}_i \quad \text{----} (3)$$

Substuting eqns (1) & (2) in (3)

$$\frac{d}{dt}\left(\sum_{i=1}^{N} m_i \dot{\bar{r}}_i \, \delta \bar{r}_i\right) = \sum_{i=1}^{N} F_i \, \delta \bar{r}_i + \delta T$$

If δw is the virtual work of the applied force.

Then, $\delta w = \sum_{i=1}^{N} F_i \, \delta \bar{r}_i$

$$\therefore \frac{d}{dt}\left(\sum_{i=1}^{N} m_i \dot{\bar{r}}_i \, \delta \bar{r}_i\right) = \delta w + \delta T$$

Integrating this equation w.r.t time between t_0 and t_1.

We get, $\int_{t_0}^{t_1} (\delta w + \delta T)\, dt = \left(\sum_{i=1}^{N} m_i \dot{\bar{r}}_i \, \delta \bar{r}_i\right)_{t_0}^{t_1}$

If we assume that the variations δ are zero at t and t_1.

Then, $\int_{t_0}^{t_1} (\delta w + \delta T)\, dt = 0$.

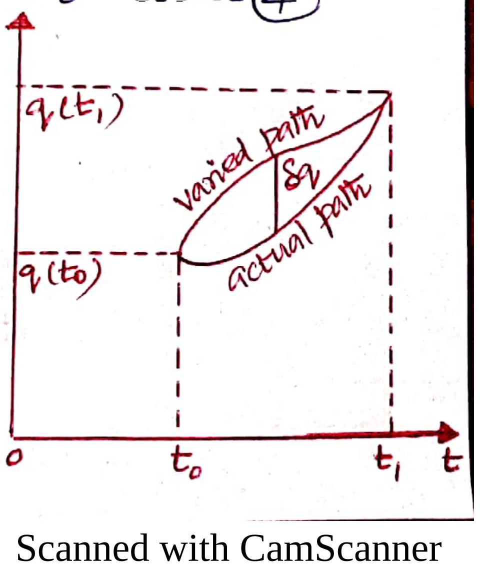
For a given virtual displacement and time the value of δw and δT are independent of the Co-ordinates.

So, Let us make a transformation to generalised Co-ordinates $q_1, q_2, \ldots, q_N$, then the Kinetic energy is the function of q's and $\dot{q}$'s.

$\therefore$ The virtual work is $\delta w = \sum_{i=1}^{N} Q_i \, \delta q_i$

where Q_i's are the applied generalised forces and δq's are zero at t_0 and t_1.

$$\int_{t_0}^{t_1} \left(\delta T + \sum_{i=1}^{N} Q_i \, \delta q_i\right) dt = 0 \quad \text{------ (4)}$$



The actual and varied path in an (N+1) dimensional space consisting of N q's and t is shown in figure.

We observe that the two end points of the actual and varied path are fixed in an extended

Configuration's space.

Equation (4) is called the generalised version of the hamilton's principle.

If we again assume that all the applied forces are derivable from the potential function $V(q,t)$.

Then $\delta w = -\sum_{i=1}^{3N} \frac{\partial V}{\partial x_j} \delta x_j = -\delta V$

$$(4) \Rightarrow \int_{t_0}^{t_1} (\delta T - \delta V)\, dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} \delta (T-V)\, dt = 0$$

For a holonomic system operations of integration and variation can be interchange.

$$\therefore \delta \int_{t_0}^{t_1} (T-V)\, dt = 0$$

If $T-V = L$, then $I = \delta \int_{t_0}^{t_1} L dt = 0$.

Where both the actual and varied paths meet the Conditions imposed by holonomic Constraints.

NOTE

We have shown that if

$$I = \int_{x_0}^{x_1} f(y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n', x)\, dx$$

The Euler lagrange's equations are

$$\frac{\partial f}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) = 0$$

ie, $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$

ie, The standard form of lagrange's equation for the necessary and sufficient Condition for $\delta I = 0$

BOOK WORK

To show that the Hamilton's principle is given by $\delta \int_{t_0}^{t_1} L\,dt = 0$ is valid only for holonomic system (OR)

To show that $\delta \int_{t_0}^{t_1} L\,dt = 0$ applied to non-holonomic System but not on variational principle.

Proof

Suppose there are n generalised co-ordinates $q_1, q_2, \ldots, q_n$ and m non-holonomic constraints is given by

$$\sum_{i=1}^{n} a_{ji}(q,t)\,\dot{q}_i + a_{jt}(q,t) = 0 \quad , j = 1, 2, \ldots, m \quad \text{------} (1)$$

Let $q_i^*(t)$ denote the actual path and $q(t)$ denote the varied path.

$$\therefore q_i = q_i^* + \delta q_i \quad \& \quad \dot{q}_i = \dot{q}_i^* + \delta \dot{q}_i \quad \text{------} (2)$$

We assume that the actual and varied paths conform to the constraints (1)

We can represent a's by Taylor's expansion

$$a_{ji}(q,t) = a_{ji}(q^*,t) + \sum_{k=1}^{n}\left(\frac{\partial a_{ji}}{\partial q_k}\right)_0 \delta q_k .$$

and $$a_{jt}(q,t) = a_{jt}(q^*,t) + \sum_{k=1}^{n}\left(\frac{\partial a_{jt}}{\partial q_k}\right)_0 \delta q_k .$$

Neglecting higher order derivates

Also from equ (1)

$$\sum_{i=1}^{n} a_{ji}(q^*,t)\,\dot{q}_1^* + a_{jt}(q^*,t) = 0 \quad , j = 1, 2, \ldots, m.$$

$\therefore$ Equation (1) becomes

$$\sum_{i=1}^{n}\left[a_{ji}(q^*,t) + \sum_{k=1}^{n}\left(\frac{\partial\ _{ji}}{\partial q_k}\right)_0 \delta q_k\right](\dot{q}_i^* + \delta \dot{q}_i)$$

$$+ a_{jt}(q^*,t) + \sum_{k=1}^{n}\left(\frac{\partial\ _{jt}}{\partial q_k}\right)_0 \delta q_k = 0$$

ie, $\sum_{i=1}^{n} a_{ji}(q^*,t)\dot{q}_i^* + a_{jt}(q^*,t) + \sum_{i=1}^{n} a_{ji}(q^*,t)\delta\dot{q}_i$

$+ \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial a_{ji}}{\partial q_k}\right)_0 \dot{q}_i^* \delta q_k + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial a_{ji}}{\partial q_k}\right)\delta q_k \delta\dot{q}_i$

$+ \sum_{k=1}^{n}\left(\frac{\partial a_{jt}}{\partial q_k}\right)_0 \delta q_k = 0$

$0 + \sum_{i=1}^{n} a_{ji}(q^*,t)\delta\dot{q}_i + \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial a_{ji}}{\partial q_k}\right)\dot{q}_i^* \delta q_k$

$+ 0 + \sum_{k=1}^{n}\left(\frac{\partial a_{jt}}{\partial q_k}\right)\delta q_k = 0$ ----- (3) [∵ Product of small quantity are neglected]

ie, $\sum_{i=1}^{n} a_{ji}(q^*,t)\delta\dot{q}_i + \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial a_{ji}}{\partial q_k}\right)\dot{q}_i^* \delta q_k$ [∵ Using eqn (1)

$+ \sum_{k=1}^{n}\left(\frac{\partial a_{jt}}{\partial q_k}\right)\delta q_k = 0$

Let δq's instantaneous constraints in which

$\sum_{i=1}^{n} a_{ji}(q^*,t)\delta q_i = 0, \quad j = 1,2,\ldots,m$ ----- (4)

Any moving constraints that stops during virtual displacements is called instantaneous constraints.

Diff. eqn (4) w.r.t time, changing the indices, we get

$\sum_{k=1}^{n} \dot{a}_{jk}(q^*,t)\delta q_k + \sum_{i=1}^{n} a_{ji}(q^*,t)\delta\dot{q}_i = 0$

where, $\dot{a}_{jk}(q^*,t) = \sum_{i=1}^{n}\left(\frac{\partial a_{jk}}{\partial q_i}\right)_0 \dot{q}_i^* + \left(\frac{\partial a_{jk}}{\partial t}\right)_0$

$\therefore \sum_{k=1}^{n}\left[\sum_{i=1}\left(\frac{\partial a_{jk}}{\partial q_i}\right)_0 \dot{q}_i^* + \left(\frac{\partial a_{jk}}{\partial t}\right)_0\right]\delta q_k + \sum_{i=1}^{n} a_{ji}(q^*,t)\delta\dot{q}_i = 0$

ie, $\sum_{i=1}^{n} a_{ji}(q^*,t)\delta\dot{q}_i + \sum_{k=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial a_{jk}}{\partial q_i}\right)_0 \dot{q}_i^* \delta q_k$

$+ \sum_{k=1}^{n}\left(\frac{\partial q_{jk}}{\partial t}\right)_0 \delta q_k = 0$ --- (5)

$$① - ⑤ \Rightarrow \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial a_{ji}}{\partial q_k} - \frac{\partial a_{jk}}{\partial q_i}\right)_0 \dot{q}_i^* \delta q_k$$

$$+ \sum_{k=1}^{n}\left(\frac{\partial a_{jt}}{\partial q_k} - \frac{\partial a_{jk}}{\partial t}\right)_0 \delta q_k = 0 \quad \text{---- ⑥}$$

In general $\dot{q}_i^* = 0$

$\therefore$ For equ ⑥ to be valid for any set of δq's.

We have, $$\left(\frac{\partial a_{ji}}{\partial q_k} - \frac{\partial a_{jk}}{\partial q_i}\right)_0 = 0, \quad i=k=1,2,\dots,n, \; j=1,2,\dots m \quad \text{---- ⑦}$$

$$\left(\frac{\partial a_{jt}}{\partial q_k} - \frac{\partial a_{jk}}{\partial t}\right)_0 = 0, \quad k=1,2,\dots,n, \; j=1,2,\dots,m \quad \text{----- ⑧}$$

In equations ⑦ & ⑧ represent the exactness conditions for the integrability of equ ①.

These conditions applying only when the constraints are holonomic.

ie, If the varied paths conform to the actual constraints and with δq's are consistent with the instantaneous contraints, then the system must be holonomic.

ie, Hamilton's principle applied to the holonomic system.

For a non-holonomic systems the varied path in which δq's are constrained by equ ④ will not be geometrically possible because the path will not conform to equ ①

NOTE

The operation of variations and integrations can be interchanged only for holonomic system and not for non-holonomic system.

THEOREM

Derive the Hamilton's canonical equations of motion

Proof

Consider the holonomic system described by the standard form of lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad , \quad i = 1, 2, \dots, n \quad \text{-----} (1)$$

The generalised mometum conjucate q_i is given by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad , \quad i = 1, 2, \dots, n \quad \text{-----} (2)$$

$$(1) \Rightarrow \frac{d}{dt}(p_i) - \frac{\partial L}{\partial q_i} = 0$$

$$\text{ie, } \dot{p}_i = \frac{\partial L}{\partial q_i} \quad , \quad i = 1, 2, \dots, n \quad \text{-----} (3)$$

Let the hamiltonian function $H(q, p, t)$ for the System is defined as

$$H(q, p, t) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q, \dot{q}, t) \quad \text{-----} (4)$$

But generalised momentum is linear in $\dot{q}$

$$\therefore p_i = \sum_{j=1}^{n} m_{ij}(q, t)\, \dot{q}_j + a_i(q, t)$$

$$\dot{q}_i = \sum_{j=1}^{n} b_{ij}(p_j - a_j),$$ where $b_{ij}(q, t)$ an element of the matrix $b = m^{-1}$.

(Since the inertia matrix m can be inverted, it is positive definite)

Now,

$$\delta H = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\delta q_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\delta p_i + \frac{\partial H}{\partial t}\delta t \quad \text{-----} (5)$$

$$(4) \Rightarrow \delta H = \sum_{i=1}^{n} p_i \delta \dot{q}_i + \sum_{i=1}^{n} \dot{q}_i \delta p_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\delta q_i - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i - \frac{\partial L}{\partial t}\delta t$$

$$= \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \sum_{i=1}^{n} \dot{q}_i \delta p_i - \sum_{i=1}^{n} \dot{p}_i \delta q_i - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i - \frac{\partial L}{\partial t}\delta t$$

$$\therefore \delta H = \sum_{i=1}^{n} \dot{q}_i \, \delta p_i - \sum_{i=1}^{n} \dot{p}_i \, \delta q_i - \frac{\partial L}{\partial t} \delta t \quad \text{-----} (6)$$

Comparing equations (5) & (6)

Since δp_i, δq_i and δt are independent.

We have,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \frac{-\partial H}{\partial q_i}, \quad i = 1, 2, \ldots, n \quad \text{-----} (7)$$

$$\& \quad \frac{\partial L}{\partial t} = \frac{-\partial H}{\partial t} \quad \text{-----} (8)$$

Equation (7) represents hamilton's canonical equation of motion.

NOTE

(1) If the consider n p's and n q's together as an matrix then the hamilton's equation can be written as a first order non-linear vector equation of the form

$$\dot{\bar{x}} = \lambda(\bar{x}, t)$$

(2) Suppose we have a holonomic system of lagrange's equation of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i', \quad \text{Where } Q_i' \text{ is that}$$

partition of the generalised applied force which is derivable from the potential function

$$\dot{p}_i = \frac{\partial L}{\partial q_i} + Q_i'$$

Then the hamilton's equations for this system are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = p_i$$

$$\therefore \dot{p}_i = \frac{-\partial H}{\partial q_i} + Q_i', \quad i = 1, 2, \ldots, n.$$

(3) A non-holonomic system described by lagrange's equation is of the form

$$\dot{p}_i = \frac{\partial L}{\partial q_i} + Q_i' + \sum_{j=1}^{m} \lambda_j a_{ji}, \quad i = 1, 2, \ldots, n.$$

$\therefore$ The corresponding hamiltonian equations are (12)

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \& \quad \dot{p}_i = \frac{-\partial H}{\partial q_i} + Q_i' + \sum_{j=1}^{m} \lambda_j a_{ji} \quad , i = 1, 2, \dots, n$$

Where m constraints are

$$\sum_{j=1}^{n} a_{ji} \dot{q}_i + a_{jt} = 0 \quad , i = 1, 2, \dots, m$$

Book work

Show that

(1) For a holonomic system to the hamiltonian function is a quadtoatic p's.

(2) For a scleronomic system H = Total energy T+V

(3) For a concervative holonomic system H has the constant value.

(4) For a conservative non-holonomic system H is constant.

(5) For a natural system H is a constant and is equal to the total energy.

Proof

(1). WKT, The generalised momentum

$$p_i = \sum_{j=1}^{n} m_{ij}(q,t)\dot{q}_j + a_i(q,t) \quad \text{-----} (1)$$

$$\therefore \sum_{i=1}^{n} p_i \dot{q}_i = \sum_{j=1}^{n}\sum_{i=1}^{n} m_{ij}\dot{q}_i\dot{q}_j + \sum_{i=1}^{n} a_i \dot{q}_i$$

ie, $\displaystyle\sum_{i=1}^{n} p_i \dot{q}_i = 2T_2 + T_1$ ----- (2)

$$\left[\begin{array}{l} \because T_2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j \ , \ T_1 = \sum_{i=1}^{n} a_i \dot{q}_i \\ T_0 = \frac{1}{2}\sum_{K=1}^{3N} m_K \left(\frac{\partial x_K}{\partial t}\right)^2 \ \& \ T = T_2 + T_1 + T_0 \\ \text{Where } T \text{ is kinetic energy} \end{array}\right] \text{-----} (3)$$

$\therefore$ The Hamiltonian function

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L$$

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ie, $H = \sum_{i=1}^{n} p_i \dot{q}_i - (T - V)$ ------ (4)

$= \sum_{i=1}^{n} p_i \dot{q}_i - (T_2 + T_1 + T_0) + V$

$= 2T_2 + T_1 - T_2 - T_1 - T_0 + V$

$\therefore H = T_2 - T_0 + V$ ------ (5)

Using matrix notation

$$T_2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j$$

ie, $T_2 = \frac{1}{2} \dot{q}_i^T m \dot{q}$ ------ (6)

$$p_i = \sum_{j=1}^{n} m_{ij} \dot{q}_j + a_i$$

$$\dot{q}_i = \sum_{j=1}^{n} b_{ij} (p_j - a_j)$$

$\Rightarrow \dot{q} = b(p-a)$ where $b = m^{-1}$

(6) $\Rightarrow T_2 = \frac{1}{2} b^T (p-a)^T m \, b (p-a)$

ie, $T_2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} p_i p_j - \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} a_i p_j + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} a_i a_j$ ------ (7)

(5) $\Rightarrow H = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} p_i p_j - \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} a_i p_j + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} a_i a_j$

$\Rightarrow H = H_2 + H_1 + H_0$.

where $H_2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} p_i p_j$, $H_1 = \frac{-1}{2} \sum \sum b_{ij} a_i p_j$

& $H_0 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} a_i a_j$

$\therefore$ The holonomic system H is a quadratic in the p's.

(2) Consider a sceleronomic system, here the transformation equation from the cartesian to the generalised co-ordinates do not contain 't' explicity.

$\therefore$ a's zero.

$\therefore T = T_2$

$$\therefore H(p,q,t) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\, p_i p_j + V(q,t)$$

Where b's function of q only.

$H_2 = T_2$, $H_1 = 0$, $H_0 = V$

$\therefore$ For a sceleronomic system $H = T+V$ total energy.

(3) Consider a conservative holonomic system.

Here, the hamiltonian function varies with time.

$$\therefore \dot{H} = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right) + \frac{\partial H}{\partial t}$$

The Hamiton's canonical equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \frac{-\partial H}{\partial q_i}$$

$$\therefore \dot{H} = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\cdot\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial H}{\partial t}$$

$$\Rightarrow \dot{H} = \frac{\partial H}{\partial t}$$

ie, The total derivative of H is its potential derivative.

$\therefore$ L does not contain 't' explicity, neither H does.

$\therefore$ A conservative holonomic system H has a constant value.

(4) Consider the conservative non-holonomic system.

Here, the Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \& \quad \dot{p}_i = \frac{-\partial H}{\partial q_i} + \sum_{j=1}^{n} \lambda_j a_{ji}$$

But the constraints for the conservative system are

$$\sum_{i=1}^{n} a_{ji}\dot{q}_i = 0, \quad j = 1,2,\dots,n.$$

$$\therefore \dot{H} = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right) + \frac{\partial H}{\partial t}$$

$$= \sum_{i=1}^{n}\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \sum_{i=1}^{n}\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\sum_{j=1}^{n}\lambda_j a_{ji}\dot{q}_i + 0$$

$$\therefore \dot{H} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial H}{\partial p_i}\lambda_j a_{ji}\dot{q}_i$$

$$\Rightarrow \dot{H} = 0 \qquad \left[\because \sum_{i=1}^{n} a_{ji}\dot{q}_i = 0\right.$$

ie, H is Constant.

(5) For a Conservative non holonomic system also

$$H = T_2 - T_0 + V$$
$$= T + V$$
$$= \text{Total energy}$$
$$\Rightarrow H = h$$

For a natural system $T_0 = 0$, $T_1 = 0$

$\therefore$ From the above equation

$$H = T_2 + V$$
$$= T + V$$
$$\Rightarrow H = h$$

$\therefore$ The natural system h is constant and it is equal to the total energy.

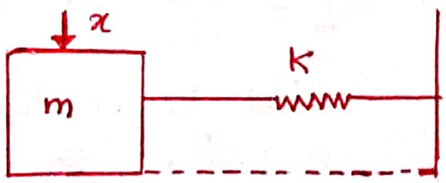
PROBLEMS

① For a mass spring system consisting of a mass m and linear spring of stiffness K.

Find the equation of motion, Using Hamiltonian procedure.

Solution

Assume that the displacement x is measured from the unstressed position of the spring.



Kinetic energy $T = \frac{1}{2}m\dot{x}^2$

potential energy $V = \frac{1}{2}Kx^2$

UNIT - V

Hamilton - Jacobi theory

Hamilton's principle function

Consider the canonical integral $I = \int_{t_0}^{t_1} L\,dt$ ------ (1) associated with Hamilton's principle.

Suppose we evaluate this integral over the actual path of the holonomic system that obeys Lagrange's equation on Hamilton's equ.

If we know the initial value q_i's and $\dot q_0$'s then further motion can be determined.

ie, q and p can be found at any final time t_1

Consider the solution $q_{i1} = q_{i1}(q_0, \dot q, t_0, t_1)$, $i = 1, 2, \dots, n$ ---- (2)

and solve for the initial velocity $\dot q_{i0}$.

If we assume that the Jacobian $\dfrac{\partial(q_{11}, q_{21}, \dots q_{n1})}{\partial(\dot q_{10}, \dot q_{20}, \dots, \dot q_{n0})} \neq 0$

we can find the unique initial velocity

$\dot q_{i0} = \eta_i(q_0, q, t_0, t_1)$, for $i = 1, 2, \dots, n$ ------- (3)

If t_1 is the running time in (2) and if we evaluate (1) as the function of (q_0, q_0, t_0, t_1), we obtained a canonical integral of the form $S(q_0, q, t_0, t_1) = \int_{t_0}^{t_1} L\,dt$

This function S is assumed to be twice differentiable in all its argument and it is known as the Hamilton's principle function "S" is the canonical integral I expressed as a function of the n points in the extended Configuration space.

2. Book work

Sm Find the complete solution of the Hamilton's problem to Hamilton's principle function

Solution

Let $S(q_0, q_1, t_0, t_1) = \int_{t_0}^{t_1} L\,dt$ ------- (1) be the Hamilton's principle function, when we apply a general non-Contemporaneous variation to be integral (1) (or) Hamilton's principle fun is the generating fun of canonical t.f (or) id...

$$\delta I = \int_{t_0}^{t_1}\left[\sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)\delta q_i\right]dt + \int_{t_0}^{t_1}\left[\left(\frac{\partial L}{\partial t}\delta t - \sum_{i=1}^{n}\frac{\partial L}{\partial \dot q_i}\dot q_i - L\right)\frac{d}{dt}\delta t\right]$$

$$\delta t - \int_{t_0}^{t_1}\sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i}\right]\delta q_i\,dt \quad \dots (2)$$

[∵ (1) eqn of the derivative of principle of variation]

For a standard holonomic form the last integral varish hamilton's function.
the total derivatives of the Hamiltonian function.

$$H(p,q,t) = \sum_{i=1}^{n} p_i \dot{q}_i - L \quad \text{is} \quad \dot{H} = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right) + \frac{\partial H}{\partial t}$$

Hamilton's canonical equations $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = \frac{-\partial H}{\partial q_i}$ yields

$$\dot{H} = \frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t} \quad [\because 2^{nd} \text{ eqn of Hamilton canonical eqn derivatives}]$$

$$② \Rightarrow \delta I = \int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{d}{dt}(p_i \delta q_i)\, dt + \int_{t_0}^{t_1} \left[-\dot{H}\delta t - H\frac{d}{dt}\delta t\right] dt$$

$$\text{ie, } \delta I = \int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{d}{dt}(p_i \delta q_i)\, dt - \int_{t_0}^{t_1} \frac{d}{dt}(H\delta t)\, dt \quad \text{----} ③$$

The principle function S is the canonical integral I expressed as the function of n points in the extended configuration space and so we can identify δs with δI.

$$\therefore \delta s = \left[\sum_{i=1}^{n} p_i \delta q_i - H\delta t\right]_{t_0}^{t_1}$$

In the differential form

$$ds = \sum_{i=1}^{n} p_{i1}\, dq_{i1} - \sum_{i=1}^{n} p_{i0}\, dq_{i0} - H_1\, dt_1 + H_0\, dt_0 \quad \text{----} ④$$

If we directly differentiate (1).

$$ds = \sum_{i=1}^{n} \frac{\partial s}{\partial q_{i1}} dq_{i1} + \sum_{i=1}^{n} \frac{\partial s}{\partial q_{i0}} dq_{i0} + \frac{\partial s}{\partial t_1} dt + \frac{\partial s}{\partial t_0} dt_0 \quad \text{----} ⑤$$

Since $(2n+2)$ arguments of the principle function can be varied independently, from equation ④ and ⑤. We have,

$$p_{i1} = \frac{\partial s}{\partial q_{i1}}, \quad p_{i0} = \frac{-\partial s}{\partial q_{i0}}, \quad i = 1, 2, \dots n \quad \text{----} ⑥$$

$$H_1 = \frac{-\partial s}{\partial t_1}, \quad H_0 = \frac{\partial s}{\partial t_0} \quad \text{-----} ⑦$$

If we assume that $\left|\frac{\partial^2 s}{\partial q_{i0}\partial q_{i1}}\right| \neq 0$, each q_{i1} is a function of a initial conditions and time from eqn ⑥

Thus $q_{i1} = q_{i1}(q_0, p_0, t_0, t_1)$ $i = 1, 2, \dots n$ ---- ⑧

is the standard solution of Lagrange's problem and gives the motion in the configuration space as the function of time.

If we substitute the result in ⑥

We get $p_{i1} = p_{i1}(q_0, p_0, t_0, t_1)$, $i = 1, 2, \dots, n$ ---- ⑨

Equations ② and ④ constitute the solution of the Hamilton's problem giving the motion in the phase-space as a function of time

3. Write a short note on pfaffian differential form?

Solution

A pfaffian form Ω in m variables $x_1, x_2, \dots, x_m$ can be written as

$$\Omega = X_1(x)\,dx_1 + X_2(x)\,dx_2 + \dots + X_m(x)\,dx_m.$$

This indeed leads to a line integral over a path in x-space.

If $C_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}$ and if all the c's are zero, the Pfaffian differential form is an exact differential.

But, in general the differential form is not exact.

4. Book work. 10m, 5

Obtain Hamilton-Jacobi equation.

Solution

Consider the Hamilton's principle function S.

WKT, ds can be expressed as a difference between two pfaffian differential form, one involving the initial values and the other final values of p's, q's and t as

$$ds = \sum_{i=1}^{n} p_{i1}\,dq_{i1} - \sum_{i=1}^{n} p_{i0}\,dq_{i0} - H_1\,dt_1 + H_0\,dt_0 \quad \text{------} ①$$

Suppose that the initial conditions are specified as n α's and n β's.

Where $\alpha_i = \alpha_i(q_{10}, q_{20}, \dots, q_{n0}, p_{10}, p_{20}, \dots, p_{n0})$ and

$\beta_i = \beta_i(q_{10}, q_{20}, \dots, q_{n0}, p_{10}, p_{20}, \dots, p_{n0})$, $i = 1, 2, \dots, n$ ---②

with the condition that $\sum_{i=1}^{n} p_{i0}\,dq_{i0} = \sum_{i=1}^{n} \beta_i\,d\alpha_i$ ---- ③

($\because$ Jacobian $\neq 0$ and by pfaffian thm)

From equation ① and ③

$$ds = \sum_{i=1}^{n} p_{i1}\,dq_{i1} - \sum_{i=1}^{n} \beta_i\,d\alpha_i - H_1\,dt_1 + H_0\,dt_0 \quad \text{-----} ④$$

Here S is considered as a function of $(q_{i1}, \alpha_i, t_1, t_0)$

$$\therefore ds = \sum_{i=1}^{n} \frac{\partial S}{\partial q_{i1}}\,dq_{i1} + \sum_{i=1}^{n} \frac{\partial S}{\partial \alpha_i}\,d\alpha_i + \frac{\partial S}{\partial t_1}\,dt_1 + \frac{\partial S}{\partial t_0}\,dt_0 \ \dots$$

If $\left|\frac{\partial^2 s}{\partial q_{i1}, \partial \alpha_i}\right| \neq 0$ then we can solve for α's in terms of $\frac{\partial s}{\partial q_{i1}}$, Where $i = 1, 2, \ldots, n$.

From eqn ④ and ⑤ $p_{i1} = \frac{\partial s}{\partial q_{i1}}$

$\therefore$ The Jacobian, $\left|\frac{\partial^2 s}{\partial q_{i1}, \partial \alpha_i}\right| = \left|\frac{\partial}{\partial \alpha_i}\left(\frac{\partial s}{\partial q_{i1}}\right)\right|$

$= \frac{\partial}{\partial \alpha_i}(p_{i1}) \Rightarrow \left|\frac{\partial^2 s}{\partial q_{i1}, \partial \alpha_i}\right| = \left|\frac{\partial(p_{11}, p_{21}, \ldots, p_{n1})}{\partial(\alpha_1, \alpha_2, \ldots, \alpha_n)}\right| \neq 0$

Where p's are considered as functions of $(q_1, \alpha_1, t_1, t_0)$.

$\therefore$ q's and α's are independent variables.

From eqn ④ and ⑤

$$-\beta_i = \frac{\partial s}{\partial \alpha_i} \quad (i = 1, 2, \ldots, n) \quad \text{-----} ⑥$$

$$p_{i1} = \frac{\partial s}{\partial q_{i1}} \quad (i = 1, 2, \ldots, n) \quad \text{-----} ⑦$$

$$H_1 = \frac{-\partial s}{\partial t_1} \quad \text{and} \quad H_0 = \frac{\partial s}{\partial t_0} \quad \text{-----} ⑧$$

If the intial time $t_0 = 0$ then $dt_0 = 0$

We measure time from this instant and so we drop 1 subscript for convenience.

$$⑤ \Rightarrow ds = \sum_{i=1}^{n} p_i \, dq_i - \sum_{i=1}^{n} \beta_i \, d\alpha_i - H dt \quad \text{-----} ⑨$$

$\therefore$ S is of the form $S(q, \alpha, t)$.

Hence its total differential takes the form

$$ds = \sum_{i=1}^{n} \frac{\partial s}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial s}{\partial \alpha_i} d\alpha_i + \frac{\partial s}{\partial t} dt \quad \text{-----} ⑩$$

If $\left|\frac{\partial^2 s}{\partial q_i \partial \alpha_j}\right| \neq 0$, then we can equate ⑨ & ⑩

$$-\beta_i = \frac{\partial s}{\partial \alpha_i}, \quad i = 1, 2, \ldots, n \quad \text{-----} ⑪$$

$$p_i = \frac{\partial s}{\partial q_i}, \quad i = 1, 2, \ldots, n \quad \text{-----} ⑫$$

$$\frac{\partial s}{\partial t} = -H \quad \text{-----} ⑬$$

Equation ⑪ can be solved for q's as functions of (α, β, t) and this provides the solution for the Lagrangian problem.

This possible because

$$\left|\frac{\partial^2 s}{\partial q_i \partial \alpha_j}\right| = \left|\frac{\partial}{\partial q_i}\left(\frac{\partial s}{\partial \alpha_j}\right)\right| = \left|\frac{\partial}{\partial q_i}(-\beta_j)\right| \Rightarrow \left|\frac{\partial^2 s}{\partial q_i \partial \alpha_j}\right| \neq 0$$

ie, $$\left|\frac{\partial(\beta_1, \beta_2, \dots, \beta_n)}{\partial(q_1, q_2, \dots, q_n)}\right| \neq 0$$

Substituting these solutions for q's in eqn ②

we get expression for p's as functions of (α, β, t) and we get the solution from the Hamilton problem

Now H is usually considered as the function of (q, p, t).

If we substitute p's from ⑫

we get from eqn ⑬

$$\frac{\partial s}{\partial t} + H\left(q, \frac{\partial s}{\partial q}, t\right) = 0$$

This is a 1^{st} order a partial differential equation and is called Hamilton Jacobi equation.

Jacobi's theorem 10m I

Statement

5M If $s(q, \alpha, t)$ is any complete solution of the Hamilton Jacobi equation $\frac{\partial s}{\partial t} + H\left(q, \frac{\partial s}{\partial q}, t\right) = 0$ and if the equations

$$-\beta_i = \frac{\partial s}{\partial \alpha_i} \quad (i = 1, 2, \dots, n)$$

$$p_i = \frac{\partial s}{\partial q_i} \quad (i = 1, 2, \dots, n)$$ (Where β's are arbitrary constant)

and used to solve for $q_i(\alpha, \beta, t)$ and $p_i(\alpha, \beta, t)$

Then these expressions provide the general solutions of the canonical equations associated with $H(q, p, t)$.

Proof

The Hamilton Jacobian equation is

$$\frac{\partial s}{\partial t} + H\left(q, \frac{\partial s}{\partial q}, t\right) = 0 \quad \text{------} ①$$

Given that $-\beta_i = \frac{\partial s}{\partial \alpha_i}$ $(i = 1, 2, \dots, n)$ -- ②

and $p_i = \frac{\partial s}{\partial q_i}$ $(i = 1, 2, \dots, n)$ ----- ③

Differentiating (1) partially w.r.t α_i

we have, $$\frac{\partial^2 s}{\partial \alpha_i \partial t} + \sum_{j=1}^{n} \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial \alpha_i} = 0 \quad \text{----} ④$$

Where p_j are considered as functions of (q, α, t)

$③ \Rightarrow \frac{\partial S}{\partial \alpha_i} = -\beta_i$, where $\frac{\partial S}{\partial \alpha_i}$ are functions of (q, α, t) and β_i are constants.

Taking the total time derivatives

We have,

$$\frac{\partial^2 S}{\partial t \, \partial \alpha_i} + \sum_{j=1}^{n} \frac{\partial^2 S}{\partial q_j \, \partial \alpha_i} \dot{q}_j = 0 \quad \text{------ ⑤}$$

$$⑤ - ④ \Rightarrow \sum_{j=1}^{n} \frac{\partial^2 S}{\partial q_j \, \partial \alpha_i} \dot{q}_j - \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} \cdot \frac{\partial p_j}{\partial \alpha_i} = 0$$

ie, $$\sum_{j=1}^{n} \frac{\partial^2 S}{\partial q_j \, \partial \alpha_i} \dot{q}_j - \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} \frac{\partial}{\partial \alpha_i}\left(\frac{\partial S}{\partial q_j}\right) = 0 \quad [\because \text{Using } ③$$

ie, $$\sum_{j=1}^{n} \left(\dot{q}_j - \frac{\partial H}{\partial p_j}\right) \frac{\partial^2 S}{\partial q_j \, \partial \alpha_i} = 0, \quad i = 1, 2, \ldots, n.$$

But $\left| \frac{\partial^2 S}{\partial q_j \, \partial \alpha_i} \right| \neq 0$

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad j = 1, 2, \ldots, n \quad \text{------ ⑥}$$

Now, differentiating (1) partially w.r.t q_j, assuming that p_i is a function of (q, α, t). We have,

$$\frac{\partial^2 S}{\partial q_j \, \partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \cdot \frac{\partial p_i}{\partial q_j} + \frac{\partial H}{\partial q_j} = 0 \quad \text{------ ⑦}$$

$$③ \Rightarrow p_j = \frac{\partial S}{\partial q_j}$$

Differentiating w.r.t time, we have

$$\dot{p}_j - \frac{\partial^2 S}{\partial t \, \partial q_j} - \sum_{i=1}^{n} \frac{\partial^2 S}{\partial q_j \, \partial q_i} \dot{q}_i = 0 \quad \text{------ ⑧}$$

$$⑦ + ⑧ \Rightarrow \dot{p}_j + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial p_i}{\partial q_j} - \sum_{i=1}^{n} \frac{\partial^2 S}{\partial q_j \, \partial q_i} \dot{q}_i + \frac{\partial H}{\partial q_j} = 0$$

ie, $$\dot{p}_j + \sum_{i=1}^{n} \dot{q}_i \frac{\partial}{\partial q_j}\left(\frac{\partial S}{\partial q_i}\right) - \sum_{i=1}^{n} \frac{\partial^2 S}{\partial q_j \, \partial q_i} \dot{q}_i + \frac{\partial H}{\partial q_j} = 0 \quad \left[\because \text{Using } ③ \,\&\, ⑥\right.$$

ie, $$\dot{p}_j = \frac{-\partial H}{\partial q_j}, \quad j = 1, 2, \ldots, n \quad \text{----- ⑨}$$

Equations ⑥ and ⑨ are the canonical equation of motion.

∴ Any complete solution of the Hamilton Jacobi equation leads to a solution of the Hamilton's problem.

1 ~~Prob~~ work Hamilton Jacobi eqn for a conservative eqn 4

5m Obtain modified Hamilton Jacobi equation and find the solution of this equation with the ignorable co-ordinates for conserving or non-conservating system.

Proof

Consider the conservative ~~Ham~~ Holonomic system whose configuration is described by n independent q's.

The ~~order~~ Hamiltonian function for the system is not an explicit function of time and it is a constant motion.

∴ The ~~other~~ ~~la~~ Euler lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

This equation has n $\dot{q}$'s.

Suppose $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_K$ are missing in this system.

Then $\frac{\partial L}{\partial q_i} = 0$ for q_i, $(i = 1, 2, \dots K)$

and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$ for q_i $(i = 1, 2, \dots K)$

ie, $\frac{d}{dt}(p_i) = 0 \Rightarrow \dot{p}_i = 0$

$\Rightarrow p_i = \alpha_i$ (constant)

Such co-ordinates $q_1, q_2, \dots, q_K$ are called ignorable co-ordinate

∴ $H(q, p) = \alpha_n = h$. ---- (1) Where h is the value of the Jacobi integral or energy integral which ~~or~~ we arbitrarily identical with α_n.

The Hamilton Jacobi equation is

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0 \quad \text{------ (2)}$$

$$\therefore \frac{\partial S}{\partial t} = -H = -\alpha_n \quad \text{------ (3)}$$

Integrating this we have,

$$S(q, \alpha, t) = -\alpha_n t + W(q, \alpha) \quad \text{------ (4)}$$

This function $W(q, \alpha)$ does not contain t explicitly.

It is called the characteristic function.

From equation (4)

$$\frac{\partial S}{\partial \alpha_i} = \frac{\partial W}{\partial \alpha_i}, \quad i = 1, 2, \dots, n \quad \text{------ (5)}$$

and $\frac{\partial s}{\partial \alpha_n} = \frac{\partial W}{\partial \alpha} - t$ ------ (6)

$$\frac{\partial s}{\partial q_i} = \frac{\partial W}{\partial q_i} \quad (i = 1, 2, \ldots, n) \quad \text{------ (7)}$$

From equations (3) and (7) are Hamiltonian Jacobi equation reduces to $H\left(q, \frac{\partial W}{\partial q}\right) = \alpha_n$ ------ (8)

Equation (8) is called the modified Hamilton Jacobi equation

Note (1)

In Jacobi's thm we have taken $-\beta_i = \frac{\partial s}{\partial \alpha_i}$ & $p_i = \frac{\partial s}{\partial q_i}$

(5) $\Rightarrow$ $-\beta_i = \frac{\partial W}{\partial \alpha_i}$, $i = 1, 2, \ldots, n$ ------ (9)

(6) $\Rightarrow$ $t + \frac{\partial s}{\partial \alpha_n} = \frac{\partial W}{\partial \alpha_n}$

ie, $t - \beta_n = \frac{\partial W}{\partial \alpha_n}$ ------ (10)

(7) $\Rightarrow$ $p_i = \frac{\partial s}{\partial q_i} = \frac{\partial W}{\partial q_i}$ $\quad i = 1, 2, \ldots n$ ------ (11)

Where β_n is initial time t_0.

Since W is not an explicit function of time.

Eqn (9) gives the path of the system in the configuration space without reference to time.

Eqn (10) gives the relation of time to position along the path.

Note (2) Hamilton Jacobi eqn for Ignorable co-ordinates

If the system has $q_1, q_2, \ldots, q_K$ as ignorable co-ordinates

then $p_i = \alpha_i \quad (i = 1, 2, \ldots, K)$.

Initially let us assume that the system is not conservative.

$\therefore$ For this $q_1, q_2, \ldots, q_K$, $p_i = \alpha_i$, $i = 1, 2, \ldots, K$.

$\therefore$ We can assume the principle function in the form

$$s(q, \alpha, t) = \sum_{i=1}^{K} \alpha_i q_i + s'(q_{K+1}, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t) \quad \text{----- (12)}$$

The Hamilton Jacobi equation in this case

$$\frac{\partial s'}{\partial t} + H\left(q_{K+1}, \ldots, q_n, \alpha_1, \ldots, \alpha_K, \frac{\partial s'}{\partial q_{K+1}}, \ldots, \frac{\partial s'}{\partial q_n}, t\right) = 0 \quad \text{--- (13)}$$

$\therefore$ The complete solution of eqn (13) involves $(n-K)$ non-additional constants exclusive of the constant moments $\alpha_1, \alpha_2, \ldots, \alpha_K$. $\left[\because \frac{\partial s}{\partial t} + H\left(q, \frac{\partial s}{\partial q}, t\right) = 0\right.$

One s' is known the solution of the motion of the system, is obtained from

$$-\beta_i = q_i + \frac{\partial s'}{\partial \alpha_i} \quad i = 1,2,\ldots,k \text{ ------ } (14) \quad \left[\because \beta_i = \frac{\partial s}{\partial \alpha_i}, i = 1,2 \ldots n\right.$$

$$S(q,\alpha,t) = \alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_k q_k + s'(q_{k+1}, \ldots, q_n, \alpha_1, \ldots, \alpha_k, t)$$

$$-\beta_i = \frac{\partial s'}{\partial \alpha_n}, \quad i = k+1, \ldots, n \text{ ------ } (15)$$

$$p_i = \alpha_i, \quad i = 1,2,\ldots,k \text{ ------ } (16)$$

and $$p_i = \frac{\partial s'}{\partial q_i} \quad i = k+1, \ldots, n \text{ ------ } (17)$$

Where $\beta_i = -q_{i0}$, $i = 1,2,\ldots,k$.

ie, β_i corresponding to ignorable co-ordinates is just the negative of the initial value of co-ordinates $[\because \beta_i = \beta_i(q_{10}, q_{20}, \ldots, q_{n0}, p_{10}, p_{20}, \ldots, p_{n0}]$.

Note (3)

Consider the conservative system to the ignorable co-ordinates $q_1, q_2, \ldots, q_k$ for this principle function takes the form

$$S(q,\alpha,t) = \sum_{i=1}^{k} \alpha_i q_i - \alpha_n t + w'(q_{k+1}, \ldots, q_n, \alpha_1, \ldots, \alpha_n) \text{ --- } (18)$$

$[\because$ Using (4)

$\therefore$ Modified Hamilton Jacobi equation becomes

$$H\left(q_{k+1}, \ldots, q_n, \alpha_1, \ldots, \alpha_k, \frac{\partial w'}{\partial q_{k+1}}, \ldots, \frac{\partial w'}{\partial q_n}\right) = \alpha_n \text{ --- } (19) \quad [\because \text{Using } (8)$$

The complete solution for this w' in this case involves $(n-1-k)$ non-additive constant.

ie, $\alpha_{k+1}, \ldots, \alpha_{n-1}$, the energy constant α_n and the constant moments $\alpha_1, \alpha_2, \ldots, \alpha_k$.

$\therefore$ The motion of the system is given by

$$-\beta_i = q_i + \frac{\partial w'}{\partial \alpha_i} \quad (i = 1,2,\ldots,k) \text{ ----- } (20)$$

$$-\beta_i = \frac{\partial w'}{\partial \alpha_i} \quad (i = k+1, \ldots, n) \text{ ----- } (21)$$

$$t - \beta_i = \frac{\partial w'}{\partial \alpha_n} \text{ ----- } (22)$$

$$p_i = \alpha_i \quad (i = 1,2,\ldots k) \text{ ----- } (23)$$

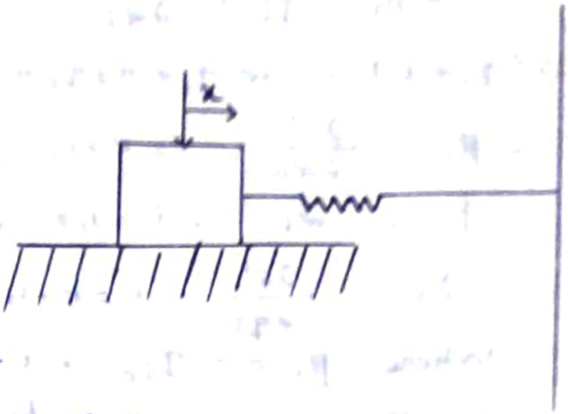
$$p_i = \frac{\partial w'}{\partial q_i}, (i = k+1, \ldots, n) \text{ ----- } (24)$$

10m II Problems

1. Solve the mass spring problem using Hamilton Jacobi equation.

Solution

The mass-spring problem belongs to a natural system.



$\therefore$ Kinetic energy $T = \frac{1}{2} m\dot{x}^2$

and potential energy $V = \frac{1}{2} kx^2$

Momentum $p = \frac{\partial T}{\partial \dot{x}} = m\dot{x}$

$H = T+V = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$

$= \frac{p^2}{2m} + \frac{1}{2} kx^2 \qquad [\because \dot{x} = p/m$

For a conservative system we can use the modified Hamilton Jacobi equation.

$$H\left(q, \frac{\partial W}{\partial q}\right) = \alpha_1$$

ie, $\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2}kx^2 = \alpha$, where α is the energy constant

$\therefore$ $\frac{\left(\frac{\partial W}{\partial x}\right)^2 + mkx^2}{2m} = \alpha \Rightarrow \left(\frac{\partial W}{\partial x}\right)^2 = 2\alpha m - kmx^2$

ie, $\left(\frac{\partial W}{\partial x}\right)^2 = 2m\left(\alpha - \frac{1}{2}kx^2\right) \Rightarrow \frac{\partial W}{\partial x} = \sqrt{2m}\sqrt{\alpha - \frac{1}{2}kx^2}$

$W = \sqrt{2m}\int_{x_0}^{x_1}\sqrt{\alpha - \frac{1}{2}kx^2}\, dx = \sqrt{2m}\sqrt{\frac{k}{2}}\int_{x_0}^{x_1}\sqrt{\frac{2\alpha}{k} - x^2}\, dx$

$\therefore W(x, \alpha) = m\omega\int_{x_0}^{x_1}\sqrt{a^2 - x^2}\, dx$, where $a = \sqrt{\frac{2\alpha}{m\omega^2}}$ and $\omega = \sqrt{\frac{k}{m}}$

$\Rightarrow \sqrt{k} = \sqrt{m}\,\omega$.

From equation (10)

$$t - \beta_n = \frac{\partial W}{\partial \alpha_n}$$

$$t - t_0 = m\omega\frac{d}{d\alpha}\int_{x_0}^{x_1}\sqrt{a^2 - x^2}\, dx$$

$$= m\omega\frac{d}{da}\left[\int_{x_0}^{x_1}\sqrt{a^2 - x^2}\, dx\right]\frac{da}{d\alpha}$$

$$t - t_0 = \frac{1}{w}\left[\cos^{-1}\frac{x_0}{a} \quad \cos^{-1}\frac{x_1}{a}\right]$$

ie, $t - t_0 = \frac{1}{w}\left[\phi - \cos^{-1}\frac{x_1}{a}\right]$, where $\phi = \cos^{-1}\frac{x_0}{a}$

$\cos\phi = \frac{x_0}{a} = \frac{x_0}{\sqrt{2\alpha/m w^2}}$

$\Rightarrow \cos\phi = \sqrt{\frac{m w^2}{2\alpha}}\, x_0.$

$$w(t - t_0) = \phi - \cos^{-1}\left(\frac{x_1}{a}\right)$$

$$\Rightarrow \cos^{-1}\left(\frac{x_1}{a}\right) = \phi - w(t - t_0)$$

$$\Rightarrow x_1 = a\cos\left[\phi - w(t - t_0)\right]$$

If x_1 is arbitrary solution

$$x = a\cos\left[\phi - w(t - t_0)\right] \quad \text{-----} \ (1)$$

Initially let $x = x_0$, when $t = t_0$. ie, $\dot{x}(t_0) = V_0$.

$$x_0 = \sqrt{\frac{2\alpha}{m w^2}}\cos\phi$$

$$\dot{x} = \sqrt{\frac{2\alpha}{m w^2}}\sin\left[w(t - t_0) - \phi\right] w$$

ie, $V_0 = \sqrt{\frac{2\alpha}{m w^2}}\sin\phi \cdot w \quad [\because t = t_0$

$= a w \sin\phi. \quad \Rightarrow \sin\phi = \frac{V_0}{a w}$

(1) $\Rightarrow x = a\left(\cos w(t - t_0)\cos\phi - \sin w(t - t_0)\sin\phi\right)$

$$= a\cos\phi \cdot \cos w(t - t_0) - a\sin\phi \cdot \sin w(t - t_0)$$

ie, $x = x_0 \cos w(t - t_0) + \frac{V_0}{w}\sin w(t - t_0)$

This is the solution of the system,

$\therefore$ This equation represents the simple harmonic motion.

The amplitute of the oscillation $a = \sqrt{x_0^2 + \frac{V_0^2}{w^2}}$

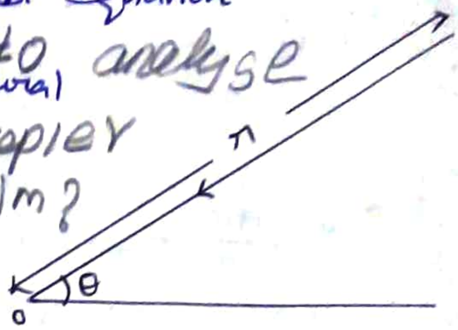
2\. Solve kepler's problem using Hamilton Jacobi equation

10m

Solution

Hamilton Jacobi method to analyse kepler plm?

WKT, The ~~Kepler's~~ kepler's problem is a natural system using the polar Co-ordinates



Kinetic energy $T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$

Potential energy $V = \frac{-M}{r}$.

Lagrangian principle $L = T - V$

ie, $L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{M}{r}$

Momentum $p_r = \frac{\partial L}{\partial \dot{r}} = \dot{r}$

and $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta}$.

For a natural system the Hamiltonian

H = Total energy $= \alpha$

$T + V = \alpha$.

$$\therefore \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{M}{r} = \alpha_t \quad \text{------} (1)$$

Here, α_t represents the constant value of the total energy.

Here the Co-ordinate θ does not appear therefore it is ignorable.

$\therefore p_\theta$ has a constant value.

Let $p_\theta = \alpha_\theta$.

(WKT, $S(q,\alpha,t) = \sum_{i=1}^{k} \alpha_i q_i - \alpha_n t + W'(q_{k+1}, \dots, q_n, \alpha_1, \dots, \alpha_n)$)

$\therefore S = -\alpha_t t + \alpha_\theta \theta + W'(r, \alpha_t, \alpha_\theta)$)

The modified Hamilton Jacobi equation is

$$H\left(q, \frac{\partial W}{\partial q}\right) = \alpha_n.$$

$$\therefore \frac{1}{2}\left(\frac{\partial W'}{\partial r}\right)^2 + \frac{1}{2r^2}\alpha_\theta^2 - \frac{M}{r} = \alpha_t \quad [\because \text{From } (1)$$

$$\Rightarrow \frac{\left(\frac{\partial W'}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} - \frac{2M}{r}}{2} = \alpha_t$$

$$\text{ie, } \left(\frac{\partial W'}{\partial r}\right)^2 = 2\alpha_t + \frac{2M}{r} - \frac{\alpha_\theta^2}{r^2} \quad \text{------} (2)$$

$$\therefore \frac{\partial W'}{\partial r} = \sqrt{2\alpha_t + \frac{2M}{r} - \frac{\alpha_\theta^2}{r^2}}$$

ie, $W' = \int_{r_0}^{r} \sqrt{2\alpha_t + \frac{2M}{r} - \frac{\alpha_\theta^2}{r^2}}\, dr$, Where r_0 is the value of the radial distance r at $t = t_0$.

7

Also $t - \beta_r = \frac{\partial W'}{\partial \alpha_r}$

$$\therefore t - t_0 = \frac{\partial W'}{\partial \alpha_t} \Rightarrow t - t_0 = \int_{r_0}^{r} \frac{1}{2\sqrt{2\alpha_t + \frac{2M}{r} - \frac{\alpha_\theta^2}{r^2}}}\, 2\cdot dr$$

From ① $p_r = \dot{r}$, $p_\theta = \alpha_\theta$

$$② \Rightarrow \dot{r}^2 + \frac{\alpha_\theta^2}{r^2} - \frac{2M}{r} = 2\alpha_t \Rightarrow \dot{r}^2 = 2\alpha_t + \frac{2M}{r} - \frac{\alpha_\theta^2}{r^2}$$

$$\therefore t - t_0 = \int_{r_0}^{r} \frac{dr}{\sqrt{\dot{r}^2}} \Rightarrow t - t_0 = \int_{r_0}^{r} \frac{dr}{\dot{r}} \quad \text{.....③}$$

$|||^{ly}$ $-\beta_i = q_i + \frac{\partial W'}{\partial \alpha_i}$ $(i = 1, 2, \ldots, k)$

$$-\theta_0 = \theta + \frac{\partial W'}{\partial \alpha_\theta} \Rightarrow \theta + \theta_0 = -\frac{\partial W'}{\partial \alpha_\theta}$$

$$\theta + \theta_0 = -\int_{r_0}^{r} \frac{1\left(\frac{-2\alpha_\theta}{r^2}\right) dr}{2\sqrt{2\alpha_t + \frac{2M}{r} - \frac{\alpha_\theta^2}{r^2}}}$$

$$\left[\because r = \frac{1}{t},\ dr = \frac{-1}{t^2}\right]$$

$$\Rightarrow \theta + \theta_0 = -\int_{r_0}^{r} \frac{-\alpha_\theta\, dr}{r\sqrt{2r^2\alpha_t + 2Mr - \alpha_\theta^2}} \quad \text{.....④}$$

Comparing ③ and ④ we have ③ gives t as a function of r. ④ gives θ as a function of r.

ie, ④ gives the shape of the orbit.

When $\theta_0 = 0$, $r_0 = r$ minimum

$$\int \frac{-dr}{\sqrt{a^2 - r^2}} = \cos^{-1}\left(\frac{r}{a}\right)$$

$\therefore$ Integrating ④ we get

$$\theta = \cos^{-1}\left[\frac{\alpha_\theta^2 - Mr}{r\sqrt{M^2 + 2\alpha_t\alpha_\theta^2}}\right]$$

$$\cos\theta = \frac{\alpha_\theta^2 - Mr}{r\sqrt{M^2 + 2\alpha_t\alpha_\theta^2}} = \frac{M\left(\frac{\alpha_\theta^2}{M} - r\right)}{rM\sqrt{1 + \frac{2\alpha_t\alpha_\theta^2}{M^2}}}$$

$$\sqrt{1 + \frac{2\alpha_t\alpha_\theta^2}{M^2}}\cos\theta = \frac{\frac{\alpha_\theta^2}{M}}{r} - 1$$

$$\frac{\frac{\alpha_\theta^2}{M}}{r} = 1 + \sqrt{1 + \frac{2\alpha_t\alpha_\theta^2}{M^2}}\cos\theta.$$

This is the equation of conic whose eccentricity

$$e = \sqrt{1 + \frac{2\alpha_t\alpha_\theta^2}{M^2}}$$

Separability

$$W = \sum_{i=1}^{n} w_i(q_i)$$

ie, It consists of the sum of 'n' functions where each functions w_i contains only one of q's.

Note

1. In this section we assume that W is a complete integral of the modified Hamilton Jacobi equation and thus contains n additive constants α's.

2. The conservative Holonomic system whose kinetic energy function contains only the squared terms of $\dot{q}$'s or p's and no product terms in this variables are called and no product terms in this variables are called orthogonal systems.

Liouville's system on separability

3. The Liouville's system is an orthogonal system which has kinetic energy and potential energy in the form

$$L\,T = \frac{1}{2}\left[\sum_{i=1}^{n} f_i(q_i)\left(\sum_{i=1}^{n} \frac{\dot{q}_i^{\,2}}{R_i(q_i)}\right)\right]$$

$$T = \frac{R_1 p_1^2 + R_2 p_2^2 + \cdots + R_n p_n^2}{2(f_1 + f_2 + \cdots + f_n)^2} \quad \text{and} \quad V = \frac{V_1(q_1) + \cdots + V_n(q_n)}{f_1(q_1) + \cdots + f_n(q_n)}$$

Where f_i, R_i and V_i are each functions of q_i.

$$\sum f_i(q_i) > 0, \quad R_i(q_i) > 0 \quad [R_i \text{ is identical with } M^{-1}]$$

(Book work)

5M Show that Liouville's conditions are sufficient to ensure separability of the given system and hence find the solution for the motion of the system.

Proof

We can show that the complete solution $W(q)$ of the modified Hamilton Jacobi equation exists and this solution has the separable form

$$W = \sum_{i=1}^{n} w_i(q_i)$$

Modified Hamilton Jacobi equation for this system can be written in the form

$$\sum_{i=1}^{n}\left[\frac{1}{2}R_i\left(\frac{\partial W}{\partial q_i}\right)^2+V_i\right]=h\sum_{i=1}^{n}f_i \quad \text{-----} (1)$$

$$\left[\because T+V=h,\quad \frac{\sum R_i P_i}{2\sum f_i}+\frac{\sum V_i}{\sum f_i}=h\right]$$

Let us group these term in each co-ordinates $q_i\ (i=1,\dots,n)$ and use $\alpha_1,\alpha_2,\dots,\alpha_n$ as separate constants.

$$\therefore \frac{1}{2}R_i\left(\frac{\partial W_i}{\partial q_i}\right)^2+V_i-hf_i=\alpha_i,\quad i=1,\dots,n \quad \text{-----} (2)$$

[$\because$ This is true for Liouville's system]

Where $\alpha_1+\alpha_2+\dots+\alpha_n=0$ ----- (3) [$\because$ From (1)]

Then, the equation (2) is integrated.

We have,

$$W=\sum_{i=1}^{n}\int\frac{1}{R_i}\sqrt{Q_i(q_i)}\,dq_i \quad \text{-----} (4)$$

Where $Q_i(q_i)=2R_i\left[hf_i(q_i)-V_i(q_i)+\alpha_i\right],\quad i=1,2,\dots,n$

This solution (4) actually contains $(n+1)$ constants namely $\alpha_1,\alpha_2,\dots,\alpha_n,h$.

But from equation (3) one α_i can be ~~de~~ eliminated leaving the required n independent constants.

$\therefore$ Equation (4) is the solution of the modified Hamilton Jacobi equation.

$\therefore$ Liouville's conditions are sufficient for the separability of an ~~orthog~~ orthonormal system.

To find the solution we will first eliminate one α_n.

From equation (3)

ie, $\alpha_n=-\alpha_1-\alpha_2-\dots-\alpha_{n-1}$.

$$\therefore \frac{\partial W}{\partial\alpha_i}=\frac{\partial W_i}{\partial\alpha_i}+\frac{\partial W_n}{\partial\alpha_n}\cdot\frac{\partial\alpha_n}{\partial\alpha_i},\quad i=1,2,\dots,n-1$$

$$\Rightarrow \frac{\partial W}{\partial\alpha_i}=\frac{\partial W_i}{\partial\alpha_i}-\frac{\partial W_{\alpha_n}}{\partial\alpha_n}$$

But $-\beta_i=\frac{\partial W}{\partial\alpha_i},\ i=1,2,\dots,n-1$ and $t-\beta_n=\frac{\partial W}{\partial\alpha_n}$.

$$\therefore \frac{\partial W}{\partial\alpha_i}=\int\frac{dq_i}{\sqrt{Q_i(q_i)}}-\int\frac{dq_n}{\sqrt{Q_n(q_n)}}=-\beta_i\quad(i=1,2,\dots,n-1) \quad \text{-----} (5)$$

$$\frac{\partial W}{\partial h}=\sum_{i=1}^{n}\int\frac{f_i\,dq_i}{\sqrt{Q_i(q_i)}}=t-\beta_n \quad \text{-----} (6)$$

Equations ⑤ and ⑥ is the solution to the Lagrangian problem and present the paths of the system in the extend configuration space.

The path in the phase-space is found by the addition equation

$$p_i = \frac{\partial W}{\partial q_i} = \frac{1}{R_i}\sqrt{Q_i(q_i)} \quad (i = 1, 2, \dots n) \quad [\because \text{From eqn } ④$$

Since β_i in eqn ⑤ is a constant along any actual paths of the system, the increments in the values of any two of given integrals must be equal for any interval of time.

$$\therefore \frac{dq_1}{\sqrt{Q_1(q_1)}} = \frac{dq_2}{\sqrt{Q_2(q_2)}} = \dots = \frac{dq_n}{\sqrt{Q_n(q_n)}} = dI.$$

10m Steckel's theorem.

II Statement

Consider an orthogonal system whose kinetic energy is given by

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{q}_i^{\,2} = \frac{1}{2}\sum_{i=1}^{n} c_i p_i^{\,2} \text{ ----- } ① \quad \text{Where } c_i(q_1, \dots, q_n) >$$

This system is separable iff (i) a non-singular $n \times n$ matrix $[\phi_{ij}(q_i)]$ and column matrix $[\psi_i(q_i)]$ exists such that

$$c^T\phi = (1, 0, 0, \dots, 0) \text{ ----- } ② \quad \text{and} \quad c^T\psi = V \text{ ----- } ③,$$ Where $V(q_1, q_2, \dots, q_n)$ is the potential energy and c is the column matrix composed by n c's.

Proof

Necessary part

Let us assume that the given orthogonal system is separable.

$\therefore$ It possesses the characteristic function

$$W(q, \alpha) = \sum_{i=1}^{n} W_i(q_i).$$

This characteristic function is a complete integral of the modified Hamilton Jacobi equation namely,

$$\frac{1}{2}\sum_{i=1}^{n} c_i\left(\frac{\partial W_i}{\partial q_i}\right)^2 + V = \alpha_1 \text{ ------ } ④$$

Where α_1 is a total energy.

As a system is separable $\left(\frac{\partial W_i}{\partial q_i}\right)^2$ is a function of $(q_i, \alpha_1, \alpha_2, \dots, \alpha_n)$ and $\alpha_1 + \alpha_2 + \dots + \alpha_n = 0$.

∴ We can choose the separation constant such that α_i's appear linearly.

Hence the most general form involving the single co-ordinate q_i is

$$\left(\frac{\partial W}{\partial q_i}\right)^2 = -2\psi_i(q_i) + 2\sum_{j=1}^{n} \phi_{ij}(q_i)\alpha_j \quad \text{----} (5)$$

Where the numerical co-efficients $\psi_i(q_i)$, $\phi_{ij}(q_i)$ are chosen for convenience.

Substuting equation (5) in (4)

$$\frac{1}{2}\sum_{i=1}^{n} c_i\left[-2\psi_i(q_i) + 2\sum_{j=1}^{n}\phi_{ij}(q_i)\alpha_j\right] + V = \alpha_1$$

Using the matrix notation we have

$$-c^T\psi + c^T\phi\alpha + V = \alpha_1.$$

Comparing the co-efficients of α

$$c^T\phi\alpha = \alpha_1$$

$$c^T\phi = (1,0,0,\dots,0)$$

This is the first condition of stackel's theorem.

Sufficient part

Let the condition be $c^T\phi = (1,0,\dots,0)$, $c^T\psi = V$.

Let '$\bar{a}$' be the column matrix defined by $a_i = \left(\frac{\partial W}{\partial q_i}\right)^2$, $i=1,\dots,n$

The modified Hamilton Jacobi equation is of the form

$H(q, \frac{\partial W}{\partial q}) = \alpha_1$ $\frac{1}{2}\sum c_i a_i + V = \alpha_1$ using matrix notation we have

$$\frac{1}{2}c^T\bar{a} + V = \alpha_1$$

$T + V = \alpha_1$ ie, $\frac{1}{2}c^T\bar{a} + c^T\psi = (1,0,0,\dots,0)\alpha$ ----- (6)

$\frac{1}{2}\sum_{i=1} c_i\left(\frac{\partial W}{\partial q}\right) + V = \alpha_1$

But $c^T = (1,0,0,\dots,0)\phi^{-1}$

$$(6) \Rightarrow (1,0,0,\dots,0)\left(\frac{1}{2}\phi^{-1}\bar{a} + \phi^{-1}\psi\right) = (1,0,0,\dots,0)\alpha$$

$$\phi^{-1}\left(\frac{\bar{a}}{2} + \psi\right) = \alpha. \Rightarrow \frac{\bar{a}}{2} + \psi = \phi\alpha.$$

This is identically with equation (5)

∴ The system is separable.

Hence the theorem.

Problem (or) discuss separability using Kepler's problem

Verify stackel's condition for Kepler's problem.

Solution

Kinetic energy $T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta)$

$\therefore T = \frac{1}{2} p_r^2 + \frac{1}{2r^2} p_\theta^2 + \frac{1}{2r^2\sin^2\theta} p_\phi^2$.

and potential energy $V = \frac{-M}{r}$

Where p_r, p_θ and p_ϕ are generalised momentum.

This is an orthogonal system.

$\therefore H = T + V = \text{Constant}$

The modified Hamilton Jacobi equation is

$$\frac{1}{2}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{1}{2r^2\sin^2\theta}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 - \frac{M}{r} = \alpha_t \quad \text{--- (1)}$$

Where the characteristic function

$$W = W_r(r) + W_\theta(\theta) + W_\phi(\phi)$$

$\therefore W$ is the separable form.

In T and V, ϕ is missing.

$\therefore \phi$ is an ignorable co-ordinate.

$\therefore p_\phi = \frac{\partial W_\phi}{\partial \phi} = \alpha_\phi$.

$\therefore W_\phi = \alpha_\phi \cdot \phi$.

Multiply eqn (1) by $2r^2$.

We have,

$$\cancel{r^2\left(\frac{\partial W_r}{\partial r}\right)^2 - 2r^2\left(\frac{M}{r} + \alpha_t\right) + \left(\frac{\partial W_\phi}{\partial \phi}\right)^2 \frac{1}{\sin^2\theta} = 0}$$

$$r^2\left(\frac{\partial W_r}{\partial r}\right)^2 - 2r^2\left(\frac{M}{r} + \alpha_t\right) + \left[\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2\right] = 0$$

ie, $$r^2\left(\frac{\partial W_r}{\partial r}\right)^2 - 2r^2\left(\frac{M}{r} + \alpha_t\right) + \left[\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta}\right] = 0$$

Here the first two terms are functions of r only and the last two terms are functions of θ only.

$\therefore$ They are equal to a separate constant.

ie, $$\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha_\theta^2 \quad \text{and} \quad r^2\left(\frac{\partial W_r}{\partial r}\right)^2 - 2r^2\left(\frac{M}{r} + \alpha_t\right) = -\alpha_\theta^2$$