Unit. I

Integer programming problem

A LPP in which all of the variables in the optimum solution are only a non-ve integers is called Integer programming problem. It is also called Integer linear programming

Pure IPP:

A LPP in which all of the variable in the optimum solution are only a non-ve integer is called pure IPP or all IPP.

Mixed IPP

A LPP if some of the variables in optimal solution are integer, the remaining variables are non-ve values is called mixed IPP.

0-1 programming problem

A LPP involves an optimum soln of all the variables are ~~called~~ either 0 or 1 is called 0-1 programming problem. It is also called standard discrete programming problem

Note:

The general IPP is max $z = cx$, subject to the constraint $Ax \le b$, $x \ge 0$ and some or all of the variables are integers

Note:

The IPP can be solved by two methods. They are

i. Cutting plane method or Gomory's fractional cut algorithm

ii. Branch and Bound method

Gomory's fractional cut algorithm or cutting plane method:

Step.1

Convert the minimization IPP into an maximization IPP.

Step.2

Find the optimum solution of the resulting IPP by using simplex method.

Step.3

i. If all $x_{B_i} \ge 0$ and are integers, an optimum integer solution is obtained

ii. If all $x_{B_i} \ge 0$ and atleast one x_{B_i} is not an integer then go to next step

Step. 4

Rewrite each x_{Bi} as $x_{Bi} = [x_{Bj}] + f_j$, where $\{x_{Bj}\}$ is the integral part of x_{Bi} and f_j is the +ve fractional of x_{Bi}, $0 \le f_j \le 1$.

Step. 5

Express each of the -ve fraction of the k^{th} row of the optimum simplex table as the sum of a -ve integer and a non -ve fraction.

Step. 6

Find the fractional cut constraint (Gomorian constraints) from the source row

$$-\sum_{j=1}^{n} f_{kj} x_j \le -f_k$$

or $$-\sum_{j=1}^{n} f_{kj} + s_j = -f_k$$

Step. 7

Add the fractional cut constraint obtained at the bottom of the optimum simplex table.

Step. 8

repeat the procedure until an optimum integer solution is obtained.

problem. 1

Find the optimum integer solution to the following LPP.

$\max z = x_1 + x_2$

Subject to the constraint

$3x_1 + 2x_2 \le 5$

$x_2 \le 2$

$x_1, x_2 \ge 0$ and are integers

Soln:

G.T $\max z = x_1 + x_2$

Subject to the constraints

$3x_1 + 2x_2 \le 5$

$x_2 \le 2$

$x_1, x_2 \ge 0$ and are integers

By introducing slack variables x_3 and x_4, the problem can be re written as

$\max z = x_1 + x_2 + 0.x_3 + 0.x_4$

Sub to

$3x_1 + 2x_2 + x_3 = 5$

$x_2 + x_4 = 2$

$x_1, x_2, x_3, x_4 \ge 0$ and are integers

Ist table:

C_B	B_A	x_B	x_1 (1)	x_2 (1)	x_3 (0)	x_4 (0)
0	x_4	5	[3]	2	1	0
0	x_5	2	0	1	0	1
$z_j - c_j$		0	-1	-1	0	0

Here the entering variable is x_1, the leaving variable is x_5 and the pivot element is 3

II nd table:

C_B	B_A	x_B	x_1 (1)	x_2 (1)	x_3 (0)	x_4 (0)
1	x_1	5/3	1	2/3	1/3	0
0	x_5	2	0	[1]	0	1
$z_j - c_j$		5/3	0	-1/3	1/3	0

$$\min\left(\frac{5/3}{2/3}, \frac{2}{1}\right) = \min\left(5/2, 2\right)$$

Here the entering variable is x_2, the leaving variable is x_5 and the pivot element is 1.

III rd table:

C_B	B_A	x_B	x_1 (1)	x_2 (1)	x_3 (0)	x_4 (0)
1	x_1	1/3	1	0	1/3	-2/3
1	x_2	2	0	1	0	1
$z_j - c_j$		7/3	0	0	1/3	1/3

Here all $z_j - c_j \geq 0$, the value of $x_1 = 1/3$, $x_2 = 2$ and $Z = 7/3$

Here $x_1 = 1/3$ we have

$$\frac{1}{3} = x_1 + \frac{1}{3}x_3 - \frac{2}{3}x_4$$

$$= x_1 + \frac{1}{3}x_3 + \left(-1 + \frac{1}{3}\right)x_4$$

The Gomorian constraint is given by

$$\frac{1}{3}x_3 + \frac{1}{3}x_4 \geq \frac{1}{3}$$

$$\Rightarrow -\frac{1}{3}x_3 - \frac{1}{3}x_4 \leq -\frac{1}{3}$$

Let s be a Gomorian slack variable, the constraint can be rewritten as

$$-\frac{1}{3}x_3 - \frac{1}{3}x_4 + s = -\frac{1}{3}$$

Here the solution is optimal, but not feasible, to obtain a feasible and optimal solution, we have to use dual simplex method.

Add this fractional cut constraint at the bottom of the above optimum simplex table we have

I st table

			1	1	0	0	0
C_B	BA	x_B	x_1	x_2	x_3	x_4	S
1	x_1	$1/3$	1	0	$1/3$	$-2/3$	0
1	x_2	2	0	1	0	1	0
0	S	$-\frac{1}{3}$	0	0	$-1/3$	$-1/3$	1
$Z_j - C_j$		$7/3$	0	0	$1/3$	$1/3$	0

$\max\left(\frac{1/3}{-1/3}, \frac{1/3}{-1/3}\right)$

$= \max(-1, -1)$

$= -1$

Here the entering variable is x_3, the leaving variable is s and the pivot element is s.

II nd table:

			1	1	0	0	0
C_B	BA	x_B	x_1	x_2	x_3	x_4	S
1	x_1	0	1	0	0	-1	1
1	x_2	2	0	1	0	1	0
0	x_3	1	0	0	1	1	-3
$Z_j - C_j$		2	0	0	0	0	1

Here all $Z_j - C_j \geq 0$, the solution is an optimum solution.

$\therefore$ The optimum solutions are $x_1^* = 0$, $x_2^* = 2$ and the optimum value $z^* = 2$.

Problem: 2

Find the optimum integer solution to the following LPP

$\max z = 2x_1 + 2x_2$

Subject to the constraints.

$5x_1 + 3x_2 \leq 8$

$2x_1 + 4x_2 \leq 8$

$x_1, x_2 \geq 0$ and are integers

Soln:

G.T $\max z = 2x_1 + 2x_2$

subject to

$5x_1 + 3x_2 \leq 8$

$2x_1 + 4x_2 \leq 8$

$x_1, x_2 \geq 0$ and are integers.

By introducing slack variables x_3 and x_4, the problem can be rewritten as

$\max z = 2x_1 + 2x_2 + 0.x_3 + 0.x_4$

Subject to

$5x_1 + 3x_2 + x_3 = 8$

$2x_1 + 4x_2 + x_4 = 8$

$x_1, x_2, x_3, x_4 \geq 0$ and are integers

I^{st} table

			2	2	0	0
C_B	BA	x_B	x_1	x_2	x_3	x_4
0	x_3	8	$\boxed{5}$	3	1	0
0	x_4	8	2	4	0	1
Z_j-C_j		0	-2	-2	0	0

$\rightarrow \min\left(\frac{8}{5}, \frac{8}{2}\right)$
$= \min(1.6, \ldots)$
$= 1.6$

Here the entering variable is x_1, the leaving variable is ... and the pivot element is 5.

II^{nd} table

			2	2	0	0
C_B	BA	x_B	x_1	x_2	x_3	x_4
2	x_1	8/5	1	3/5	1/5	0
0	x_4	24/5	0	$\boxed{14/5}$	-2/5	1
Z_j-C_j		16/5	0	-4/5	2/5	0

$\min\left(\frac{8/5}{3/5}, \frac{24/5}{14/5}\right)$
$= \min\left(\frac{8}{3}, \frac{24}{14}\right)$
$= \min(2.67, 1.\ldots)$
$= 1.7$

Here the entering variable is x_2, the leaving variable is ... and the pivot element is 14/5.

III^{rd} table:

			2	2	0	0
C_B	BA	x_B	x_1	x_2	x_3	x_4
2	x_1	4/7	1	0	2/7	-3/14
2	x_2	12/7	0	1	-1/7	5/14
Z_j-C_j		32/7	0	0	2/7	2/7

Here all $Z_j - C_j \geq 0$ the value of $x_1 = 4/7$, $x_2 = 12/7$ and $Z = 32/7$

Here $x_1 = \frac{4}{7} = 0 + \frac{4}{7}$

$x_2 = \frac{12}{7} = 1 + \frac{5}{7}$

Here $\max\left(\frac{4}{7}, \frac{5}{7}\right) = \frac{5}{7}$

Consider the second row

$$\frac{12}{7} = x_2 - \frac{1}{7}x_3 + \frac{5}{14}x_4$$

$$\Rightarrow 1 + \frac{5}{7} = x_2 + \left(-1 + \frac{6}{7}\right)x_3 + \frac{5}{14}x_4$$

The Gomorian constraint is given by

$$\frac{6}{7}x_3 + \frac{5}{14}x_4 \geq 5/7$$

$$\Rightarrow -\frac{6}{7}x_3 - \frac{5}{14}x_4 \leq -\frac{5}{7}$$

Let s_1 be a Gomorian slack variable, the constraint can be rewritten as

$$-\frac{6}{7}x_3 - \frac{5}{14}x_4 + s_1 = -5/7$$

Here the solution is optimal but not feasible, to obtain a feasible and optimal solution we have to use dual simplex method.

Add this fractional cut Gomorian at the bottom of the above optimum simplex table we have

Ist table:

			2	2	0	0	0
C_B	BA	x_B	x_1	x_2	x_3	x_4	S_1
2	x_1	$4/7$	1	0	$2/7$	$-3/14$	0
2	x_2	$12/7$	0	1	$-1/7$	$5/14$	0
0	S	$-5/7$	0	0	$\boxed{-6/7}$	$-5/14$	1
$Z_j - C_j$		$32/7$	0	0	$2/7$	$2/7$	0

$\max\left(\frac{2/7}{-6/7}, \frac{2/7}{-5/14}\right) = \max\left(-\frac{1}{3}, -\frac{4}{5}\right)$

Here the entering variable is x_3, the leaving variable is S and the pivot element is S

IInd table

			2	2	0	0	0
C_B	BA	x_B	x_1	x_2	x_3	x_4	S_1
2	x_1	$1/3$	1	0	0	$-1/3$	$1/3$
2	x_2	$11/6$	0	1	0	$5/12$	$-1/6$
0	x_3	$5/6$	0	0	1	$5/12$	$-7/6$
$Z_j - C_j$		$13/3$	0	0	0	$1/6$	$1/3$

Here all $Z_j - C_j \geq 0$, the ~~solution is an optimum solution but not feasible~~ value of $x_1 = 1/3$, $x_2 = 11/6$, $x_3 = 5/6$ and $Z = 13/3$

Here
$$x_1 = 1/3 = 0 + 1/3$$
$$x_2 = 11/6 = 1 + 5/6$$
$$x_3 = 5/6 = 0 + 5/6$$

Here $\max\left(\frac{1}{3}, 5/6, 5/6\right) = 5/6$

Consider the second row

$$\frac{11}{6} = x_2 + 5/12\, x_4 - \frac{1}{6} S_1$$

$$\Rightarrow 1 + 5/6 = x_2 + 5/12\, x_4 + (-1 + 5/6) S_1$$

The Gomorian constraint is given by

$$5/12\, x_4 + 5/6\, S_1 \geq 5/6$$

$$\Rightarrow -5/12\, x_4 - 5/6\, S_1 \leq -5/6$$

Now we introduce a Gomorian slack variable S_2, the constraint ~~equation~~ can be rewritten

$$-5/12\, x_4 - 5/6\, S_1 + S_2 = -5/6$$

Here the solution is optimal but not feasible, to obtain a feasible and optimal solution we have to use dual simplex method.

Add this fractional cut Gomorian at the bottom of the above optimum simplex table we have

Ist table

			2	2	0	0	0	0
C_B	B_N	x_B	x_1	x_2	x_3	x_4	S_1	S_2
2	x_1	$1/3$	1	0	0	$-1/6$	$1/3$	0
2	x_2	$11/6$	0	1	0	$5/12$	$-1/6$	0
0	x_3	$5/6$	0	0	1	$5/12$	$-7/6$	0
0	S_2	$-5/6$	0	0	0	$\boxed{-5/12}$	$-5/6$	1
	Z_j-C_j	$13/3$	0	0	0	$1/6$	$1/3$	0

$\max\left(\frac{\frac{1}{6}}{\frac{-5}{12}}, \frac{\frac{1}{3}}{-5/6}\right)$

$= \max\left(-\frac{1}{6}\times\frac{12}{5}, \frac{1}{3}\ldots\right)$

$= \max\left(-\frac{2}{5}, -\frac{2}{5}\right)$

$= -\frac{2}{5}$

Here the entering variable is x_4, the leaving variable is S_2 and the pivot element is $-5/12$

IInd table:

			2	2	0	0	0	0
C_B	B_N	x_B	x_1	x_2	x_3	x_4	S_1	S_2
2	x_1	1	1	0	0	0	1	$-4/5$
2	x_2	1	0	1	0	0	-1	1
0	x_3	0	0	0	1	0	-2	1
0	x_4	2	0	0	0	1	2	$-12/5$
	Z_j-C_j	4	0	0	0	0	0	$2/5$

Here all $Z_j - C_j \geq 0$, the solution is an optimal and feasible solution.

$\therefore$ The optimum solutions are $x_1^* = 1$, $x_2^* = 1$ and the optimum value $Z^* = 4$.

problem:
~~Home work~~:

Solve the following IPP

minimize $Z = -2x_1 - 3x_2$

subject to

$2x_1 + 2x_2 \leq 7$

$x_1 \leq 2$

$x_2 \leq 2$

$x_1, x_2 \geq 0$ and are integers

Soln:

G.T $\min Z = -2x_1 - 3x_2$

s.t

$2x_1 + 2x_2 \leq 7$

$x_1 \leq 2$

$x_2 \leq 2$

$x_1, x_2 \geq 0$ and are integers.

Using minimize theorem we have

Minimize theorem
$\min z = -\max(-z)$

$$\max(-z) = \max z^* = 2x_1 + 3x_2$$

Subject to

$$2x_1 + 2x_2 \le 7$$
$$x_1 \le 2$$
$$x_2 \le 2$$

$x_1, x_2 \ge 0$ and are integers

By introducing slack variables x_3, x_4 and x_5, the problem can be rewritten as

$$\max z^* = 2x_1 + 3x_2 + 0.x_3 + 0.x_4 + 0.x_5$$

Subject to

$$2x_1 + 2x_2 + x_3 = 7$$
$$x_1 + x_4 = 2$$
$$x_2 + x_5 = 2.$$

$x_1, x_2, x_3, x_4, x_5 \ge 0$ and are integers.

I st table:

			2	3	0	0	0	
C_B	BA	x_B	x_1	x_2	x_3	x_4	x_5	
0	x_3	7	2	2	1	0	0	$\min\left(\frac{7}{2}, \frac{2}{1}\right)$
0	x_4	2	1	0	0	1	0	$= \min(3.5, 2)$
0	x_5	2	0	$\boxed{1}$	0	0	1 →	$= 2$
$z_j - c_j$		0	−2	−3 ↑	0	0	0	

Here the entering variable is x_2, the leaving variable is x_5 and the pivot element is 1.

II nd table:

				2	3	0	0	0	
	C_B	BA	x_B	x_1	x_2	x_3	x_4	x_5	
$-2\times$ III rd row + 1st row	0	x_3	3	$\boxed{2}$	0	1	0	−2 →	$\min\left(\frac{3}{2}, \frac{2}{1}\right)$
II nd row	0	x_4	2	1	0	0	1	0	$= \min(1.5, 2)$ $= 1.5$
III rd row ÷ 1	3	x_2	2	0	1	0	0	1	
	$z_j - c_j$		6	−2 ↑	0	0	0	3	

Here the entering variable is x_1, the leaving variable is x_3 and the pivot element is 2

III rd table

			2	3	0	0	0
C_B	BA	x_B	x_1	x_2	x_3	x_4	x_5
2	x_1	3/2	1	0	1/2	0	−1
0	x_4	1/2	0	0	−1/2	1	1
3	x_2	2	0	1	0	0	1
$z_j - c_j$		9	0	0	3/2	0	1

Here all $z_j - c_j \ge 0$, the value of $x_1 = 3/2$, $x_2 = 2$, $x_4 = 1/2$ and $z = 9$

Here $x_1 = \frac{3}{2} = 1 + \frac{1}{2}$

$x_4 = \frac{1}{2} = 0 + \frac{1}{2}$

Here $\max\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$

Consider the first row

$$\frac{3}{2} = x_1 + \frac{1}{2}x_3 - x_5$$

$$\Rightarrow 1 + \frac{1}{2} = x_1 + \frac{1}{2}x_3 - x_5$$

The Gomorian constraint is given by

$$\frac{1}{2}x_3 \geq \frac{1}{2}$$

$$\Rightarrow -\frac{1}{2}x_3 \leq -\frac{1}{2}$$

Now we introduce a Gomorian slack variable S_1, the constraint can be rewritten as

$$-\frac{1}{2}x_3 + S_1 = -\frac{1}{2}$$

Here the solution is optimal but not feasible, to obtain a feasible and optimal solution, we have to use dual simplex method.

Add this fractional cut constraint at the bottom of the above optimum simplex table we have

I^{st} table:

			3	2	0	0	0	0
C_B	BA	x_B	x_1	x_2	x_3	x_4	x_5	S
2	x_1	$3/2$	1	0	$1/2$	0	-1	0
0	x_4	$1/2$	0	0	$-1/2$	1	1	0
3	x_2	2	0	1	0	0	1	0
0	S_1	$-\frac{1}{2}$	0	0	$\boxed{-\frac{1}{2}}$	0	0	1 →
$Z_j - C_j$		9	0	0	$3/2$ ↑	0	1	0

Here the entering variable is x_3, the leaving variable is S_1 and the pivot element is $-\frac{1}{2}$

II^{nd} table:

				3	2	0	0	0	0
	C_B	BA	x_B	x_1	x_2	x_3	x_4	x_5	S_1
I^{st} row	2	x_1	1	1	0	0	0	-1	1
$-\frac{1}{2} \times IV^{th}$ row + II^{nd} row	0	x_4	1	0	0	0	1	1	-1
III^{rd} row	3	x_2	2	0	1	0	0	1	0
IV^{th} row $\div -\frac{1}{2}$	0	x_3	1	0	0	1	0	0	-2
	$Z_j - C_j$		8	0	0	0	0	1	2

Here all $Z_j - C_j \geq 0$, the the solution is an optimum solution.

∴ The optimum solutions are $x_1^* = 1$, $x_2^* = 2$ and the optimum value $Z^* = -Z^* = -8$

Home work

Find the optimum integer solution to the following linear programming problem.

$$\max z = x_1 + 2x_2$$

Subject to

$$2x_2 \le 7$$
$$x_1 + x_2 \le 7$$
$$2x_1 \le 11$$

$x_1, x_2 \ge 0$ and are integers.

Gomory's mixed integer programming method

The integer programming problem is first solved as a continuous LPP by ignoring the integrality condition. If the values of the integer constrained variables are integers, then the current solution is an optimum solution.

From the source row

$$\sum_{j=1}^{n} a_{kj} x_j = x_{B_k}$$

$$\Rightarrow \sum_{j=1}^{n} ([a_{kj}] + f_{kj}) x_j = [x_{B_k}] + f_k$$

$$\Rightarrow \sum_{j=1}^{n} f_{kj} x_j \ge f_k$$

ie $$\sum_{j \in J^+} f_{kj} x_j + \left(\frac{f_k}{f_{k-1}}\right) \sum_{j \in J^+} f_{kj} x_j \ge f_k$$

$$\Rightarrow -\sum_{j \in J^+} f_{kj} x_j - \left(\frac{f_k}{f_{k-1}}\right) \sum_{j \in J^-} f_{kj} x_j \le -f_k$$

$$\Rightarrow -\sum_{j \in J^+} f_{kj} x_j - \left(\frac{f_k}{f_{k-1}}\right) \sum_{j \in J^-} f_{kj} x_j + S_k = -f_k$$

Where S_k = Gomorian slack

$$J^+ = \{J \mid f_{kj} \ge 0\}$$
$$J^- = \{J \mid f_{kj} < 0\}$$

Add this secondary constraint at the bottom of the optimum simplex table and use dual simplex method to obtain the new feasible optimal solution. Repeat the procedure until the values of the integer restricted variables are integers in the optimum solution obtained.

Problem:

Solve the following mixed LPP

max $Z = x_1 + x_2$

subject to

$2x_1 + 5x_2 \le 16$

$6x_1 + 5x_2 \le 30$

$x_1, x_2 \ge 0$ and x_1 is a non -ve integer.

Soln:

G.T max $Z = x_1 + x_2$

subject to

$2x_1 + 5x_2 \le 16$

$6x_1 + 5x_2 \le 30$

$x_1, x_2 \ge 0$ and x_1 is a non -ve integer

By introducing slack variables x_3 and x_4 the problem can be rewritten as

max $z = x_1 + x_2 + 0.x_3 + 0.x_4$

subject to

$2x_1 + 5x_2 + x_3 = 16$

$6x_1 + 5x_2 + x_4 = 30$

$x_1, x_2, x_3, x_4 \ge 0$ and x_1 is a non -ve integer.

I^{st} table:

			1	1	0	0
C_B	BA	X_B	x_1	x_2	x_3	x_4
0	x_3	16	2	5	1	0
0	x_4	30	$\boxed{6}$	5	0	1
$Z_j - C_j$		0	-1	-1	0	0

$\rightarrow$ min $\left(\frac{16}{2}, \frac{30}{6}\right)$ = min (8, 5) = 5

Here the entering variable is x_1, the leaving variable is x_4 and the pivot element is 6

II^{nd} table:

				1	1	0	0
	C_B	BA	X_B	x_1	x_2	x_3	x_4
$-2 \times II^{nd}$ row $+ I^{st}$ row	0	x_3	6	0	$\boxed{10/3}$	1	$-1/3$
II^{nd} row $\div 6$	1	x_1	5	1	$5/6$	0	$1/6$
	$Z_j - C_j$		5	0	$-1/6$	0	$1/6$

$\rightarrow$ min $\left(\frac{6}{10/3}, \frac{5}{5/6}\right)$

$= \min\left(\frac{18}{10}, \frac{30}{5}\right)$

$= \min\left(\frac{9}{5}, 6\right)$

$= \frac{9}{5}$

Here the entering variable is x_2 the leaving variable is x_3 and the pivot element is $\frac{10}{3}$

III^{rd} table:

				1	1	0	0
	C_B	BA	X_B	x_1	x_2	x_3	x_4
I^{st} row $\div \frac{10}{3}$	1	x_2	$9/5$	0	1	$3/10$	$-1/10$
$-\frac{5}{6} \times I^{st}$ row $+ II^{nd}$ row	1	x_1	$7/2$	1	0	$-1/4$	$1/4$
	$Z_j - C_j$		$53/10$	0	0	$1/20$	$3/20$

Here all $Z_j - C_j \ge 0$, the solution is an optimum and not feasible

Here $x_1 = 7/2$, $x_2 = 9/5$ and $Z = 53/10$

Given x_1 is a non-ve integer, we choose the second row

$$\frac{7}{2} = x_1 - \frac{1}{4}x_3 + \frac{1}{4}x_4$$

$$\Rightarrow 3 + \frac{1}{2} = x_1 - \frac{1}{4}x_3 + \frac{1}{4}x_4$$

The Gomorian constraint is given by

$$\left(\frac{\frac{1}{2}}{\frac{1}{2}-1}\right)\left(-\frac{1}{4}\right)x_3 + \frac{1}{4}x_4 \geq \frac{1}{2}$$

$$\Rightarrow +\frac{1}{4}x_3 + \frac{1}{4}x_4 \geq \frac{1}{2}$$

$$\Rightarrow -\frac{1}{4}x_3 - \frac{1}{4}x_4 \leq -\frac{1}{2}$$

By introducing a Gomorian slack variable s_1 the constraint can be rewritten as

$$-\frac{1}{4}x_3 - \frac{1}{4}x_4 + s_1 = -\frac{1}{2}$$

Add the Gomorian constraint at the bottom of the above optimum simplex table we have

Istable:

			1	1	0	0	0
C_B	BV	x_B	x_1	x_2	x_3	x_4	s_1
1	x_2	9/5	0	1	3/10	-1/10	0
1	x_1	7/2	1	0	-1/4	1/4	0
0	s_1	$-\frac{1}{2}$	0	0	$\boxed{-1/4}$	-1/4	1
	$Z_j - C_j$	53/10	0	0	1/20	3/20	0

$$\max\left(\frac{\frac{1}{20}}{-1/4}, \frac{\frac{3}{20}}{-1/4}\right)$$
$$= \max\left(-\frac{4}{20}, -\frac{12}{20}\right)$$
$$= \max\left(-\frac{1}{5}, -\frac{3}{5}\right)$$
$$= -\frac{1}{5}$$

∴ Here the entering variable is x_3, the leaving variable is s_1 and the pivot element is $-1/4$

IInd table:

			1	1	0	0	0
C_B	B_A	x_B	x_1	x_2	x_3	x_4	S_1
1	x_2	$6/5$	0	1	0	$-3/5$	$6/5$
1	x_1	-4	1	0	0	$1/2$	-1
0	x_3	2	0	0	1	1	-4
	$z_j - c_j$	$26/5$	0	0	0	$1/10$	$1/5$

$\frac{1}{4} \times$ IIIrd row + Ist row

IIIrd row $\div -\frac{1}{4}$

Since all $z_j - c_j \geq 0$, the solution is an optimal solution.

$\therefore$ The optimum solutions are $x_1^* = 4$, $x_2^* = 6/5$ and the optimum value $z^* = 26/5$.

problem:

Solve the following mixed IPP.

$\min z = x_1 - 3x_2$

Subject to

$x_1 + x_2 \leq 5$

$-2x_1 + 4x_2 \leq 11$

$x_1, x_2 \geq 0$ and x_2 is an integer

soln:

G.T $\min z = x_1 - 3x_2$

Subject to

$x_1 + x_2 \leq 5$

$-2x_1 + 4x_2 \leq 11$

$x_1, x_2 \geq 0$ and x_2 is an integer.

Using minimize theorem, we have

$\max(-z) = \max z^* = -x_1 + 3x_2$

S.T

$x_1 + x_2 \leq 5$

$-2x_1 + 4x_2 \leq 11$

$x_1, x_2 \geq 0$ and x_2 is an integer.

By introducing slack variable x_3 and x_4, the problem can be rewritten as

$\max z^* = -x_1 + 3x_2 + 0.x_3 + 0.x_4$

S.T $x_1 + x_2 + x_3 = 5$

$-2x_1 + 4x_2 + x_4 = 11$

$x_1, x_2, x_3, x_4 \geq 0$ and x_2 is an integer

Ist table:

			-1	3	0	0
C_B	B_A	x_B	x_1	x_2	x_3	x_4
0	x_3	5	1	1	1	0
0	x_4	11	2	$\boxed{4}$	0	1
	$z_j - c_j$	0	1	-3	0	0

$\min\left(\frac{5}{1}, \frac{11}{4}\right) = \frac{11}{4} \rightarrow$

$\uparrow$

Here the entering variable is x_2, the leaving variable is x_4, the pivot element is 4.

II nd table:

			-1	3	0	0
C_B	B_A	x_B	x_1	x_2	x_3	x_4
0	x_3	9/4	$\boxed{3/2}$	0	1	-1/4
3	x_2	1/4	-1/2	1	0	1/4
$z_j - c_j$		3/4	-1/2	0	0	3/4

$-1 \times$ II nd row + I st row

II nd row $\div 4$

Here the entering variable is x_1, the leaving variable is x_3 and the pivot element is $3/2$

III rd table:

			-1	3	0	0
C_B	B_A	x_B	x_1	x_2	x_3	x_4
-1	x_1	3/2	1	0	2/3	-1/6
3	x_2	7/2	0	1	1/3	1/6
$z_j - c_j$		9	0	0	1/2	2/3

I st row $\div \frac{3}{2}$

$\frac{1}{2} \times$ I st row + II nd row

Here all $z_j - c_j \geq 0$, the solution is an optimum and not feasible.

Here $x_1 = 3/2$, $x_2 = 7/2$ and $z = 9$

Given x_2 is a non-ve integer, we choose the second row

$$\cancel{1/6 = x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4}$$

$$\frac{7}{2} = x_2 + \frac{1}{3}x_3 + \frac{1}{6}x_4$$

$$\Rightarrow 3 + \frac{1}{2} = x_2 + \frac{1}{3}x_3 + \frac{1}{6}x_4$$

The Gomorian constraint is given by

$$\frac{1}{3}x_3 + \frac{1}{6}x_4 \geq \frac{1}{2}$$

$$\Rightarrow -\frac{1}{3}x_3 - \frac{1}{6}x_4 \leq -\frac{1}{2}$$

By introducing a Gomorian slack variable S_1, the constraint can be rewritten as

$$-\frac{1}{3}x_3 - \frac{1}{6}x_4 + S_1 = -\frac{1}{2}$$

Add the Gomorian constraint at the bottom of the above optimum simplex table we have

I st table:

			-1	3	0	0	0
C_B	B_A	x_B	x_1	x_2	x_3	x_4	S_1
-1	x_1	3/2	1	0	2/3	-1/6	0
3	x_2	7/2	0	1	1/3	1/6	0
0	S_1	-1/2	0	0	$\boxed{-1/3}$	-1/6	1
$z_j - c_j$		9	0	0	1/2	2/3	0

$$\max\left(\frac{1/2}{-1/3}, \frac{2/3}{-1/6}\right) = \max(-1, -4) = -1$$

Here the entering variable is x_3, the leaving variable is S_1 and the pivot element is $-1/3$

IIIrd table:

		$C_j \rightarrow$	-1	3	0	0	0
	C_B	B.V	x_B	x_1	x_2	x_3	x_4	s_1
$-\frac{2}{3} \times$ IIIrd row + Ist row	-1	x_1	1/2	1	0	0	-1/2	2
$-\frac{1}{3} \times$ IIIrd row + IInd row	3	x_2	3	0	1	0	0	1
IIIrd row $\div -\frac{1}{3}$	0	x_3	3/2	0	0	1	1/2	-3
		$Z_j - C_j$	17/2	0	0	0	1/2	1

Here all $Z_j - C_j \geq 0$, the solution is an optimum solution

∴ The optimum solutions are $x_1^* = \frac{1}{2}$, $x_2^* = 3$ and the optimum value $Z^* = -(-Z) \div -Z^* = -\frac{17}{2}$.

Home work:

Solve the following mixed integer programming problem.

~~minimize $z = 10x_1 + 9x_2$ subject to $x_1 \leq 8$, $x_2 \leq 10$, $5x_1 + 3x_2$~~

maximize $z = 4x_1 + 6x_2 + 2x_3$

subject to

$4x_1 - 4x_2 \leq 5$

$-x_1 + 6x_2 \leq 5$

$-x_1 + x_2 + x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$ and

x_1, x_3 are integers.

Branch and Bound method

This method is applicable to both pure (all) as well as mixed integer programming problems and involves the continuous version of the problem.

Let the given LPP be

maximize $z = CX$

subject to

$AX \leq b$

$X \geq 0$ and integers.

The given problem, is first solved as a continuous LPP by ignoring the integrality condition. If in the optimal solution some one of the variables, say x_r is not an integer then

$x_r^* < x_r < x_r^* + 1$, where x_r^* and $x_r^* + 1$ are consecutive non-ve integers.

Hence any feasible integer value of x_r must must satisfy one of the two conditions

$x_r \leq x_r^*$ or $x_r \geq x_r^* + 1$

These two conditions are mutually exclusive and hence both cannot be amended in the LPP simultaneously. By adding these two conditions separately to the continuous LPP, we form different subproblems.

Sub problem 1

$\max Z = CX$

subject to

$AX \leq b$

$x_r \leq x_r^*$

and $X \geq 0$

Sub problem 2

$\max Z = CX$

subject to

$AX \leq b$

$x_r \geq x_r^* + 1$

and $X \geq 0$.

Thus we have branched or partitioned the original problem into two subproblems. Geometrically it means that the branching process eliminates that portion of the feasible region that contains no feasible integer solution. Each of these sub-problems is then solved separately as a LPP.

<u>problem:</u>

Use Branch and bound technique to solve the following.

$\max Z = x_1 + 4x_2$

subject to

$2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_1, x_2 \geq 0$ and are integers

<u>soln:</u>

G.T $\max Z = x_1 + 4x_2$

subject to

$2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_1, x_2 \geq 0$ and are integers.

consider $2x_1 + 4x_2 = 7$ —— ①

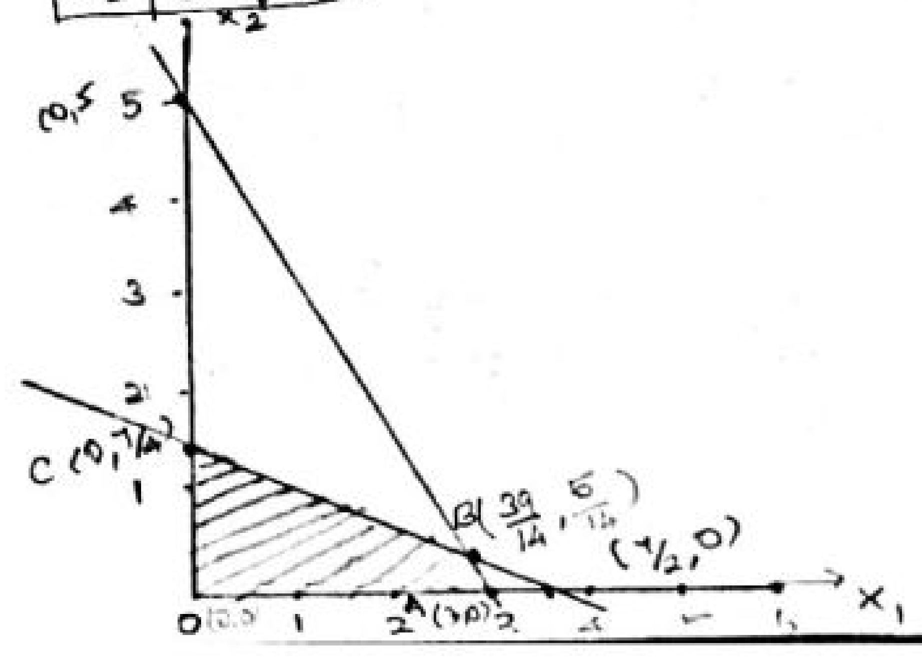
now

x_1	0	$7/2$
x_2	$7/4$	0

consider $5x_1 + 3x_2 = 15$ —— ②

now

x_1	0	3
x_2	5	0



① × 5 $\quad 10x_1 + 20x_2 = 35$ —— ③

② × 2 $\quad 10x_1 + 6x_2 = 30$ —— ④

③ − ④ $\quad 14x_2 = 5$

$x_2 = 5/14$.

$x_2 = 5/14$ in ① we get

$2x_1 + 4(5/14) = 7$

$\Rightarrow 2x_1 = 7 - \frac{10}{7} = \frac{39}{7}$

$\Rightarrow x_1 = \frac{39}{14}$

At $O(0,0)$, $Z = 0 + 4(0) = 0$.

$A(3,0)$, $Z = 3 + 4(0) = 3$

$B\left(\frac{39}{14}, \frac{5}{14}\right)$, $Z = \frac{39}{14} + 4\left(\frac{5}{14}\right) = \frac{59}{14}$

$C\left(0, \frac{7}{4}\right)$; $Z = 0 + 4\left(\frac{7}{4}\right) = 7$.

∴ The optimum solution is $x_1 = 0$; $x_2 = \frac{7}{4}$ and max $Z = 7$

Here $x_2 = \frac{7}{4}$, the problem should be branched into two subproblems

Now $x_2 = \frac{7}{4} \Rightarrow 1 < x_2 < 2$

$\Rightarrow x_2 \le 1$ or $x_2 \ge 2$

Sub problem: 1

max $Z = x_1 + 4x_2$

subject to

$2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_2 \le 1$

$x_1, x_2 \ge 0$.

Consider $2x_1 + 4x_2 = 7$ —— ①

Now

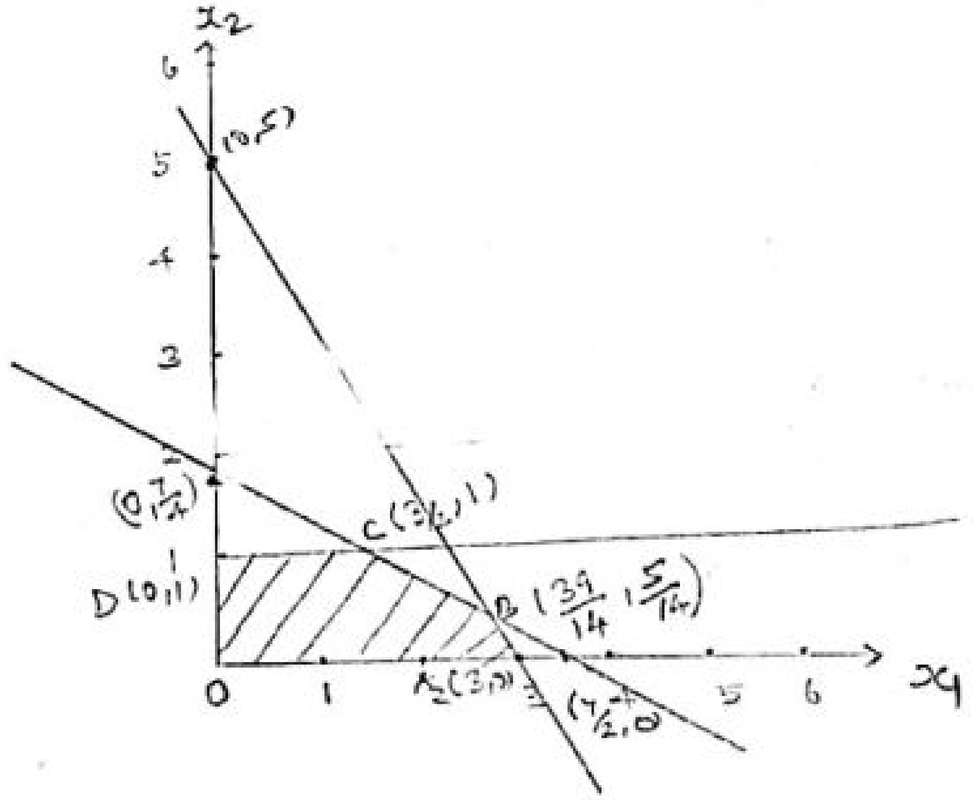
x_1	0	7/2
x_2	7/4	0

Consider $5x_1 + 3x_2 = 15$ —— ②

Now

x_1	0	3
x_2	5	0

Consider $x_2 = 1$ —— ③



Sub ③ in ① we get

$$2x_1 + 4(1) = 7$$

$$\Rightarrow x_1 = \frac{7-4}{2} = \frac{3}{2}$$

Now $O(0,0)$, $Z = 0 + 4(0) = 0$

$A(3,0)$; $Z = 3 + 4(0) = 3$

$B(\frac{39}{14}, \frac{5}{14})$; $Z = \frac{39}{14} + 4(\frac{5}{14}) = \frac{59}{14}$

$C(3/2, 1)$; $Z = 3/2 + 4(1) = \frac{11}{2}$

$D(0,1)$; $Z = 0 + 4(1) = 4$.

$\therefore$ The optimum solution is $x_1 = \frac{3}{2}$, $x_2 = 1$, and max $Z = \frac{11}{2}$

Here $x_1 = \frac{3}{2}$, this sub problem can be ~~branched be~~ branched again.

<u>Sub problem : 2</u>

max $Z = x_1 + 4x_2$

s.t

$$2x_1 + 4x_2 \le 7$$
$$5x_1 + 3x_2 \le 15$$
$$x_2 \ge 2$$
$$x_1, x_2 \ge 0.$$

Consider $2x_1 + 4x_2 = 7$

Now

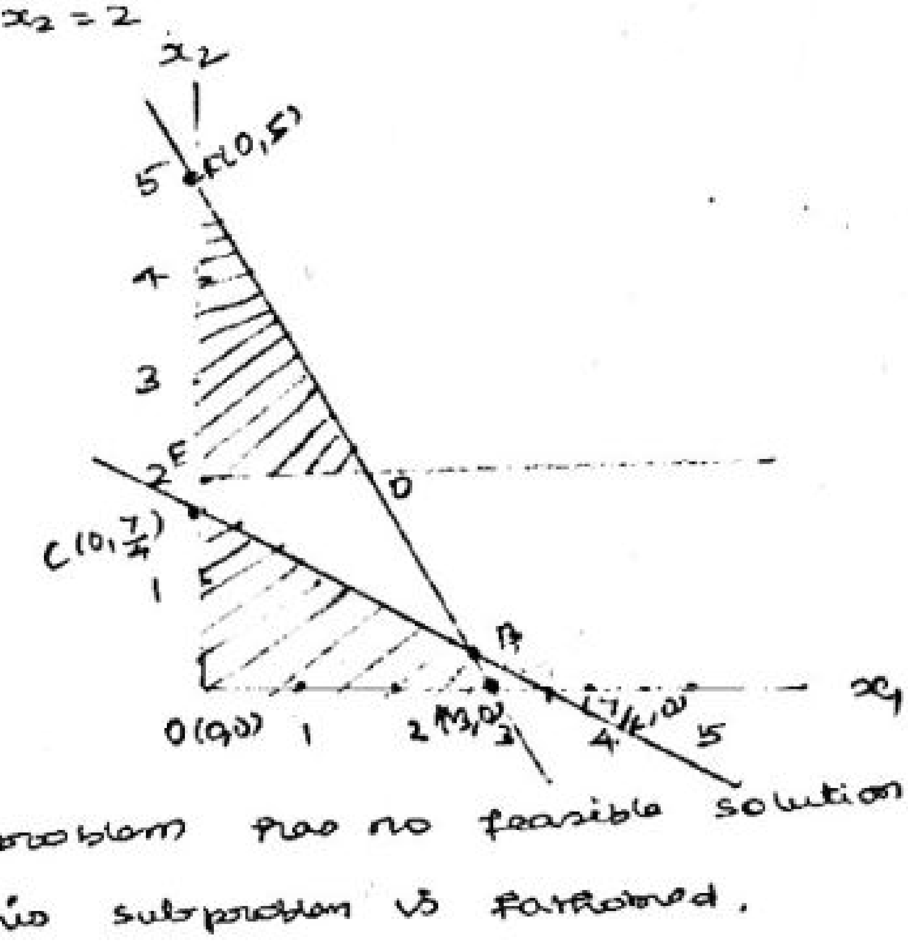
x_1	0	7/2
x_2	7/4	0

Consider $5x_1 + 3x_2 = 15$

Now

x_1	0	3
x_2	5	0

Consider $x_2 = 2$



$\therefore$ The problem has no feasible solution.

Hence this subproblem is fathomed.

In sub problem : 1, $x_1 = \frac{3}{2} \Rightarrow 1 \le x_1 \le 2$

$\Rightarrow x_1 \le 1$ or $x_1 \ge 2$.

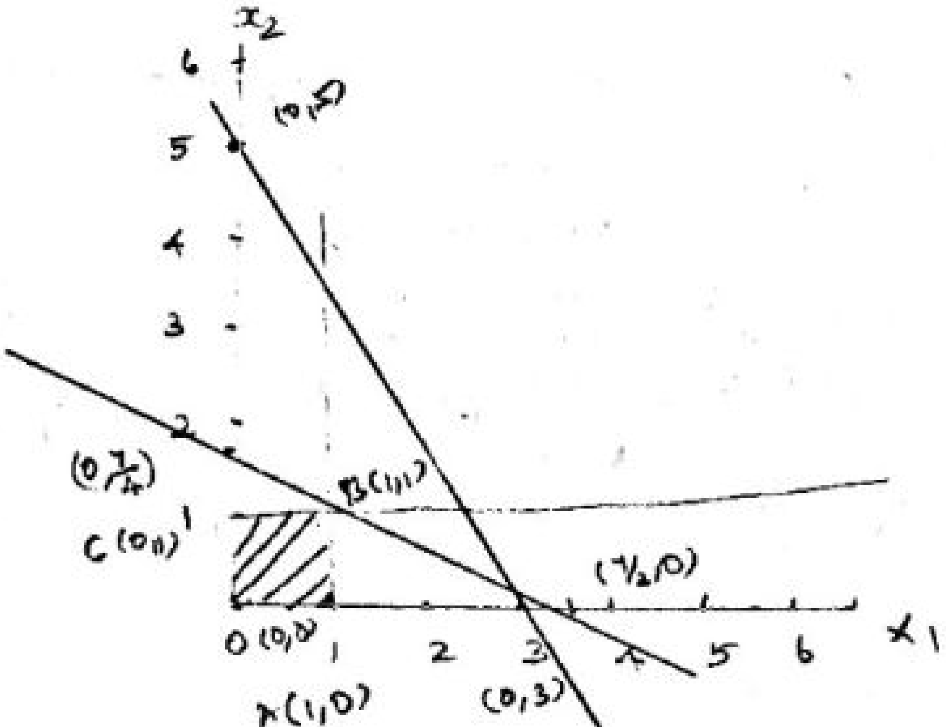
Subproblem: 3

$$\max z = x_1 + 4x_2$$

subject to

$$2x_1 + 4x_2 \le 7$$
$$5x_1 + 3x_2 \le 15$$
$$x_2 \le 1$$
$$x_1 \le 1$$
$$x_1, x_2 \ge 0.$$

Now.



At $O(0,0)$; $z = 0 + 4(0) = 0$

$A(1,0)$; $z = 1 + 4(0) = 1$

$B(1,1)$; $z = 1 + 4(1) = 5$

$C(0,1)$; $z = 0 + 4(1) = 4.$

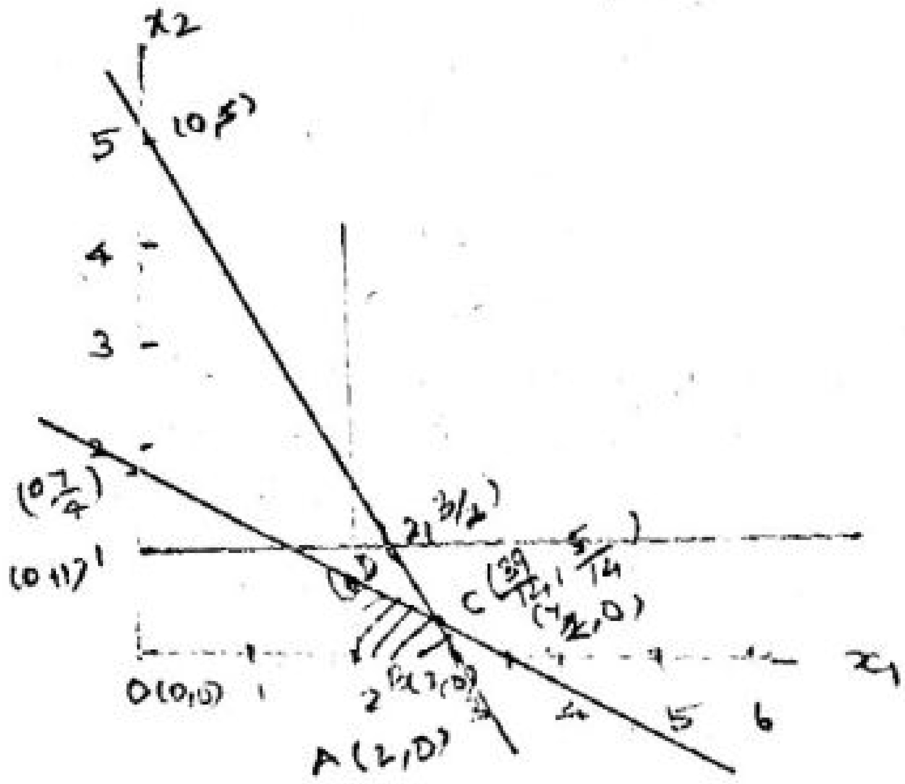
$\therefore$ The optimum solutions are $x_1 = 1$, $x_2 = 1$ and $\max z = 5$

Subproblem: 4

$$\max z = x_1 + 4x_2$$

s.t

$$2x_1 + 4x_2 \le 7$$
$$5x_1 + 3x_2 \le 15$$
$$x_2 \le 1$$
$$x_1 \ge 2.$$
$$x_1, x_2 \ge 0.$$



consider $2x_1 + 4x_2 = 7$. --- ①

$x_1 = 2$ --- ②

Sub ② in ① we get

$2x_1 + 4(2) = 7$

$\cancel{2x_1 = 7-8}$

$\cancel{x_1 = -\frac{1}{2}}$

$2(2) + 4x_2 = 7$

$\Rightarrow 4x_2 = 7-4$

$\Rightarrow x_2 = \frac{3}{4}$.

~~∴ The optimum solution is $x_1 = 2, x_2 = 3/4$ an~~

Now $A(2,0)$, $z = 2 + 4(0) = 2$

$B(3,0)$; $z = 3 + 4(0) = 3$

$C(\frac{39}{14}, \frac{5}{14})$; $z = \frac{39}{14} + 4(\frac{5}{14}) = \frac{59}{14}$

$D(2, \frac{3}{4})$; $z = 2 + 4(\frac{3}{4}) = 5$

∴ The optimum solution is $x_1 = 2$, $x_2 = \frac{3}{4}$ and max $z = 5$

In subproblem 4, $x_2 = \frac{3}{4} \Rightarrow 0 \le x_2 \le 1 \Rightarrow x_2 \le 0$ or $x_2 \ge 1$.

Subproblem: 5

max $z = x_1 + 4x_2$

s.t

$2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_2 \le 1$

$x_1 \ge 2$

$x_2 \le 0$.

$x_1, x_2 \ge 0$.

Here $x_2 \le 0$, is not possible, ∴ $x_2 = 0$., Hence $x_1 = 3$

∴ The optimum solution is $x_1 = 3$, $x_2 = 0$ and max $z = 3$.

This subproblem is fathomed.

Subproblem: 6

max $z = x_1 + 4x_2$.

s.t

$2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_2 \le 1$

$x_1 \ge 2$

$x_2 \ge 1$.

$x_1, x_2 \ge 0$.

Here $x_2 \le 1$ and $x_2 \ge 1$, is not possible and no feasible soln; Hence this subproblem is also fathomed.

Among the available integer valued solution, the best integer solution is subproblem: 3.

∴ Hence the optimum integer solution is $x_1^* = 1$, $x_2^* = 1$ and the optimum value $z^* = 5$.

Problem 12

Use Branch and Bound method to solve the following

maximize $z = 2x_1 + 2x_2$

s.t

$5x_1 + 3x_2 \le 8$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0$ and are Integer

Soln:

G.T $\max z = 2x_1 + 2x_2$

s.t $5x_1 + 3x_2 \le 8$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0$ and are Integers.

consider $5x_1 + 3x_2 = 8$ ——— (1)

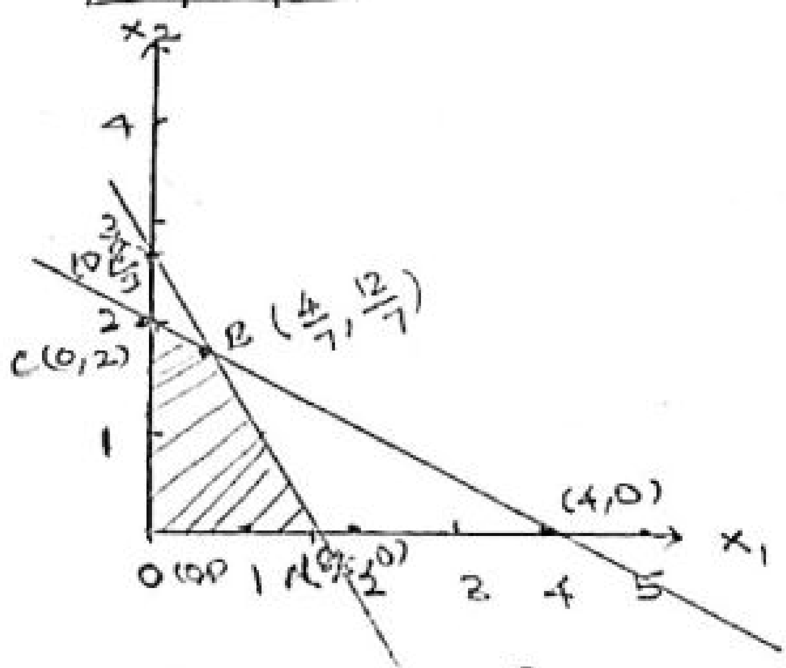
Now

x_1	0	8/5
x_2	8/3	0

consider $x_1 + 2x_2 = 4$ ——— (2)

Now

x_1	0	4
x_2	2	0



(1) × 1 , $5x_1 + 3x_2 = 8$ ——— (3)

(2) × 5 $5x_1 + 10x_2 = 20$ ——— (4)

(3) − (4) $-7x_2 = -12$

$x_2 = \frac{-12}{-7}$

$\boxed{x_2 = \frac{12}{7}}$

$x_2 = \frac{12}{7}$ in (2) we get

$x_1 + 2\left(\frac{12}{7}\right) = 4$

$\Rightarrow x_1 = 4 - \frac{24}{7}$

$\boxed{x_1 = \frac{4}{7}}$

At $O(0,0)$; $Z = 2(0) + 2(0) = 0.$

$A(8/5, 0)$; $Z = 2(8/5) + 2(0) = \frac{16}{5}$

$B(\frac{4}{7}, \frac{12}{7})$; $Z = 2(\frac{4}{7}) + 2(\frac{12}{7}) = \frac{8}{7} + \frac{24}{7} = \frac{32}{7}$

$C(0, 2)$; $Z = 2(0) + 2(2) = 4$

$\therefore$ The optimum solution is $x_1 = \frac{4}{7}$, $x_2 = \frac{12}{7}$ and max $Z = \frac{32}{7}$

Here $x_2 = \frac{12}{7}$, the problem should be branched into two subproblems.

Now $x_2 = \frac{12}{7} \Rightarrow 1 < x_2 < 2$

$\Rightarrow x_2 \le 1$ or $x_2 \ge 2$.

Subproblem : 1

$$\max Z = 2x_1 + 2x_2$$

s.t

$$5x_1 + 3x_2 \le 8$$
$$x_1 + 2x_2 \le 4$$
$$x_2 \le 1$$
$$x_1, x_2 \ge 0.$$

Consider $5x_1 + 3x_2 = 8$ —— ①

Now

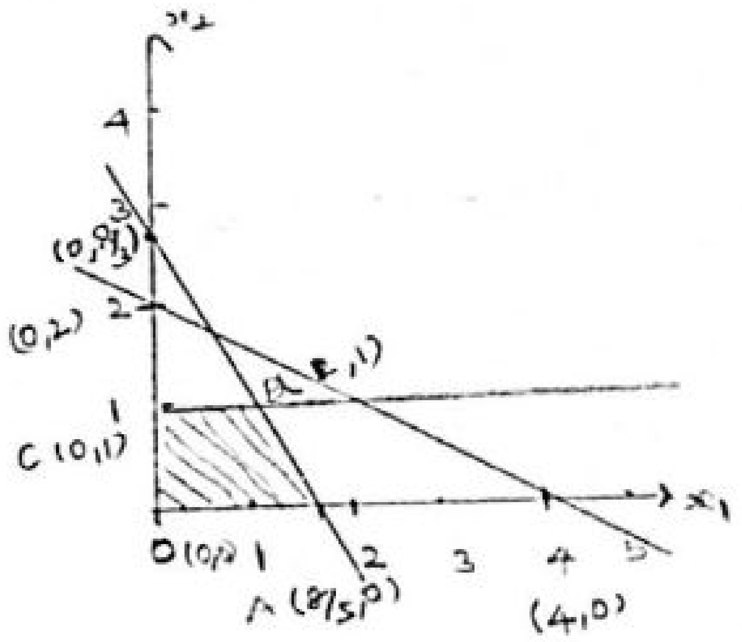
x_1	0	8/5
x_2	8/3	0

Consider $x_1 + 2x_2 = 4$ —— ②

Now

x_1	0	4
x_2	2	0

Consider $x_2 = 1$ —— ③



Sub ③ in ① we get

$\cancel{1 + 2x_2 = 4}$

$\cancel{\Rightarrow 2x_2 = 4 - 1}$

$\cancel{\Rightarrow 2x_2 = 3}$

$\cancel{\Rightarrow x_2 = 3/2}$

$\cancel{x_1 + 2(1) = 4}$

$\cancel{\Rightarrow x_1 = 4 - 2}$

$\cancel{\Rightarrow x_1 = 2}$

$5x_1 + 3(1) = 8$

$\Rightarrow 5x_1 = 8 - 3$

$\Rightarrow x_1 = \frac{5}{5}$

$\Rightarrow x_1 = 1$

At $O(0,0)$; $z = 2(0) + 2(0) = 0$

$A(8/3, 0)$; $z = 2(8/3) + 2(0) = 16/3$

$B(1,1)$; $z = 2(1) + 2(1) = 4$

$C(0,1)$; $z = 2(0) + 2(1) = 2$

∴ The optimum solutions are $x_1 = 1$, $x_2 = 1$, and max $z = 4$

This subproblem is fathomed

<u>Subproblem: 2</u>

$$\max z = 2x_1 + 2x_2$$

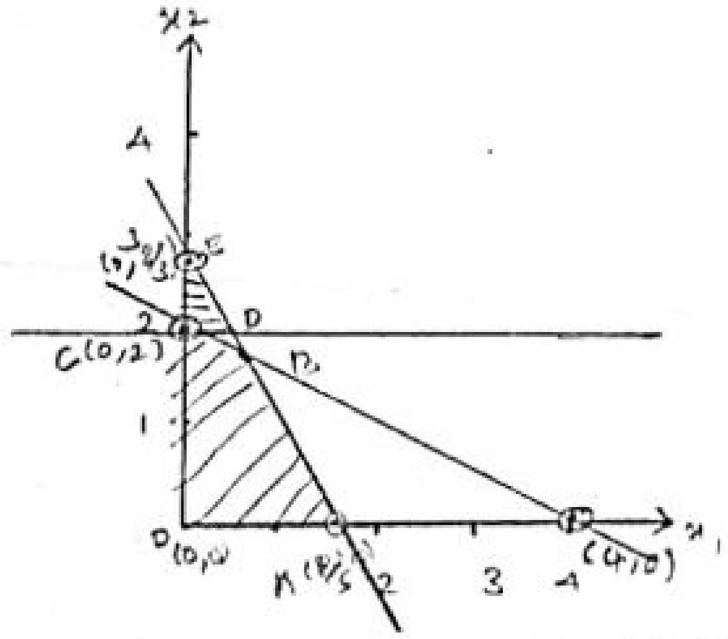
s.t

$$5x_1 + 3x_2 \le 8$$

$$x_1 + 2x_2 \le 4$$

$$x_2 \ge 2$$

$$x_1, x_2 \ge 0.$$



Here the problem meet one common point at $(0,2)$

∴ The optimum solutions are $x_1 = 0$, $x_2 = 2$ and max $z = 4$.

∴ The subproblem is also fathomed.

∴ The subproblem (1) and subproblem (2) has the same optimum value max $z = 4$.

∴ The optimum solution are $x_1^* = 1$, $x_2^* = 1$ or $x_1^* = 0$, $x_2^* = 2$ and the optimum value $z^* = 4$.

<u>problem:</u>

Use Branch and Bound method to solve the following IPP.

$$\text{minimize } z = 4x_1 + 3x_2$$

Subject to

$$5x_1 + 3x_2 \ge 30.$$

$$x_1 \le 4$$

$$x_2 \le 6$$

$x_1, x_2 \ge 0$ and are integers.

Soln:

G.T

Min $z = 4x_1 + 3x_2$

Subject to

$5x_1 + 3x_2 \geq 30$

$x_1 \leq 4$

$x_2 \leq 6$

$x_1, x_2 \geq 0$ and are integers

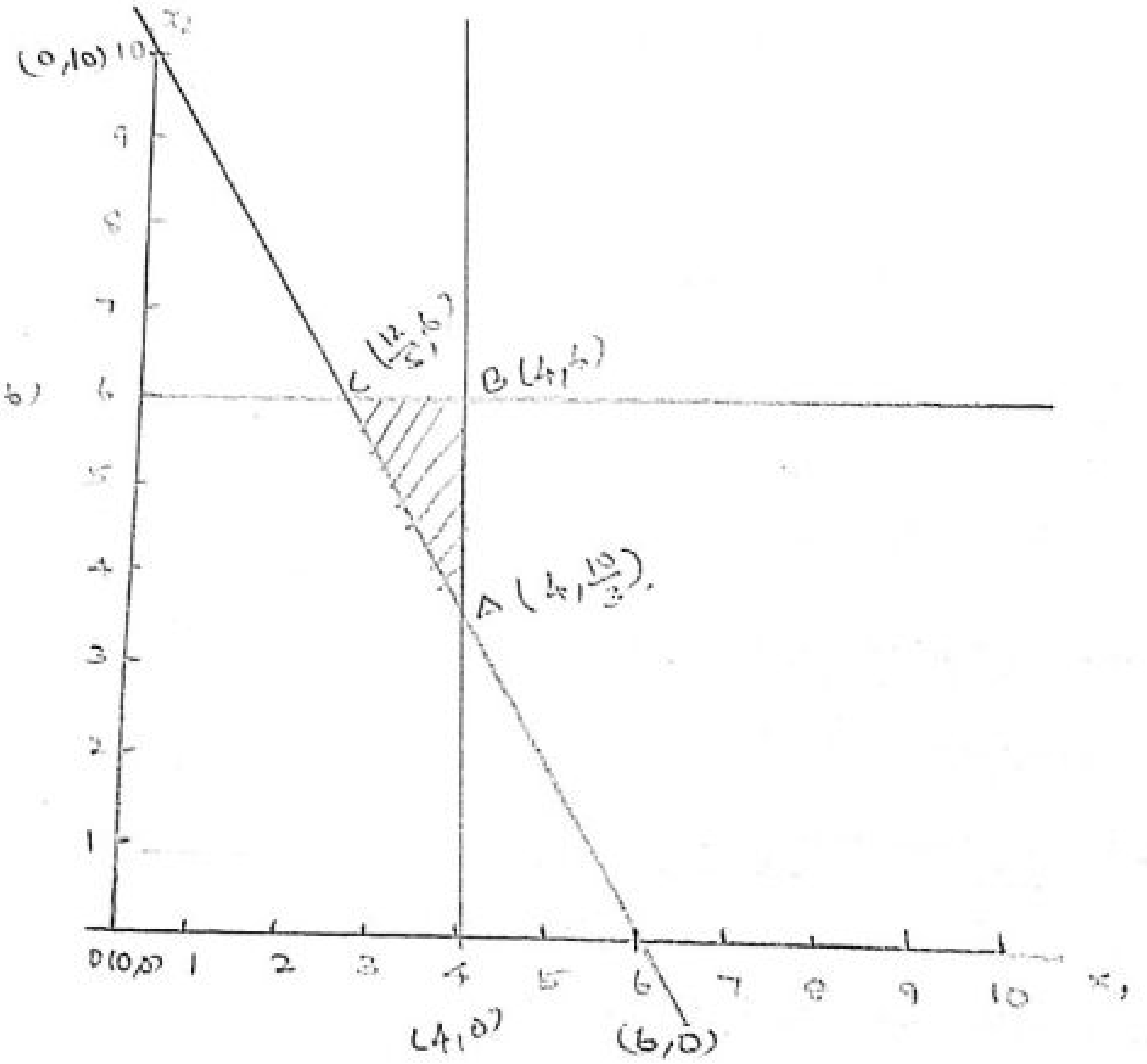
Consider $5x_1 + 3x_2 = 30$. —— (1)

Now

x_1	0	6
x_2	10	0

Consider $x_1 = 4$ —— (2)

Consider $x_2 = 6$ —— (3)



sub (2) in (1) we get

$5(4) + 3x_2 = 30$

$\Rightarrow 3x_2 = 30 - 20$

$x_2 = \frac{10}{3}$

sub (3) in (1) we get

$5x_1 + 3(6) = 30$

$\Rightarrow 5x_1 = 30 - 18$

$\Rightarrow x_1 = \frac{12}{5}$

At $A(4, \frac{10}{3})$, $z = 4(4) + 3(\frac{10}{3}) = 26.$

$B(4,6)$; $z = 4(4) + 3(6) = 34$

$C(\frac{12}{5}, 6)$; $z = 4(\frac{12}{5}) + 3(6) = \frac{48}{5} + 18 = 9.6 + 18 = 27.6$

The optimum solution is $x_1 = 4$, $x_2 = \frac{10}{3}$ and $\min z = 26$.

Here $x_2 = \frac{10}{3}$, the problem should be branched into two subproblem

Now $x_2 = \frac{10}{3} \Rightarrow 3 \le x_2 \le 4 \Rightarrow x_2 \le 3$ or $x_2 \ge 4$.

Sub problem : 1

$$\min Z = 4x_1 + 3x_2$$

Subject to

$$5x_1 + 3x_2 \ge 30$$
$$x_1 \le 4$$
$$x_2 \le 6$$
$$x_2 \le 3$$
$$x_1, x_2 \ge 0.$$

Here $x_2 \le 6$ and $x_2 \le 3$ is not possible., the solution is not feasible.

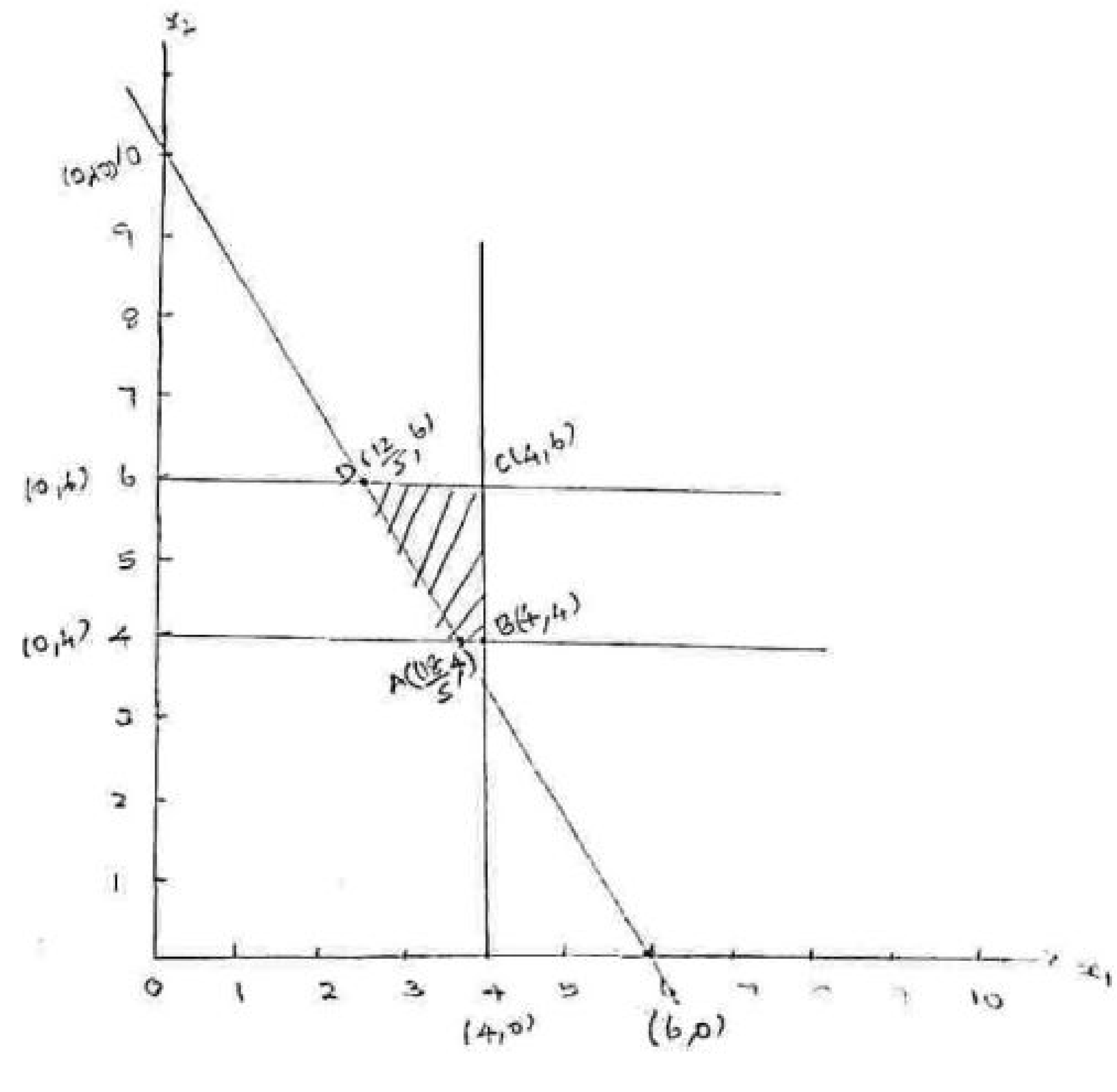
Hence this Subproblem is fathomed.

Subproblem : 2

$$\min z = 4x_1 + 3x_2$$

Subject to

$$5x_1 + 3x_2 \ge 30$$
$$x_1 \le 4$$
$$x_2 \le 6$$
$$x_2 \ge 4$$
$$x_1, x_2 \ge 0.$$



Solving $5x_1 + 3x_2 = 30$ --- (1)

$x_2 = 4$ --- (2)

Sub (2) in (1) we get

$5x_1 + 3(4) = 30$

$\Rightarrow 5x_1 = 30 - 12$

$\Rightarrow x_1 = \frac{18}{5}$

At $A(\frac{18}{5}, 4)$; $z = 4(\frac{18}{5}) + 3(4) = \frac{72}{5} + 12 = \frac{132}{5} = 26.4$

$B(4, 4)$; $z = 4(4) + 3(4) = 28$

$C(4, 6)$; $z = 4(4) + 3(6) = 34$

$D(\frac{12}{5}, 6)$; $z = 4(\frac{12}{5}) + 3(6) = \frac{48}{5} + 18 = 9.6 + 18 = 27.6$

$\therefore$ The optimum solution is $x_1 = \frac{18}{5}$, $x_2 = 4$ and $\min z = 26.4$

Here $x_1 = \frac{18}{5}$, the problem can be branched into two sub problems.

Now $x_1 = \frac{18}{5} \Rightarrow 3 \le x_1 \le 4 \Rightarrow x_1 \le 3$ or $x_1 \ge 4$.

Subproblem : 3

$\min z = 4x_1 + 3x_2$

subject to

$5x_1 + 3x_2 \ge 30$

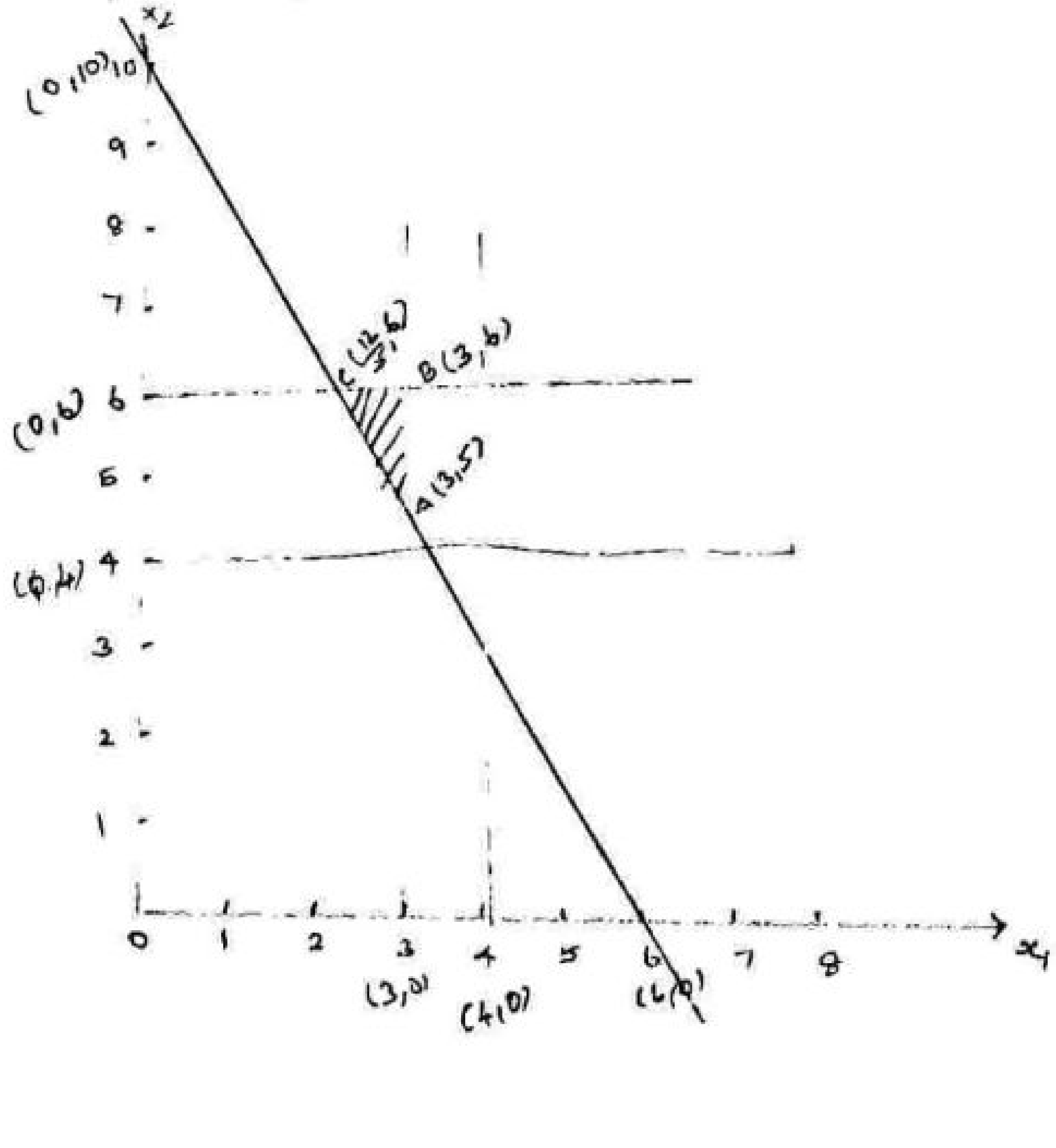
$x_1 \le 4$

$x_2 \le 6$

$x_2 \ge 4$

$x_1 \le 3$

$x_1, x_2 \ge 0$.



consider $5x_1 + 3x_2 = 30$ —①

$x_1 = 3$ —②

sub ② in ① $5(3) + 3x_2 = 30$

$\Rightarrow 3x_2 = 30 - 15$

$x_2 = \frac{15}{3} = 5.$

At $A(3,5)$; $z = 4(3) + 3(5) = 27$

$B(3,6)$; $z = 4(3) + 3(6) = 30$

$C(\frac{12}{5}, 6)$; $z = 4(\frac{12}{5}) + 3(6) = \frac{48}{5} + 18 = 9.6 + 18 = 27.6.$

∴ optimum solutions are $x_1 = 3$, $x_2 = 5$ and $\min z = 27.$

Subproblem: 4

$\min z = 4x_1 + 3x_2$

subject to

$5x_1 + 3x_2 \geq 30$

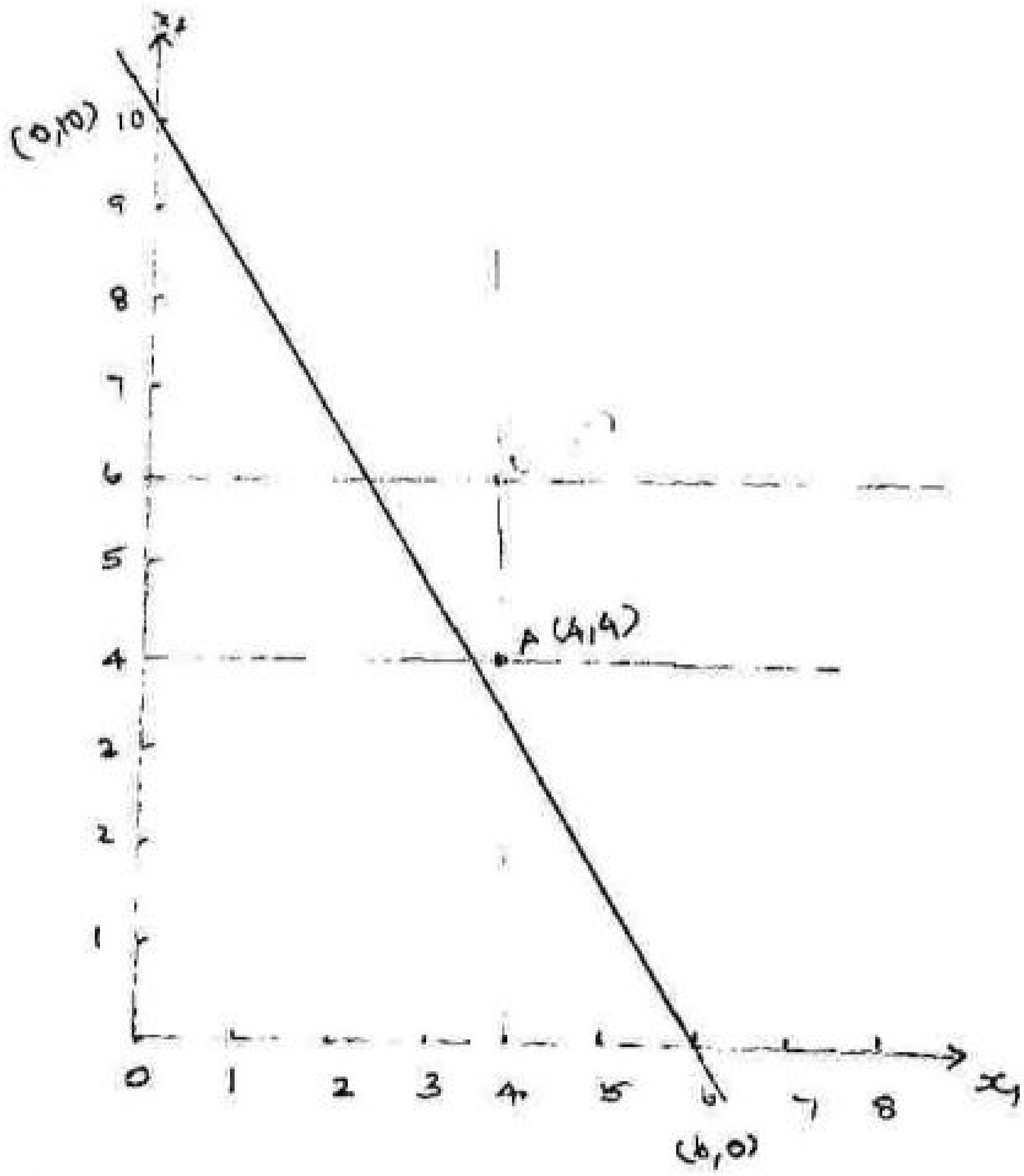
$x_1 \leq 4$

$x_2 \leq 6$

$x_2 \geq 4$

$x_1 \geq 4$

$x_1, x_2 \geq 0.$



At $A(4,4)$; $z = 4(4) + 3(4) = 28$

∴ The optimum solutions are $x_1 = 4$, $x_2 = 4$ and $\min z = 28$

Here both subproblem: 3 and subproblem: 4, the minimum value of z is 27.

∴ The optimum solutions are $x_1 = 3$, $x_2 = 5$ and the optimum value $z^* = 27.$

consider $5x_1 + 3x_2 \geq 30$ —①

$x_1 = 3$ —②

sub ② in ① $5(3) + 3x_2 = 30$

$\Rightarrow 3x_2 = 30 - 15$

$x_2 = \frac{15}{3} = 5.$

At $A(3,5)$; $z = 4(3) + 3(5) = 27$

$B(3,6)$; $z = 4(3) + 3(6) = 30$

$C(\frac{12}{5}, 6)$; $z = 4(\frac{12}{5}) + 3(6) = \frac{48}{5} + 18 = 9.6 + 18 = 27.6.$

∴ optimum solutions are $x_1 = 3$, $x_2 = 5$ and min $z = 27$.

<u>subproblem: 4</u>

min $z = 4x_1 + 3x_2$

subject to

$5x_1 + 3x_2 \geq 30$

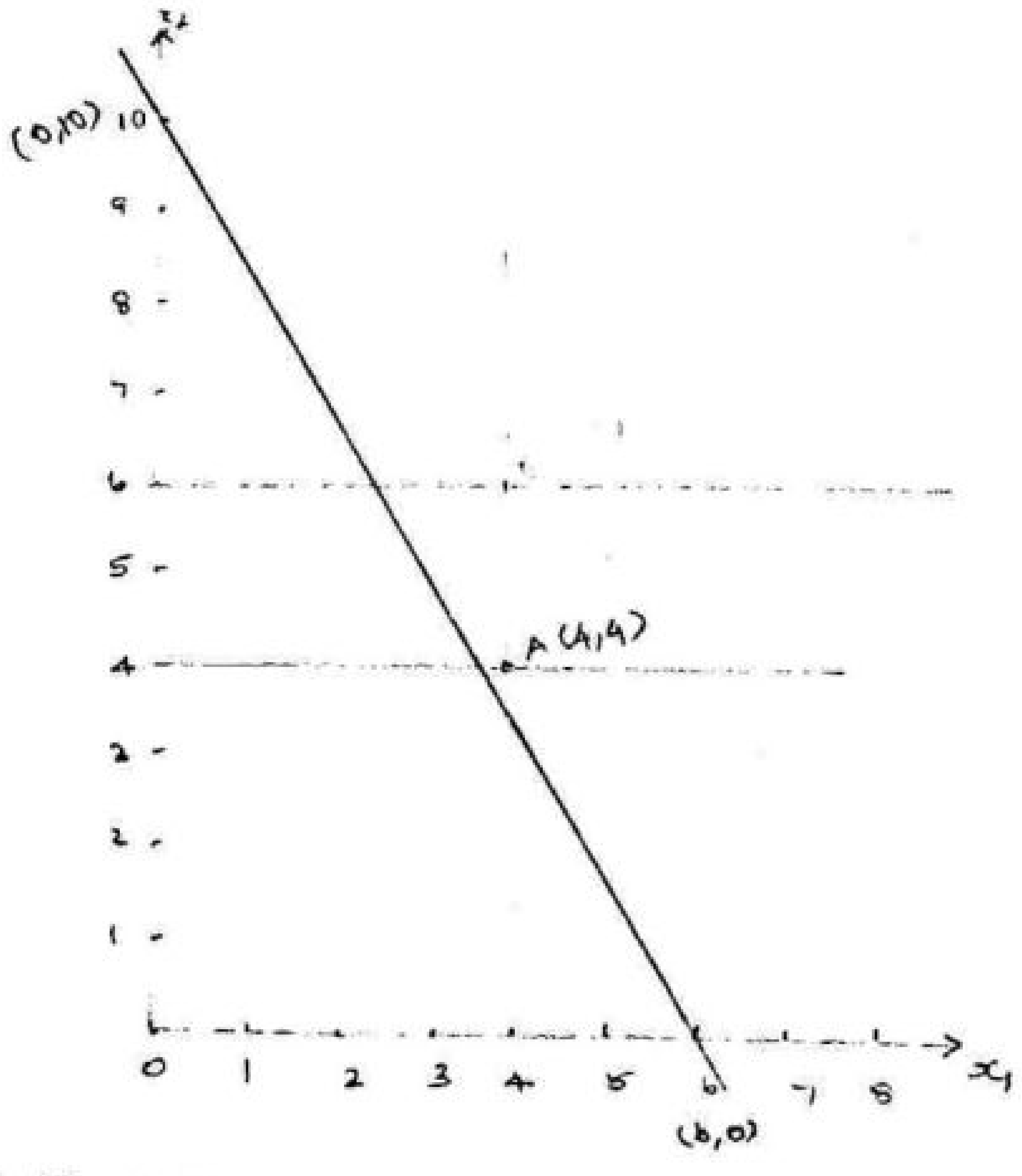
$x_1 \leq 4$

$x_2 \leq 6$

$x_2 \geq 4$

$x_1 \geq 4$

$x_1, x_2 \geq 0.$



At $A(4,4)$; $z = 4(4) + 3(4) = 28$

∴ The optimum solutions are $x_1 = 4$, $x_2 = 4$ and min $z = 28$

Here both subproblem: 3 and subproblem: 4, the minimum value of z is 27.

∴ The optimum solutions are $x_1 = 3$ $x_2 = 5$ and the optimum value $z^* = 27$.

Home work:

Use Branch and Bound method to solve the following

maximize $z = 5x_1 + 4x_2$

Subject to

$x_1 + x_2 \leq 5$

$10x_1 + 6x_2 \leq 45$

$x_1, x_2 \geq 0$ and are integers.

Zero - one linear programming

A LPP with decision variable restricted to values of zero or one are called 0-1 linear programming or Binary linear programming.

Additive algorithm to 0-1 LPP

The 0-1 LPP uses only two operations both additive and subtraction. then it is called Additive algorithm

problem:

Solve the following LPP.

maximize $f = 3x_1 + 2x_2 - 5x_3$

Subject to

$x_1 + x_2 + x_3 \leq 4$

$7x_1 + 3x_3 \leq 8$

$11x_1 - 6x_2 \geq 3$

$x_1, x_2, x_3 = 0$ or 1

Soln:

G.T max $f = 3x_1 + 2x_2 - 5x_3$

Subject to

$x_1 + x_2 + x_3 \leq 4$

$7x_1 + 3x_3 \leq 8$

$11x_1 - 6x_2 \geq 3$

$x_1, x_2, x_3 = 0$ or 1.

Step:1

Convert the maximum objective function into a minimum objective function.

$\min z = -3x_1 - 2x_2 + 5x_3$ — ①

Here $c_1 = -3 < 0$, $c_2 = -2 < 0$.

replace $x_1 = 1 - x_1'$, $x_2 = 1 - x_2'$ in ① we get

$\min z = -3(1 - x_1') - 2(1 - x_2') + 5x_3$

$= -3 + 3x_1' - 2 + 2x_2' + 5x_3$

$\Rightarrow \min z = 3x_1' + 2x_2' + 5x_3 + 5$

Eliminating the constant in the objective function

$\min z' = 3x_1' + 2x_2' + 5x_3$, where $z' = z + 5$

The constraints becomes

$$x_1+x_2+x_3\le 4 \Rightarrow (1-x_1')+(1-x_2')+x_3\le 4$$
$$\Rightarrow -x_1'-x_2'+x_3+2\le 4$$
$$\Rightarrow -x_1'-x_2'+x_3\le 4-2$$
$$\Rightarrow -x_1'-x_2'+x_3\le 2$$
$$\Rightarrow -x_1'-x_2'+x_3+s_1=2$$

$$7x_1+3x_3\le 8 \Rightarrow 7(1-x_1')+3x_3\le 8$$
$$\Rightarrow 7-7x_1'+3x_3\le 8$$
$$\Rightarrow -7x_1'+3x_3\le 8-7$$
$$\Rightarrow -7x_1'+3x_3\le 1$$
$$\Rightarrow -7x_1'+3x_3+s_2=1$$

$$11x_1-6x_2\ge 3 \Rightarrow -11x_1+6x_2\le -3$$
$$\Rightarrow -11(1-x_1')+6(1-x_2')\le -3$$
$$\Rightarrow \cancel{-11+11x_1'+6x_2\le -3}$$
$$\Rightarrow \cancel{11x_1'+6x_2\le -3+11}$$
$$\Rightarrow \cancel{11x_1'}-11+11x_1'+6-6x_2'\le -3$$
$$\Rightarrow 11x_1'-6x_2'-5\le -3$$
$$\Rightarrow 11x_1'-6x_2'\le -3+5$$
$$\Rightarrow 11x_1'-6x_2'+s_3=2$$

$\therefore x_1', x_2', x_3 \ge 0$ or 1

Let $x_1'=x_2'=x_3=0$; $Z'=0$.

$\therefore (s_1^0, s_2^0, s_3^0)=(2,1,2)=(b_1,b_2,b_2)$ (say)

$$\therefore A=\begin{pmatrix}-1 & -1 & 1\\ -7 & 0 & 3\\ 11 & -6 & 0\end{pmatrix}$$

Here all b_i's ≥ 0, the current solution is feasible.

i.e. $x_1'=x_2'=x_3=0$.

Here $x_1^*=1-x_1'=1-0=1$

$x_2=1-x_2'=1-0=1$

$x_3=0$.

$\therefore Z=3(1)+2(1)-5(0)=5$.

$\therefore$ The optimum solutions are $x_1^*=1$, $x_2^*=1$, $x_3^*=0$

and the optimum value $Z^*=5$

Unit - II

Dynamic programming

Operations Research:-

Operations Research is a technique for devoting to observing, understanding and Predicting the behaviour of a purposeful manmechine system.

Deterministic Dynamic programming:-

A Dynamic programming model is a recursive equation linking the different stages of the problem in a manner that each stage is an optimum feasible solution is also optimum and feasible for the entire problem.

Recursive nature of computations in dynamic Programming:-

An optimum solution of one subproblem is used as an input to the next subproblem, these subproblem are normally linked by a common constraints, the feasibility of these constraints maintained is called Recursive.

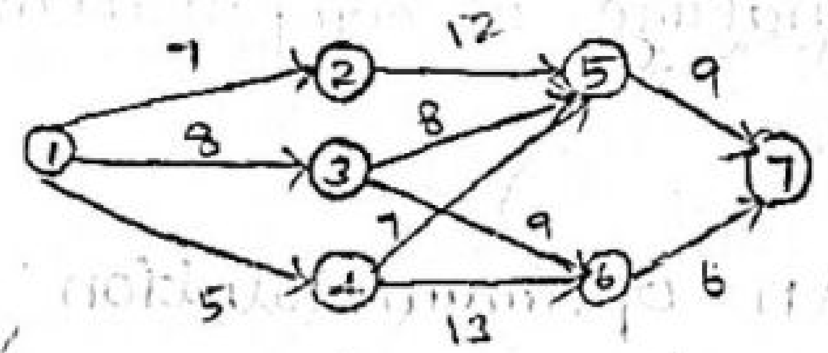
Forward Recursion:-

Forward Recursion is a computation from the initial stage to final stage. Let $f_i(x_i)$ be the shortest distance to the node x_i at stage i and $d(x_{i-1}, x_i)$ be the distance from node x_{i-1} to node x_i then the forward recursion equation is $f_i(x_i) = \min\limits_{\substack{\text{all routs} \\ \text{of } (x_{i-1}, x_i)}} \left[f_{i-1}(x_{i-1}) + d(x_{i-1}, x_i)\right]$

and $f_0(x_0) = 0$.

Problem:- [Shortest routes problem]

Suppose that the shortest highway between two cities, the network provides a possible routes between the starting city at node 1 and the destination city at node 7.

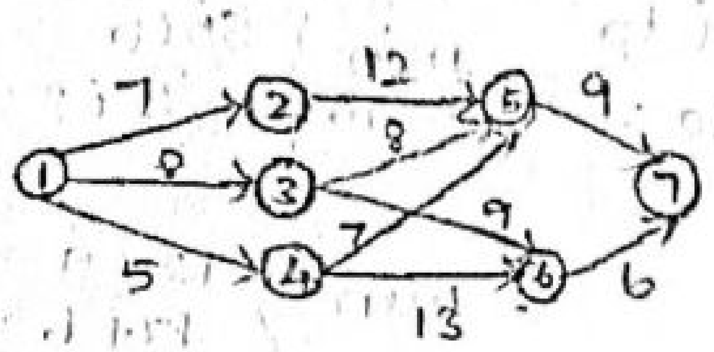


Find the shortest distance and the shortest routes.

Soln

3

Given that



Given that 1 is the initial node and 7 is the final node.

Stage 1 :-

Shortest distance from node 1 to node 2 = 7

Shortest distance from node 1 to node 3 = 8

Shortest distance from node 1 to node 4 = 5

Stage 2 :-

$$\left.\begin{array}{c}\text{shortest distance from node } i \\ \text{to node } 5\end{array}\right\} = \min_{i=2,3,4} \left\{ \begin{array}{c}\text{SD to} \\ \text{node } i\end{array} + \begin{array}{c}\text{distance} \\ \text{travelled} \\ \text{from node } i \\ \text{to node } 5\end{array} \right\}$$

$$= \min_{i=2,3,4} \left\{ \begin{array}{l} 7 + 12 = 19 \\ 8 + 8 = 16 \\ 5 + 7 = 12 \end{array} \right\}$$

$$= 12$$

shortest distance from node 1 to node 5 = 12

$$\text{SD from node } i \text{ to node } 6 = \min_{i=3,4} \left\{ \begin{array}{c}\text{SD to} \\ \text{node } i\end{array} + \begin{array}{c}\text{distance} \\ \text{travelled} \\ \text{from node } i \\ \text{to node } 6\end{array} \right\}$$

$$= \min \left\{ \begin{array}{l} 8 + 9 = 17 \\ 5 + 13 = 18 \end{array} \right\}$$

$$= 17$$

$\therefore$ SD from node 1 to node 6 = 17.

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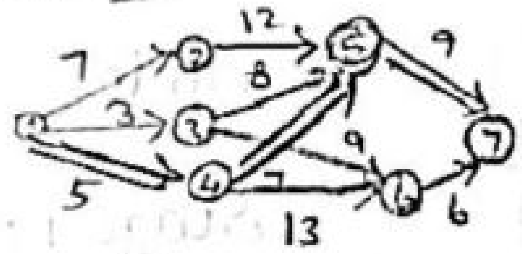
Stage : 3

$$\text{SD from node } i \text{ to node } 7 = \min_{i=5,6} \left\{ \begin{array}{l} \text{SD to node } i + \text{distance travelled from node } i \text{ to node } 7 \end{array} \right\}$$

$$= \min \left\{ \begin{array}{l} 12 + 9 = 21 \\ 17 + 6 = 23 \end{array} \right\}$$

$$= 21. \quad \underline{\underline{\text{Ans}}}$$



SD from node 1 to node 7 = 21

Shortest distance = 21 KM

Shortest routes is $1 \rightarrow 4 \rightarrow 5 \rightarrow 7$.

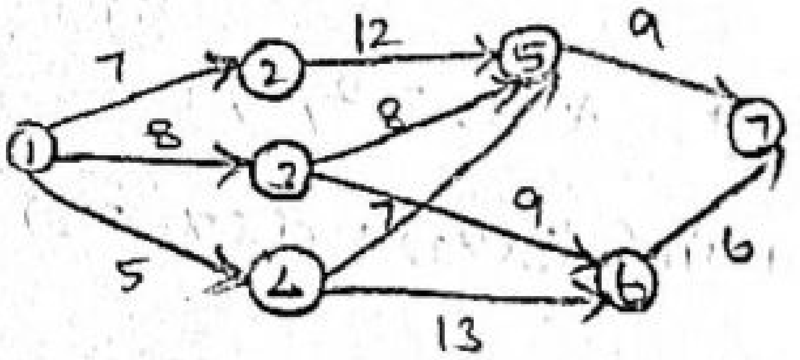
Backward recursion:-

Backward recursion is a computation from the final stage to initial stage. Let $f_i(x_i)$ be the shortest distance to node x_i at stage i, $d(x_i, x_{i+1})$ be the distance from node x_i and x_{i+1}, then the backward recursive equation is

$$f_i(x_i) = \min_{\substack{\text{all routes} \\ \text{of } (x_i, x_{i+1})}} \left[f_{i+1}(x_{i+1}) + d(x_i, x_{i+1}) \right]$$

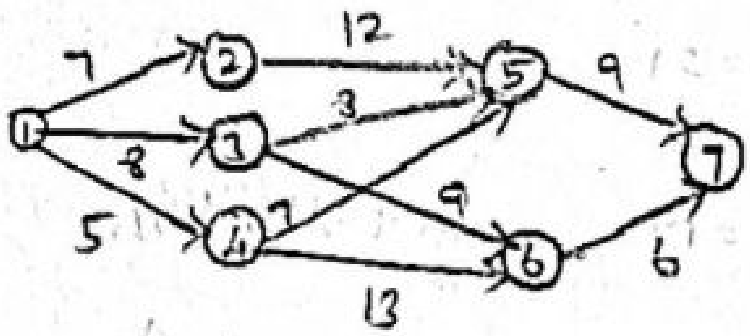
and $f_{i+1}(x_{i+1}) = 0$.

Problem:-
Using Backward recursion method to find the shortest distance and the shortest routes for the following.



Soln

Given that



Here 1 is the initial node and 7 is the final node

Wkt $f_i(x_i) = \min\limits_{\substack{\text{all routes} \\ \text{of } (x_i, x_{i+1})}} \left\{ f_{i+1}(x_{i+1}) + d(x_i, x_{i+1}) \right\}$

and $f_{i+1}(x_{i+1}) = 0.$

Here $f_4(x_4) = 0$

Stage:- 3

Here $x_4 = 7$ & $x_3 = 5$ & 6

$f_3(x_3) = \min\left(f_4(x_4) + d(x_3, x_4)\right)$

x_3	$f_4(x_4) + d(x_3, x_4)$ $x_4 = 7$	optimum solution $f_3(x_3)$	x_4^*
5	$0+9=9$	9	7
6	$0+6=6$	6	7

Stage: 2

Here $x_3 = 5, 6$. & $x_2 = 2, 3, 4$

$$f_2(x_2) = \min(f_3(x_3) + d(x_2, x_3))$$

x_3 / x_2	$f_3(x_3)+d(x_2,x_3)$: $x_3 = 5$	$x_3 = 6$	optimum solution $f_2(x_2)$	x_3^*
2	$9+12=21$	-	21	5
3	$9+8=17$	$6+9=15$	15	6
4	$9+7=16$	$6+13=19$	16	5

Stage: 1

Here $x_2 = 2, 3, 4$ and $x_1 = 1$

Here $f_1(x_1) = \min(f_2(x_2) + d(x_1, x_2))$.

x_1	x_2: 2	3	4	optimum soln. $f_1(x_1)$	x_2^*
1	$21+7=28$	$15+8=23$	$16+5=21$	21	4

From stage 1, node 1 linked to node 4, stage 2, node 4 linked to node 5 and stage 3, node 5 linked to node 7.

$\therefore$ The shortest router is $1 \to 4 \to 5 \to 7$ and the SD is 21 KM.

SD - Shortest distance

Selected dynamic programming application:- (7)

knapsack (or) Flyaway (or) Cargo loading model.

The knapsack model deals, with the situation of the soldier a general resource allowcation model in which the single limited resource is assigned, to a number of alternatives with the objective of maximum of total returns

The jet pilot must determined the most valuable items to take abroad, a jet is called flyaway problems.

Cargoloading problem in which a vessel with limited volume or weighted capacity is loaded with the most valuable cargo items.

Note:-

The general problem is of the form

$max\ z = r_1 m_1 + r_2 m_2 + \dots + r_n m_n$.

Subject to the constraints:-

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$w_1 m_1 + w_2 m_2 + \dots + w_n m_n \le W.$

$m_1, m_2, \dots m_n \ge 0$ and are integers

The simplest way to determine a recussive equation in two steps.

Step: 1

$$f_i(x_i) = \max_{\substack{m_i = 0,1,2 \dots [\frac{W}{w_i}] \\ i = 1 \text{ to } n}} (r_i m_i + f_i(x_{i+1}))$$

and $f_{n+1}(x_{n+1}) = 0$

Step: 2

Let $x_i - x_{i+1} = w_i m_i$

$\Rightarrow x_{i+1} = x_i - w_i m_i$

$$\therefore f_i(x_i) = \max_{\substack{m_i = 0,1,2,\dots [\frac{W}{w_i}] \\ i = 1 \text{ to } n.}} [r_i m_i + f_{i+1}(x_i - w_i m_i)]$$

Here $[\frac{W}{w_i}]$ is the largest integer $\le \frac{W}{w_i}$

less then are equal to

Problem:- (9)

The four tons vessel can be loaded with 1 (or) more of 3 items ~~3 items~~

The following table gives the unit weight ~~unit~~ w_i in tons and the unit revenue in thousands of dollars r_i for item i

item i	w_i (tons)	r_i (thousands of dollars)
1	2	31
2	3	47
3	1	14

How should the vessal be loaded to maximize the total return?

Soln

GIT

item i	w_i (tons)	r_i (thousands of dollars)
1	2	31
2	3	47
3	1	14

GIT $W = 4$ tons.

Stage : 3

Here $m_3 = \left[\frac{W}{w_3}\right] = \left[\frac{4}{1}\right] = 4$

$m_3 = 0, 1, 2, 3, 4$

Here $x_3 = 0, 1, 2, 3, 4$.

wkt $f_3(x_3) = \max_{m_3 = 0, 1, 2, 3, 4} (r_3 m_3 + f_4(x_4))$

$m_3 = 0$

Here $f_4(x_4) = 0$

Here $r_3 = 14$) $f_4(x_4) + 14 m_3$ optimum soln

x_3 \ m_3	0	1	2	3	4	$f_3(x_3)$	m_3^*
0	0+0+0=0					0	0
1		0+14=14				14	1
2			0+28=28			28	2
3				0+42=42		42	3
4					0+56=56	56	4

Stage 2

wkt $m_2 = \left[\frac{W}{w_2}\right] = \left[\frac{4}{3}\right] = [1.333] \Rightarrow$

$\therefore m_2 = 0, 1$.

Here $x_2 = 0, 1, 2, 3, 4$

Here $f_2(x_2) = \max(r_2 m_2 + f_3(x_2 - w_2 m_2))$

$47 m_2 + f_3(x_2 - 3_2 m_2)$.

x_2	m_2 0	1	optimum soln $f_2(x_2)$	m_2^*
0	0+0=0	–	0	0
1	0+14=14.	–	14	0
2	0+28=28	–	28	0
3	0+42=42	47+0=47.	47	1
4	0+56=56	47+14=61	61	1

Stage: 1

Here $m_1 = \left[\frac{W}{w_1}\right] = \left[\frac{4}{2}\right] = [2] = 2$

$\therefore m_1 = 0, 1, 2$

Here $x_1 = 0, 1, 2, 3, 4$

Here $f_1(x_1) = \max(r_1 m_1 + f_2(x_1 - w_1 m_1))$

$31 m_1 + f_2(x_1 - 2m_1)$

x_1	m_1 0	1	2	optimum soln $f_1(x_1)$	m_1^*
0	0+0=0	–	–	0	0
1	0+14=14	–	–	14	0
2	0+28=28	31+0=31	–	31	1
3	0+47=47	31+14=45	–	47	0
4	0+61=61	31+28=59	62+0=62	62	2

Here $x_{i+1} = x_i - w_i m_i$

Here. $x_1 = 4, m_1 = 2$

Here $x_2 = x_1 - w_1 m_1 = 4 - 2(2) = 0.$

$x_2 = 0 \Rightarrow m_2 = 0$

Here $x_3 = x_2 - w_2 m_2 = 0 - 3(0) = 0.$

$x_3 = 0 \Rightarrow m_3 = 0$

∴ the optimum solution are.

$m_1^* = 2, m_2^* = 0, m_3^* = 0$

and the optimum value $ 62000

Problem:-

determine the value of u_1, u_2 and u_3 to the following maximize $u_1 u_2 u_3$

sub to the constraint

$u_1 + u_2 + u_3 = 10.$

$u_1, u_2, u_3 \geq 0$

soln

GT max $u_1 u_2 u_3$

Subject to the constraint

$u_1 + u_2 + u_3 = 10$

$u_1, u_2, u_3 \geq 0$

Here the dynamic programming problem can be treated as a three stage problem. with state variable x_i and returns $f_i(x_i)$ we get,

let $x_3 = u_1 + u_2 + u_3 = 10$

$x_2 = u_1 + u_2 = x_3 - u_3$

$x_1 = u_1 = x_2 - u_2$.

Let $f_1(x_1) = \max u_1 = \max(x_2 - u_2)$.

$f_2(x_2) = \max (u_1 u_2)$

$= \max (u_2 (x_2 - u_2))$.

$= \max (u_2 f_1(x_1))$

$f_3(x_3) = \max (u_1 u_2 u_3)$.

$\Rightarrow f_3(x_3) = \max (u_3 f_2(x_2))$.

Consider $u_2(x_2 - u_2) = u_2 x_2 - u_2^2$

Diff the above w.r.t u_2 and equating to 0.

we get,

$x_2(1) - 2u_2 = 0$.

$\Rightarrow 2u_2 = x_2$

$\Rightarrow u_2 = x_2/2$

maxima: $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} < 0$ = -ve

minima: $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} > 0$ = +ve

$f_2(x_2) = \max \left(\frac{x_2}{2} (x_2 - x_2/2) \right)$.

$= \max \left(\frac{x_2}{2} \cdot \frac{x_2}{2} \right)$

$f_2(x_2) = \frac{x_2^2}{4}$

Now, $f_3(x_3) = \max \left(u_3 \frac{x_2^2}{4} \right)$.

$f_3(x_3) = \max \left(\frac{u_3}{4} (x_3 - u_3)^2 \right)$

diff $\frac{u_2}{x_1}(x_3-u_3)^2$ w.r.t u_3 and equating to 0

we get,

$\frac{1}{x_1}\{(x_3-u_3)^2(1)+u_3\,2(x_3-u_3)(0-1)\}=0$

$\Rightarrow (x_3-u_3)^2-2u_3(x_3-u_3)=0\ (x_1)$

$\Rightarrow (x_3-u_3)[x_3-u_3-2u_3]=0$

$\Rightarrow ((x_3-u_3)(x_3-3u_3))=0$

$\Rightarrow x_3-u_3=0$ (or) $x_3-3u_3=0$

$\Rightarrow x_3=u_3$ (or) $u_3=\frac{x_3}{3}$

Here $u_3\neq x_3$, $\therefore u_3=\frac{x_3}{3}$.

$\Rightarrow u_3=10/3$

Here $u_2=\frac{x_2}{2} \Rightarrow x_2=2u_2$

$\Rightarrow x_3-u_3=2u_2$

$\Rightarrow 10-\frac{10}{3}=2u_2$

$\Rightarrow \frac{20}{3}=2u_2$

$\Rightarrow u_2=\frac{10}{3}$

And $u_1=x_2-u_2$

$=2u_2-u_2$

$=u_2$

$u_1 = 10/3$.

$\therefore$ The optimum solutions are

$$u_1^* = 10/3, \quad u_2^* = 10/3, \quad u_3^* = 10/3$$

and the optimum value

$$\max\left(\frac{10}{3} \cdot \frac{10}{3} \cdot \frac{10}{3}\right) = \frac{1000}{27}$$

Problem:-

Solve the following dynamic programming Problem. minimize (or) min $z = y_1^2 + y_2^2 + y_3^2$

subject to the constraint.

$$y_1 + y_2 + y_3 \geq 15$$

$$y_1, y_2, y_3 \geq 0$$

Soln

GT $\min z = y_1^2 + y_2^2 + y_3^2$

Subject to the constraint

$$y_1 + y_2 + y_3 \geq 15$$

$$y_1, y_2, y_3 \geq 0$$

Here the dynamic programming problem can be treated as a three stage problem with state variable x_i and returns $f_i(x_i)$ we get.

Let $x_3 = y_1 + y_2 + y_3 \geq 15$

$$x_2 = y_1 + y_2 = x_3 - y_3$$

$$x_1 = y_1 = x_2 - y_2$$

Let $f_1(x_1) = \min y_1^2 = \min ((x_2 - y_2)^2)$ (11)

$f_2(x_2) = \min(y_1^2 + y_2^2)$.

$= \min(y_2^2 + (x_2 - y_2)^2)$

$= \min(y_2^2 + f_1(x_1))$

$f_3(x_3) = \min(y_1^2 + y_2^2 + y_3^2)$.

$f_3(x_3) = \min(y_3^2 + f_2(x_2))$

Since, the given function is minimum

consider $f_2(x_2) = \min(y_2^2 + (x_2 - y_2)^2)$.

diff $y_2^2 + (x_2 - y_2)^2$ w.r.t "y_2" and equating to 0 we get.

$2y_2 + 2(x_2 - y_2)(0-1) = 0$

$\Rightarrow 2y_2 - 2x_2 + 2y_2 = 0.$

$\Rightarrow 4y_2 - 2x_2 = 0.$

$\Rightarrow y_2 = \frac{2x_2}{4}$

$\Rightarrow y_2 = x_2/2$

consider $f_3(x_3) = \min(y_3^2 + f_2(x_2))$.

$= \min(y_3^2 + (\frac{x_2}{2})^2 + (x_2 - \frac{x_2}{2})^2)$

$= \min(y_3^2 + \frac{x_2^2}{4} + \frac{x_2^2}{4})$

$= \min(y_3^2 + \frac{x_2^2}{2})$.

$$f_3(x_3) = \min\left(y_3^2 + \frac{(x_3-y_3)^2}{2}\right)$$

Consider $y_3^2 + \frac{(x_3-y_3)^2}{2}$, diff w.r.t "$y_3$" and equating to 0 we get,

$$2y_3 + \frac{1}{2}\,2(x_3-y_3)(0-1)=0$$

$\Rightarrow 2y_3 - x_3 + y_3 = 0.$

$\Rightarrow 3y_3 - x_3 = 0$

$\Rightarrow y_3 = x_3/3$

(minimum $\frac{dy}{dx}=0, \frac{d^2y}{dx^2}>0$ +ve

Since the objective function is minimum.

$\therefore x_3 = 15.$

$\therefore y_3 = \frac{x_3}{3} = \frac{15}{3} = 5$

Here $y_2 = \frac{x_2}{2} = \frac{1}{2}(x_3-y_3) = \frac{1}{2}(15-5) = \frac{10}{2} = 5$

$\therefore y_2 = 5$

Here $y_1 = x_2 - y_2 = 2y_2 - y_2$

$= y_2 = 5$

$y_1 = 5$

$\therefore \min z = 5^2 + 5^2 + 5^2 = 75$

The optimum solutions are

$y_1^* = 5,\ y_2^* = 5,\ y_3^* = 5.$

and the optimum value $z^* = 75$.

use dynamic programming problem to show that z

$z = p_1 \log p_1 + p_2 \log p_2 + \dots + p_n \log p_n$

Subject to the constraint.

$p_1 + p_2 + \dots + p_n = 1$

$p_i \geq 0, \; i = 1, 2, \dots, n.$

is minimum when $p_1 = p_2 = p_3 = \dots = p_n = 1/n$

Soln (or) divide unity into n parts so as to maximize the quantity $\sum p_i \log p_i$.

Let $\min z = p_1 \log p_1 + p_2 \log p_2 + \dots + p_n \log p_n$

Subject to.

$p_1 + p_2 + p_3 + \dots + p_n = 1$

$p_1, p_2, \dots p_n \geq 0$

Let $f_n(1) = \min(p_1 \log p_1 + p_2 \log p_2 + \dots + p_n \log p_n)$

$= \min \sum_{i=1}^{n} p_i \log p_i$

I-stage:-

Put $n=1$ & $p_1 = 1$

$f_1(1) = \min p_1 \log p_1$

$= 1 \log 1$

$f_1(1) = 0$

II - stage:

Put $n=2$ & put $p_1 = x$, $p_2 = 1-x$,

$p_1 + p_2 = 1, \quad 0 \leq x \leq 1.$

$f_2(1) = \min(p_1 \log p_1 + p_2 \log p_2)$

$$= \min_{0 \le x \le 1} (x \log x + (1-x)\log(1-x))$$

$$= \min_{0 \le x \le 1} (x \log x + f_1(1-x))$$

Illy.

$$f_3(1) = \min_{0 \le x \le 1} (x \log x + f_2(1-x))$$

In general

$$f_n(1) = \min_{0 \le x \le 1} (x \log x + f_{n-1}(1-x))$$

Consider $f_2(1) = \min(x \log x + (1-x)\log(1-x))$.

Since, it is minimum, diff $x \log x + (1-x)\log(1-x)$

w.r.t "x" and equating to 0 we get,

$$\cancel{x} \cdot \frac{1}{\cancel{x}} + \log x(1) + \cancel{(1-x)}\frac{1}{\cancel{1-x}}(0-1)$$

$$+ \log(1-x)(0-1) = 0.$$

$$\Rightarrow \cancel{1} + \log x - \cancel{1} - \log(1-x) = 0$$

$$\Rightarrow \log_e \frac{x}{1-x} = 0 \qquad \Big|\ \because \log M - \log N = \log \frac{M}{N}$$

$$\Rightarrow \frac{x}{1-x} = e^0$$

$$\Rightarrow x = 1(1-x)$$

$$\Rightarrow x + x = 1$$

$$\Rightarrow \boxed{x = 1/2}$$

Scanned with CamScanner

$f_2(1) = \min\left(\frac{1}{2}\log\frac{1}{2} + (1-\tfrac{1}{2})\log(1-\tfrac{1}{2})\right)$

$= \min\left(\frac{1}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\log\tfrac{1}{2}\right)$

$f_2(1) = 2\left(\frac{1}{2}\log\tfrac{1}{2}\right)$.

$\therefore$ the optimal policy is $(\tfrac{1}{2}, \tfrac{1}{2})$ and ~~$f_2(1)=$~~

$f_2(1) = 2(\tfrac{1}{2}\log\tfrac{1}{2})$.

|||ly for $n=3$ the optimal Policy is $(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})$ and.
$f_3(1) = 3(\tfrac{1}{3}\log\tfrac{1}{3})$.

|||ly for $n=m$ the optimal policy is $(\frac{1}{m}, \frac{1}{m}, \ldots \frac{1}{m})$

and
$f_m(1) = m\left(\frac{1}{m}\log\tfrac{1}{m}\right)$

|||ly for $n=m+1$, ~~the~~

$$f_{m+1}(1) = \min_{0\le x\le 1}\left(x\log x + f_m(1-x)\right)$$

$$f_{m+1}(1) = \min_{0\le x\le 1}\left(x\log x + m\left(\frac{1-x}{m}\log\frac{(1-x)}{m}\right)\right)$$

Consider $f_{m+1}(1)$, Diff $x\log x + (1-x)\log\frac{(1-x)}{m}$

w.r.t "x" and equating to 0 we get,

$$1.\log x + x.\frac{1}{x} + (1-x)\frac{1}{\frac{1-x}{m}}\cdot\frac{1}{m}(0-1) + \log\frac{(1-x)}{m}(0-1) = 0$$

$$\Rightarrow \boxed{\log x + \cancel{1}} - (1-\cancel{x})\frac{\cancel{m}}{\cancel{1-x}}\cdot\frac{1}{\cancel{m}} - \log\frac{(1-x)}{m} = 0$$

$\Rightarrow \quad \log x + 1 - 1 - \log \frac{1-x}{m} = 0$

$$\Rightarrow \quad \log\left(\frac{x}{\frac{1-x}{m}}\right) = 0$$

$$\Rightarrow \quad \log_e \frac{mx}{(1-x)} = 0$$

$$\Rightarrow \quad \frac{mx}{1-x} = e^0$$

$$\Rightarrow \quad mx = (1-x)$$

$$\Rightarrow \quad mx + x = 1$$

$$\Rightarrow \quad x = \frac{1}{m+1}$$

$$\therefore f_{m+1}(1) = \min\left(\frac{1}{m+1}\log\frac{1}{m+1} + m\left(\frac{\left(1-\frac{1}{m+1}\right)}{m}\log\frac{\left(1-\frac{1}{m+1}\right)}{m}\right)\right)$$

$$= \min\left(\frac{1}{m+1}\log\frac{1}{m+1} + m\left(\frac{m+1-1}{m(m+1)}\log\frac{m+1-1}{m(m+1)}\right)\right)$$

$$= \min\left(\frac{1}{m+1}\log\frac{1}{m+1} + \frac{m}{m+1}\log\frac{1}{m+1}\right)$$

$$= \min\left(\left(\frac{1}{m+1}\left(\log\frac{1}{m+1}\right)\right)(m+1)\right)$$

$$= \min\left((m+1)\left(\frac{1}{m+1}\log\frac{1}{m+1}\right)\right)$$

$$f_{m+1}(1) = (m+1)\left(\frac{1}{m+1}\log\frac{1}{m+1}\right)$$

$\therefore$ For $n = m+1$ the optimal policy is

$\left(\frac{1}{m+1}, \frac{1}{m+1}, \ldots \frac{1}{m+1}\right)$ and $f_{m+1}(1) = (m+1)\left(\frac{1}{m+1} \log \frac{1}{m+1}\right)$

In general, the optimal policy is $\left(\frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n}\right)$

and $f_n(1) = n\left(\frac{1}{n} \log \frac{1}{n}\right)$

$\therefore$ the optimal solution is

$P_1^* = P_2^* = P_3^* = \ldots\ldots = P_n^* = \frac{1}{n}$ and the

optimum value $z = n\left(\frac{1}{n} \log \frac{1}{n}\right)$

use Bellman's optimality principle to maximize $b_1x_1 + b_2x_2 + \ldots b_nx_n$

where $x_1 + x_2 + \ldots + x_n = c$, c is a +ve constant

& $x_1, x_2 \ldots x_n \geq 0$

Soln:

~~max z~~

According to the condition of the given

Problem. the dynamic programming problem is

of the form.

$$\max z = b_1x_1 + b_2x_2 + \ldots + b_nx_n$$

Subject to the

Constraint $x_1 + x_2 + \ldots + x_n = c$

$$x_1, x_2 \ldots x_n \geq 0$$

Let $f_n(c) = \max (b_1x_1 + b_2x_2 + \cdots + b_nx_n)$

$$= \max \sum_{i=1}^{n} b_ix_i$$

Step: 1

Let $n=1$ and put $x_1 = c$,

$$f_1(c) = \max_{x_1=c} (b_1x_1) = b_1c$$

Step: 2

Let $n=2$ and put $x_2 = z$ and $x_1 = c-z$,

$$x_1 + x_2 = c$$

$$f_2(c) = \max_{0 \le z \le c} (b_1x_1 + b_2x_2)$$

$$= \max_{0 \le z \le c} (\underline{b_1(c-z)} + b_2z)$$

$$f_2(c) = \max_{0 \le z \le c} (b_2z + f_1(c-z))$$

|||ly

$$f_3(c) = \max_{0 \le z \le c} (b_3z + f_2(c-z))$$

$$\vdots$$

$$f_n(c) = \max_{0 \le z \le c} (b_nz + f_{n-1}(c-z))$$

consider $f_2(c) = \max\limits_{0 \le z \le c} (b_1(c-z) + b_2z)$

$$= \max_{0 \le z \le c} (b_1c - b_1z + b_2z)$$

$$= \max_{0 \le z \le c} (b_1c + z(b_2 - b_1))$$

since, $f_2(c)$ is maximum

$b_2 - b_1$ is +ve and $z = c$

$$\therefore f_2(c) = \max_{0 \le z \le c} (b_1 c + c(b_2 - b_1))$$

$$= \max_{0 \le z \le c} (\cancel{b_1 c} + b_2 c - \cancel{b_1 c})$$

$$f_2(c) = b_2 c$$

$\therefore$ The optimum policy is.

$$x_1 = 0, \; x_2 = c \;\&\; f_2(1) = b_2 c$$

$||^{ly}$

for $n=3$, the optimum policy is

$$x_1 = 0, \; x_2 = 0, \; x_3 = c \;\&\; f_3(c) = b_3 c.$$

for $n = m$, the optimum policy is.

$$x_1 = x_2 = x_3 = \dots = x_{m-1} = 0 \;/\; x_m = c$$

and $f_m(c) = b_m(c)$.

Put $n = m+1$, $f_{m+1}(c) = \max\limits_{0 \le z \le c} (b_{m+1} z + b_m (c - z))$

$$= \max_{0 \le z \le c} (b_{m+1} z + b_m c - b_m z)$$

$$f_{m+1}(c) = \max_{0 \le z \le c} (b_m c + z(b_{m+1} - b_m))$$

since, $f_{m+1}(c)$ is maximum, $b_{m+1} - b_m$ is +ve and $z = c$.

$$\therefore f_{m+1}(c) = \max_{0 \le z \le c} (b_m c + z(b_{m+1} - b_m))$$

$$= \max_{0 \le z \le c} (b_m c + c b_{m+1} - c b_m)$$

$$f_{m+1}(c) = b_{m+1} c.$$

$\therefore$ The optimum policy is.

$$x_1 = x_2 = \dots = x_m = 0 \quad \& \quad x_{m+1} = c$$

$$f_{m+1}(c) = b_{m+1} c$$

In general, the optimum policy is.

$$x_1 = x_2 = \dots = x_{n-1} = 0 \quad \& \quad x_n = c$$

$$f_n(c) = b_n c$$

work force size model:-

Let x_i be the actual number of labourers employed in a week i.

Let $c_1(x_i - b_i)$ be the cost maintaining an excess labour force $x_i - b_i$ and $c_2(x_i - x_{i-1})$. be the cost hearing additional labourers $(x_i - x_{i-1})$, no additional cost is incurred when empoloyment is disconnected.

The dynamic programming recurrasive equation is of the form

$$f_i(x_{i-1}) = \min_{x_i > b_i} \left[c_1(x_i - b_i) + c_2(x_i - x_{i-1}) + f_{i+1}(x_i) \right]$$

$i = 1, 2, \dots n$

and $f_{n+1}(x_n) = 0$

The computations ~~state~~ starts at stage n with $x_n = b_n$ and terminates at stage 1.

Problem:-

A construction contractor estimates the size of the work force needed over the next 5 weeks to be fixed 5, 7, 8, 4 and 6 workers respectively. excess labour kept on the force will cost \$ 300 per worker per week and the new hearing in any week in a fixed cost of \$400 + \$200 per worker per week. Find the optimum cost.

Sol:

Here $b_1 = 5$, $b_2 = 7$, $b_3 = 8$, $b_4 = 4$, $b_5 = 6$

GT $c_1(x_i - b_i) = 3(x_i - b_i)$

$c_2(x_i - x_{i-1}) = 4 + 2(x_i - x_{i-1})$

W.k.t

$$f_i(x_{i-1}) = \min_{x_i > b_i} \left(c_1(x_i - b_i) + c_2(x_i - x_{i-1}) + f_{i+1}(x_i) \right)$$

and $f_{n+1}(x_n) = 0$

c_1 & c_2 are cost function in 100's of dollars (27)

Stage: 5

Here $f_6(x_5) = 0$ & $b_5 = 6 = x_5$

$c_1(x_5 - b_5) + c_2(x_5 - x_4) + f_6(x_5)$ optimum solution

x_4	$x_5 = 6$	$f_5(x_4)$	x_5^*
4	$3(6-6)$ $3(0)+4+2(6-4)+0 = 8$	8	6
5	$3(0)+4+2(6-5)+0 = 6.$	6	6
6.	$4+2(6-6) \Rightarrow 0.$ $3(0)+0+0 = 0$	0	6.

Stage: 4

Here $b_4 = 4 = x_4$

$c_1(x_4 - b_4) + c_2(x_4 - x_3) + f_5(x_4)$ optimum solution

x_3	$x_4 = 4$	5	6	$f_4(x_3)$	x_4^*
8	$3(4-4)+0+8 = 8$	$3(5-4)+0+6 = 9$	$3(6-4)+0+0 = 6$	6	6

Stage: 3

Here $b_3 = 8$

$c_1(x_3 - b_3) + c_2(x_3 - x_2) + f_4(x_3)$ optimum solution

x_2	$x_3 = 8$	$f_3(x_2)$	x_3^*
7	$3(8-8)+4+2(8-7)+6 = 12$	12	8
8	$3(8-8)+0+6 = 6.$	6	8

Stage : 2   Here $b_2 = 7$.

$c_1(x_2 - b_2) + c_2(x_2 - x_1) + f_3(x_2)$

$x_2 =$	7	8	optimum solution (min) $f_2(x_1)$	x_2^*
x_1				
5	$3(7-7)+4+2(7-5)+12 = 20$	$3(8-7)+4+2(8-5)+6 = 19$	19	8
6	$3(7-7)+4+2(7-6)+12 = 18$	$3(8-7)+4+2(8-6)+6 = 17$	17	8
7	$3(7-7)+4+2(7-7)+12 = 12$	$3(8-7)+4+2(8-7)+6 = 15$	12	7
8	$3(7-7)+0+12 = 12$	$3(8-7)+0+6 = 9$.	9	8

Stage : 1

Here $b_1 = 5$

$c_1(x_1 - b_1) + c_2(x_1 - x_0) + f_2(x_1)$.

x_1	5	6	7	8	optimum solution $f_1(x_0)$	x_1^*
x_0						
0	$3(5-5)+4+2(5-0)+19 = 33$	$3(6-5)+4+2(6-0)+17 = 36$	$3(7-5)+4+2(7-0)+12 = 36$	$3(8-5)+4+2(8-0)+9 = 38$	33	5.

Here $x_0 = 0 \rightarrow x_1 = 5 \rightarrow x_2 = 8 \rightarrow x_3 = 8 \rightarrow x_4 = 6 \rightarrow x_5 = 6$

$\therefore$ The optimum solutions are $x_1^* = 5$, $x_2^* = 8$, $x_3^* = 8$, $x_4^* = 6$

$x_5^* = 6$. and the optimum value $\$(33 \times 100) = 3300$

$\$3300$

Inventory model

Problem of dimensionality:-

The state at any stage is said to be two dimensional because it consists of two element both weight and volume.

Problem:-

~~Ace~~ Acme manufacturing and produces two Products. The daily capacity of the manufacturing Process is 430 units, product 1 requires 2 min/unit and product 2 requires 1 min / unit there is no limit on the amount produced on product 1, but the maximum daily demand for the product 2, is 230 units. The unit profit of product 1 is \$ 2 and that of product 2 is \$ 5. Find the optimum soln for dynamic programming problem.

Soln:-

Let x_1 & x_2 are product 1 & Product 2 respectively,

The mathematical form of the above Problem is

$\max z = 2x_1 + 5x_2$

Subject to the constraint.

$2x_1 + x_2 \le 430$

$x_2 \le 230$

$x_1, x_2 \ge 0$

Stage : 2

Let $f_2(v_2, w_2) = \max_{0 \le x_2 \le (v_2, w_2)} (5x_2)$

Here, maximum of $5x_2$ occurs at x_2 ;

$x_2 = \min(v_2, w_2)$.

The solution of Stage 2 is ~~(v_2, w_2)~~.

~~$f_2(v_2, w_2)$~~ . ~~5 min~~.

State	$f_2(v_2, w_2)$	x_2
(v_2, w_2)	$5\min(v_2, w_2)$	$\min(v_2, w_2)$

Stage : 1

Let $f_1(v_1, w_1) = \max_{0 \le 2x_1 \le v_1} (2x_1 + f_2(v_1 - 2x_1, w_1))$

$= \max_{0 \le 2x_1 \le v_1} (2x_1 + 5\min(v_1 - 2x_1, w_1))$

Put $v_1 = 430$, $w_1 = 230$

~~min~~ It follows that

$\min(430 - 2x_1, 230)$

$$= \begin{cases} 230, & 0 \le x_1 \le 100 \\ 430 - 2x_1, & 100 \le x_1 \le 215 \end{cases}$$

and $f_1(430, 230) = \max\limits_{0 \le x_1 \le 215} (2x_1 + 5\min(430 - 2x_1, 230))$

$$= \max \begin{cases} 2x_1 + 1150 & 0 \le x_1 \le 100 \\ -8x_1 + 2150 & 100 \le x_1 \le 215 \end{cases}$$

The optimum value of $f_1(430, 230)$ occurs at $x_1 = 100$ we get,

State	$f_1(v_1, w_1)$	x_1
(430, 230)	1350	100

$\therefore$ The optimum value of x_2 is $V_2 = V_1 - 2x_1$

$= 430 - 2(100)$

$\boxed{V_2 = 230}$

$w_2 = w_1 - 0$

$w_2 = 230 - 0$

$\boxed{w_2 = 230}$

$\therefore x_2 = \min(V_2, w_2) = \min(230, 230)$

$\boxed{x_2 = 230}$

$\therefore$ The optimum solution of $x_1^* = 100$, $x_2^* = 230$

& $Z^* = \$1350$

ADVANCED OPERATIONS RESEARCH

Operations Research

Operations Research is an experimental and applied science devoted to observing, understanding and predicting the behaviour of a purposeful manmachine system.

Unit III (Decision Theory and Games)

Game:

The competitive situation is called as a game. The following properties are..

i, There are finite number of participants called players.

ii, Every player has finite number of strategies available to him.

iii, Every game results in an outcome.

Number of players:

A game involves only two players then it is called a two person game. The number of players are more than two, the game is called n-person game.

Two person zero sum game:

A game with only two person is said to be two person zero sum game if the gain of one player is equal to the loss of the other.

pay-off matrix:

When a player selected their particular strategies in terms of gains or losses (payoff) can be represented in the form of a matrix is called pay-off matrix.

Example:

Consider two person tossing a coin simultaneously. player B pays Rs 7 to A if (H, H) occurs and Rs 4 if (T, T) occurs. otherwise player A pays Rs 3 to B.

The A's payoff matrix is

$$\text{Player A}\quad \begin{array}{c|cc} & \multicolumn{2}{c}{\text{Player B}} \\ & H & T \\ H & 7 & -3 \\ T & -3 & 4 \end{array}$$

The B's payoff matrix is

$$\text{player B}\quad \begin{array}{c|cc} & \multicolumn{2}{c}{\text{player A}} \\ & H & T \\ H & -7 & 3 \\ T & 3 & -4 \end{array}$$

Maximin value:

A player selects the strategies which maximizes his minimum gain is called maximin value. It is also called lower value and it is denoted by $\underline{v}$

Minimax value

A player selects his strategies which minimizes his maximum loss is called minimax value. It is also called upper value and it is denoted by $\overline{v}$

Note:

In a payoff matrix □ represents the minimum payoff in each row and ○ represents the maximum payoff in each column.

Saddle point:

The position in the payoff matrix, the maximum of the row minima coincides with the minimum of the column maxima. is called saddle point. It is also called equilibrium point.

Note:

The value of the game and the saddle point is not unique. The value of the game is denoted by v.

Strictly determinable game:

A game is said to be a strictly determinable if $\underline{v} = \overline{v} = v$.

Fair game:

A game is said to be a fair game if $\underline{v} = \overline{v} = 0$.

Note:

The value of the game is satisfies $\underline{v} \le v \le \overline{v}$

problem:

Determine the following two person zero sum game are strictly determinable and fair.

$$\text{player A} \quad \overset{\text{Player B}}{\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}}$$

Soln:

G.T $\quad \text{player A} \overset{\text{Player B}}{\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}}$

The payoff matrix is

	B_1	B_2	row minima
A_1	5	$\boxed{0}$	0
A_2	$\boxed{0}$	2	0
column maxima	5	2	

WKT $\quad \underline{v}$ = maximin value

= max (row minima)

$\underline{v} = 0$

WKT $\quad \overline{v}$ = minimax value

= mini (column maxima).

$\overline{v} = 2$

Here $\underline{v} \neq \overline{v}$, the given game is not strictly determinable and not fair.

problem:

Solve the following game whose payoff matrix is

$$\text{player A} \quad \overset{\text{player B}}{\begin{pmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{pmatrix}}$$

Soln:

G.T

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{pmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{pmatrix} \end{array}$$

The payoff matrix is

	B_1	B_2	B_3	B_4	B_5	Row minima
A_1	9	3	1	8	$\boxed{0}$	0
A_2	6	5	$\boxed{4}$	6	7	4
A_3	$\boxed{2}$	4	3	3	8	2
A_4	5	6	2	2	$\boxed{1}$	1
column maxima	9	6	4	8	8	

WKT $\underline{v}$ = maximin value

= max (row minima)

= 4

$\overline{v}$ = minimax value

= min (column maxima)

= 4.

Here $\underline{v} = \overline{v} = 4 = v$

Here the saddle point is (A_2, B_3).

∴ the best strategies for the player A is A_2 and B is B_3.

∴ The value of the game $v = 4$.

Home work:

Determine the following two person zero sum game are strictly determinable and fair.

$$\begin{array}{cc} & \text{Player B} \\ \text{player A} & \begin{pmatrix} 0 & 2 \\ -1 & 4 \end{pmatrix} \end{array}$$

Optimal strategy:

The particular plan by which a player optimizes his gains or losses, without knowing the competitors strategies is called optimal strategy.

Pure Strategy:

Pure strategy is a decision rule which is always used by the players, to select the objectives is to maximize gains or minimize losses.

Mixed strategy:

The objectives of the players is to maximize expected gains or to minimize expected losses by making a solution with fixed probabilities.

Mixed strategy can be divided into two models. They are

(i) Graphical method

(ii) Linear programming method.

Graphical method:

The graphical method is useful for the game, where the payoff matrix is of size $2\times n$ or $m\times 2$; the game with mixed strategies that has only two pure strategies for one of the players in the two person zero sum game.

problem:

Solve the following game graphically

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{pmatrix} 1 & 3 & 11 \\ 8 & 5 & 2 \end{pmatrix} \end{array}$$

soln:

$$\text{G.T} \quad \begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{pmatrix} 1 & 3 & 11 \\ 8 & 5 & 2 \end{pmatrix}_{2\times 3} \end{array}$$

The payoff matrix is

	B_1	B_2	B_3	row minima
A_1	$\boxed{1}$	3	11	1
A_2	8	5	$\boxed{2}$	2
column maxima	8	5	11	

WKT $\underline{v}$ = maximin value

= max (row minima)

= 2

$\overline{v}$ = minimax value

= min (column maxima)

= 5.

Here $\underline{v} \neq \overline{v}$, not a saddle point.

Let the player A plays the mixed strategy $S_A = \begin{pmatrix} A_1 & A_2 \\ P_1 & P_2 \end{pmatrix}$, $P_1+P_2=1$

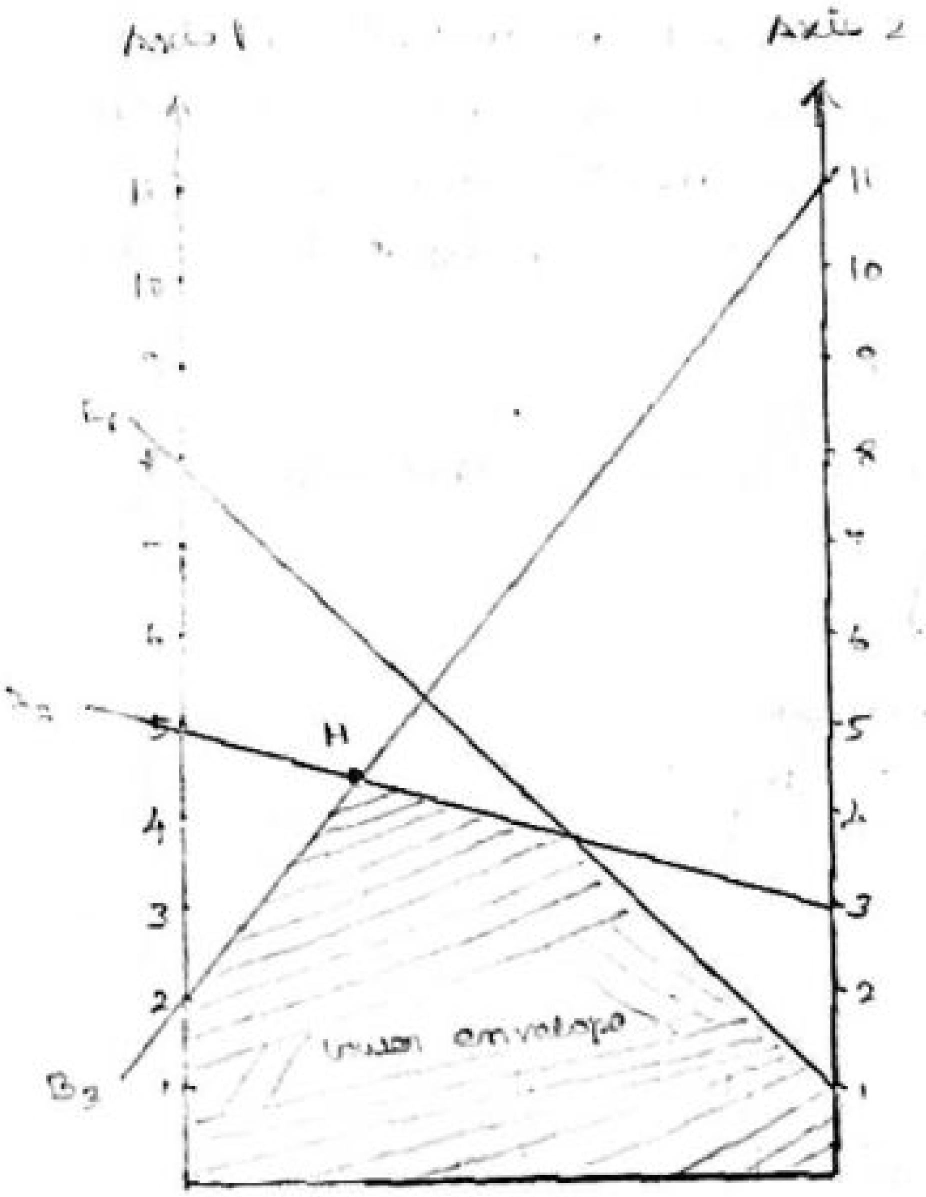
against player B.

Let
$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline P_1 & 1 & 2 & 11 \\ P_2 & 8 & 5 & 2 \end{array}$$

The A's expected payoffs against B's pure move are given by

B's pure move	A's expected payoff
B_1	$1.P_1 + 8.P_2$
B_2	$3.P_1 + 5P_2$
B_3	$11P_1 + 2P_2$

The graph is of the form.



Let ~~be~~ H be the highest point in the lower envelop intersecting both B_2 and B_3.

$\therefore$ The original 2×3 game can be reduced as a 2×2 game.

$$\therefore \begin{array}{c|cc} & B_2 & B_3 \\ \hline A_1 & 3 & 11 \\ A_2 & 5 & 2 \end{array}$$

Here $S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$, $p_1+p_2=1$ and $S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 0 & q_1 & q_2 \end{pmatrix}$ $q_1+q_2=1$

Now the reduced payoff matrix is

$$\begin{array}{c|cc} & q_1 & q_2 \\ \hline p_1 & 3 & 11 \\ p_2 & 5 & 2 \end{array}$$

let $E(p,q)$ be the expected payoff function.

ie $E(P,q) = 3p_1q_1 + 5p_2q_1 + 11p_1q_2 + 2p_2q_2$

$= 3p_1q_1 + 5(1-p_1)q_1 + 11p_1(1-q_1) + 2(1-p_1)(1-q_1)$

$= 3p_1q_1 + 5q_1 - 5p_1q_1 + 11p_1 - 11p_1q_1 + 2 - 2p_1 - 2q_1 + 2p_1q_1$

$\because p_1+p_2=1 \Rightarrow p_2 = 1-p_1$

$\because q_1+q_2=1 \Rightarrow q_2 = 1-q_1$

$= -11p_1q_1 + 3q_1 + 9p_1 + 2$

$= -11\left(p_1q_1 - \frac{3}{11}q_1 - \frac{9}{11}p_1\right) + 2$

$= -11\left[\left(p_1-\frac{3}{11}\right)\left(q_1-\frac{9}{11}\right) + \frac{27}{121}\right] + 2$

$= -11\left[\left(p_1-\frac{3}{11}\right)\left(q_1-\frac{9}{11}\right)\right] + \cancel{11}\cdot\frac{27}{\cancel{121}_{11}} + 2$

$= -11\left(\left(p_1-\frac{3}{11}\right)\left(q_1-\frac{9}{11}\right)\right) + \frac{27}{11} + 2$

$\left(p_1-\frac{3}{11}\right)\left(q_1-\frac{9}{11}\right) = p_1q_1 - \frac{3}{11}q_1 - \frac{9}{11}p_1 + \frac{27}{121}$

$\Rightarrow \left(p_1-\frac{3}{11}\right)\left(q_1-\frac{9}{11}\right) - \frac{27}{121} = p_1q_1 - \frac{3}{11}q_1 - \frac{9}{11}p_1$

$\therefore E(P,q) = -11\left(\left(p_1-\frac{3}{11}\right)\left(q_1-\frac{9}{11}\right)\right) + \frac{49}{11}$

consider $p_1 - \frac{3}{11} = 0 \Rightarrow p_1 = \frac{3}{11}$

WKT $p_1+p_2 = 1 \Rightarrow p_2 = 1-p_1 = 1-\frac{3}{11} = \frac{8}{11}$

consider $q_1 - \frac{9}{11} = 0 \Rightarrow q_1 = \frac{9}{11}$

WKT $q_1+q_2=1 \Rightarrow q_2 = 1-q_1 = 1-\frac{9}{11} = \frac{2}{11}$

$\therefore$ The optimum strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 \\ \frac{3}{11} & \frac{8}{11} \end{pmatrix}; \quad S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 0 & \frac{9}{11} & \frac{2}{11} \end{pmatrix}$$

and the value of the game $v = \frac{49}{11}$.

<u>problem</u>:

Solve the following game graphically

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{pmatrix} 2 & 2 & 3 & -2 \\ 4 & 3 & 2 & 6 \end{pmatrix} \end{array}$$

<u>soln</u>:

$$\begin{array}{ccc} & & \text{player B} \\ \text{G.T} & \text{player A} & \begin{pmatrix} 2 & 2 & 3 & -2 \\ 4 & 3 & 2 & 6 \end{pmatrix} \end{array}$$

The payoff matrix is

	B_1	B_2	B_3	B_4	Row minima
A_1	2	2	3	$\boxed{-2}$	-2
A_2	4	3	$\boxed{2}$	6	1
Column maxima	4	3	3	6	

WKT $\underline{v}$ = maximin value
= max (row minima)
= 2
$\overline{v}$ = minimax value
= min (column maxima)
= 3

Here $\underline{v} \neq \overline{v}$, not a saddle point

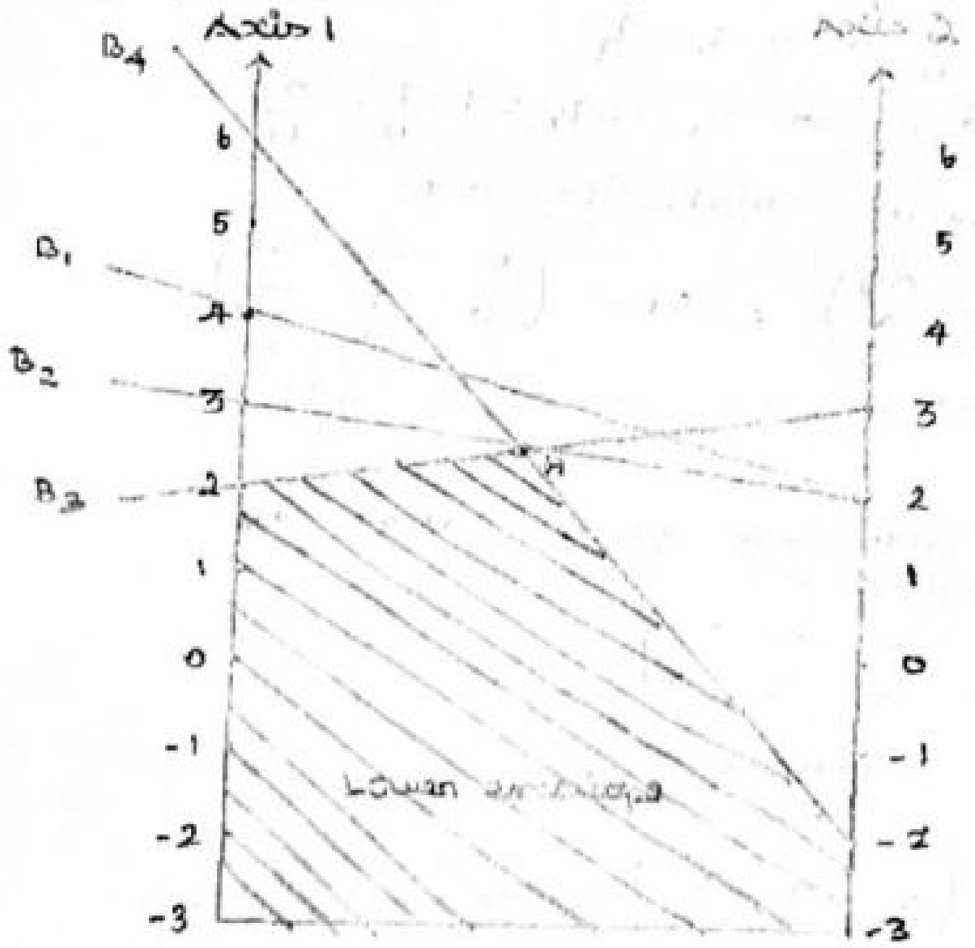
Let the player A, plays the mixed strategy $S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$, $p_1 + p_2 = 1$
against player B.

$$\text{ie} \quad \begin{array}{c c} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{array}{c} p_1 \\ p_2 \end{array} & \begin{pmatrix} 2 & 2 & 3 & -2 \\ 4 & 3 & 2 & 6 \end{pmatrix} \end{array}$$

The A's expected payoff against B's puremove are given by

B's puremove	A's expected payoff
B_1	$2P_1 + 4P_2$
B_2	$2P_1 + 3P_2$
B_3	$3P_1 + 2P_2$
B_4	$-2P_1 + 6P_2$

The graph is of the form.



Let H be the highest point in the lower envelope intersecting both B_3 and B_4

$\therefore$ The original 2x4 game can be reduced as a 2x2 game

$$\therefore \begin{array}{c|cc} & B_3 & B_4 \\ \hline A_1 & 3 & -2 \\ A_2 & 2 & 6 \end{array}$$

Here $S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}_{p_1+p_2=1}$ and $S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & q_1 & q_2 \end{pmatrix}_{q_1+q_2=1}$

Here the reduced payoff matrix is

$$\begin{array}{c|cc} & q_1 & q_2 \\ \hline p_1 & 3 & -2 \\ p_2 & 2 & 6 \end{array}$$

Let $E(p,q)$ be the expected payoff function

ie $E(p,q) = 3p_1q_1 - 2p_1q_2 + 2p_2q_1 + 6p_2q_2$

$\because p_1+p_2=1 \Rightarrow p_2 = 1-p_1$

$q_1+q_2=1 \Rightarrow q_2 = 1-q_1$

$= 3p_1q_1 - 2p_1(1-q_1) + 2(1-p_1)q_1 + 6(1-p_1)(1-q_1)$

$= 3p_1q_1 - 2p_1 + \cancel{2p_1q_1} + 2q_1 - \cancel{2p_1q_1} + 6 - 6p_1 - 6q_1 + 6p_1q_1$

$= 9p_1q_1 - 8p_1 - 4q_1 + 6$

$= 9\left(p_1q_1 - \frac{8}{9}p_1 - \frac{4}{9}q_1\right) + 6$

$= 9\left[\left(p_1 - \frac{4}{9}\right)\left(q_1 - \frac{8}{9}\right) - \frac{32}{81}\right] + 6$

$= 9\left(p_1 - \frac{4}{9}\right)\left(q_1 - \frac{8}{9}\right) + \cancel{9}\left(\frac{-32}{\cancel{81}\,9}\right) + 6$

$\left(p_1 - \frac{4}{9}\right)\left(q_1 - \frac{8}{9}\right) = p_1q_1 - \frac{4}{9}q_1 - \frac{8}{9}p_1 + \frac{32}{81}$

$\Rightarrow \left(p_1 - \frac{4}{9}\right)\left(q_1 - \frac{8}{9}\right) - \frac{32}{81} = p_1q_1 - \frac{4}{9}q_1 - \frac{8}{9}p_1$

$\therefore E(p,q) = 9\left(p_1 - \frac{4}{9}\right)\left(q_1 - \frac{8}{9}\right) + \frac{22}{9}$

Consider $p_1 - \frac{4}{9} = 0 \Rightarrow p_1 = \frac{4}{9}$

WKT $p_1 + p_2 = 1 \Rightarrow p_2 = 1 - p_1 = 1 - \frac{4}{9} = \frac{5}{9}$

Consider $q_1 - \frac{8}{9} = 0 \Rightarrow q_1 = \frac{8}{9}$

WKT $q_1 + q_2 = 1 \Rightarrow q_2 = 1 - q_1 = 1 - \frac{8}{9} = \frac{1}{9}$

$\therefore$ The optimum strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 \\ \frac{4}{9} & \frac{5}{9} \end{pmatrix} ; \quad S_B = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & \frac{8}{9} & \frac{1}{9} \end{pmatrix}$$

and the value of the game $v = \frac{22}{9}$.

<u>Home work</u>:

Solve the following game graphically

i) Player B

player A $\begin{pmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{pmatrix}$

ii) player B

player A $\begin{pmatrix} 1 & 3 & -3 & 7 \\ 2 & 5 & 4 & -6 \end{pmatrix}$

problem:

Solve the following game graphically

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{pmatrix} 1 & 6 \\ 4 & 5 \\ 5 & 3 \end{pmatrix} \end{array}$$

Soln:

$$\begin{array}{cc} & \text{Player B} \\ \text{G.T player A} & \begin{pmatrix} 1 & 6 \\ 4 & 5 \\ 5 & 3 \end{pmatrix}_{3\times 2} \end{array}$$

The payoff matrix is

	B_1	B_2	row minima
A_1	$\boxed{1}$	6	1
A_2	$\boxed{4}$	5	4
A_3	5	$\boxed{3}$	3
column maxima	5	6	

WKT $\underline{v}$ = maximin value

= max (row minima)

= 4

$\overline{v}$ = minimax value

= min (column maxima)

= 5.

Here $\underline{v} \neq \overline{v}$, not a saddle point

Let the player B plays the mixed strategy

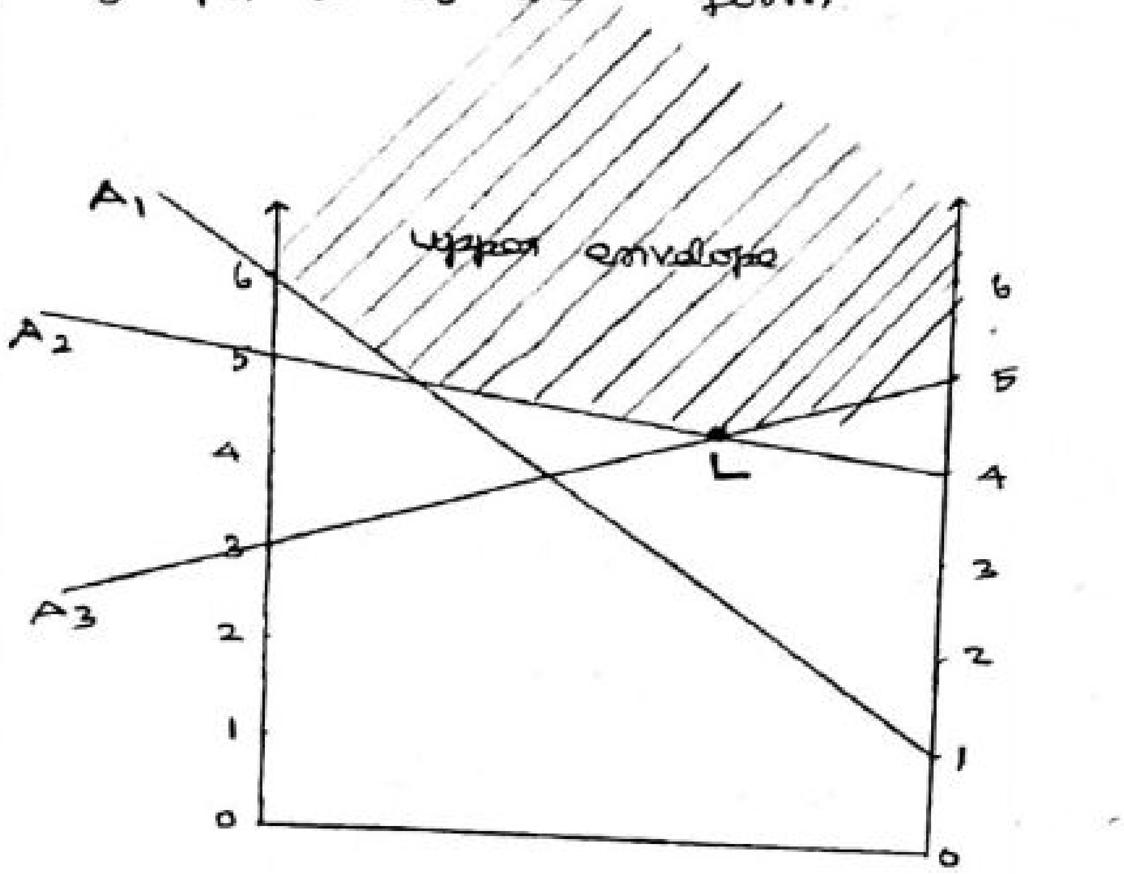
$S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$, $q_1+q_2=1$ against player A

$$\text{ie} \quad \begin{array}{c|cc} & q_1 & q_2 \\ \hline A_1 & 1 & 6 \\ A_2 & 4 & 5 \\ A_3 & 5 & 3 \end{array}$$

The B's expected payoff against A's pure move are given by

A's pure move	B's expected payoff
A_1	$1.q_1 + 6q_2$
A_2	$4q_1 + 5q_2$
A_3	$5q_1 + 3q_2$

The graph is of the form



Let L be the lowest point in the upper envelope intersecting both A_2 and A_3

The original 3×2 game, can be reduced as a 2×2 game.

$$\begin{array}{c c} & \begin{matrix} B_1 & B_2 \end{matrix} \\ \begin{matrix} A_2 \\ A_3 \end{matrix} & \begin{pmatrix} 4 & 5 \\ 5 & 3 \end{pmatrix} \end{array}$$

Here $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & p_1 & p_2 \end{pmatrix}_{p_1+p_2=1}$ and $S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}_{q_1+q_2=1}$

Here the reduced payoff matrix is $\begin{array}{c c} & \begin{matrix} q_1 & q_2 \end{matrix} \\ \begin{matrix} p_1 \\ p_2 \end{matrix} & \begin{pmatrix} 4 & 5 \\ 5 & 3 \end{pmatrix} \end{array}$

Let $E(p,q)$ be the expected payoff function

ie $E(p,q) = 4p_1q_1 + 5p_1q_2 + 5p_2q_1 + 3p_2q_2$

$= 4p_1q_1 + 5p_1(1-q_1) + 5(1-p_1)q_1 + 3(1-p_1)(1-q_1)$

$= 4p_1q_1 + 5p_1 - 5p_1q_1 + 5q_1 - 5p_1q_1 + 3 - 3p_1 - 3q_1 + 3p_1q_1$

$= -3p_1q_1 + 2p_1 + 2q_1 + 3$

$= -3\left(p_1q_1 - \frac{2}{3}p_1 - \frac{2}{3}q_1\right) + 3$

$= -3\left(\left(p_1 - \frac{2}{3}\right)\left(q_1 - \frac{2}{3}\right) - \frac{4}{9}\right) + 3$

$p_1 + p_2 = 1$
$\Rightarrow p_2 = 1 - p_1$
$q_1 + q_2 = 1$
$\Rightarrow q_2 = 1 - q_1$

$\left(p_1 - \frac{2}{3}\right)\left(q_1 - \frac{2}{3}\right)$
$= p_1q_1 - \frac{2}{3}q_1 - \frac{2}{3}p_1 + \frac{4}{9}$
$\Rightarrow \left(p_1 - \frac{2}{3}\right)\left(q_1 - \frac{2}{3}\right) - \frac{4}{9}$
$= p_1q_1 - \frac{2}{3}q_1 - \frac{2}{3}p_1$

$$= -3\left(\left(p_1 - \frac{2}{3}\right)\left(q_1 - \frac{2}{3}\right)\right) - 3\left(-\frac{4}{9}\right) + 3$$

$$= -3\left[\left(p_1 - \frac{2}{3}\right)\left(q_1 - \frac{2}{3}\right)\right] + \frac{4}{3} + 3$$

$$\therefore E(p, q) = -3\left[\left(p_1 - \frac{2}{3}\right)\left(q_1 - \frac{2}{3}\right)\right] + \frac{13}{3}$$

consider $p_1 - \frac{2}{3} = 0 \Rightarrow p_1 = \frac{2}{3}$

WKT $p_1 + p_2 = 1 \Rightarrow p_2 = 1 - p_1 = 1 - \frac{2}{3} = \frac{1}{3}$

consider $q_1 - \frac{2}{3} = 0 \Rightarrow q_1 = \frac{2}{3}$

WKT $q_1 + q_2 = 1 \Rightarrow q_2 = 1 - q_1 = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore$ The optimum strategies are

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \quad S_B = \begin{pmatrix} B_1 & B_2 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

and the value of the game $v = \frac{13}{3}$.

Home work:

Solve the following game graphically.

(i)
$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{pmatrix} 4 & 0 \\ 3 & 1 \\ 1 & 2 \end{pmatrix} \end{array}$$

(ii)
$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{pmatrix} \end{array}$$

The Linear programming method

A two person zero sum game can also be solved by Linear programming approach. The major advantage of using Linear programming techniques is that it ~~solves~~ solves mixed strategy game of any size.

Slack variable

A variable is said to be slack if it added to the left hand side of the constraints and convert them into equality.

EX:

WKT $4 < 7 \Rightarrow 4 + 3 = 7$

Surplus variable

A variable is said to be surplus if it subtracted to the left hand side of the constraint and convert them into equality

EX

WKT $9 > 5 \Rightarrow 9 - 4 = 5$

Problem:

Solve the following game by using simplex method.

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{pmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{pmatrix} \end{array}$$

Soln:

$$\begin{array}{ccc} & \text{player B} & \\ \text{GT} \quad \text{player A} & \begin{pmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{pmatrix} & 3\times3 \end{array}$$

Since some of the entries in the payoff matrix are negative, we add a suitable constant $c = 4$ to each term.

The payoff matrix can be rewritten as

$$\begin{array}{ccccc} & & & \text{player B} & \\ & & B_1 & B_2 & B_3 \\ \text{player A} & A_1 & 5 & 3 & 7 \\ & A_2 & 7 & 9 & 1 \\ & A_3 & 10 & 6 & 2 \end{array} \quad 3\times3$$

Here $S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$; $p_1 + p_2 + p_3 = 1$

and $S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$, $q_1 + q_2 + q_3 = 1$

$$\begin{pmatrix} 5 & 3 & 7 \\ 7 & 9 & 1 \\ 10 & 6 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \le \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The LPP for player B is

$$\text{minimum } \gamma = \text{maximum } \frac{1}{\gamma} = x_1 + x_2 + x_3$$

subject to the constraints

$$5x_1 + 3x_2 + 7x_3 \le 1$$
$$7x_1 + 9x_2 + x_3 \le 1$$
$$10x_1 + 6x_2 + 2x_3 \le 1$$
$$x_1, x_2, x_3 \ge 0. \text{ and } x_i = \frac{q_i}{\gamma}, \; i = 1, 2, 3.$$

By introducing slack variables x_4, x_5 and x_6 the above problem can be rewritten as

$$\text{Max } \frac{1}{\gamma} = x_1 + x_2 + x_3 + 0.x_4 + 0.x_5 + 0.x_6$$

S.T

$$5x_1 + 3x_2 + 7x_3 + x_4 = 1$$
$$7x_1 + 9x_2 + x_3 + x_5 = 1$$
$$10x_1 + 6x_2 + 2x_3 + x_6 = 1$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0.$$

I st table:

			1	1	1	0	0	0
C_B	BA	x_B	x_1	x_2	x_3	x_4	x_5	x_6
0	x_4	1	5	3	$\boxed{7}$	1	0	0
0	x_5	1	7	9	1	0	1	0
0	x_6	1	10	6	2	0	1	0
$Z_j - C_j$		0	−1	−1	−1	0	0	0

$\rightarrow \min\left(\frac{1}{7}, \frac{1}{1}, \frac{1}{2}\right) = \frac{1}{7}$

Here the entering variable is x_3, the leaving variable is x_4 and the pivot element is 7.

II nd table:

				1	1	1	0	0	0
	C_B	BA	x_B	x_1	x_2	x_3	x_4	x_5	x_6
I st row ÷ 7	1	x_3	1/7	5/7	3/7	1	1/7	0	0
−1×I st row + II nd row	0	x_5	6/7	$\frac{44}{7}$	$\boxed{\frac{60}{7}}$	0	−1/7	1	0
−2×I st row + III rd row	0	x_6	5/7	$\frac{60}{7}$	$\frac{36}{7}$	0	−2/7	0	1
	$Z_j - C_j$		1/7	−2/7	−4/7	0	1/7	0	0

$\rightarrow \min\left(\frac{1/7}{3/7}, \frac{6/7}{60/7}, \frac{5/7}{36/7}\right) = \min\left(\frac{1}{3}, \frac{1}{10}, \ldots\right) = \frac{1}{10}$

Here the entering variable is x_2, the leaving variable is x_5 and the pivot element is $\frac{60}{7}$

III rd table

			1	1	1	0	0	0	
	C_B	BA	x_B	x_1	x_2	x_3	x_4	x_5	x_6
$-\frac{2}{?} \times$ III row + I st row	1	x_3	$1/10$	$2/5$	0	1	$3/20$	$-1/20$	0
II nd row $\div \frac{60}{7}$	1	x_2	$1/10$	$11/15$	1	0	$-1/60$	$7/60$	0
$-\frac{3}{?} \times$ II nd row + III rd row	0	x_6	$1/5$	$\frac{24}{5}$	0	0	$-1/5$	$-3/5$	1
	$Z_j - C_j$		$1/5$	$2/15$	0	0	$2/15$	$1/15$	0

Here all $Z_j - C_j \geq 0$, the solution is an optimum solution.

Here $\frac{1}{r} = \frac{1}{5} \Rightarrow r = 5$.

$\therefore$ The value of the game $v = r - c = 5 - 4 = 1$.

Here $x_i = \frac{q_i}{r} \Rightarrow q_i = x_i r$

$\therefore$ The optimum strategies for player B is

$q_1^* = x_1 r = 0 \times 5 = 0$, $q_2^* = x_2 r = 1/10 \times 5 = 1/2$

$q_3^* = x_3 r = 1/10 \times 5 = 1/2$

$$\therefore S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

$\therefore$ The optimum strategies for player A is

$P_1^* = 2/15 \times 5 = 2/3$, $P_2^* = 1/15 \times 5 = \frac{1}{3}$, $P_3^* = 0 \times 5 = 0$

$$\therefore S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$$ and the value of the game

$v = 1$

Problem:

Solve the following game using simplex method

$$\begin{array}{cc} & \text{player B} \\ \text{player A} & \begin{pmatrix} 2 & -2 & 3 \\ -3 & 5 & -1 \end{pmatrix} \end{array}$$

Soln:

$$\begin{array}{ccc} & & \text{player B} \\ \text{G.T} & \text{player A} & \begin{pmatrix} 2 & -2 & 3 \\ -3 & 5 & -1 \end{pmatrix} \end{array}$$

Since some of the entries in the payoff matrix are negative, we add a suitable constant $C = 4$ to each term.

The payoff matrix can be rewritten as

$$\begin{array}{cc} & \text{player } B \\ \text{player } A & \begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 6 & 2 & 7 \\ A_2 & 1 & 9 & 3 \end{array} \end{array}$$

Here $S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$, $p_1+p_2=1$ and $S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$

$q_1+q_2+q_3=1$

The LPP for player B is

minimum γ = maximum $\frac{1}{\gamma} = x_1+x_2+x_3$

s.t

$6x_1+2x_2+7x_3 \le 1$

$x_1+9x_2+3x_3 \le 1$

$x_1, x_2, x_3 \ge 0$. and $x_i = \frac{q_i}{\gamma}, i=1,2,3$

$$\begin{pmatrix} 6 & 2 & 7 \\ 1 & 9 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \le \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

By introducing slack variables x_4 and x_5 the above problem can be rewritten as

$\max \frac{1}{\gamma} = x_1+x_2+x_3+0.x_4+0.x_5$

s.t $6x_1+2x_2+7x_3+x_4=1$

$x_1+9x_2+3x_3+x_5=1$

$x_1, x_2, x_3, x_4, x_5 \ge 0$.

1st table:

			1	1	1	0	0
C_B	BA	x_B	x_1	x_2	x_3	x_4	x_5
0	x_4	1	$\boxed{6}$	2	7	1	0
0	x_5	1	1	9	3	0	1
z_j-c_j		0	−1	−1	−1	0	0

$\rightarrow \min\left(\frac{1}{6}, \frac{1}{1}\right) = \frac{1}{6}$

Here the entering variable is x_1, the leaving variable is x_4 and the pivot element is 6.

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IInd table:

			1	1	1	0	0	
	C_B	B_A	x_B	x_1	x_2	x_3	x_4	x_5
Ist row ÷ 6	1	x_1	1/6	1	1/3	7/6	1/6	0
-1×Ist row + IInd row	0	x_5	5/6	0	$\boxed{26/3}$	11/6	-1/6	1
	$Z_j - C_j$		1/6	0	-2/3	1/6	1/6	0

$\min\left(\frac{1/6}{1/3}, \frac{5/6}{26/3}\right) = \min\left(1/2, \frac{5}{52}\right) = 5/52$

Here the entering variable is x_2 the leaving variable is x_5 and the pivot element is $26/3$.

IIIrd table:

			1	1	1	0	0	
	C_B	B_A	x_B	x_1	x_2	x_3	x_4	x_5
$-\frac{1}{3}$ × 3rd row + Ist row	1	x_1	7/52	1	0	57/52	9/52	-1/26
IInd row ÷ $\frac{26}{3}$	1	x_2	5/52	0	1	11/52	-1/52	3/26
	$Z_j - C_j$		3/13	0	0	4/13	2/13	1/13

Here all $Z_j - C_j \geq 0$, the solution is an optimum solution.

Here $\frac{1}{r} = \frac{3}{13} \Rightarrow r = \frac{13}{3}$.

∴ The value of the game $v = r - c = \frac{13}{3} - 4 = \frac{1}{3}$

∴ The optimum strategies for player B is

$q_1^* = x_1 r = 7/52 \times 13/3 = 7/12$

$q_2^* = x_2 r = 5/52 \times \frac{13}{3} = 5/12$, $q_3^* = x_3 r = 0 \times \frac{13}{3} = 0$

∵ $x_i = \frac{q_i}{r}$
$\Rightarrow q_i = x_i r$

$$\therefore S_B = \begin{pmatrix} B_1 & B_2 & B_3 \\ 7/12 & 5/12 & 0 \end{pmatrix}$$

∴ The optimum strategies for player A is

$p_1^* = 2/13 \times 13/3 = \frac{2}{3}$, $p_2^* = 1/13 \times 13/3 = 1/3$

∴ $S_A = \begin{pmatrix} A_1 & A_2 \\ 2/3 & 1/3 \end{pmatrix}$ and the value of the game $v = 1/3$

Home work:

Solve the following game using simplex method

i, player A, player B $\begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{pmatrix}$

ii, player A, player B $\begin{pmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{pmatrix}$

~~slack variable~~

~~A variable is said to be slack, if it is added to the left hand side of the constraints and convert into equality~~

Decision Analysis

Several available options can be selected by a decision maker considered to be best according to some predesignated standard is called decision making

The decision making process involves the following steps.

i, Determine the various alternative course of action from which the final decision is to be made.

ii, Identify the possible outcome called events or states of outcome

iii, Determine the payoff function which describe the consequences resulting from the different combinations of the acts and events.

Decision under uncertainty

Decision making under uncertainty involves alternative actions whose payoff depends on the states of nature

The lack of information has led to the development of the following criteria for analysing and decision problem.

i, criteria of pessimism

ii, criteria of optimism

iii, Laplace criteria

iv, savage

v, Hurwicz

ii, Criteria of ~~pessimism~~ pessimism (Maximin gain criteria or minimax loss criteria)

The decision maker achieves the greatest possible payoff or lowest possible cost, the following steps are

i, Choose minimum value in each row

ii, Select the decision alternative which has the maximum of the minimum payoffs

iii, Loss table is given choose the maximum in each row and selects the decision alternative which has the minimum of the maximum losses.

Problem:

A businessman has three alternatives open to him each of which can be followed by any of the four possible events. The conditional payoff (in Rs) for each action event combination are given below.

Alternatives	payoff conditional on events			
	A	B	C	D
x	8	0	-10	6
y	-4	12	18	-2
z	14	6	0	8

Determine which alternative should the businessman choose if he adopts the maximin criteria.

Soln

For the given payoff matrix the minimum payoff for each alternatives are

$X = -10, \; Y = -4, \; Z = 0.$

Here the maximum of ~~the~~ these minimum payoff is 0.

∴ The alternative z is selected according to the maximin principle.

iii, Criteria of optimism (Maximax gain criteria or minimin loss criteria)

In this criteria we assume that the state of nature is on the side of the decision maker. This criteria is called criteria of optimism

The decision maker select, that the particular strategy which corresponds to the maximum of the maximum payoff for each strategy.

The decision maker selects that the particular strategy which correspons to the minimum of the minimum payoff for each strategy.

problem:

A businessman has three alternatives open to him each of which can be followed by any of the four possible events.. The conditional payoff (in Rs) for each action event combinations are given below.

Alternatives	payoff conditional on events.			
	A	B	C	D
X	8	0	-10	6
Y	-4	12	18	-2
Z	14	6	0	8

Determine which alternatives should the businessman choose if he adopts the maximax criteria.

Soln:

For the given payoff matrix the maximum payoff for each alternatives are

$X = 8, \quad Y = 18, \quad Z = 14.$

Since the maximum of these maximum payoff is 18

→ The alternative Y is selected according to maximax principle..

iii, The Laplace criteria

The Laplace criteria gives all the information by assigning equal probabilities. The following steps are

i, calculate the average of each row.

ii, Select the alternative which have the maximum average in the case of profit matrix and minimum average in the case of loss matrix.

problem:

Consider the following payoff in terms of yearly net profits for each decision alternatives

	N_1	N_2	N_3
D_1	150	250	350
D_2	450	250	200
D_3	100	180	290

Which decision is to chosen on the basis of Laplace criteria.

Soln:

G.T

	N_1	N_2	N_3
D_1	150	250	350
D_2	450	250	200
D_3	100	180	290

Now

	N_1	N_2	N_3	row average
D_1	150	250	350	250
D_2	450	250	200	$\boxed{300}$
D_3	100	180	290	190

In a profit matrix 300 is the row average

Hence by Laplace criteria the decision D_2 is chosen.

Savage criteria:

The savage criteria is based on the concept of opportunity loss or regret, ~~It is also called Regret criteria.~~ to select the course of action that minimizes the maximum loss. It is also called Regret criteria.

Notes:

(i) regret payoff = maximum payoff − i^{th} payoff

problem:

A business man has three alternatives open to him each of which can be followed by any one of the four possibility. The conditional payoffs (in Rs) for each action event combination are given below

Alternative	payoffs combination on events.			
	A	B	C	D
X	8	0	−10	6
Y	−4	12	18	−3
Z	14	6	0	8

Determine the regret payoff amount.

soln:

WKT regret payoff = maximum payoff − (ith payoff)

Alternative	Payoff Amount A	B	C	D	Regret payoff Amount A	B	C	D	Maximum regret
X	8	0	-10	6	6	12	28	2	28
Y	-4	12	18	-2	18	0	0	10	18*
Z	14	6	0	8	0	6	18	0	18*
Max payoff	14	12	18	8					

Since alternatives Y and Z both corresponds to the minimal of the maximum possible regret.

∴ The decision maker may choose either Y or Z.

Hurwicz criteria:

This criteria provides a balance between extreme pessimism and the extreme optimism is made by weighting them with certain degrees of optimism and pessimism.

Note:

Let α be the degree of optimism and $1-\alpha$ be the degree of pessimism and $0 \le \alpha \le 1$.

Let H be the Hurwicz quantity we have

$H = \alpha \times \text{maximum} + (1-\alpha) \times \text{minimum}$ for each alternatives.

problem:

Using Hurwicz principle to obtain an optimum solution to the following and $\alpha = 0.5$

Customer Category	Supply level A_1	A_2	A_3	A_4
E_1	7	12	20	27
E_2	10	9	10	25
E_3	23	20	14	23
E_4	32	24	21	17

soln:

G.T $\alpha = 0.5$

WKT $H = \alpha \times \max + (1-\alpha) \times \min$

$= 0.5 \times \max + (1-0.5) \times \min$

$= 0.5 \times \max + 0.5 \times \min$

$= 0.5 \times (\max + \min)$

New	Alternative	Maximum payoff	Minimum payoff	h
	A_1	32	7	$0.5\times(32+7) = 19.5$
	A_2	24	9	$0.5\times(24+9) = 16.5$
	A_3	21	10	$0.5\times(21+10) = 15.5$
	A_4	27	17	$0.5\times(27+17) = 22$

∴ The optimum solution is to choose the alternative A_3.

Decision under Risk

A decision maker selects several possible options whose probabilities of occurrence can be stated is called decision under risk.

This is divided into three major methods. They are

i, Expected monetary value (EMV) criteria

ii, Expected opportunity Loss (EOL)

iii, Expected value of perfect information.

EMV criteria:

A given course of action is the expected value of the conditional payoff for the action is called EMV. The following steps are

i, List the conditional profit for each action event combinations.

ii, For each action determines the expected conditional payoff

iii, Determine EMV for each action.

iv, Choose the action which corresponds to the original EMV.

Unit-IV - Inventory models

Inventory:-

An inventory can be defined as a stock of goods which must be carried out in order to ensure smooth and efficient running of a business

ABC Inventory System:-

The ABC system is a simple procedure that can be used to isolate the item that require special attention in terms of inventory control.

class A items represents small quantities of expensive items and must be subject to tight inventory control.

class B items are next in order where a moderate form of inventory control can be applied.

2

Class c items should be given the lowest ~~priority~~ priority in the application of any form of inventory control.

order quantity:-

order quantity represents the optimum amount that should be ordered every time, an order is ~~plas~~ placed and may vary with time depending on the situation.

The time interval of order can be divided into two parts they are

(i) periodic review case

(ii) continuous review case.

(i) periodic review case:-

Receive a new order of the amount speci-fied by the order quantity at equal interval of time.

(ii) continuous review case:-

When the inventory level reaches the reorder point place a new order whose size equals to the order quantity.

Purchasing cost:-

The commodity unit price becomes depends on the size of order is called purchasing cost.

When the unit price of the item decreases with the increase of order quantity is called quantity discount (or) price break.

~~Setup cost (k)~~:-

~~This is the cost associated with the~~

~~Setting up of~~

Setup cost (k):-

The cost associated with obtaining goods through purchasing or manufacturing (or) ordering are ~~known as~~ called setup cost. ~~which~~ it is denoted by k.

Holding cost :-

The cost associated with carrying (or) holding the goods in stock is called holding (or) carrying cost. It is denoted by h.

Shortage cost:- (c_2)

The fanality cost that are inaccured in a result of shostage are called shortage cost (or) Stockout cost it is denoted by c_2.

lead time :-

The time between placing of an order and it's arrival of stock is called lead time. it is also called delivary Lags.

stock replenishment:-

An inventory system may operates with delivary lags the actual replenishment may occur instantaneously (or) uniformly.

Instantaneous stock replenishment:-

Instantaneous stock replenishment may occur when the stock is purchased ~~from~~ the outside source.

Uniform stock replishment may occurs

uniform stock replenishement may occurs when the product is manufactured locally with in the organisation.

5

Time horizon :-

The period over which the inventory level will be controlled is called time horizon.

Number of supply echelons:-

A Staring point ~~are~~ is organised that one point acts as a supply point for ~~are~~ the other point. this type of operations may be replaced as different levels. so ~~show~~ that a demand point may again become a new ~~sy~~ supply point is called multi echelons system.

deterministic model:-

In an inventory model demand is assumed to be fixed and completely pre-determined.

Such a system is called economic ~~economic~~ lot size system.

it is also called economic order quantity (EOQ)

(b)

Single System Static model (EOQ)
or classic EOQ model

Purchasing problem with no shortages:-

In this model we have to find optimum order quantity y^*, minimum average cost C_{min} and optimum time interval t^* the following conditions are satisfy.

(i) demand (D) is uniform.

(ii) shortages are not allowed ($C_2=0$)

(iii) Holding cost (h)

(iv) Set-up cost (K)

(v) production rate is infinite.

Proof

WKT optimum time interval $t^* = \frac{y^*}{D}$

WKT Average stock $= \frac{y}{2}$

Now, Total cost per unit time $= TCU(y)$

$= $ setupcost / unit time + Holding cost / unit time

$= K/_{y/D} + h(y/2)$

$$TCU(y) = \frac{KD}{y} + \frac{hy}{2} \rightarrow (1)$$

Diff (1) w.r.t "y" we get,

$$\frac{d}{dy}(TCU(y)) = KD\left(-1/y^2\right) + \frac{h}{2}(1)$$

$$\frac{d}{dy}(TCU(y)) = KD\left(\frac{-1}{y^2}\right) + \frac{h}{2}(1)$$

$$\frac{d}{dy}(TCU(y)) = -\frac{KD}{y^2} + \frac{h}{2}$$

Since, the cost is minimum.

$$\frac{d}{dy}(TCU(y)) = 0$$

$$\text{(ie.)} -\frac{KD}{y^2} + \frac{h}{2} = 0$$

$$\Rightarrow \frac{h}{2} = \frac{KD}{y^2}$$

$$y^2 = \frac{2KD}{h}.$$

$$y = \sqrt{\frac{2KD}{h}}$$

∴ The optimum order quantity,

$$y^* = \sqrt{\frac{2KD}{h}}$$

$$C_{min} \cancel{TC(min)} = \sqrt{2KDh}$$

1) A manufacturer has to supply his customer 600 units of his production per year, shortages are not allowed and the storage cost amount is to Rs 0.60 per unit per year. The setup cost per run is Rs 80 Find the optimum run size, optimum time interval and minimum average costs.

Soln (8)

GT Demand $D = 600$ units / year.

Shortage cost $C_2 = 0$

Storage cost $h = $ RS 0.60 / unit / year

Setup cost $k = $ RS 80 / unit.

WKT optimum order quantity

$$y^* = \sqrt{\frac{2kD}{h}}$$

$$= \sqrt{\frac{2(80)(600)}{0.60}}$$

$$= \sqrt{\frac{2(80)(600)(100)}{0.60 \times 100}}$$

$$= \sqrt{\frac{2(80)(600)(100)}{60}}$$

$$= \sqrt{160000}$$

$$y^* = 400 \text{ units.}$$

optimum time interval $= t^* = \frac{y^*}{D} = \frac{400}{600} = \frac{2}{3} = 0.67$ year

minimum average cost $C_{min} = \sqrt{2kDh}$

$$= \sqrt{2 \times 80 \times 600 \times 0.60}$$

$$= \sqrt{2 \times 80 \times 360}$$

$$= \sqrt{57,600}$$

$C_{min} = $ RS 240/-

Problem

2) The daily demand for a commodity is approximately 100 units/day. every time an order is placed, a fixed cost (K) $100/unit is incurred the daily holding cost (h) per unit inventory is $0.02. If the lead time is 12 days, determine economic lot size & the reorder point.

Soln

GT Demand $D = 100$ units / day

setup cost $K = \$100$ / unit

holding cost $h = \$0.02$ / unit / day.

lead time $L = 12$ days.

W.k.t Economic lot size = EOQ

= optimum order quantity,

i.e, $$y^* = +\sqrt{\frac{2KD}{h}}$$

$$= +\sqrt{\frac{2 \times 100 \times 100}{0.02}}$$

$$= +\sqrt{\frac{2 \times 100 \times 100 \times 100}{0.02 \times 100}}$$

$$= +\sqrt{\frac{\cancel{2}(1000000)}{\cancel{2}}}$$

$y^* = +1000$ units.

Wkt reorder point = optimum time interval.

$$(ii)\ t^* = \frac{y^*}{D}$$

$$= \frac{1000}{100}$$

$$t^* = 10 \text{ days.}$$

Here the lead time is 12 days and the cycle of reordered length is 10 days i.e, 12 − 10 = 2 days

excess

∴ The demand for 2 days = 2 × 100 = 200 units

3) The daily demand for a commodity is approximately 100 units, every time an order is placed a fixed cost (K) \$100 unit is incurred the daily holding cost (h) per unit inventory is \$ 0.02.
If the lead time is (i) 23 days, (ii) 8 day
(iii) 10 days, determine economic lot size & the reordered point.

Soln :-

GT demand D = 100 units | day

Setup cost K = \$ 100 | unit.

holding cost h = \$ 0.02 | unit | day

wkt Economic lot size = EOQ = optimum order quantity.

$(x)\ y^* = +\sqrt{\frac{2kD}{h}}$

$= +\sqrt{\frac{2 \times 100 \times 100}{0.02} \times \frac{100}{100}}$

$= +\sqrt{\frac{2(1000000)}{2}}$

$= +\sqrt{1000000}$

$y^* = 1000$ units

w.k.t reorder point = optimum time interval

(ie) $t^* = \frac{y^*}{D}$

$= \frac{1000}{100}$

$t^* = 10$ days

(i) Here the lead time is 23 days and the cycle of reorder length is 10 days. then two cycle can be completed, $23 - 20 = 3$ days excess. $\Rightarrow 3 \times 100 = 300$ days

(ii) Here the lead time is 8 days and one cycle can not be completed. then $8 \times 100 = 800$ units

(iii) Here the lead time is 10 day and the one cycle ~~ten cycle~~ and day can be completed, 0 excess and no excess demand

$\therefore$ 10 days $= 1000$ units.

Obtain the formula for EOQ in single item static mod

· single item static model with price breaks.

In this model the purchasing price per unit may depend on the size of the quantity purchased. this suituation is usually occurs a discrete price break (or) quantity discount

In this case the purchasing price should be consider in the inventory model.

consider an inventory model with instanta stock replenishment and no shortage

Assume that the cost per unit is c_1 for $y < q$ and c_2 for $y \geq q$ ~~for y~~ where $c_1 > c_2$ & q is the quantity above which a price break is granted

W.k.t
Total cost = Purchasing cost + set up cost + Holding cost

W.k.t
Total cost per unit time

$$TCU_1(y) = Dc_1 + \frac{KD}{y} + \frac{hy}{2}, \quad y < q$$

$$TCU_2(y) = Dc_2 + \frac{KD}{y} + \frac{hy}{2}, \quad y \geq q.$$

Let y_m be the quantity at which the minimum value of TCU_1 & TCU_2 occurs

$$\text{ie } y_m = \min(TCU_1, TCU_2) = \sqrt{\frac{2KD}{h}}$$

The optimum order quantity y^* depends on q, the price break point falls w.r.t zone I, II & III respectively.

These zones are defined by determining q_1 & y_m from the equation

$TCU_1(y) = TCU_2(q_1)$

The solution of the equation depending whether q falls in zone I (or) II (or) III, the optimum order quantity

$$y^* = \begin{cases} y_m & 0 \le q \le y_m \quad \text{zone-I} \\ q & y_m \le q \le q_1 \quad \text{zone-II} \\ y_m & q \ge q_1 \quad \text{zone-III} \end{cases}$$

1) Consider the inventory model with the following information: $K = \$10$, $h = \$1$, $D = 5$ units, $C_1 = \$2$, $C_2 = \$1$ and $q = 15$ units. compute y_m & $TCU(y^*)$

<u>soln</u>

GT $K = \$10$, $h = \$1$, $D = 5$ units, $C_1 = \$2$, $C_2 = \$1$

and $q = 15$ units.

$$y^* = \begin{cases} y_m & 0 \le q \le y_m \quad \text{zone I} \\ q & y_m \le q \le q_1 \quad \text{zone II} \\ y_m & q \ge q_1 \quad \text{zone III} \end{cases}$$

WKT $y_m = +\sqrt{\dfrac{2KD}{h}}$

$$= +\sqrt{\frac{2\times(10)\times 5}{1}}$$

$y_m = 10$ units.

Here, $y_m < q$ the result is in zone II (or) zone III

Here the value of q_1 is computed.

Here $TCU_1(y_m) = TCU_2(q_1)$

$$\Rightarrow DC_1 + \frac{KD}{y_m} + \frac{hy_m}{2} = DC_2 + \frac{KD}{q_1} + \frac{hq_1}{2}$$

$$\Rightarrow 5 \cdot 2 + \frac{10 \cdot 5}{10} + \frac{1 \cdot 10}{2} = 5 \cdot 1 + \frac{10 \cdot 5}{q_1} + \frac{1 \cdot q_1}{2}$$

$\Rightarrow 10+5+5 = 5+\frac{50}{q_1}+\frac{q_1}{2}$

$\Rightarrow 15 = \frac{50}{q_1}+\frac{q_1}{2}$

($\times 2q_1$ on both sides)

$\Rightarrow 30q_1 = 100+q_1^2$

$\Rightarrow q_1^2-30q_1+100=0$

Here $q_1 = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \frac{-(-30)\pm\sqrt{(-30)^2-4(1)(100)}}{2(1)}$

$= \frac{30\pm\sqrt{500}}{2}$

$= \frac{30\pm 22.36}{2}$

$= \frac{30+22.36}{2}$ (or) $\frac{30-22.36}{2}$

$= \frac{52.36}{2}$ (or) $\frac{7.64}{2}$

$q_1 = 26.18$ (or) 3.82

$q_1=26.18$ (or) $q_1=3.82$

Here q_1 is selected as a large value, $q_1=26.18$

$\therefore 10<15<26.18 \Rightarrow y_m \leq q \leq q_1$

$\therefore$ The result is in zone-II

$y^* = q = 15$

$\therefore TCU_2(y^*) = TCU_2(15)$

$$= Dc_2 + \frac{KD}{y^*} + \frac{hy^*}{2}$$

$$= 5.1 + \frac{10.5}{15} + \frac{1.15}{2}$$

$$= 5 + \frac{10}{3} + \frac{15}{2}$$

$$= 5 + 3.33 + 7.50$$

$\therefore TCU(y^*) = \$15.83$

Consider an inventory model with the following information.

$K = \$10$, $h = \$1$, $D = 5$ units, $c_1 = \$2$, $c_2 = \$1$.

(i) $q = 5$ units & (ii) $q = 30$ units. compute y_m & $TCU(y^*)$

Soln.

GT, $K = \$10$, $h = \$1$, $D = 5$ units, $c_1 = \$2$, $c_2 = \$1$

(i) $q = 5$ units.

WKT $y_m = +\sqrt{\frac{2KD}{h}}$

$$= +\sqrt{\frac{2(10)(5)}{1}}$$

$$= +\sqrt{100}$$

$y_m = 10$ units

Here $0 \le 5 \le 10 \Rightarrow 0 \le q \le y_m$

The result is in zone - I

$\therefore y^* = y_m = 10$

$$TCU_2(y^*) = TCU_2(10) = DC_2 + \frac{KD}{y^*} + \frac{hy^*}{2}$$

$$= 5.1 + \frac{10.5}{10} + \frac{1.10}{2}$$

$$= 5 + 5 + 5$$

$$TCU(y^*) = \$15$$

—x—

(ii) $q = 30$ units

$y_m < q$. $10 < 30$ the result is in zone II & III

Here the value of q_1 is computed

Here $TCU_1(y_m) = TCU_2(q_1)$

$$DC_1 + \frac{KD}{y_m} + \frac{hy_m}{2} = DC_2 + \frac{KD}{q_1} + \frac{hq_1}{2}$$

$$5.2 + \frac{10.5}{10} + \frac{1.10}{2} = 5.1 + \frac{10.5}{q_1} + \frac{1.q_1}{2}$$

$$10 + 5 + \cancel{5} = \cancel{5} + \frac{50}{q_1} + \frac{q_1}{2}$$

$$15 = \frac{50}{q_1} + \frac{q_1}{2}$$

$$30q_1 = 100 + q_1^2$$

$$q_1^2 - 30q_1 + 100 = 0$$

$$q_1 = \frac{-(-30) \pm \sqrt{(-30)^2 - 4(1)(100)}}{2(1)}$$

$$q_1 = \frac{30 \pm \sqrt{900 - 400}}{2}$$

$$q_1 = \frac{30 \pm 22.36}{2}$$

$$q_1 = \frac{30 + 22.36}{2} \quad (\text{or}) \quad q_1 = \frac{30 - 22.36}{2}$$

$$q_1 = \frac{52.36}{2} \quad (\text{or}) \quad \frac{7.64}{2}$$

$$q_1 = 26.18 \quad (\text{or}) \quad q_1 = 3.82$$

Here q_1 is selected as a large value $q_1 = 26.18$.

$$30 > 26.18 \Rightarrow q \geq q_1$$

$\therefore$ The result is in zone-III

$$y^* = y_m = 10$$

$$TCU_2(y^*) = TCU_2(10)$$

$$= \frac{DC_2}{\cancel{}} + \frac{KD}{y^*} + \frac{hy^*}{2}$$

$$= 5.1 + \frac{10.5}{10} + \frac{1.10}{2}$$

$$= 5 + 5 + 5$$

$$TCU_2(y^*) = 15$$

$\therefore$ $TCU(y^*) = \$15$ // Ans.

(19) Multi-item static model with storage limitation

In this model an inventory system including $n \geq 1$ items that are computing for a limited storage space.

Let A be a maximum storage area available for n items.

assumethat the storage area requirment per unit of the i^{th} item is a_i

Let y_i be the amount ordered of the i^{th} item, the storage requirment constraints becomes

$$a_1 y_1 + a_2 y_2 + \cdots + a_n y_n = \sum_{i=1}^{n} a_i y_i \leq A.$$

Here each item is replenish~~ment~~ed instant-aneously and there is no quantity discount & no shortages are allowed.

Let D_i = demand

k_i - setupcost

h_i - holdingcost.

(20)

$\therefore$ the minimum total cost of the n items are

$$\min TCU(y_1, y_2, \dots y_n) = \sum_{i=1}^{n}\left(\frac{k_iD_i}{y_i} + \frac{h_iy_i}{2}\right)$$

Subject to constraint,

$$\sum_{i=1}^{n} a_iy_i \leq A$$

$$y_1, y_2, y_3 \dots y_n \geq 0$$

Wkt, $y_i^* = \sqrt{\frac{2k_iD_i}{h_i}} \rightarrow (1)$

The result is formulating the langrangian function is $L(\lambda, y_1, y_2 \dots y_n) = \underset{TCU}{\min TCU}(y_1, y_2 \dots y_n) - \lambda\left(\sum_{i=1}^{n} a_iy_i - A\right)$

$$= \sum_{i=1}^{n}\left(\frac{k_iD_i}{y_i} + \frac{h_iy_i}{2}\right) - \lambda\left(\sum_{i=1}^{n} a_iy_i - A\right) \rightarrow (2)$$

Diff (2) P.w.r.t "y_i" we get,

$$\frac{\partial L}{\partial y_i} = k_iD_i\left(\frac{-1}{y^2}\right) + \frac{h_i}{2}(1) - \lambda a_i(1)$$

$$= \frac{-k_iD_i}{y_i^2} + \frac{h_i}{2} - \lambda a_i$$

Since it is minimum

$$\frac{\partial L}{\partial y_i} = 0$$

$$\therefore \frac{-k_iD_i}{y_i^2} + \frac{h_i}{2} - \lambda a_i = 0$$

$$\Rightarrow \frac{k_i D_i}{y_i^2} = \frac{h_i}{2} - \lambda a_i$$

$$\Rightarrow \frac{k_i D_i}{y_i^2} = \frac{h_i - 2\lambda a_i}{2}$$

$$\Rightarrow \frac{2k_i D_i}{y_i^2} = h_i - 2\lambda a_i$$

$$\Rightarrow y_i^2 = \frac{2k_i D_i}{h_i - 2\lambda a_i}$$

$$\Rightarrow y_i^* = +\sqrt{\frac{2k_i D_i}{h_i - 2\lambda a_i}}$$

Here y_i^* is dependent on λ^*

if $\lambda^* = 0$, y_i^* gives the solution of the unconstraint case.

if $\lambda^* < 0$, for the minimization case by trying successive negative value of λ^*, the value of λ^* should result in simultaneous value of y_i^* & satisfies the given constraints in equality sense. $\therefore$ the value of λ^* gives the correct area of y^*

Consider the

Problem (22)

Consider the inventory model with three items $(n=3)$

The parameter of the problem are shown in the table below.

Item i	k_i (\$)	D_i (units)	h_i (\$)	a_i (ft^2)
1	10	2	0.3	1
2	5	3	0.1	1
3	15	4	0.2	1

assume that the total available storage area is $A = 25\ ft^2$

find y_1^*, y_2^*, y_3^*.

Soln

GT

item i	k_i (\$)	D_i (units)	h_i (\$)	a_i (ft^2)
1	10	2	0.3	1
2	5	3	0.1	1
3	15	4	0.2	1

GT $A = 25\ ft^2$

WKT $y_i^* = \sqrt{\dfrac{2k_iD_i}{h_i - 2\lambda a_i}}$

23 Now,

$A = 25$

$y_1 + 3y_2 + y_3 - 25$

λ	$y_1^* = \sqrt{\frac{2(10)(2)}{0.3 - 2\lambda(1)}}$	$y_2^* = \sqrt{\frac{2(5)(2)}{0.1 + 2\lambda(1)}}$	$y_3^* = \sqrt{\frac{2(15)(4)}{0.2 - 2\lambda(1)}}$	$\sum_{i=1}^{n} a_i y_i - A$
0	11.54	17.32	24.49	28.35
-0.05	10	12.25	20.	17.25
-0.10	8.94	10	17.32	11.25
-0.15	8.16	8.66.	15.49.	7.31
-0.20	7.56	7.75	14.14	4.45
-0.25	7.07.	7.07.	13.09	2.23
-0.30	6.67	6.55	12.25	0.47
-0.35	6.32	6.12	11.55	-1.01

$\therefore$ The storage constraints is satisfy in equality sense' for the value of λ between -0.30 & -0.35.

$\therefore \lambda^* = -0.35$

$\therefore$ The optimum solution are $y_1^* = 6.32$, $y_2^* = 6.12$ & $y_3^* = 11.55$.

Problem:

Consider the inventory problem with 3 iteams ($n=3$). The perametre of the problem on show in the Problem below.

(24)

Item	k_i (\$)	D_i (units)	h_i (\$)	a_i (ft^2)
1	10	2	0.3	1
2	5	3	0.1	1
3	15	4	0.2	1

assume that the total available storage area are (i) $A = 45 ft^2$, (ii) $A = 30 ft^2$ & (iii) $A = 20 ft^2$ find the value of y_1^*, y_2^* & y_3^* forall.

<u>soln</u>

GT

Item	k_i (\$)	D_i (units)	h_i (\$)	a_i (ft^2)
1	10	2	0.3	1
2	5	3	0.1	1
3	15	4	0.2	1

GT $A = 45 ft^2$

Wkt

$$y_i^* = \sqrt{\frac{2 k_i D_i}{h_i - 2\lambda a_i}}$$

Scanned with CamScanner

NOW,

$A = 25, \quad A = 30, \quad A = 20$

λ	$y_1^* = \sqrt{\frac{2(10)(2)}{0.3 - 2\lambda(1)}}$	$y_2^* = \sqrt{\frac{2(15)(3)}{0.1 - 2\lambda(1)}}$	$y_3^* = \sqrt{\frac{2(15)(1)}{0.2 - 2\lambda(1)}}$	$\sum a_i y_i - 20$	$\sum a_i y_i - 30$
0	11.54	17.32	24.49 53.35	33.35	23.35
−0.05	10	12.25	20 42.25	22.25	12.25
−0.10	8.94	10	17.32 36.26	16.26	6.26
−0.15	8.17	8.66	15.49 32.32	12.32	2.32
−0.20	7.56	7.75	14.14 29.45	9.45	−0.55
−0.25	7.07	7.07	13.09	7.23	
−0.30	6.67	6.55	12.25	5.47	
−0.35	6.32	6.12	11.55	3.99	
−0.40	6.03	5.77	10.95	2.75	
−0.45	5.77	5.48	10.44	1.69	
−0.50	5.55	5.22	10	0.77	
−0.55	5.35	5	9.61	−0.04	

$\sum a_i y_i -$ 8.85 −2.65

(i) The storage constraints is satisfy in equality sense for the value of λ between -0.50 & -0.55

$\therefore \lambda^* = -0.55$

$\therefore$ The optimum solution are $y_1^* = 5.35$, $y_2^* = 5$

$y_3^* = 9.61$

(ii) The storage constraints is satisfy in equality sense for the value of λ between -0.15 & -0.20

$\therefore \lambda^* = -0.20$

$\therefore$ The optimum solution are $y_1^* = 7.56$, $y_2^* = 7.75$

$y_3^* = 19.14$

(iii) The storage constraints is satisfy in equality sense for the value of λ between $\cancel{-0.25}$ -0 & -0.05

$\therefore \lambda^* = -0.05$

$\therefore$ The optimum solution are $\cancel{y_1^* = 8.94, y_2^* = 10.8}$

$\cancel{y_3^* = 17.32}$. $y_1^* = 10$, $y_2^* = 10.25$, $y_3^* = 20.3$

Dynamic EOQ model:-

Dynamic deterministic demand represents material requirement. This model presents 2 ideas. They are,

(i) The inventory level is reviewed periodically over a finite number of equal periods

The demand per period through deterministic is dynamic if it ~~various~~ varied from one period to the next.

No setup model:-

This model involves a planning Horizon with n-equal periods.

Each period has a limited production capacity that can be included ~~so~~ several.

A current period may produce more than its immediate demand to satisfied the demand for later period. An inventory holding cost must be charged

The general assumption of the model are

(i) No setupcost is incurred in any period

(ii) No shortages is allowed.

(iii) The unit production cost in any period is either a constrant (or) increase the marginal cost

(iv) The unit holding cost in any period is a constant.

Problem:-

(2) metalco ~~co~~ produces draft deflection for use in home fire places during the month of december

to march. The demand starts slow, peaks, in the middle of the ~~season~~ season & tapper's of towards the end because of the popularity of the product. Metalco may use overtime to satisfies the demand. The following table provides the production capacity and demand for the four winter months.

month (units)	capacity		Demand (units)
	Regular (units)	overtime (units)	
1	90	50	100
2	100	60	190
3	120	80	210
4	110	70	160

unit production cost in any period is \$ 6 during regular time and \$ 9 during overtime. Holding cost per unit per month is \$ 0.1. Find the associated total cost.

GT The production cost (Regular) = \$ 6 /unit
The production cost (overtime) = \$ 9 /unit
Holding cost = \$ 0.1 /unit/month

Soln GT

month (units)	capacity Regular (units)	overtime (units)	demand (units)
1	90	50	100
2	100	60	190
3	120	80	210
4	110	70	160

The cumulative supply and demand for the following

	cumulative	
month	supply	demand.
1	$90+50=140$	100
2	$140+100+60=300$	$100+190=290$
3	$300+120+80=500$	$290+210=500$
4	$500+110+70=680$	$500+160=660$

Here cumulative supply $\neq$ cumulative demand,

i.e, $680 \neq 660$

Here we introduce a dummy destination whose demand is 20.

$\therefore$ This problem is a balanced one.

Let R_i & O_i be the regular time & the overtime for the month i, $i=1,2,3,4$.

	1	2	3	4	Surplus	
R_1	6 (90)	6.1	6.2	6.3	0	90
O_1	9 (10)	9.1 (30)	9.2 (10)	9.3	0	50→40
R_2		6 (100)	6.1	6.2	0	100
O_2		9 (60)	9.1	9.2	0	60
R_3			6 (120)	6.1	0	120
O_3			9 (80)	9.1	0	80
R_4				6 (110)	0	110
O_4				9 (50)	0 (20)	50→2
	100 → 10	190 → 90 → 30	210 → 90 → 10	160 → 50	20	

∴ The optimum solution is in the followi[ng]

table.

Period	Production schedule.
R_1	Produce 90 units for period 1
O_1	Produce 50 units, 10 units for per[iod] ~~20 units~~ 30 units for perio[d] 10 units for perio[d]
R_2	Produce 100 units for period

O_2	Produce 60 units for period 2
R_3	Produce 120 units for period 3
O_3	Produce 80 units for period 3
R_4	Produce 110 units for period 4
D_4	Produce 50 units for period 4 with 20 units idle capacity

The total associated cost $= \$(90\times6 + 10\times9 + 30\times9.1 + 10\times9.2 + 100\times6 + 60\times9 + 120\times6 + 80\times9 + 110\times6 + 50\times9 + 20\times0)$

$= \$4685$

Problem

metako co. produces draft deflection for use in home fire place during the month of december to march. The demand starts slow, peaks in the middle of the season and tappers of towards the end because of the popularity of the product. metako co. may use overtime to satisfies the demand. The following table provides the production capacity and demand for the four winter months.

months (units)	capacity		demand units
	Regular time (units)	overtime (units)	
1	90	50	100
2	100	60	190
3	120	80	210
4	110	70	160

assume that the unit production cost & the holding cost is given in the following.

months.	Production cost		Holding cost
	Regular time (units)	overtime (units)	
1	5.00	7.50	0.10
2	3.00	4.50	0.15
3	4.00	6.00	0.12
4	1.00	1.50	0.20.

Find the total associated cost.

Soln

GT

month	capacity		demand
	Regular time	overtime	
1	90	50	100
2	100	60	190
3	120	80	210
4	110	70	160

GT

month	Production cost Regular time	overtime	Holding cost
1	5.00	7.50	0.10
2	3.00	4.50	0.15
3	4.00	6.00	0.12
4	1.00	1.50	0.20

The cumulative ~~sy~~ supply and demand for the following

month	cumulative supply	demand
1	90+50=140	100
2	140+100+60=300	100+190=290
3	300+120+80=500	290+210=500
4	500+10+70=680	500+160=660

Here cumulative supply ≠ cumulative demand

i.e, 680 ≠ 660

Here we introduce a dummy destination whose demand is 20.

∴ The problem is a balanced one

(34)

Let R_i & O_i be the regular time & the over time for the month i, $i = 1, 2, 3, 4$.

	1	2	3	4	surplus	
R_1	5 . 90	5.10	5.20	5.30	0	90
O_1	7.50 10	7.60 30	7.70 10	7.80	0	50 → 40 → 10
R_2		3 100	3.15	3.30	0	100
O_2		4.50 60	4.65	4.80	0	60
R_3			4.12 120	4.[illegible]	0	120
O_3			6 80	6.12	0	80
R_4				1 110	0	110
O_4				1.50 50	0 20	50 → 20
	100 → 10	190 → 90 → 30	210 → 90 → 10	160 → 50	20	

∴ The optimum solution is in the following table.

Period	Production schedule
R_1	Produce 90 units for period 1
O_1	Produce 50 units, 10 units for period 2 & 10 units for period 3
R_2	Produce 100 units for period

O_2 Produce 60 units for period 2

R_3 Produce 120 units for period 3

O_3 Produce 80 units for period 3

R_4 Produce 110 units for period 4

R_4 Produce 50 units for period 4 with 20 units idle capacity.

$\therefore$ The total associated cost $= \$ (90\times5 + 7.50\times10 + 7.60\times2$
$+ 3\times100 + 4.50\times60 + 7.70\times10$
$+ 4\times120 + 6\times80 + 1\times110$
$+ 1.50\times50 + 0\times20)$

$= \$ 2545$

Setup model.

In this model no shortages is allowed and a setup cost is ~~included~~ incurred each time a new Production cost is started. This model can be divided into 2 methods. They are

(i) Exact dynamic programming algorithm

(ii) Heuristic method.

Exact dynamic programming algorithm.

In this model no shortages are allowed

Z_i - amount ordered, D_i - demand for perio

x_i - inventory at the start of period i. (34)

k_i - setup cost in period i.

H_i - ~~Unit holding~~ inventory holding cost from period i to $i+1$

The associated production cost function for period i is $c_i(z_i) = \begin{cases} 0 & , z_i = 0 \\ k_i + c_i(z_i) & , z_i > 0 \end{cases}$

The Holding cost per period i is.

$$x_{i+1} = x_i + z_i - D_i$$

The state at period is defined as x_{i+1}, the end of period inventory level $0 \leq x_{i+1} \leq D_{i+1} + D_{i+2} + \cdots D_n$

Let $f_i(x_{i+1})$ be the minimum inventory cost for period $1, 2, \ldots i$. The end of the period inventory is x_{i+1}

therefore, $f_1(x_2) = \min\limits_{z_1 = D_1 + x_2 - x_1} (c_1(z_1) + h_1 x_2)$

$$f_i(x_{i+1}) = \min_{0 \leq z_i \leq D_i + x_{i+1}} \left(c_i(z_i) + h_i x_{i+1} + f_{i-1}(x_{i+1} + D_i - z_i) \right)$$

Problem :-

The following table provides the data for 3 Period inventory situation

Period i	D_i (units)	k_i (\$)	h_i (\$)
1	3	3	1
2	2	7	3
3	4	6	2

The demand occurs ~~the~~ in discrete units and the starting inventory is $x_1 = 1$ unit. The unit production cost is \$10 for 1st 3 units & \$20 for each additional units determine the optimal inventory policy.

Soln

GT

Period i	D_i (units)	k_i (\$)	h_i (\$)
1	3	3	1
2	2	7	3
3	4	6	2

The mathematical form is,

$$C_i(z_i) = \begin{cases} 10z_i, & 1 \leq z_i \leq 3 \\ 30 + 20(z_i - 3), & z_i \geq 4 \end{cases}$$

Stage : 1

G.T $x_1 = 1$ units.

$D_1 = 3$

Wkt,

$$0 \le x_{i+1} \le D_{i+1} + D_{i+2} + \dots + D_n$$

Here

$$0 \le x_2 \le D_2 + D_3$$

$$\Rightarrow 0 \le x_2 \le 2 + 4 = 6$$

(ie) $0 \le x_2 \le 6$.

Wkt,

$$f_1(x_2) = \min_{z_1 = D_1 + x_2 - x_1} \left(C_1(z_1) + h_1 x_2 \right)$$

Wkt, $x_{i+1} = x_i + z_i - D_i$

$$\Rightarrow z_i = x_{i+1} - x_i + D_i$$

Here,

$$z_1 = x_2 - x_1 + D_1$$

$$= x_2 - 1 + 3$$

$$z_1 = 2 + x_2$$

Now,

x_2	$h_1 x_2$	z_1: 2	3	4	5	6	7	8	optimum cost $f_1(x_2)$	z_1^*
		$c_1(z_1) = x_1 + c_1(z_1)$ 23	33	53	73	93	113	133		
		$C_1(z_1) + h_1 x_2$								
0	0	23+0=23							23	2
1	1		33+1=34						34	3
2	2			53+2=55					55	4
3	3				73+3=76				76	5
4	4					93+4=97			97	6
5	5						113+5=118		118	7
6	6							133+6 =139	139	8

Stage-2, Here $D_2 = 2$, $0 \le x_3 \le D_3 \Rightarrow 0 \le x_3 \le 4$.

Here $0 \le z_i \le D_i + x_{i+1} \Rightarrow \boxed{0 \le z_2 \le D_2 + x_3}$

$\boxed{h_2 = 3}$ $\Rightarrow 0 \le z_2 \le 2 + x_3$.

~~$c_3(z_3) = 16 + c_2(z_2)$~~ $c_i(z_i) = 10 z_i$, $z_i \le 2$; $c_i(z_i) = 30 + 20(z_i - 2)$, $z_i \ge 4$

$c_2(z_2) + h_2 x_3 + f_1(x_3 + D_2 - z_2)$

x_3	$h_2 x_3$	z_2: 0	1	2	3	4	5	6	optimum soln min $f_2(x_3)$	z_2^*
		$c_2 z_2$: 0	17	27	37	57	77	97		
0	0	0+0+55=55	17+0+34=51	27+0+23 =50	–	–	–	–	50	2
1	3	0+3+76=79	17+3+55=75	27+3+34 =64	37+3+23=63	–	–	–	63	3
2	6	0+6+97=103	17+6+76=99	27+6+55 =88	37+6+34=77	57+6+23 =86	–	–	77	3
3	9	0+9+118=127	17+9+97=123	27+9+76 =112	37+9+55=101	57+9+34 =100	77+9+23 =109	–	100	4
4	12	0+12+139=151	17+12+118=147	27+12+97 =136	37+12+76 =125	57+12+55 =124	77+12+34 =123	97+12+23 =132	123	5

Stage: 3 Here $D_3 = 4$, $x_4 = 0$ & Here $0 \le z_i \le D_i + x_{i+1} \Rightarrow 0 \le z_3 \le D_3 + x_4$

$\Rightarrow 0 \le z_3 \le 4 + 0 = 4$

$c_3(z_3) + h_3 x_4 + f_2(x_4 + D_3 - z_3)$

x_4	$h_3 x_4$	z_3: 0	1	2	3	4	optimum solution min $f_3(x_4)$	z_3^*
		$c_3(z_3)$: 0	16	26	36	56		
0	0	0+0+123= 123	16+0+100 =116	26+0+77 =103	36+0+63 =99	56+0+50 =106	99	3

$\therefore x_4 = 0 \rightarrow z_3 = 3.$

WKT $x_{i+1} = x_i - D_i + z_i$

$x_i = x_{i+1} - z_i + D_i$

Put $i = 3$,

$x_3 = x_4 - z_3 + D_3.$

$x_3 = 0 - 3 + 4 = 1$

$\therefore x_3 = 1 \rightarrow z_2 = 3$

Put $i = 2$,

$x_2 = x_3 - z_2 + D_2$

$x_2 = 1 - 3 + 2 = 0$

$x_2 = 0 \rightarrow z_1 = 2$

$\therefore$ The optimum solution are

$z_1^* = 2,\ z_2^* = 3,\ z_3^* = 3$ &

the optimum value \$99

2) Find the optimum solution for the following four. Period inventory model.

Period i	Demand D_i (units)	setup Cost k_i (\$)	Holding cost h_i (\$)
1	5	5	1
2	2	7	1
3	3	9	1
4	3	7	1

(4.1) The unit production cost is \$ 1 each for 1st 8 units and \$2 each for additional units. determine the optimal inventory policy.

Soln

GT

Period i	Demand (units)	k_i (\$)	h_i (\$).
1	5	5	1
2	2	7	1
3	3	9	1
4	3	7	1

The mathematical form is.

$$c_i(z_i) = \begin{cases} 1 \cdot z_i & 0 \leq z_i \leq 6 \\ 6 + 2(z_i - 6) & z_i \geq 7 \end{cases}$$

Stage: 1

Here $x_1 = 0$, $D_1 = 5$

wkt, $0 \leq x_{i+1} \leq D_{i+1} + D_{i+2} + \dots + D_n$

Here $0 \leq x_2 \leq D_2 + D_3 + D_4$.

$\Rightarrow 0 \leq x_2 \leq 2 + 3 + 3 = 8$

$\Rightarrow 0 \leq x_2 \leq 8$

wkt $x_{i+1} = x_i + z_i - D_i$

$z_i = x_{i+1} - x_i + D_i$

$z_1 = x_2 - x_1 + D_1$

$z_1 = x_2 - 0 + 5$

$z_1 = x_2 + 5 \Rightarrow z_1 = x_2 + 5$

$\boxed{h_1 = 1}$

x_2	$h_1 x_2$	z_1	5	6	7	8	9	10	11	12	13	Optimum soln.	
x_2	$h_1 x_2$	$[C_1(z_1) = h_1 + g(z_1)]$	10	11	13	15	17	19	21	23	25	$\min f_1(x_2)$	z_1^*
		$C_1(z_1) + h_1 x_2$											
0	0		10+0=10									10	5
1	1			11+1=12								12	6
2	2				13+2=15							15	7
3	3					15+3=18						18	8
4	4						17+4 =21					24	9
5	5							19+5 =24				24	10
6	6								21+6 =27			27	11
7	7									23+7 =30		30	12
8	8										25+8 =33	33	13

(43)

Stage: 2

$0 \le x_{i+1} \le D_{i+1} + D_{i+2} + \dots + D_n$

Here. $D_2 = 2$, wkt $0 \le x_3 \le D_3 + D_4 \Rightarrow 0 \le x_3 \le 3+3 = 6 \Rightarrow 0 \le x_3 \le 6$

Here, $0 \le z_i \le D_i + x_{i+1} \Rightarrow 0 \le z_2 \le D_2 + x_3 \Rightarrow 0 \le z_2 \le 2 + x_3$

x_3	z_2 :	0	1	2	3	4	5	6	7	8	optimum solut	
	$[C_2(z_2) = k_2 + c_2(z_2)]$	0	8	9	10	11	12	13	15	17.	$\min f_2(x_3)$	z_2^*
	$h_2 x_3$	$c_2(z_2) + h_2 x_3 + f_1(x_3 + D_2 - z_2)$.										
0	0	0+0+15 =15	8+0+12 =20	9+0+10 =19	—	—	—	—	—	—	15	0.
1	1	0+1+18 =19.	8+1+15 =24	9+1+12 =22	10+1+10 =21	—	—	—	—	—	19	0
2	2	0+2+21 =23	8+2+18 =28	9+2+15 =26	10+2+12 =24	11+2+10 =23	—	—	—	—	23	0
3	3	0+3+24 =27	8+3+21 =32	9+3+18 =30	10+3+15 =28	11+3+12 =26	12+3+10 =25	—	—	—	25	5
4	4	0+4+27 =31	8+4+24 =36	9+4+21 =34	10+4+18 =32	11+4+15 =30	12+4+12 =28	13+4+10 =27	—	—	27	6
5	5	0+5+30 =35	8+5+27 =40	9+5+24 =38	10+5+21 =36	11+5+18 =34	12+5+15 =32	13+5+12 =30	15+5+10 =30	—	30	6
6	6	0+6+33 =39	8+6+30 =44	9+6+27 =42	10+6+24 =40	11+6+21 =38	12+6+18 =36	13+6+15 =34	15+6+12 =33	17+6+10 =33	33	7

Stage : 3

Here, $D_3 = 3$, where $0 \le x_4 \le D_4 \Rightarrow 0 \le x_4 \le 3$

Here $0 \le z_i \le D_i + x_{i+1} \Rightarrow 0 \le z_3 \le D_3 + x_4 \Rightarrow 0 \le z_3 \le 3 + x_4$

$\boxed{h_3 = 1}$

x_4	$h_3 x_4$	z_3: 0	1	2	3	4	5	6	optimum soln.	
	$C_3(z_3) = k_3 + c_3(z_3)$	0	10	11	12	13	14	15	$\min f_3(x_4)$	z_3^*
		$C_3(z_3) + h_3 x_4 + f_2(x_4 + D_3 - z_3)$								
0	0	0+0+25 =25	10+0+23 =33	11+0+19 =30	12+0+15 =27	13+0+[illegible]	–	–	25	0
1	1	0+1+27 =28	10+1+25 =36	11+1+23 =35	12+1+19 =32	13+1+15 =29	–	–	28	0
2	2	0+2+30 =32	10+2+27 =39	11+2+25 =38	12+2+23 =37	13+2+19 =34	14+2+15 =31	–	31	5
3	3	0+3+33 =36	10+3+30 =43	11+3+27 =41	12+3+25 =40	13+3+23 =39	14+3+19 =36	15+3+15 =33	33	6

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<u>Stage 4:</u>

Here $D_4 = 3$ WKT $0 \le x_5 \le D_5$ $\boxed{k_4 = 7}$

$\therefore x_5 = 0$

Here $0 \le z_i \le D_i + x_{i+1} \Rightarrow 0 \le z_4 \le D_4 + x_5$

$\Rightarrow 0 \le z_4 \le 3 + 0 = 3 \Rightarrow 0 \le z_4 \le 3$

	z_4:	0	1	2	3		
	$[C_4(z_4) = k_4 + c_4(z_4)]$	0	8	9	10	Optimum solution	
x_5	$h_4 x_5$	$c_4 z_4 + h_4 x_5 + f_3(x_5 + D_4 - z_4)$				$\min f_4(x_5)$	z_4^*
0	0	$0+0+33=33$	$8+0+31 = 39$	$9+0+28 = 37$	$10+0+25 = 35$	33	0

$x_5 = 0 \rightarrow z_4 = 0$

wkt

$$x_{i+1} = x_i - D_i + z_i$$

$$x_i = x_{i+1} + D_i - z_i$$

Put $i=4$,

$$x_4 = x_5 + D_4 - z_4$$

$$x_4 = 0 + 3 + 0 \Rightarrow x_4 = 3,$$

$$x_4 = 3 \rightarrow z_3 = 6$$

Put $i=3$

$$x_3 = x_4 + D_3 - z_3$$

$$x_3 = 3 + 3 - 6 = 0$$

$$x_3 = 0 \rightarrow z_2 = 0$$

Put $i=2$,

$$x_2 = x_3 + D_2 - z_2$$

$$x_2 = 0 + 2 - 0 \Rightarrow x_2 = 2$$

$$x_2 = 2 \rightarrow z_1 = 7$$

∴ The optimum solution are

$$z^*_1 = 7, \quad z^*_2 = 0, \quad z^*_3 = 6, \quad z^*_4 = 0$$

The optimum value \$ 38 //

Probabilistic model in inventory
(or)
Dynamic programming algorithm with constant (or) decreasing marginal costs.

The dynamic programming model holds to promise inrucucing the volume of computation. the unit production cost and the holding cost are non-increasing function of the production ~~cost~~ quantity ~~and~~ or the inventory level and the quantity discount is allowed. ~~The problem can be solved by the~~

The problem can be solved by the following conditions.

(i) when $x_1 = 0$, it is optimal to satisfies the demand in any period i, either from new production (or) entering inventory,

i.e., $z_i x_i = 0$

if $x_i > 0$, the amount can be ~~sonitted~~ writtenoff ~~of~~ from the demand of the successive period.

(ii) The optimal production quantity z_i for period i must either be 0 to satisfy the exact demand for one or more continuous succeeding periods

Problem:-

(U) (N/D) (A) [illegible]

A four period inventory model operates the following data:

Period i	demand (D_i) (units)	setup cost (K_i) \$
1	76	98
2	26	114
3	90	185
4	67	70

The initial inventory $x_1 = 15$. the unit production cost \$2 & the unit holding cost \$1 for all periods find the optimal policy.

Soln

GT

Period i	demand (D_i) (units)	setup cost (K_i) (\$)
1	76	98
2	26	114
3	90	185
4	67	70

GT $x_1 = 15$ units.

Production cost = \$2

Holding cost = \$1.

Here $D_1 = 76 - 15 = 61$ units.

Stage: 1 ($h=1$)

x_2	$h_1 x_2$	z_1: 61	61+26 = 87	87+90 = 177	177+67 = 244
		$c_1(z_1)$: 2×61+98 = 220	2×87+98 = 272	2×177+98 = 452	2×244+98 = 586
		$c_1(z_1)+h_1(x_2)$			
0	0	220+0=220			
26	26		272+26 = 298		
116	116			452+116 = 568	
183	183				586+183 = 769

Optimum solution.

min $f_1(x_2)$	z_1^*
220	61.
298	87
568	177
769	244

~~Stage: 2~~

~~Here $D_2 = 26$.~~

~~| | | z_2 | 0 | 26 | 116 | 182 | optimum soln |~~
~~| x_2 | h_2x_2 | $c_2(z_2)$ | $2\times 26+114 = 166$ | $2\times116+114 = 346$ | $2\times183+114 = 480$ | | |~~

Stage: 2

Here $D_2 = 26$.

		z_2	0	26	116	183	optimum soln	
x_2	h_2x_2	$c_2(z_2)$	0	$2\times26+114 = 166$	$2\times116+114 = 346$	$2\times183+114 = 480$	$\min f_2(x_2)$	z_2^*
		$c_2(z_2) + h_2x_2 + f_1(x_2+D_2-z_2)$						
0	0		$0+0+298 = 298$	$166+0+220 = 386$	–	–	298	0
90	90		$0+90+568 = 658$	–	$346+90+220 = 656$	–	656	116
157	157		$0+157+769 = 926$	–	–	$480+157+220 = 857$	857	183

Stage: 3

Here $D_3 = 90$.

		z_3	0	90	157	optimum soln	
x_4	h_3x_4	$c_3(z_3)$	0	$2\times90+185 = 365$	$2\times157+185 = 499$	$\min f_3(x_4)$	z_3^*
		$c_3(z_3) + h_3x_4 + f_2(x_4+D_3-z_3)$					
0	0		$0+0+656 = 656$	$365+0+298 = 663$	–	656	0
67	67		$0+67+857 = 924$	–	$499+67+298 = 864$	864	157

Stage: 4

Here $D_4 = 67$.

		z_4	0	67	optimum soln	
x_5	$h_4(x_5)$	$c_4(z_4)$	0	$2\times67+70 = 204$	$\min f_4(x_5)$	z_4^*
		$c_4(z_4) + h_4x_5 + f_3(x_5+D_4-z_4)$				
0	0		$0+0+864 = 864$	$204+0+\cancel{656} = 860$	860	67

$x_5 = 0 \le x_5 \le x_4$

$x_5 = 0 \rightarrow z_4 = 67$

Wrt

$x_{i+1} = x_i - D_i + z_i$

$i = 4,$

$x_5 = x_4 - D_4 + z_4$

$x_4 = x_5 + D_4 - z_4$

$x_4 = 0 + 67 - 67$

$x_4 = 0 \rightarrow z_3 = 0$

$i = 3,$

$x_3 = x_4 + D_3 - z_3$

$x_3 = 0 + 90 - 0$

$x_3 = 90 \rightarrow z_2 = 116,$

$i = 2$

$x_2 = x_3 + D_2 - z_2$

$x_2 = 90 + 26 - 116.$

$x_2 = 0 \rightarrow z_1 = \cancel{220}\ 61$

$i = 1,$

$x_1 = x_2 + D_1 - z_1$

$x_1 = 0 + 61 - 61$

$x_1 = 0$

The optimum solution are $z_1^* = 61, z_2^* = 116, z_3^* = 0, z_4 = 67$

The optimum value is 868

1) Find the optimum inventory policy for the following 5-Period model.

Period	D_i	k_i (\$)
1	50	80
2	70	70
3	100	60
4	30	80
5	60	60

The unit production cost is \$10 for all period the unit holding cost is \$1 / period.

Soln

GT

Period i	demand (D_i) (units)	Setup cost k_i (\$)
1	50	80
2	70	70
3	100	60
4	30	80
5	60	60

GT, Production cost = \$10

Holding cost = \$1

Here $x_1 = 0$, $D_1 = 50$

~~$D_1 = 50$~~

Stage 1

x_2	$h_1 x_2$	z_1 = 50	120	220	250	310	optimum soln	
		$c(z_1)$ 580	1280	2230	2580	3180	$f_1(x_2)$	z_1^*
		$c_1(z_1) + h_1 x_2$						
0	0	0+580=580					580	50
70	70		70+1280 =1350				1350	120
170	170			2230+170 =2400			2400	220
200	200				2580+200 =2780		2780	250
260	260					260+3180 =3440	3440	310

Stage:2 $D_2 = 70$

x_3	$h_2 x_3$	z_2	0	70	170	200	260	Optimum Soln.	
		$c_2(z_2)$	0	770	1770	2070	2670		
			$c_2(z_2)+h_2x_3+f_1(x_3+D_2-z_2)$					$\min f_2(x_3)$	z_2^*
0	0		0+0+1350 =1350	770+0+580 =1350	–	–	–	1350	70
100	100		0+100+2450 =2550	–	1770+100+580 =2450	–	–	2450	170
130	130		0+130+2780 =2910	–	–	2070+130+580 =2840	–	2840	200
~~190~~	~~190~~		~~0+190+3440~~ =3630	–	–	–	2670+190+580 =3440	~~3440~~	~~260~~

Stage:3 $D_3 = 100$

x_4	$h_3 x_4$	z_3	0	100	130	190	Optimum Soln	
		$c_3(z_3)$	0	1060	1360	1960		
			$c_3(z_3)+h_3x_4+f_2(x_4+D_3-z_3)$				$\min f_3(x_4)$	z_3^*
0	0		0+0+2450 =2450	1060+0+1350 =2410	–	–	2410	100
30	30		0+30+2840 =2870	–	1360+30+1350 =2740	–	2740	130
90	90		0+90+3440 =3530	–	–	1960+90+1350 =3400	3400	190

Stage:4 $D_4 = 30$

x_5	$h_4 x_5$	z_4	0	30	90	Optimum Soln	
		$c_4(z_4)$	0	380	980		
			$c_4(z_4)+h_4x_5+f_3(x_5+D_4-z_4)$			$\min f_4(x_5)$	z_4^*
0	0		0+0+2740 =2740	380+0+2410 =2790	–	2740	0
60	60		0+60+3400 =3460	–	980+60+2410 =3480	3460	90

Stage:5 $D_4 = 60$

x_6	$h_5 x_6$	z_5	0	60	Optimum Soln	
		$c_5(z_5)$	0	660		
			$c_5(z_5)+h_5x_6+f_4(x_6+D_5-z_5)$		$\min f_5(x_6)$	z_5^*
0	0		0+0+3460 =3460	660+0+2740 =3400	3400	60

$x_6 = 0 \rightarrow z_5 = 60$

Note

$$x_i = x_{i+1} + D_i - z_i$$

$i=5$, $$x_5 = x_6 + D_5 - z_5$$

$$x_5 = 0 + 60 - 60 = 0 \Rightarrow x_5 = 0 \rightarrow z_4 = 0$$

$$x_4 = x_5 + D_4 - z_4$$
$$= 0 + 30 - 0 = 30 \Rightarrow x_4 = 30 \rightarrow z_3 = 130$$

$$x_3 = x_4 + D_3 - z_3$$
$$= 30 + 100 - 130 = 0 \Rightarrow x_3 = 0 \rightarrow z_2 = 70$$

$$x_2 = x_3 + D_2 - z_2$$
$$= 0 + 70 - 70 = 0 \Rightarrow x_2 = 0 \rightarrow z_1 = 50$$

$\therefore$ The optimum solutions are,

$z_1^* = 50$, $z_2^* = 70$, $z_3^* = 130$. $z_4^* = 0$

$z_5^* = 60$

The optimum value is \$3400

$75 = 60$., optimum value \$ 860.

Sliver Heuristic method:- (55)

In an inventory situation the unit Production cost is constant and identical for all periods is called Heuristic. The Heuristic identifies the successive feature product whose demand can be fixed from the demand of the current period.

Procedure :-

Let $TC(i,t)$ be the associated setup cost and holding cost for the same period $i, i+1 \ldots t$

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ii) $TC(i,t) = \begin{cases} k_i & i = t \\ k_i + h_i D_{i+1} + (h_i + h_{i+1}) D_{i+2} + \dots + \left(\sum_{r=i}^{t-1} h_r\right) D_t & i < t \end{cases}$

Let us assume that the associated cost per Period $TCU(i,t) = \frac{TC(i,t)}{t-i+1}$. Then the given current period i, the Heuristic determines t^* that minimizes $TCU(i,t)$.

Problem

2M 2007

Find the optimum inventory policy for the following 6 period inventory situation.

Period i	Demand D_i (units)	k_i (\$)	h_i (\$)
1	10	20	1
2	15	17	1
3	7	10	1
4	20	18	3
5	13	5	1
6	25	50	1

Soln

The unit production cost is \$2 for all periods.

Soln

Given

Period i	D_i (units)	K_i (\$)	h_i (\$).
1	10	20	1.
2	15	17	1
3	7	10	1.
4	20	18	3
5	13	5	1
6	25	50	1.

Given Production cost = \$2.

WKT
$$TC(i,t) = \begin{cases} k_i & i=t \\ k_i + h_i D_{i+1} + (h_i + h_{i+1}) D_{i+2} + \dots + \left(\sum_{i=1}^{t} h_i\right) D_i & i<t \end{cases}$$

$$TCU(i,t) = \frac{TC(i,t)}{t-i+1}$$

Stage: 1

Here $i=1$,

Now, $TC(1,1) = k_1 = 20$

$TC(1,2) = k_1 + h_1 D_2$

$= 20 + 1 \cdot 15$

$TC(1,2) = 35$

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$t=1$

Period i	D_i	$TC(1,t)$	$TCU(1,t) = \frac{TC(1,t)}{t-1+1}$
1	10	20	$\frac{20}{1} = 20$
2	15	$20+1.15=35$	$\frac{35}{2} = 17.5$
3	7	$20+1.15+(1+1)7 = 49$,	$\boxed{\frac{49}{3} = 16.33}$
4	20	$49+(1+1+1)20 = 109$	$\frac{109}{4} = 27.25$

$\therefore t^* = 3$, The order $= 10+15+7 = 32$ units in period 3, ~~the period 1 to 3~~.

Stage: 2

Here $i = t^*+1 = 3+1 = 4$, $K_4 = 18$

Period i	D_i	$TC(4,t)$	$TCU(4,t) = \frac{TC(4,t)}{t-4+1}$
4	20	18	$\boxed{\frac{18}{1} = 18}$
5	13	$18+3\cdot13 = 57$	$\frac{57}{2} = 28.5$

$\therefore t^* = 4$, The order $= 20$ units in period 4

Stage: 3

Here $i = t^*+1 = 4+1 = 5$, $K_5 = 5$

(59)

period i	D_i	$TC(5,t)$	$TCU(5,t) = \frac{TC(5,t)}{t-5+1}$
5	13	5	$\boxed{\frac{5}{1} = 5}$
6	25	$5+1.25 = 30$	$\frac{30}{2} = 15$

$\therefore$ $t^* = 5$, The order = 13 units in period 5

from the last

$t^* = 6$, The order = 25 units in period 6.

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Unit : 5

Non Linear programming Algorithm

Unconstrained non linear algorithm

The unconstrained non linear algorithm can be divided into two models. They are

1) Direct search algorithm
2) Gradient algorithm.

Direct search method:

The interval or interval of uncertainty is known to include the optimum is identified.

In this method is to present the method is called ~~dichotomous~~ dichotomous search method.

Suppose $f(x)$ is maximum. Define two points x_1 and x_2 w.r.t a and b such that the interval $a \le x \le x_2$ and $x_1 \le x \le b$ overlap a finite amount Δ.

The following 3 cases to find the value of $f(x_1)$ and $f(x_2)$

(i) If $f(x_1) > f(x_2)$, $\quad x_L = a \le x = x^* \le x_2$

(ii) If $f(x_1) < f(x_2)$, $\quad x_1 < x = x^* \le b = x_R$

(iii) If $f(x_1) = f(x_2)$, $\quad x_1 < x = x^* < x_2$.

Here
$$x_1 = x_L + \frac{(x_R - x_L - \Delta)}{2} = \frac{x_L + x_R - \Delta}{2}$$

$$x_2 = x_L + \frac{(x_R - x_L + \Delta)}{2} = \frac{x_L + x_R + \Delta}{2}$$

~~and~~ $\Delta = x_2 - x_1$ and $x_R + x_L = x_1 + x_2$

Δ is chosen very small. Continuing in this manner the limit can narrow the interval, then we find the average of x_L and x_R we get the required result.

problem:

Using Dichotomous search method to find the value of x in the following

$$\text{Max } f(x) = \begin{cases} 3x, & 0 \le x \le 2 \\ -\frac{x}{3} + \frac{20}{3}, & 2 \le x \le 3. \end{cases}$$

Soln:

$$\text{G.T max } f(x) = \begin{cases} 3x, & 0 \le x \le 2 \\ -\frac{x}{3} + \frac{20}{3}, & 2 \le x \le 3 \end{cases}$$

Let x_L and x_R be the left and right boundaries of the current interval. i.e $x_L = 0$ and $x_R = 3$.

Let x_1 and x_2 be two points.

Let $\Delta = 0.001$

WKT $x_1 = x_L + \frac{(x_R - x_L - \Delta)}{2} = \frac{x_L + x_R - \Delta}{2}$

$x_2 = x_L + \frac{(x_R - x_L + \Delta)}{2} = \frac{x_L + x_R + \Delta}{2}$

WKT (i) $f(x_1) > f(x_2)$, $x_L = a \leq x = x^* \leq x_2$ (ii) $f(x_1) < f(x_2)$, $x_1 \leq x = x^* \leq b = x_R$

Table for Dichotomous search method

x_L	x_R	$x_1 = \frac{x_L + x_R - 0.001}{2}$	$x_2 = \frac{x_L + x_R + 0.001}{2}$	$f(x_1)$	$f(x_2)$
0	3	1.49950	1.50050	4.49850	4.50150
1.49950	3	2.24925	2.25025	5.91692	5.91658
1.49950	2.25025	1.87438	1.87538	5.62314	5.62613
1.87438	2.25025	2.06182	2.06282	5.97940	5.97906
1.87438	2.06282	1.96810	1.96910	5.90430	5.90730
1.96810	2.06282	2.01496	2.01596	5.99501	5.99468
1.96810	2.01596	1.99153	1.99253	5.97459	5.97759
1.99153	2.01596	2.00325	2.00425	5.99892	5.99859
1.99153	2.00424	1.99738	1.99838	5.99214	5.99514
1.99738	2.00424	2.00031	2.00131	5.99989	5.99956

From this table $x_L = 1.99738$ and $x_R = 2.00424$

WKT $x_L \leq x^* \leq x_R$

i.e. $1.99738 \leq x^* \leq 2.00424$

WKT $x^* = \frac{x_R + x_L}{2}$

$= \frac{2.00424 + 1.99738}{2}$

$= \frac{4.00162}{2}$

$x^* = 2$ (approximately)

∴ The optimum solution is $x^* = 2$.

Home work:

Find the maximum of the following function by Dichotomous search method, when $\Delta = 0.05$.

$$f(x) = \begin{cases} 4x & 0 \leq x \leq 2 \\ 4-x & 2 \leq x \leq 4. \end{cases}$$

Gradient method:

This is the method for optimizing function that are twice continuous differentiable.

The idea is to generate successive point of the functions. This method is called gradient method or Steepest ascent method.

Suppose, $f(x)$ is a maximum function. Let x_0 be an initial point. Let $\nabla f(x_k)$ be the gradient of f at x_k.

Here $\frac{\partial f}{\partial P}$ is maximized as a given point at path p.

Let $x_{k+1} = x_k + r_k \nabla f(x_k)$, where r_k is a optimal stepsize at x_k.

Let $h(r)$ be a single variable function defined by $h(r) = f(x_{k+1}) = f(x_k + r \nabla f(x_k))$, where $r_k = r$

The procedure terminates when two successive trial points x_k and x_{k+1} is approximately equal.

$\therefore r\nabla f(x_k) = 0$

But $r \neq 0$, $\nabla f(x_k) = 0$ is satisfied at x_k.

Problem:-

1. Verify the optimum solution $(1/3, 4/3)$ for the following.

10m

$$\text{Max } f(x_1, x_2) = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

Soln

GT $\text{Max } f(x_1, x_2) = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$.

WKT

$$x_{k+1} = x_k + r\nabla f(x_k)$$

$$h(r) = f(x_{k+1}) = f(x_k + r\nabla f(x_k))$$

Now,

$$\nabla f(x) = \nabla f(x_1, x_2) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)$$

$$= \left(4 + 0 - 4x_1 - 2x_2(1) - 0,\ 0 + 6 - 0 - 2x_1(1) - 4x_2\right)$$

$$\Rightarrow \nabla f(x) = (4 - 4x_1 - 2x_2,\ 6 - 2x_1 - 4x_2)$$

Step: 1

let $x_0 = (1, 1)$ be an initial value

Now, $\nabla f(x_0) = \nabla f(1, 1) = (-2, 0)$.

Now, $x_1 = x_0 + r\nabla f(x_0)$

$= (1, 1) + r(-2, 0)$

$= (1, 1) + (-2r, 0)$

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$= (1-2r, 1+0)$

$x_1 = (1-2r, 1)$

Here $h(r) = f(x_1) = f(\underset{x_1}{1-2r}, \underset{x_2}{1})$

$= 4(1-2r) + 6(1) - 2(1-2r)^2$
$- 2(1-2r)(1) - 2(1)^2$

$= 4 - 8r + \cancel{6} - \cancel{2} - 8r^2 + 8r$
$- \cancel{2} + 4r - \cancel{2}$

$h(r) = -8r^2 + 4r + 4.$

using dichotomous search method the value of $r = 1/4$.

$\therefore x_1 = (1 - 2(1/4), 1)$

$x_1 = (1/2, 1)$

Step:2

$\nabla f(x_1) = \nabla f(1/2, 1) = (4 - 4(1/2) - 2(1), 6 - 2(1/2) - 4(1))$

$\nabla f(x_1) = (0, 1)$

wkt $\therefore x_2 = x_1 + r \nabla f(x_1).$

$= (1/2, 1) + r(0, 1)$

$= (1/2, 1) + (0, r)$

$x_2 = (1/2, 1+r).$

(i)

wkt $h(r) = f(x_2) = f(\overset{x_1}{1/2}, \overset{x_2}{1+r})$

$= 4(1/2) + 6(1+r) + 2(1/2)^2 - 2(1/2)(1+r)$
$- 2(1+r)^2.$

$= 2 + 6 + 6r - \frac{1}{2} - 1 - r - 2 - 2r^2 - 4r$

$h(r) = -2r^2 + r + 9/2.$

using dichotomous search method the value of

$r = 1/4.$

$\therefore x_2 = (1/2, 1 + 1/4) = (1/2, 5/4).$

Step:3

now, $\nabla f(x_2) = \nabla f(1/2, 5/4) = (4, 4(1/2) - 2(5/4),$
$6 - 2(1/2) - 4(\frac{5}{4}))$

$\nabla f(x_2) = (-1/2, 0).$

wkt $x_3 = x_2 + r \nabla f(x_2)$

$= (1/2, 5/4) + r(-1/2, 0)$

$= (1/2, 5/4) + (-\frac{r}{2}, 0)$

$x_3 = (\frac{1-r}{2}, 5/4)$

wkt $h(r) = f(x_3) = f(\overset{x_1}{\frac{1-r}{2}}, \overset{x_2}{\frac{5}{4}})$

$= 4(\frac{1-r}{2}) + 6(5/4) - 2(\frac{1-r}{2})^2$
$- 2(\frac{1-r}{2})(5/4) - 2(5/4)^2$

using dichotomous search method, the value of $r = 1/4$ (8)

$$x_3 = \left(\frac{1-1/4}{2}, 5/4\right)$$

$$x_3 = \left(\frac{3}{8}, 5/4\right)$$

Step: 4

Now, $\nabla f(x_3) = \nabla f(3/8, 5/4)$

$$= \left(4 - 4(3/8) - 2(5/4), 6 - 2(3/8) - 4(5/4)\right)$$

$$= \left(4 - 3/2 - 5/2, 6 - \frac{3}{4} - 5\right)$$

$$\nabla f(x_3) = (0, 1/4)$$

wkt. $x_4 = x_3 + r \nabla f(x_3)$

$$= (3/8, 5/4) + r(0, 1/4)$$

$$= (3/8, 5/4) + \left(0, \frac{r}{4}\right)$$

$$x_4 = \left(\frac{3}{8}, \frac{5+r}{4}\right) \text{ (or) } \left(3/8, \frac{r+5}{4}\right)$$

wkt

$$h(r) = f(x_4) = f\left(3/8, \frac{r+5}{4}\right)$$

$$= 4(3/8) + 6\left(\frac{r+5}{2}\right) - 2(3/8)^2 - 2(3/8)\left(\frac{r+5}{4}\right) - 2\left(\frac{r+5}{4}\right)^2$$

using dichotomous search method the value of $r = 1/4$

$\therefore x_4 = \left(3/8, \frac{1/4 + 5}{4}\right) = \left(3/8, \frac{21}{16}\right)$

step:5

Now, $\nabla f(x_4) = \nabla f\left(3/8, \frac{21}{16}\right) = \left(4 - 4(3/8) - 2\left(\frac{21}{16}\right),\ 6 - 2(3/8) - 4\left(\frac{21}{16}\right)\right)$

$= \left(4 - \frac{33}{8}, 0\right)$

$\nabla f(x_4) = (-1/8, 0)$.

w.k.t

$x_5 = x_4 + r \nabla f(x_4)$.

$= \left(3/8, \frac{21}{16}\right) + r\left(-1/8, 0\right)$.

$= \left(3/8, \frac{21}{16}\right) + \left(-\frac{r}{8}, 0\right)$.

$x_5 = \left(\frac{3-r}{8}, \frac{21}{16}\right)$

w.k.t

$h(r) = f(x_5) = f\left(\frac{3-r}{8}, \frac{21}{16}\right)$

$= 4\left(\frac{3-r}{8}\right) + 6\left(\frac{21}{16}\right) - 2\left(\frac{3-r}{8}\right)^2 - 2\left(\frac{3-r}{8}\right)\left(\frac{21}{16}\right) - 2\left(\frac{21}{16}\right)^2$

using dichotomous search method the value of

$r = 1/4$

$\therefore x_5 = \left(\frac{3 - \frac{1}{4}}{8}, \frac{21}{16}\right)$.

$x_5 = \left(\frac{11}{32}, \frac{21}{16}\right)$

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Step: 6

Now, $\nabla f(x_5) = \left(\frac{11}{32}, \frac{21}{16}\right)$.

$$= \left(4 - 4\left(\frac{11}{32}\right) - 2\left(\frac{21}{16}\right), 6 - 2\left(\frac{11}{32}\right) - 4\left(\frac{21}{16}\right)\right)$$

$$= \left(0, \frac{1}{16}\right)$$

wkt

$$x_6 = x_5 + r \nabla f(x_5)$$

$$= \left(\frac{11}{32}, \frac{21}{16}\right) + r\left(0, \frac{1}{16}\right)$$

$$= \left(\frac{11}{32}, \frac{21}{16}\right) + \left(0, \frac{r}{16}\right)$$

$$x_6 = \left(\frac{11}{32}, \frac{21+r}{16}\right)$$

$$h(r) = f(x_6) = f\left(\frac{11}{32}, \frac{21+r}{16}\right)$$

$$= 4\left(\frac{11}{32}\right) + 6\left(\frac{21+r}{16}\right) - 2\left(\frac{11}{32}\right)^2 - 2\left(\frac{11}{32}\right)\left(\frac{21+r}{16}\right) - 2\left(\frac{21+r}{16}\right)^2$$

using dichotomous search method the value of

$r = 1/4$,

$$x_6 = \left(\frac{11}{32}, \frac{21 + 1/4}{16}\right)$$

$$= \left(\frac{11}{32}, \frac{85}{64}\right)$$

$$X_6 = \left(\frac{11}{32}, \frac{85}{64} \right).$$

$$X_6 = (0.3438, 1.3281)$$

$$X_6 = (0.3333, 1.3333). \quad \text{(approximately)}$$

$$X_6 = \left(\frac{1}{3}, \frac{4}{3} \right).$$

The optimum solution is.

$x_1 = 1/3$ and $x_2 = 4/3$.

— α —

Constrained Algorithm

The general constrained non linear programming problem is defined by

$$\max z \text{ or } \min z = f(x)$$
s.t
$$g(x) \le b_i$$
$$x \ge 0.$$

$f(x)$ and $g(x)$ is a non linear functions.

The constrained Algorithm can be divided into two methods. They are

i. Indirect method
ii. Direct methods.

Indirect method:

Indirect method is a method for solving the non linear program by dealing with one or more linear program derived from the original program. This method include separable, quadratic and chance constrained program.

Direct method:

Direct method is a method to deals with the original program.

Separable programming

A function $f(x_1, x_2 .. x_n)$ is separable if it can be expressed as the sum of n single variable function $f_1(x_1), f_2(x_2) ... f_n(x_n)$

ie $f(x_1, x_2 .. x_n) = f_1(x_1) + f_2(x_2) + .. + f_n(x_n)$

problem:

Using separable programming method to find the optimum solution for the following.

$$\max z = x_1 + x_2^4$$
s.t
$$3x_1 + 2x_2^2 \le 9$$
$$x_1, x_2 \ge 0$$

soln:

G.T $\max z = x_1 + x_2^4$

s.t
$$3x_1 + 2x_2^2 \le 9$$
$$x_1, x_2 \ge 0.$$

consider the separable function

$$f_1(x_1) = x_1, \quad f_2(x_2) = x_2^4, \quad g_1(x_1) = 3x_1, \quad g_2(x_2) = 2x_2^2$$

The function $f_1(x_1)$ and $g_1(x_1)$ are linear function, x_1 is treated as one variable.

consider $f_2(x_2)$ and $g_2(x_2)$. We assume four break points $a_{2k} = 0, 1, 2, 3$, for $k = 1, 2, 3, 4$ respectively, and the value of x_2 cannot exceed 3

K	a_{2k}	$f_2(a_{2k}) = a_{2k}^4$	$g_2(a_{2k}) = 2a_{2k}^2$
1	0	0	0
2	1	1	2
3	2	16	8
4	3	81	18

$\therefore f_2(x_2) = w_{21} f_2(a_{21}) + w_{22} f_2(a_{22}) + w_{23} f_2(a_{23}) + w_{24} f_2(a_{24})$

$= 0.w_{21} + 1.w_{22} + 16w_{23} + 81w_{24}$

$f_2(x_2) = w_{22} + 16w_{23} + 81w_{24}$

$|||^{ly}$ $g_2(x_2) = w_{21} g_2(a_{21})' + w_{22} g_2(a_{22}) + w_{23} f_2(a_{23}) + w_{24} f_2(a_{24})$

$= 0.w_{21} + 2w_{22} + 8w_{23} + 18w_{24}$

$= 2w_{22} + 8w_{23} + 18w_{24}$

$\therefore$ The problem is of the form

$\max z = x_1 + w_{22} + 16w_{23} + 81w_{24}$.

s.t

$3x_1 + 2w_{22} + 8w_{23} + 18w_{24} \leq 9$

$w_{21} + w_{22} + w_{23} + w_{24} = 1$

$x_1, w_{21}, w_{22}, w_{23}, w_{24} \geq 0$.

By introducing slack variable s_1, the problem can be rewritten as

$\max z = x_1 + w_{22} + 16w_{23} + 81w_{24} + 0.s_1 + 0.w_{21}$.

s.t

$3x_1 + 2w_{22} + 8w_{23} + 18w_{24} + s_1 = 9$

$w_{21} + w_{22} + w_{23} + w_{24} = 1$.

$x_1, w_{21}, w_{22}, w_{23}, w_{24}, s_1 \geq 0$, here w_{21} is an artificial variable.

1st table:

			1	1	16	81	0	0
C_B	BA	x_B	x_1	w_{22}	w_{23}	w_{24}	s_1	w_{21}
0	s_1	9	3	2	8	18	1	0
0	w_{21}	1	0	1	$\boxed{1}$	1	0	1
$z_j - c_j$		0	-1	-1	-16	-81	0	0

Here s_1 is a slack variable and w_{21} is currently a basic variable and +ve. The required basic condition dictates it must leave before w_{24} can enter the solution. The next entering variable is w_{23}, w_{21} leaves and the pivot element is 1.

IInd table:

			1	1	16	81	0	0	
	c_B	B_A	x_B	x_1	w_{22}	w_{23}	w_{24}	s_1	w_{21}
$-8\times$ IInd row $+$ Ist row	0	s_1	1	3	-6	0	$\boxed{10}$	1	-8 → $\min(\frac{1}{10}, \frac{1}{1}) = \frac{1}{10}$
IInd row $\div 1$	16	w_{23}	1	0	1	1	1	0	1
	$z_j - c_j$		16	-1	15	0	-65 ↑	0	16

Here the entering variable is w_{24}, the leaving variable is s_1 and the pivot element is 10.

IIIrd table:

			1	1	16	81	0	0	
	c_B	B_A	x_B	x_1	w_{22}	w_{23}	w_{24}	s_1	w_{21}
Ist row $\div 10$	81	w_{24}	$1/10$	$\frac{3}{10}$	$-3/5$	0	1	$1/10$	$-4/5$
$-1\times$ Ist row $+$ 2nd row	16	w_{23}	$9/10$	$-3/10$	$9/5$	1	0	$-1/10$	$9/5$
	$z_j - c_j$		$45/2$	$19/2$	-21	0	0	$13/2$	-36

The remaining variable w_{21} and w_{22} for the entering variable. w_{21} is not adjacent to w_{23} and w_{24}. Ill^{ly} w_{22} cannot enter because w_{24} cannot leave.

∴ The optimum solutions are $x_1 = 0$,

$x_2 = 0\,w_{21} + 1.w_{22} + 2w_{23} + 3w_{24} = 0.0 + 1.0 + 2(9/10) + 3(\frac{1}{10}) = \frac{21}{10} = 2.1$

i.e. $x_1^* = 0$ and $x_2^* = 2.1$ and the optimum value $z^* = 0 + (2.1)^4 = 19.45$

Problem:

Use separable programming method to solve

$\max z = x_1 + x_2^2$

Subject to

$3x_1 + 2x_2^2 \le 9$

$x_1, x_2 \ge 0$.

Soln:

G.T $\max z = x_1 + x_2^2$

Subject to

$3x_1 + 2x_2^2 \le 9$

$x_1, x_2 \ge 0$

consider the separable function $f_1(x_1) = x_1$, $f_2(x_2) = x_2^2$
$g_1(x_1) = 3x_1$, $g_2(x_2) = 2x_2^2$. The function $f_1(x_1)$ and $g_1(x_1)$ are linear function, x_1 is treated as one variable

consider $f_2(x_2)$ and $g_2(x_2)$ we assume two break points $a_{2k} = 0,1$ for $k = 1,2$ respectively and the value of x_2 cannot exceed 1

k	a_{2k}	$f_2(a_{2k}) = a_{2k}^2$	$g_2(a_{2k}) = 2a_{2k}^2$
1	0	0	0
2	1	1	2

$f_2(x_2) = w_{21} f_2(a_{21}) + w_{22} f_2(a_{22})$

$= 0 \cdot w_{21} + 1 \cdot w_{22}$

$\Rightarrow f_2(x_2) = w_{22}$

$\|^{ly}$

$g_2(x_2) = w_{21} g_2(a_{21}) + w_{22} g_2(a_{22})$

$= 0 \cdot w_{21} + 2 \cdot w_{22}$

$\Rightarrow g_2(x_2) = 2w_{22}$

The problem is of the form

$\max z = x_1 + w_{22}$

s.t

$3x_1 + 2w_{22} \leq 9$

$w_{21} + w_{22} = 1$

$x_1, w_{21}, w_{22} \geq 0$

By introducing slack variable s_1, the problem can be rewritten as

$\max z = x_1 + w_{22} + 0 \cdot s_1 + 0 \cdot w_{21}$

s.t $3x_1 + 2w_{22} + s_1 = 9$

$w_{21} + w_{22} = 1$

$x_1, w_{21}, w_{22}, s_1 \geq 0$, Here w_{21} is an artificial variable

1st table

			1	1	0	0
C_B	B_V	x_B	x_1	w_{22}	s_1	w_{21}
0	s_1	9	3	2	1	0
0	w_{21}	1	0	$\boxed{1}$	0	1
$z_j - c_j$		0	-1	-1	0	0

$\min\left(\frac{9}{2}, \frac{1}{1}\right)$
$= \min(4.5, 1)$
$= 1$

(14)

Here the entering variable is w_{22}, the leaving variable is w_{21} and the pivot element is 1

II nd table

			1	1	0	0	
	C_B	BA	x_B	x_1	w_{22}	s_1	w_{21}
$-2\times$ II nd row $+ 2^{a}$ row	0	s_1	7	$\boxed{3}$	0	1	-2 $\rightarrow$
II nd row $\div 1$	1	w_{22}	1	0	1	0	1
	$z_j - c_j$		1	-1 $\uparrow$	0	0	1

Here the entering variable is x_1, the leaving variable is s_1 and the pivot element is 3

III rd table:

			1	1	0	0
C_B	BA	x_B	x_1	w_{22}	s_1	w_{21}
1	x_1	$7/3$	1	0	$1/3$	$-2/3$
1	w_{22}	1	0	1	0	1
$z_j - c_j$		$10/3$	0	0	$1/3$	$1/3$

$\therefore$ The optimum solutions are $x_1 = 7/3$, $x_2 = 0 \cdot w_{21} + 1 \cdot w_{22}$
$= w_{22} = 1$

ie $x_1^* = 7/3$ and $x_2^* = 1$ and the optimum value $z^* = 7/3 + 1^2 = 10/3$

Separable convex programming

Problem:
Using separable convex programming to solve

$\max z = x_1 - x_2$

Subject to

$3x_1^4 + x_2 \leq 243$

$x_1 + 2x_2^2 \leq 32$

$x_1 \geq 2.1$

$x_2 \geq 3.5$

Soln:

G.T $\max z = x_1 - x_2$

s.t

$3x_1^4 + x_2 \leq 243$

$x_1 + 2x_2^2 \leq 32$

$x_1 \geq 2.1$

$x_2 \geq 3.5$

The separable functions are

$f_1(x_1) = x_1, \quad f_2(x_2) = -x_2$

$g_{11}(x_1) = 3x_1^4, \quad g_{12}(x_2) = x_2, \quad g_{21}(x_1) = x_1, \quad g_{22}(x_2) = 2x_2^2$

Here the function $f_1(x_1)$, $f_2(x_2)$, $g_{12}(x_2) = x_2$, $g_{21}(x_1) = x_1$ are already linear

These functions satisfy the convexity condition required from minimization problem.

The range of the variable x_1 is $0 \le x_1 \le 3$ and x_2 is $0 \le x_2 \le 4$.

Let $K_1 = 3$ and $K_2 = 4$, the slope corresponding to the separable functions are determined as follows.

For $j = 1$

K	a_{1K}	$g_{11}(a_{1K}) = 3a_{1K}^4$	r_{11K}	x_{1K}
0	0	0	–	–
1	1	3	3	x_{11}
2	2	48	45	x_{12}
3	3	243	195	x_{13}

For $j = 2$

K	a_{2K}	$g_{22}(a_{2K}) = 2a_{2K}^2$	r_{22K}	x_{2K}
0	0	0	–	–
1	1	2	2	x_{21}
2	2	8	6	x_{22}
3	3	18	10	x_{23}
4	4	32	14	x_{24}

The problem is of the form

$\max Z = x_1 - x_2$

Subject to

$3x_{11} + 45x_{12} + 195x_{13} + x_2 \le 243$

$x_1 + 2x_{21} + 6x_{22} + 10x_{23} + 14x_{24} \le 32$

$x_1 \ge 2.1$

$x_2 \ge 3.5$

Here $x_1 = x_{11} + x_{12} + x_{13}$

$x_2 = x_{21} + x_{22} + x_{23} + x_{24}$

$0 \le x_{1k} \le 1, \quad k = 1 \text{ to } 3$

$0 \le x_{2k} \le 1, \quad k = 1 \text{ to } 4$

$x_1, x_2 \ge 0$

$\therefore$ The optimum solution is

$x_1 = 2.98, \quad x_2 = 3.5$

$\therefore \quad x_{11} = 1, \quad x_{12} = 1, \quad x_{13} = 0.98$

$x_{21} = 1, \quad x_{22} = 1, \quad x_{23} = 1, \quad x_{24} = 0.5.$

and the optimum value $z^* = 2.98 - 3.5 = -0.52$

Quadratic programming

The general form of a quadratic programming problem is

$$\max z = CX + X^T DX$$

s.t

$$AX \le b$$
$$X \ge 0.$$

where $X = (x_1, x_2, \dots x_n)^T$

$C = (c_1, c_2 \dots c_n)$

$b = (b_1, b_2 \dots b_n)^T$

$$\therefore A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$D = \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ d_{21} & d_{22} & \dots & d_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ d_{n1} & d_{n2} & \dots & d_{nn} \end{pmatrix}$$

The problem can be rewritten as

$$\max z = CX + X^T DX$$

Subject to

$$G(X) = \begin{pmatrix} A \\ -I \end{pmatrix} X - \begin{pmatrix} b \\ 0 \end{pmatrix} \le 0.$$

Let $\lambda = (\lambda_1 \lambda_2 \dots \lambda_n)^T$

$U = (\mu_1 \mu_2 \dots \mu_n)^T$ be the Lagrange's multipliers corresponding to the constraints $AX - b \le 0$ and $-X \ge 0$ resply

The necessary condition may be combined as

$$\begin{pmatrix} -2D & A^T & -I & 0 \\ A & 0 & 0 & I \end{pmatrix}\begin{pmatrix} x \\ \lambda \\ \mu \\ S \end{pmatrix} = \begin{pmatrix} C^T \\ b \end{pmatrix}$$

$$\mu_j x_j = 0 = \lambda_i S_i \;, \forall \; i \text{ and } j$$

$$\lambda, U, x, S \geq 0.$$

The solution of the system is obtained by using phase I of the two phase method.

Here $\lambda_i S_i = \mu_j x_j = 0$.

This means that λ_i and S_i cannot be +ve simultaneously

problem:

Find the optimum solution to the following

$$\max z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

s.t

$$x_1 + 2x_2 \leq 2$$

$$x_1, x_2 \geq 0.$$

Soln:

G.T. $\max z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

s.t

$$x_1 + 2x_2 \leq 2$$

$$x_1, x_2 \geq 0.$$

The problem can be rewritten as

$$\max z = (4 \quad 6)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1 \quad x_2)\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

s.t

$$(1 \quad 2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq 2$$

$$x_1, x_2 \geq 0.$$

Again the problem can be rewritten as

$$\underset{3\times 6}{\begin{pmatrix} 4 & 2 & 1 & -1 & 0 & 0 \\ 2 & 4 & 2 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}} \underset{6\times 1}{\begin{pmatrix} x_1 \\ x_2 \\ \lambda \\ \mu_1 \\ \mu_2 \\ s \end{pmatrix}} = \underset{3\times 1}{\begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}}$$

$$\Rightarrow \quad 4x_1 + 2x_2 + \lambda - \mu_1 + 0\mu_2 + 0.S = 4$$

$$2x_1 + 4x_2 + 2\lambda + 0\,\mu_1 - 1.\mu_2 + 0.S = 6$$

$$x_1 + 2x_2 + 0.\lambda + 0.\mu_1 + 0.\mu_2 + S = 2$$

$$x_1, x_2, \lambda, \mu_1, \mu_2, S \geq 0$$

By introducing artificial variable A_1 and A_2 the problem can be rewritten as

$$4x_1 + 2x_2 + \lambda - \mu_1 + 0.\mu_2 + 0.S + A_1 = 4$$

$$2x_1 + 4x_2 + 2\lambda + 0\,\mu_1 - 1.\mu_2 + 0.S + A_2 = 6.$$

$$x_1 + 2x_2 + 0\,\lambda + 0.\mu_1 + 0.\mu_2 + S = 6$$

$x_1, x_2, \lambda, \mu_1, \mu_2, S, A_1, A_2 \geq 0$.

I st table:

C_B	$B.A$	X_B	x_1 (6)	x_2 (6)	λ (3)	μ_1 (-1)	μ_2 (-1)	A_1 (0)	A_2 (0)	S (0)	
0	A_1	4	$\boxed{4}$	2	1	-1	0	1	0	0	$\rightarrow$ $\min(\frac{4}{4}, \frac{6}{2}, \frac{2}{1})$
0	A_2	6	2	4	2	0	-1	0	1	0	$= \min(1, 3, 2)$
0	S	2	1	2	0	0	0	0	0	1	$= 1$
$Z_j - C_j$		10	-6 $\uparrow$	-6	-3	1	1	0	0	0	

Here the entering variable is x_1, the leaving variable is A_1 and the pivot element is 4.

II nd table:

	C_B	$B.A$	X_B	x_1 (6)	x_2 (6)	λ (3)	μ_1 (-1)	μ_2 (-1)	A_1 (0)	A_2 (0)	S (0)	
I st row ÷ 4	6	x_1	1	1	1/2	1/4	-1/4	0	1/4	0	0	$\min(\frac{1}{1/2}, \frac{4}{3}, \frac{1}{3/2})$
$-2 \times$ I st row + II nd row	0	A_2	4	0	3	3/2	1/2	-1	-1/2	1	0	$= \min(2, 1.3, 0.67)$ $= 0.67$
$-1 \times$ I st row + III rd row	0	S	1	0	$\boxed{3/2}$	-1/4	1/4	0	-1/4	0	1	$\rightarrow$
	$Z_j - C_j$		4	0	-3 $\uparrow$	-3/2	-1/2	1	3/2	0	0	

Here the entering variable is x_2, the leaving variable is S and the pivot element is 3/2

III rd table:

	C_B	$B.A$	X_B	x_1 (6)	x_2 (6)	λ (3)	μ_1 (-1)	μ_2 (-1)	A_1 (0)	A_2 (0)	S (0)	
$-\frac{1}{2} \times$ III rd row + I st row	6	x_1	2/3	1	0	1/3	-1/3	0	1/3	0	-1/3	$\min(\frac{2/3}{1/3}, \frac{2}{2})$
$-3 \times$ III rd row + II nd row	0	A_2	2	0	0	$\boxed{2}$	0	-1	0	1	-2	$\rightarrow$ $= \min(2, 1)$ $= 1$
III rd row ÷ $\frac{3}{2}$	6	x_2	2/3	0	1	-1/6	1/6	0	-1/6	0	2/3	
	$Z_j - C_j$		2	0	0	-2 $\uparrow$	0	0	1	0	2	

Here the entering variable is λ, the leaving variable is A_2 and the pivot element is 2.

IV th table:

	C_B	$B.A$	X_B	x_1 (6)	x_2 (6)	λ (3)	μ_1 (-1)	μ_2 (-1)	A_1 (0)	A_2 (0)	S (0)
$-\frac{1}{3} \times$ II nd row + I st row	6	x_1	1/3	1	0	0	-1/3	1/6	1/3	-1/6	0
II nd row ÷ 2	3	λ	1	0	0	1	0	-1/2	0	1/2	-1
$\frac{1}{6} \times$ II nd row + III rd row	6	x_2	5/6	0	1	0	1/6	-1/12	-1/6	1/12	1/2
	$Z_j - C_j$		0	0	0	0	0	0	1	1	0

Here all $Z_j - C_j \geq 0$. The solution is an optimum solution. The optimum solutions are $x_1^* = 1/3$, $x_2^* = 5/6$ and the optimum value $Z^* = 4(1/3) + 6(5/6) - 2(1/3)^2 - 2(1/3)(5/6) - 2(5/6)^2 = 4.16$

Linear combination method

Using linear combination method to obtain an optimum solution for the following

$$\max f(x) = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

subject to

$$x_1 + 2x_2 \leq 2$$
$$x_1, x_2 \geq 0.$$

Soln:

G.T $\max f(x) = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

s.t $x_1 + 2x_2 \leq 2$

$x_1, x_2 \geq 0.$

WKT gradient of $f(x)$ is

$$\nabla f(x) = (4 - 4x_1 - 2x_2,\ 6 - 2x_1 - 4x_2)$$

Let $x_0 = (1/2, 1/2)$ be an initial point.

Iteration: 1

Now $\nabla f(x_0) = \nabla f(1/2, 1/2)$

$= (4 - 4(1/2) - 2(1/2),\ 6 - 2(1/2) - 4(1/2))$

$= (1, 3)$

∴ The associated linear program $w_1 = x_1 + 3x_2$, subject to the original problem.

put $x_1 = 0$, $x_2 = 1$, the optimum solution $x^* = (0, 1)$

∴ The value of w_1 at x_0 and x^* equals to 2 and 3 resply.

Let $x_1 = x_0 + r(x^* - x_0)$

$= (1/2, 1/2) + r[(0,1) - (1/2, 1/2)]$

$= (1/2, 1/2) + r(-1/2, 1/2)$

$= (1/2, 1/2) + \left(-\frac{r}{2}, \frac{r}{2}\right)$

$= \left(\frac{1-r}{2}, \frac{1+r}{2}\right)$

WKT $f(r) = f(x_1)$

$= f\left(\frac{1-r}{2}, \frac{1+r}{2}\right)$

$= \cancel{4}^{2}\frac{(1-r)}{\cancel{2}} + \cancel{6}^{3}\frac{(1+r)}{\cancel{2}} - 2\left(\frac{1-r}{2}\right)^2 - \cancel{2}\left(\frac{1-r}{\cancel{2}}\right)\left(\frac{1+r}{2}\right) - 2\left(\frac{1+r}{2}\right)^2$

$= 2 - 2r + 3 + 3r - \frac{1}{2}(r^2 - 2r + 1) - \frac{1}{2}(1 - r^2) - \frac{1}{2}(r^2 + 2r + 1)$

$= 5 + r - \frac{1}{2}(r^2 - \cancel{2r} + 1 + 1 - r^2 + r^2 + \cancel{2r} + 1)$

$$= 5 + r - \frac{1}{2}(r^2 + 3)$$

$$= \frac{10 + 2r - r^2 - 3}{2}$$

$$f_1(r) = -\frac{1}{2}(r^2 - 2r - 7)$$

using dichotomous search method the value of $r = 1$

$$\therefore x_1 = \left(\frac{1-1}{2}, \frac{1+1}{2} \right) = (0,1)$$

$$\therefore f(x_1) = f(0,1) = 4(0) + 6(1) - 2(0)^2 - 2(0)(1) - 2(1)^2$$

$$= 4$$

<u>Iteration: 2</u>

$$\text{Now } \nabla f(x_1) = \nabla f(0,1)$$

$$= (4 - 4(0) - 2(1),\ 6 - 2(0) - 4(1))$$

$$\nabla f(x_1) = (2, 2)$$

$\therefore$ The objective function of a new LPP is $w_2 = 2x_1 + 2x_2$

put $x_1 = 2$ and $x_2 = 0$, the optimum solution $x^* = (2,0)$

$\therefore$ The value of w_2 at x_1 and x^* are 2 and 4.

$$\text{WKT } x_2 = x_1 + r(x^* - x_1)$$

$$= (0,1) + r((2,0) - (0,1))$$

$$= (0,1) + r(2,-1)$$

$$= (0,1) + (2r, -r)$$

$$x_2 = (2r, 1-r)$$

$$\text{WKT } f(r) = f(x_2)$$

$$= f(2r, 1-r)$$

$$= 4(2r) + 6(1-r) - 2(2r)^2 - 2(2r)(1-r) - 2(1-r)^2$$

$$= 8r + 6 - 6r - 8r^2 - 4r + 4r^2 - 2 - 2r^2 + 4r$$

$$f(r) = -6r^2 + 2r + 4$$

using dichotomous search method the value of $r = 1/6$

$$\therefore x_2 = (2(1/6),\ 1 - \tfrac{1}{6})$$

$$= (1/3,\ 5/6)$$

$$\therefore f(x_2) = f(1/3, 5/6) = 4(1/3) + 6(5/6) - 2(1/3)^2 - 2(1/3)(5/6) - 2(5/6)^2$$

$$= 4.16.$$

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Iteration 3

Now $\nabla f(X_2) = \nabla f(1/3, 5/6)$

$$= \left(4 - 4(1/3) - 2(5/6),\ 6 - 2(1/3) - 4(5/6)\right)$$

$$= (1, 2)$$

The corresponding objective function is $w_2 = x_1 + 2x_2$.

∴ The optimum solution of this problem yields alternatively solution $x^* = (0,1)$ and $x^* = (2,0)$

∴ The value of w_3 for both points equals its value at x_2

∴ The approximate optimum solution is

$x_2 = (1/3, 5/6)$ with $f(x_2) = 4.16$.

SUMT Algorithm (Sequential unconstrained maximization technique)

SUMT algorithm is based on transforming the constrained program an equivalent unconstrained problem.

Consider the function

$P(x,t) = f(x) + t\left[\sum_{i=1}^{n} 1/g_i(x) - \sum_{j=1}^{n} 1/x_j\right]$, t is a non-ve parameter. The second summation sign accounts for the non-negativity constraints.

Since $g_i(x_i)$ is convex, $1/g_i(x_i)$ is concave

∴ $P(x,t)$ is concave in x.

∴ $P(x,t)$ possesses a unique maximum

∴ The optimization of the original constrained problem is equivalent to optimization of $P(x,t)$