CORE COURSE II

REAL ANALYSIS

Objectives:

1. To give the students a thorough knowledge of the various aspects of Real line and Metric Spaces which is imperative for any advanced learning in Pure Mathematics.
2. To train the students in problem-solving as a preparatory for competitive exams.

UNIT I

Basic Topology: Finite, Countable and Uncountable Sets – Metric spaces – Compact sets – Perfect sets – Connected sets.
Numerical Sequences and Series: Sequences – Convergence – Subsequences - Cauchy Sequences – Upper and Lower Limits - Some Special Sequences – Tests of convergence – Power series – Absolute convergence – Addition and multiplication of series – Rearrangements.

UNIT II

Continuity: Limits of functions – Continuous functions – continuity and Compactness – Continuity and connectedness – Discontinuities – Monotonic functions – Infinite limits and limits at infinity. Differentiation: Derivative of a real function – Mean value Theorems - Intermediate value theorem for derivatives – L'Hospital's Rule – Taylor's Theorem – Differentiation of vector valued functions.

UNIT III

Riemann – Stieltjes Integral: Definition and Existence – Properties – Integration and Differentiation – Integration of vector valued functions.

UNIT IV

Sequences and series of functions: Uniform Convergence and Continuity – Uniform Convergence and Differentiation – Equicontinuous families of functions – The Stone – Weierstrass Theorem.

UNIT V

Functions of several variables: Linear Transformations - Differentiation – The Contraction Principle – The Inverse Function Theorem - The Implicit Function Theorem.

TEXT BOOKS

[1] Walter Rudin , Principles of Mathematical Analysis, Third Edition, Mcgraw Hill, 1976.

UNIT – I	Chapters 2 and 3
UNIT – II	Chapters 4 and 5
UNIT – III	Chapter 6
UNIT – IV	Chapter 7
UNIT – V	Chapter 9, Sections 9.1 to 9.29

REFERENCES

1. Tom P. Apostol, Mathematical Analysis, Narosa Publishing House, New Delhi, 1985.
2. A.J. White, Real Analysis : An Introduction, Addison Wesley Publishing Co., Inc. 1968.
3. Serge Lang, Analysis I & II, Addison-Wesley Publishing Company, Inc. 1969.
4. N.L.Carothers, Real Analysis, Cambridge University press, Indian edition, 2013.

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2 ... SYSTEMS

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REAL ANALYSIS

UNIT- I

FINITE, COUNTABLE, AND UNCOUNTABLE SETS

We begin this section with a definition of the function concept.

2.1 Definition Consider two sets A and B, whose elements may be any objects whatsoever, and suppose that with each element x of A there is associated, in some manner, an element of B, which we denote by $f(x)$. Then f is said to be a *function* from A to B (or a *mapping* of A into B). The set A is called the *domain* of f (we also say f is defined on A), and the elements $f(x)$ are called the *values* of f. The set of all values of f is called the *range* of f.

2.2 Definition Let A and B be two sets and let f be a mapping of A into B. If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$. We call $f(E)$ the *image* of E under f. In this notation, $f(A)$ is the range of f. It is clear that $f(A) \subset B$. If $f(A) = B$, we say that f maps A *onto* B. (Note that, according to this usage, *onto* is more specific than *into*.)

If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the *inverse image* of E under f. If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$

such that $f(x) = y$. If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A, then f is said to be a 1-1 (*one-to-one*) mapping of A into B. This may also be expressed as follows: f is a 1-1 mapping of A into B provided that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, $x_1 \in A$, $x_2 \in A$.

(The notation $x_1 \neq x_2$ means that x_1 and x_2 are distinct elements; otherwise we write $x_1 = x_2$.)

2.3 Definition If there exists a 1-1 mapping of A *onto* B, we say that A and B can be put in 1-1 *correspondence*, or that A and B have the same *cardinal number*, or, briefly, that A and B are *equivalent*, and we write $A \sim B$. This relation clearly has the following properties:

It is reflexive: $A \sim A$.
It is symmetric: If $A \sim B$, then $B \sim A$.
It is transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$.

Any relation with these three properties is called an *equivalence relation*.

2.4 Definition For any positive integer n, let J_n be the set whose elements are the integers $1, 2, \ldots, n$; let J be the set consisting of all positive integers. For any set A, we say:

(*a*) A is *finite* if $A \sim J_n$ for some n (the empty set is also considered to be finite).
(*b*) A is *infinite* if A is not finite.
(*c*) A is *countable* if $A \sim J$.
(*d*) A is *uncountable* if A is neither finite nor countable.
(*e*) A is *at most countable* if A is finite or countable.

Countable sets are sometimes called *enumerable*, or *denumerable*.

For two finite sets A and B, we evidently have $A \sim B$ if and only if A and B contain the same number of elements. For infinite sets, however, the idea of "having the same number of elements" becomes quite vague, whereas the notion of 1-1 correspondence retains its clarity.

2.5 Example Let A be the set of all integers. Then A is countable. For, consider the following arrangement of the sets A and J:

$$A: \quad 0, 1, -1, 2, -2, 3, -3, \ldots$$
$$J: \quad 1, 2, 3, 4, 5, 6, 7, \ldots$$

We can, in this example, even give an explicit formula for a function f from J to A which sets up a 1-1 correspondence:

$$f(n) = \begin{cases} \dfrac{n}{2} & (n \text{ even}), \\ -\dfrac{n-1}{2} & (n \text{ odd}). \end{cases}$$

2.6 Remark A finite set cannot be equivalent to one of its proper subsets. That this is, however, possible for infinite sets, is shown by Example 2.5, in which J is a proper subset of A.

In fact, we could replace Definition 2.4(b) by the statement: A is infinite if A is equivalent to one of its proper subsets.

2.7 Definition By a *sequence*, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$, for $n \in J$, it is customary to denote the sequence f by the symbol $\{x_n\}$, or sometimes by $x_1, x_2, x_3, \ldots$. The values of f, that is, the elements x_n, are called the *terms* of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a *sequence in* A, or a *sequence of elements of* A.

Note that the terms $x_1, x_2, x_3, \ldots$ of a sequence need not be distinct.

Since every countable set is the range of a 1-1 function defined on J, we may regard every countable set as the range of a sequence of distinct terms. Speaking more loosely, we may say that the elements of any countable set can be "arranged in a sequence."

Sometimes it is convenient to replace J in this definition by the set of all nonnegative integers, i.e., to start with 0 rather than with 1.

2.8 Theorem *Every infinite subset of a countable set A is countable.*

Proof Suppose $E \subset A$, and E is infinite. Arrange the elements x of A in a sequence $\{x_n\}$ of distinct elements. Construct a sequence $\{n_k\}$ as follows:

Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Having chosen $n_1, \ldots, n_{k-1}$ $(k = 2, 3, 4, \ldots)$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

Putting $f(k) = x_{n_k}$ $(k = 1, 2, 3, \ldots)$, we obtain a 1-1 correspondence between E and J.

The theorem shows that, roughly speaking, countable sets represent the "smallest" infinity: No uncountable set can be a subset of a countable set.

2.9 Definition Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω which we denote by E_α.

The set whose elements are the sets E_α will be denoted by $\{E_\alpha\}$. Instead of speaking of sets of sets, we shall sometimes speak of a collection of sets, or a family of sets.

The *union* of the sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation

$$S = \bigcup_{\alpha \in A} E_\alpha. \tag{1}$$

If A consists of the integers $1, 2, \ldots, n$, one usually writes

$$S = \bigcup_{m=1}^{n} E_m \tag{2}$$

or

$$S = E_1 \cup E_2 \cup \cdots \cup E_n. \tag{3}$$

If A is the set of all positive integers, the usual notation is

$$S = \bigcup_{m=1}^{\infty} E_m. \tag{4}$$

The symbol ∞ in (4) merely indicates that the union of a *countable* collection of sets is taken, and should not be confused with the symbols $+\infty$, $-\infty$, introduced in Definition 1.23.

The *intersection* of the sets E_α is defined to be the set P such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$. We use the notation

$$P = \bigcap_{\alpha \in A} E_\alpha, \tag{5}$$

or

$$P = \bigcap_{m=1}^{n} E_m = E_1 \cap E_2 \cap \cdots \cap E_n, \tag{6}$$

or

$$P = \bigcap_{m=1}^{\infty} E_m, \tag{7}$$

as for unions. If $A \cap B$ is not empty, we say that A and B *intersect*; otherwise they are *disjoint*.

2.10 Examples

(a) Suppose E_1 consists of 1, 2, 3 and E_2 consists of 2, 3, 4. Then $E_1 \cup E_2$ consists of 1, 2, 3, 4, whereas $E_1 \cap E_2$ consists of 2, 3.

(*b*) Let A be the set of real numbers x such that $0 < x \le 1$. For every $x \in A$, let E_x be the set of real numbers y such that $0 < y < x$. Then

(i) $E_x \subset E_z$ if and only if $0 < x \le z \le 1$;

(ii) $\bigcup_{x \in A} E_x = E_1$;

(iii) $\bigcap_{x \in A} E_x$ is empty;

(i) and (ii) are clear. To prove (iii), we note that for every $y > 0$, $y \notin E_x$ if $x < y$. Hence $y \notin \bigcap_{x \in A} E_x$.

2.11 Remarks Many properties of unions and intersections are quite similar to those of sums and products; in fact, the words sum and product were sometimes used in this connection, and the symbols Σ and Π were written in place of $\bigcup$ and $\bigcap$.

The commutative and associative laws are trivial:

(8) $$A \cup B = B \cup A; \qquad A \cap B = B \cap A.$$

(9) $$(A \cup B) \cup C = A \cup (B \cup C); \qquad (A \cap B) \cap C = A \cap (B \cap C).$$

Thus the omission of parentheses in (3) and (6) is justified.

The distributive law also holds:

(10) $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

To prove this, let the left and right members of (10) be denoted by E and F, respectively.

Suppose $x \in E$. Then $x \in A$ and $x \in B \cup C$, that is, $x \in B$ or $x \in C$ (possibly both). Hence $x \in A \cap B$ or $x \in A \cap C$, so that $x \in F$. Thus $E \subset F$.

Next, suppose $x \in F$. Then $x \in A \cap B$ or $x \in A \cap C$. That is, $x \in A$, and $x \in B \cup C$. Hence $x \in A \cap (B \cup C)$, so that $F \subset E$.

It follows that $E = F$.

We list a few more relations which are easily verified:

(11) $$A \subset A \cup B,$$

(12) $$A \cap B \subset A.$$

If 0 denotes the empty set, then

(13) $$A \cup 0 = A, \qquad A \cap 0 = 0.$$

If $A \subset B$, then

(14) $$A \cup B = B, \qquad A \cap B = A.$$

2.12 Theorem *Let $\{E_n\}$, $n = 1, 2, 3, \ldots$, be a sequence of countable sets, and put*

(15) $$S = \bigcup_{n=1}^{\infty} E_n.$$

Then S is countable.

Proof Let every set E_n be arranged in a sequence $\{x_{nk}\}$, $k = 1, 2, 3, \ldots$, and consider the infinite array

(16) $$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

in which the elements of E_n form the nth row. The array contains all elements of S. As indicated by the arrows, these elements can be arranged in a sequence

(17) $$x_{11};\ x_{21}, x_{12};\ x_{31}, x_{22}, x_{13};\ x_{41}, x_{32}, x_{23}, x_{14};\ \ldots$$

If any two of the sets E_n have elements in common, these will appear more than once in (17). Hence there is a subset T of the set of all positive integers such that $S \sim T$, which shows that S is at most countable (Theorem 2.8). Since $E_1 \subset S$, and E_1 is infinite, S is infinite, and thus countable.

Corollary *Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Put*

$$T = \bigcup_{\alpha \in A} B_\alpha.$$

Then T is at most countable.

For T is equivalent to a subset of (15).

2.13 Theorem *Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \ldots, a_n)$, where $a_k \in A$ $(k = 1, \ldots, n)$, and the elements $a_1, \ldots, a_n$ need not be distinct. Then B_n is countable.*

Proof That B_1 is countable is evident, since $B_1 = A$. Suppose B_{n-1} is countable $(n = 2, 3, 4, \ldots)$. The elements of B_n are of the form

(18) $$(b, a) \qquad (b \in B_{n-1}, a \in A).$$

For every fixed b, the set of pairs (b, a) is equivalent to A, and hence countable. Thus B_n is the union of a countable set of countable sets. By Theorem 2.12, B_n is countable.

The theorem follows by induction.

Corollary *The set of all rational numbers is countable.*

Proof We apply Theorem 2.13, with $n = 2$, noting that every rational r is of the form b/a, where a and b are integers. The set of pairs (a, b), and therefore the set of fractions b/a, is countable.

In fact, even the set of all algebraic numbers is countable (see Exercise 2).

That not all infinite sets are, however, countable, is shown by the next theorem.

2.14 Theorem *Let A be the set of all sequences whose elements are the digits* 0 *and* 1. *This set A is uncountable.*

The elements of A are sequences like $1, 0, 0, 1, 0, 1, 1, 1, \ldots$.

Proof Let E be a countable subset of A, and let E consist of the sequences $s_1, s_2, s_3, \ldots$. We construct a sequence s as follows. If the nth digit in s_n is 1, we let the nth digit of s be 0, and vice versa. Then the sequence s differs from every member of E in at least one place; hence $s \notin E$. But clearly $s \in A$, so that E is a proper subset of A.

We have shown that every countable subset of A is a proper subset of A. It follows that A is uncountable (for otherwise A would be a proper subset of A, which is absurd).

The idea of the above proof was first used by Cantor, and is called Cantor's diagonal process; for, if the sequences $s_1, s_2, s_3, \ldots$ are placed in an array like (16), it is the elements on the diagonal which are involved in the construction of the new sequence.

Readers who are familiar with the binary representation of the real numbers (base 2 instead of 10) will notice that Theorem 2.14 implies that the set of all real numbers is uncountable. We shall give a second proof of this fact in Theorem 2.43.

METRIC SPACES

2.15 Definition A set X, whose elements we shall call *points*, is said to be a *metric space* if with any two points p and q of X there is associated a real number $d(p, q)$, called the *distance* from p to q, such that

(a) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$;
(b) $d(p, q) = d(q, p)$;
(c) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

Any function with these three properties is called a *distance function*, or a *metric*.

2.16 Examples The most important examples of metric spaces, from our standpoint, are the euclidean spaces R^k, especially R^1 (the real line) and R^2 (the complex plane); the distance in R^k is defined by

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| \qquad (\mathbf{x}, \mathbf{y} \in R^k). \tag{19}$$

By Theorem 1.37, the conditions of Definition 2.15 are satisfied by (19).

It is important to observe that every subset Y of a metric space X is a metric space in its own right, with the same distance function. For it is clear that if conditions (a) to (c) of Definition 2.15 hold for $p, q, r \in X$, they also hold if we restrict p, q, r to lie in Y.

Thus every subset of a euclidean space is a metric space. Other examples are the spaces $\mathscr{C}(K)$ and $\mathscr{L}^2(\mu)$, which are discussed in Chaps. 7 and 11, respectively.

2.17 Definition By the *segment* (a, b) we mean the set of all real numbers x such that $a < x < b$.

By the *interval* $[a, b]$ we mean the set of all real numbers x such that $a \leq x \leq b$.

Occasionally we shall also encounter "half-open intervals" $[a, b)$ and $(a, b]$; the first consists of all x such that $a \leq x < b$, the second of all x such that $a < x \leq b$.

If $a_i < b_i$ for $i = 1, \ldots, k$, the set of all points $\mathbf{x} = (x_1, \ldots, x_k)$ in R^k whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ $(1 \leq i \leq k)$ is called a *k-cell*. Thus a 1-cell is an interval, a 2-cell is a rectangle, etc.

If $\mathbf{x} \in R^k$ and $r > 0$, the *open* (or *closed*) *ball* B with center at $\mathbf{x}$ and radius r is defined to be the set of all $\mathbf{y} \in R^k$ such that $|\mathbf{y} - \mathbf{x}| < r$ (or $|\mathbf{y} - \mathbf{x}| \leq r$).

We call a set $E \subset R^k$ *convex* if

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in E$$

whenever $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $0 < \lambda < 1$.

For example, *balls are convex*. For if $|\mathbf{y} - \mathbf{x}| < r$, $|\mathbf{z} - \mathbf{x}| < r$, and $0 < \lambda < 1$, we have

$$\begin{aligned} |\lambda \mathbf{y} + (1 - \lambda)\mathbf{z} - \mathbf{x}| &= |\lambda(\mathbf{y} - \mathbf{x}) + (1 - \lambda)(\mathbf{z} - \mathbf{x})| \\ &\leq \lambda |\mathbf{y} - \mathbf{x}| + (1 - \lambda)|\mathbf{z} - \mathbf{x}| < \lambda r + (1 - \lambda) r \\ &= r. \end{aligned}$$

The same proof applies to closed balls. It is also easy to see that k-cells are convex.

2.18 Definition Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X.

(a) *A neighborhood* of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$, for some $r > 0$. The number r is called the *radius* of $N_r(p)$.
(b) A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$.
(c) If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E.
(d) E is *closed* if every limit point of E is a point of E.
(e) A point p is an *interior* point of E if there is a neighborhood N of p such that $N \subset E$.
(f) E is *open* if every point of E is an interior point of E.
(g) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.
(h) E is *perfect* if E is closed and if every point of E is a limit point of E.
(i) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.
(j) E is *dense in* X if every point of X is a limit point of E, or a point of E (or both).

Let us note that in R^1 neighborhoods are segments, whereas in R^2 neighborhoods are interiors of circles.

2.19 Theorem *Every neighborhood is an open set.*

Proof. Consider a neighborhood $E = N_r(p)$, and let q be any point of E. Then there is a positive real number h such that

$$d(p, q) = r - h.$$

For all points s such that $d(q, s) < h$, we have then

$$d(p, s) \leq d(p, q) + d(q, s) < r - h + h = r,$$

so that $s \in E$. Thus q is an interior point of E.

2.20 Theorem *If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.*

Proof Suppose there is a neighborhood N of p which contains only a finite number of points of E. Let $q_1, \ldots, q_n$ be those points of $N \cap E$, which are distinct from p, and put

$$r = \min_{1 \leq m \leq n} d(p, q_m)$$

[we use this notation to denote the smallest of the numbers $d(p, q_1), \ldots, d(p, q_n)$]. The minimum of a finite set of positive numbers is clearly positive, so that $r > 0$.

The neighborhood $N_r(p)$ contains no point q of E such that $q \neq p$, so that p is not a limit point of E. This contradiction establishes the theorem.

Corollary *A finite point set has no limit points.*

2.21 Examples Let us consider the following subsets of R^2:

(a) The set of all complex z such that $|z| < 1$.
(b) The set of all complex z such that $|z| \leq 1$.
(c) A nonempty finite set.
(d) The set of all integers.
(e) The set consisting of the numbers $1/n$ ($n = 1, 2, 3, \ldots$). Let us note that this set E has a limit point (namely, $z = 0$) but that no point of E is a limit point of E; we wish to stress the difference between having a limit point and containing one.
(f) The set of all complex numbers (that is, R^2).
(g) The segment (a, b).

Let us note that (d), (e), (g) can be regarded also as subsets of R^1.

Some properties of these sets are tabulated below:

	Closed	*Open*	*Perfect*	*Bounded*
(a)	No	Yes	No	Yes
(b)	Yes	No	Yes	Yes
(c)	Yes	No	No	Yes
(d)	Yes	No	No	No
(e)	No	No	No	Yes
(f)	Yes	Yes	Yes	No
(g)	No		No	Yes

In (g), we left the second entry blank. The reason is that the segment (a, b) is not open if we regard it as a subset of R^2, but it is an open subset of R^1.

2.22 Theorem *Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α. Then*

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c). \tag{20}$$

Proof Let A and B be the left and right members of (20). If $x \in A$, then $x \notin \bigcup_\alpha E_\alpha$, hence $x \notin E_\alpha$ for any α, hence $x \in E_\alpha^c$ for every α, so that $x \in \bigcap E_\alpha^c$. Thus $A \subset B$.

Conversely, if $x \in B$, then $x \in E_\alpha^c$ for every α, hence $x \notin E_\alpha$ for any α, hence $x \notin \bigcup_\alpha E_\alpha$, so that $x \in (\bigcup_\alpha E_\alpha)^c$. Thus $B \subset A$.

It follows that $A = B$.

2.23 Theorem *A set E is open if and only if its complement is closed.*

Proof First, suppose E^c is closed. Choose $x \in E$. Then $x \notin E^c$, and x is not a limit point of E^c. Hence there exists a neighborhood N of x such that $E^c \cap N$ is empty, that is, $N \subset E$. Thus x is an interior point of E, and E is open.

Next, suppose E is open. Let x be a limit point of E^c. Then every neighborhood of x contains a point of E^c, so that x is not an interior point of E. Since E is open, this means that $x \in E^c$. It follows that E^c is closed.

Corollary *A set F is closed if and only if its complement is open.*

2.24 Theorem

(*a*) *For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.*
(*b*) *For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.*
(*c*) *For any finite collection $G_1, \ldots, G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.*
(*d*) *For any finite collection $F_1, \ldots, F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.*

Proof Put $G = \bigcup_\alpha G_\alpha$. If $x \in G$, then $x \in G_\alpha$ for some α. Since x is an interior point of G_α, x is also an interior point of G, and G is open. This proves (*a*).

By Theorem 2.22,

$$\left(\bigcap_\alpha F_\alpha\right)^c = \bigcup_\alpha (F_\alpha^c), \tag{21}$$

and F_α^c is open, by Theorem 2.23. Hence (*a*) implies that (21) is open so that $\bigcap_\alpha F_\alpha$ is closed.

Next, put $H = \bigcap_{i=1}^n G_i$. For any $x \in H$, there exist neighborhoods N_i of x, with radii r_i, such that $N_i \subset G_i$ $(i = 1, \ldots, n)$. Put

$$r = \min(r_1, \ldots, r_n),$$

and let N be the neighborhood of x of radius r. Then $N \subset G_i$ for $i = 1, \ldots, n$, so that $N \subset H$, and H is open.

By taking complements, (*d*) follows from (*c*):

$$\left(\bigcup_{i=1}^n F_i\right)^c = \bigcap_{i=1}^n (F_i^c).$$

2.25 Examples In parts (*c*) and (*d*) of the preceding theorem, the finiteness of the collections is essential. For let G_n be the segment $\left(-\frac{1}{n}, \frac{1}{n}\right)$ $(n = 1, 2, 3, \ldots)$. Then G_n is an open subset of R^1. Put $G = \bigcap_{n=1}^\infty G_n$. Then G consists of a single point (namely, $x = 0$) and is therefore not an open subset of R^1.

Thus the intersection of an infinite collection of open sets need not be open. Similarly, the union of an infinite collection of closed sets need not be closed.

2.26 Definition If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X, then the *closure* of E is the set $\bar{E} = E \cup E'$.

2.27 Theorem *If X is a metric space and $E \subset X$, then*

(*a*) *$\bar{E}$ is closed,*
(*b*) *$E = \bar{E}$ if and only if E is closed,*
(*c*) *$\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.*

By (*a*) and (*c*), $\bar{E}$ is the *smallest* closed subset of X that contains E.

Proof

(*a*) If $p \in X$ and $p \notin \bar{E}$ then p is neither a point of E nor a limit point of E. Hence p has a neighborhood which does not intersect E. The complement of $\bar{E}$ is therefore open. Hence $\bar{E}$ is closed.

(*b*) If $E = \bar{E}$, (*a*) implies that E is closed. If E is closed, then $E' \subset E$ [by Definitions 2.18(*d*) and 2.26], hence $\bar{E} = E$.

(*c*) If F is closed and $F \supset E$, then $F \supset F'$, hence $F \supset E'$. Thus $F \supset \bar{E}$.

2.28 Theorem *Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.*

Compare this with the examples in Sec. 1.9.

Proof If $y \in E$ then $y \in \bar{E}$. Assume $y \notin E$. For every $h > 0$ there exists then a point $x \in E$ such that $y - h < x < y$, for otherwise $y - h$ would be an upper bound of E. Thus y is a limit point of E. Hence $y \in \bar{E}$.

2.29 Remark Suppose $E \subset Y \subset X$, where X is a metric space. To say that E is an open subset of X means that to each point $p \in E$ there is associated a positive number r such that the conditions $d(p, q) < r$, $q \in X$ imply that $q \in E$. But we have already observed (Sec. 2.16) that Y is also a metric space, so that our definitions may equally well be made within Y. To be quite explicit, let us say that E is *open relative to* Y if to each $p \in E$ there is associated an $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$. Example 2.21(*g*) showed that a set finite

$A = \{1, 2, 3, 4, 5\}$

may be open relative to Y without being an open subset of X. However, there is a simple relation between these concepts, which we now state.

2.30 Theorem *Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.*

Proof Suppose E is open relative to Y. To each $p \in E$ there is a positive number r_p such that the conditions $d(p, q) < r_p$, $q \in Y$ imply that $q \in E$. Let V_p be the set of all $q \in X$ such that $d(p, q) < r_p$, and define

$$G = \bigcup_{p \in E} V_p.$$

Then G is an open subset of X, by Theorems 2.19 and 2.24.

Since $p \in V_p$ for all $p \in E$, it is clear that $E \subset G \cap Y$.

By our choice of V_p, we have $V_p \cap Y \subset E$ for every $p \in E$, so that $G \cap Y \subset E$. Thus $E = G \cap Y$, and one half of the theorem is proved.

Conversely, if G is open in X and $E = G \cap Y$, every $p \in E$ has a neighborhood $V_p \subset G$. Then $V_p \cap Y \subset E$, so that E is open relative to Y.

COMPACT SETS

2.31 Definition By an *open cover* of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.

2.32 Definition A subset K of a metric space X is said to be *compact* if every open cover of K contains a *finite* subcover.

More explicitly, the requirement is that if $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}.$$

The notion of compactness is of great importance in analysis, especially in connection with continuity (Chap. 4).

It is clear that every finite set is compact. The existence of a large class of infinite compact sets in R^k will follow from Theorem 2.41.

We observed earlier (in Sec. 2.29) that if $E \subset Y \subset X$, then E may be open relative to Y without being open relative to X. The property of being open thus depends on the space in which E is embedded. The same is true of the property of being closed.

Compactness, however, behaves better, as we shall now see. To formulate the next theorem, let us say, temporarily, that K is compact relative to X if the requirements of Definition 2.32 are met.

$I = \{a_1, a_2 \ldots a_n\}$

2.33 Theorem *Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.*

By virtue of this theorem we are able, in many situations, to regard compact sets as metric spaces in their own right, without paying any attention to any embedding space. In particular, although it makes little sense to talk of *open* spaces, or of *closed* spaces (every metric space X is an open subset of itself, and is a closed subset of itself), it does make sense to talk of *compact* metric spaces.

Proof Suppose K is compact relative to X, and let $\{V_\alpha\}$ be a collection of sets, open relative to Y, such that $K \subset \bigcup_\alpha V_\alpha$. By theorem 2.30, there are sets G_α, open relative to X, such that $V_\alpha = Y \cap G_\alpha$, for all α; and since K is compact relative to X, we have

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} \tag{22}$$

for some choice of finitely many indices $\alpha_1, \ldots, \alpha_n$. Since $K \subset Y$, (22) implies

$$K \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}. \tag{23}$$

This proves that K is compact relative to Y.

Conversely, suppose K is compact relative to Y, let $\{G_\alpha\}$ be a collection of open subsets of X which covers K, and put $V_\alpha = Y \cap G_\alpha$. Then (23) will hold for some choice of $\alpha_1, \ldots, \alpha_n$; and since $V_\alpha \subset G_\alpha$, (23) implies (22).

This completes the proof.

2.34 Theorem *Compact subsets of metric spaces are closed.*

SM

Proof Let K be a compact subset of a metric space X. We shall prove that the complement of K is an open subset of X.

Suppose $p \in X$, $p \notin K$. If $q \in K$, let V_q and W_q be neighborhoods of p and q, respectively, of radius less than $\frac{1}{2}d(p, q)$ [see Definition 2.18(*a*)]. Since K is compact, there are finitely many points $q_1, \ldots, q_n$ in K such that

$$K \subset W_{q_1} \cup \cdots \cup W_{q_n} = W.$$

If $V = V_{q_1} \cap \cdots \cap V_{q_n}$, then V is a neighborhood of p which does not intersect W. Hence $V \subset K^c$, so that p is an interior point of K^c. The theorem follows.

2.35 Theorem *Closed subsets of compact sets are compact.*

SM

Proof Suppose $F \subset K \subset X$, F is closed (relative to X), and K is compact. Let $\{V_\alpha\}$ be an open cover of F. If F^c is adjoined to $\{V_\alpha\}$, we obtain an open cover Ω of K. Since K is compact, there is a finite subcollection Φ of Ω which covers K, and hence F. If F^c is a member of Φ, we may remove it from Φ and still retain an open cover of F. We have thus shown that a finite subcollection of $\{V_\alpha\}$ covers F.

Corollary *If F is closed and K is compact, then $F \cap K$ is compact.*

Proof Theorems 2.24(*b*) and 2.34 show that $F \cap K$ is closed; since $F \cap K \subset K$, Theorem 2.35 shows that $F \cap K$ is compact.

2.36 Theorem *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.*

Proof Fix a member K_1 of $\{K_\alpha\}$ and put $G_\alpha = K_\alpha^c$. Assume that no point of K_1 belongs to every K_α. Then the sets G_α form an open cover of K_1; and since K_1 is compact, there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that $K_1 \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$. But this means that

$$K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n}$$

is empty, in contradiction to our hypothesis.

Corollary *If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ ($n = 1, 2, 3, \ldots$), then $\bigcap_1^\infty K_n$ is not empty.*

2.37 Theorem *If E is an infinite subset of a compact set K, then E has a limit point in K.*

Proof If no point of K were a limit point of E, then each $q \in K$ would have a neighborhood V_q which contains at most one point of E (namely, q, if $q \in E$). It is clear that no finite subcollection of $\{V_q\}$ can cover E; and the same is true of K, since $E \subset K$. This contradicts the compactness of K.

2.38 Theorem *If $\{I_n\}$ is a sequence of intervals in R^1, such that $I_n \supset I_{n+1}$ ($n = 1, 2, 3, \ldots$), then $\bigcap_1^\infty I_n$ is not empty.*

Proof If $I_n = [a_n, b_n]$, let E be the set of all a_n. Then E is nonempty and bounded above (by b_1). Let x be the sup of E. If m and n are positive integers, then

$$a_n \le a_{m+n} \le b_{m+n} \le b_m,$$

so that $x \le b_m$ for each m. Since it is obvious that $a_m \le x$, we see that $x \in I_m$ for $m = 1, 2, 3, \ldots$.

2.39 Theorem *Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$ ($n = 1, 2, 3, \ldots$), then $\bigcap_1^\infty I_n$ is not empty.*

Proof Let I_n consist of all points $\mathbf{x} = (x_1, \ldots, x_k)$ such that

$$a_{n,j} \le x_j \le b_{n,j} \qquad (1 \le j \le k; n = 1, 2, 3, \ldots),$$

and put $I_{n,j} = [a_{n,j}, b_{n,j}]$. For each j, the sequence $\{I_{n,j}\}$ satisfies the hypotheses of Theorem 2.38. Hence there are real numbers x_j^* ($1 \le j \le k$) such that

$$a_{n,j} \le x_j^* \le b_{n,j} \qquad (1 \le j \le k; n = 1, 2, 3, \ldots).$$

Setting $\mathbf{x}^* = (x_1^*, \ldots, x_k^*)$, we see that $\mathbf{x}^* \in I_n$ for $n = 1, 2, 3, \ldots$. The theorem follows.

2.40 Theorem *Every k-cell is compact.*

Proof Let I be a k-cell, consisting of all points $\mathbf{x} = (x_1, \ldots, x_k)$ such that $a_j \le x_j \le b_j$ ($1 \le j \le k$). Put

$$\delta = \left\{ \sum_1^k (b_j - a_j)^2 \right\}^{1/2}.$$

Then $|\mathbf{x} - \mathbf{y}| \le \delta$, if $\mathbf{x} \in I$, $\mathbf{y} \in I$.

Suppose, to get a contradiction, that there exists an open cover $\{G_\alpha\}$ of I which contains no finite subcover of I. Put $c_j = (a_j + b_j)/2$. The intervals $[a_j, c_j]$ and $[c_j, b_j]$ then determine 2^k k-cells Q_i whose union is I. At least one of these sets Q_i, call it I_1, cannot be covered by any finite subcollection of $\{G_\alpha\}$ (otherwise I could be so covered). We next subdivide I_1 and continue the process. We obtain a sequence $\{I_n\}$ with the following properties:

(*a*) $I \supset I_1 \supset I_2 \supset I_3 \supset \cdots$;
(*b*) I_n is not covered by any finite subcollection of $\{G_\alpha\}$;
(*c*) if $\mathbf{x} \in I_n$ and $\mathbf{y} \in I_n$, then $|\mathbf{x} - \mathbf{y}| \le 2^{-n}\delta$.

By (*a*) and Theorem 2.39, there is a point $\mathbf{x}^*$ which lies in every I_n. For some α, $\mathbf{x}^* \in G_\alpha$. Since G_α is open, there exists $r > 0$ such that $|\mathbf{y} - \mathbf{x}^*| < r$ implies that $\mathbf{y} \in G_\alpha$. If n is so large that $2^{-n}\delta < r$ (there is such an n, for otherwise $2^n \le \delta/r$ for all positive integers n, which is absurd since R is archimedean), then (*c*) implies that $I_n \subset G_\alpha$, which contradicts (*b*).

This completes the proof.

The equivalence of (*a*) and (*b*) in the next theorem is known as the Heine-Borel theorem.

2.41 Theorem *If a set E in R^k has one of the following three properties, then it has the other two:*

(*a*) *E is closed and bounded.*
(*b*) *E is compact.*
(*c*) *Every infinite subset of E has a limit point in E.*

Proof If (*a*) holds, then $E \subset I$ for some k-cell I, and (*b*) follows from Theorems 2.40 and 2.35. Theorem 2.37 shows that (*b*) implies (*c*). It remains to be shown that (*c*) implies (*a*).

If E is not bounded, then E contains points $\mathbf{x}_n$ with

$$|\mathbf{x}_n| > n \qquad (n = 1, 2, 3, \ldots).$$

The set S consisting of these points $\mathbf{x}_n$ is infinite and clearly has no limit point in R^k, hence has none in E. Thus (*c*) implies that E is bounded.

If E is not closed, then there is a point $\mathbf{x}_0 \in R^k$ which is a limit point of E but not a point of E. For $n = 1, 2, 3, \ldots$, there are points $\mathbf{x}_n \in E$ such that $|\mathbf{x}_n - \mathbf{x}_0| < 1/n$. Let S be the set of these points $\mathbf{x}_n$. Then S is infinite (otherwise $|\mathbf{x}_n - \mathbf{x}_0|$ would have a constant positive value, for infinitely many n), S has $\mathbf{x}_0$ as a limit point, and S has no other limit point in R^k. For if $\mathbf{y} \in R^k$, $\mathbf{y} \neq \mathbf{x}_0$, then

$$\begin{aligned} |\mathbf{x}_n - \mathbf{y}| &\geq |\mathbf{x}_0 - \mathbf{y}| - |\mathbf{x}_n - \mathbf{x}_0| \\ &\geq |\mathbf{x}_0 - \mathbf{y}| - \frac{1}{n} \geq \frac{1}{2}|\mathbf{x}_0 - \mathbf{y}| \end{aligned}$$

for all but finitely many n; this shows that $\mathbf{y}$ is not a limit point of S (Theorem 2.20).

Thus S has no limit point in E; hence E must be closed if (*c*) holds.

We should remark, at this point, that (*b*) and (*c*) are equivalent in any metric space (Exercise 26) but that (*a*) does not, in general, imply (*b*) and (*c*). Examples are furnished by Exercise 16 and by the space $\mathscr{L}^2$, which is discussed in Chap. 11.

2.42 Theorem (Weierstrass) *Every bounded infinite subset of R^k has a limit point in R^k.*

Proof Being bounded, the set E in question is a subset of a k-cell $I \subset R^k$. By Theorem 2.40, I is compact, and so E has a limit point in I, by Theorem 2.37.

PERFECT SETS

2.43 Theorem *Let P be a nonempty perfect set in R^k. Then P is uncountable.*

Proof Since P has limit points, P must be infinite. Suppose P is countable, and denote the points of P by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$. We shall construct a sequence $\{V_n\}$ of neighborhoods, as follows.

Let V_1 be any neighborhood of $\mathbf{x}_1$. If V_1 consists of all $\mathbf{y} \in R^k$ such that $|\mathbf{y} - \mathbf{x}_1| < r$, the closure $\bar{V}_1$ of V_1 is the set of all $\mathbf{y} \in R^k$ such that $|\mathbf{y} - \mathbf{x}_1| \leq r$.

Suppose V_n has been constructed, so that $V_n \cap P$ is not empty. Since every point of P is a limit point of P, there is a neighborhood V_{n+1} such that (i) $\bar{V}_{n+1} \subset V_n$, (ii) $\mathbf{x}_n \notin \bar{V}_{n+1}$, (iii) $V_{n+1} \cap P$ is not empty. By (iii), V_{n+1} satisfies our induction hypothesis, and the construction can proceed.

Put $K_n = \bar{V}_n \cap P$. Since $\bar{V}_n$ is closed and bounded, $\bar{V}_n$ is compact. Since $\mathbf{x}_n \notin K_{n+1}$, no point of P lies in $\bigcap_1^\infty K_n$. Since $K_n \subset P$, this implies that $\bigcap_1^\infty K_n$ is empty. But each K_n is nonempty, by (iii), and $K_n \supset K_{n+1}$, by (i); this contradicts the Corollary to Theorem 2.36.

Corollary *Every interval $[a, b]$ $(a < b)$ is uncountable. In particular, the set of all real numbers is uncountable.*

2.44 The Cantor set The set which we are now going to construct shows that there exist perfect sets in R^1 which contain no segment.

Let E_0 be the interval $[0, 1]$. Remove the segment $(\frac{1}{3}, \frac{2}{3})$, and let E_1 be the union of the intervals

$$[0, \tfrac{1}{3}]\ [\tfrac{2}{3}, 1].$$

Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$[0, \tfrac{1}{9}], [\tfrac{2}{9}, \tfrac{3}{9}], [\tfrac{6}{9}, \tfrac{7}{9}], [\tfrac{8}{9}, 1].$$

Continuing in this way, we obtain a sequence of compact sets E_n, such that

(*a*) $E_1 \supset E_2 \supset E_3 \supset \cdots$;
(*b*) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set

$$P = \bigcap_{n=1}^{\infty} E_n$$

is called the *Cantor set*. P is clearly compact, and Theorem 2.36 shows that P is not empty.

No segment of the form

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right), \tag{24}$$

where k and m are positive integers, has a point in common with P. Since every segment (α, β) contains a segment of the form (24), if

$$3^{-m} < \frac{\beta - \alpha}{6},$$

P contains no segment.

To show that P is perfect, it is enough to show that P contains no isolated point. Let $x \in P$, and let S be any segment containing x. Let I_n be that interval of E_n which contains x. Choose n large enough, so that $I_n \subset S$. Let x_n be an endpoint of I_n, such that $x_n \neq x$.

It follows from the construction of P that $x_n \in P$. Hence x is a limit point of P, and P is perfect.

One of the most interesting properties of the Cantor set is that it provides us with an example of an uncountable set of measure zero (the concept of measure will be discussed in Chap. 11).

CONNECTED SETS

2.45 Definition Two subsets A and B of a metric space X are said to be *separated* if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty, i.e., if no point of A lies in the closure of B and no point of B lies in the closure of A.

A set $E \subset X$ is said to be *connected* if E is *not* a union of two nonempty separated sets.

2.46 Remark Separated sets are of course disjoint, but disjoint sets need not be separated. For example, the interval $[0, 1]$ and the segment $(1, 2)$ are *not* separated, since 1 is a limit point of $(1, 2)$. However, the segments $(0, 1)$ and $(1, 2)$ *are* separated.

The connected subsets of the line have a particularly simple structure:

2.47 Theorem *A subset E of the real line R^1 is connected if and only if it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.*

Proof If there exist $x \in E$, $y \in E$, and some $z \in (x, y)$ such that $z \notin E$, then $E = A_z \cup B_z$ where

$$A_z = E \cap (-\infty, z), \qquad B_z = E \cap (z, \infty).$$

Since $x \in A_z$ and $y \in B_z$, A and B are nonempty. Since $A_z \subset (-\infty, z)$ and $B_z \subset (z, \infty)$, they are separated. Hence E is not connected.

To prove the converse, suppose E is not connected. Then there are nonempty separated sets A and B such that $A \cup B = E$. Pick $x \in A$, $y \in B$, and assume (without loss of generality) that $x < y$. Define

$$z = \sup (A \cap [x, y]).$$

By Theorem 2.28, $z \in \bar{A}$; hence $z \notin B$. In particular, $x \le z < y$.

If $z \notin A$, it follows that $x < z < y$ and $z \notin E$.

If $z \in A$, then $z \notin \bar{B}$, hence there exists z_1 such that $z < z_1 < y$ and $z_1 \notin B$. Then $x < z_1 < y$ and $z_1 \notin E$.

EXERCISES

1. Prove that the empty set is a subset of every set.
2. A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that
$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$
Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with
$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$
3. Prove that there exist real numbers which are not algebraic.
4. Is the set of all irrational real numbers countable?
5. Construct a bounded set of real numbers with exactly three limit points.
6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?
7. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.
   (*a*) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^n \bar{A}_i$, for $n = 1, 2, 3, \ldots$.
   (*b*) If $B = \bigcup_{i=1}^\infty A_i$, prove that $\bar{B} \supset \bigcup_{i=1}^\infty \bar{A}_i$.
   Show, by an example, that this inclusion can be proper.
8. Is every point of every open set $E \subset R^2$ a limit point of E? Answer the same question for closed sets in R^2.
9. Let E° denote the set of all interior points of a set E. [See Definition 2.18(*e*); E° is called the *interior* of E.]
   (*a*) Prove that E° is always open.
   (*b*) Prove that E is open if and only if $E^\circ = E$.
   (*c*) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.
   (*d*) Prove that the complement of E° is the closure of the complement of E.
   (*e*) Do E and $\bar{E}$ always have the same interiors?
   (*f*) Do E and E° always have the same closures?

10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

11. For $x \in R^1$ and $y \in R^1$, define

$$d_1(x, y) = (x - y)^2,$$
$$d_2(x, y) = \sqrt{|x - y|},$$
$$d_3(x, y) = |x^2 - y^2|,$$
$$d_4(x, y) = |x - 2y|,$$
$$d_5(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

Determine, for each of these, whether it is a metric or not.

12. Let $K \subset R^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

13. Construct a compact set of real numbers whose limit points form a countable set.

14. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

15. Show that Theorem 2.36 and its Corollary become false (in R^1, for example) if the word "compact" is replaced by "closed" or by "bounded."

16. Regard Q, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in Q$ such that $2 < p^2 < 3$. Show that E is closed and bounded in Q, but that E is not compact. Is E open in Q?

17. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?

18. Is there a nonempty perfect set in R^1 which contains no rational number?

19. (*a*) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

(*b*) Prove the same for disjoint open sets.

(*c*) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.

(*d*) Prove that every connected metric space with at least two points is uncountable. *Hint:* Use (*c*).

20. Are closures and interiors of connected sets always connected? (Look at subsets of R^2.)

21. Let A and B be separated subsets of some R^k, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

for $t \in R^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

(*a*) Prove that A_0 and B_0 are separated subsets of R^1.

(*b*) Prove that there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.

(*c*) Prove that every convex subset of R^k is connected.

22. A metric space is called *separable* if it contains a countable dense subset. Show that R^k is separable. *Hint:* Consider the set of points which have only rational coordinates.

23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a *countable* base. *Hint:* Take all neighborhoods with rational radius and center in some countable dense subset of X.

24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint:* Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \ldots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = 1/n$ $(n = 1, 2, 3, \ldots)$, and consider the centers of the corresponding neighborhoods.

25. Prove that every compact metric space K has a countable base, and that K is therefore separable. *Hint:* For every positive integer n, there are finitely many neighborhoods of radius $1/n$ whose union covers K.

26. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. *Hint:* By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a *countable* subcover $\{G_n\}$, $n = 1, 2, 3, \ldots$. If no finite subcollection of $\{G_n\}$ covers X, then the complement F_n of $G_1 \cup \cdots \cup G_n$ is nonempty for each n, but $\bigcap F_n$ is empty. If E is a set which contains a point from each F_n, consider a limit point of E, and obtain a contradiction.

27. Define a point p in a metric space X to be a *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E.

Suppose $E \subset R^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. *Hint:* Let $\{V_n\}$ be a countable base of R^k, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary:* Every countable closed set in R^k has isolated points.) *Hint:* Use Exercise 27.

29. Prove that every open set in R^1 is the union of an at most countable collection of disjoint segments. *Hint:* Use Exercise 22.

30. Imitate the proof of Theorem 2.43 to obtain the following result:

If $R^k = \bigcup_1^\infty F_n$, where each F_n is a closed subset of R^k, then at least one F_n has a nonempty interior.

Equivalent statement: If G_n is a dense open subset of R^k, for $n = 1, 2, 3, \ldots$, then $\bigcap_1^\infty G_n$ is not empty (in fact, it is dense in R^k).

(This is a special case of Baire's theorem; see Exercise 22, Chap. 3, for the general case.)

3

NUMERICAL SEQUENCES AND SERIES

As the title indicates, this chapter will deal primarily with sequences and series of complex numbers. The basic facts about convergence, however, are just as easily explained in a more general setting. The first three sections will therefore be concerned with sequences in euclidean spaces, or even in metric spaces.

CONVERGENT SEQUENCES

3.1 Definition A sequence $\{p_n\}$ in a metric space X is said to *converge* if there is a point $p \in X$ with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$. (Here d denotes the distance in X.)

In this case we also say that $\{p_n\}$ converges to p, or that p is the limit of $\{p_n\}$ [see Theorem 3.2(*b*)], and we write $p_n \to p$, or

$$\lim_{n \to \infty} p_n = p.$$

If $\{p_n\}$ does not converge, it is said to *diverge*.

It might be well to point out that our definition of "convergent sequence" depends not only on $\{p_n\}$ but also on X; for instance, the sequence $\{1/n\}$ converges in R^1 (to 0), but fails to converge in the set of all positive real numbers [with $d(x, y) = |x - y|$]. In cases of possible ambiguity, we can be more precise and specify "convergent in X" rather than "convergent."

We recall that the set of all points p_n $(n = 1, 2, 3, \ldots)$ is the *range* of $\{p_n\}$. The range of a sequence may be a finite set, or it may be infinite. The sequence $\{p_n\}$ is said to be *bounded* if its range is bounded.

As examples, consider the following sequences of complex numbers (that is, $X = R^2$):

(*a*) If $s_n = 1/n$, then $\lim_{n\to\infty} s_n = 0$; the range is infinite, and the sequence is bounded.

(*b*) If $s_n = n^2$, the sequence $\{s_n\}$ is unbounded, is divergent, and has infinite range.

(*c*) If $s_n = 1 + [(-1)^n/n]$, the sequence $\{s_n\}$ converges to 1, is bounded, and has infinite range.

(*d*) If $s_n = i^n$, the sequence $\{s_n\}$ is divergent, is bounded, and has finite range.

(*e*) If $s_n = 1$ $(n = 1, 2, 3, \ldots)$, then $\{s_n\}$ converges to 1, is bounded, and has finite range.

We now summarize some important properties of convergent sequences in metric spaces.

3.2 Theorem *Let $\{p_n\}$ be a sequence in a metric space X.*

(*a*) *$\{p_n\}$ converges to $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many n.*

(*b*) *If $p \in X$, $p' \in X$, and if $\{p_n\}$ converges to p and to p', then $p' = p$.*

(*c*) *If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.*

(*d*) *If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$.*

Proof (*a*) Suppose $p_n \to p$ and let V be a neighborhood of p. For some $\varepsilon > 0$, the conditions $d(q, p) < \varepsilon$, $q \in X$, imply $q \in V$. Corresponding to this ε, there exists N such that $n \ge N$ implies $d(p_n, p) < \varepsilon$. Thus $n \ge N$ implies $p_n \in V$.

Conversely, suppose every neighborhood of p contains all but finitely many of the p_n. Fix $\varepsilon > 0$, and let V be the set of all $q \in X$ such that $d(p, q) < \varepsilon$. By assumption, there exists N (corresponding to this V) such that $p_n \in V$ if $n \ge N$. Thus $d(p_n, p) < \varepsilon$ if $n \ge N$; hence $p_n \to p$.

(*b*) Let $\varepsilon > 0$ be given. There exist integers N, N' such that

$$n \ge N \quad \text{implies} \quad d(p_n, p) < \frac{\varepsilon}{2},$$

$$n \ge N' \quad \text{implies} \quad d(p_n, p') < \frac{\varepsilon}{2}.$$

Hence if $n \ge \max(N, N')$, we have

$$d(p, p') \le d(p, p_n) + d(p_n, p') < \varepsilon.$$

Since ε was arbitrary, we conclude that $d(p, p') = 0$.

(*c*) Suppose $p_n \to p$. There is an integer N such that $n > N$ implies $d(p_n, p) < 1$. Put

$$r = \max\{1, d(p_1, p), \ldots, d(p_N, p)\}.$$

Then $d(p_n, p) \le r$ for $n = 1, 2, 3, \ldots$.

(*d*) For each positive integer n, there is a point $p_n \in E$ such that $d(p_n, p) < 1/n$. Given $\varepsilon > 0$, choose N so that $N\varepsilon > 1$. If $n > N$, it follows that $d(p_n, p) < \varepsilon$. Hence $p_n \to p$.

This completes the proof.

For sequences in R^k we can study the relation between convergence, on the one hand, and the algebraic operations on the other. We first consider sequences of complex numbers.

3.3 Theorem *Suppose $\{s_n\}$, $\{t_n\}$ are complex sequences, and $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$. Then*

(*a*) $\lim_{n\to\infty} (s_n + t_n) = s + t$;

(*b*) $\lim_{n\to\infty} cs_n = cs$, $\lim_{n\to\infty} (c + s_n) = c + s$, *for any number c*;

(*c*) $\lim_{n\to\infty} s_n t_n = st$;

(*d*) $\lim_{n\to\infty} \dfrac{1}{s_n} = \dfrac{1}{s}$, *provided $s_n \ne 0$ $(n = 1, 2, 3, \ldots)$, and $s \ne 0$.*

Proof

(*a*) Given $\varepsilon > 0$, there exist integers N_1, N_2 such that

$$n \ge N_1 \quad \text{implies} \quad |s_n - s| < \frac{\varepsilon}{2},$$

$$n \ge N_2 \quad \text{implies} \quad |t_n - t| < \frac{\varepsilon}{2}.$$

If $N = \max(N_1, N_2)$, then $n \geq N$ implies

$$|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \varepsilon.$$

This proves (*a*). The proof of (*b*) is trivial.

(*c*) We use the identity

(1) $$s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s).$$

Given $\varepsilon > 0$, there are integers N_1, N_2 such that

$$n \geq N_1 \quad \text{implies} \quad |s_n - s| < \sqrt{\varepsilon},$$
$$n \geq N_2 \quad \text{implies} \quad |t_n - t| < \sqrt{\varepsilon}.$$

If we take $N = \max(N_1, N_2)$, $n \geq N$ implies

$$|(s_n - s)(t_n - t)| < \varepsilon,$$

so that

$$\lim_{n \to \infty} (s_n - s)(t_n - t) = 0.$$

We now apply (*a*) and (*b*) to (1), and conclude that

$$\lim_{n \to \infty} (s_n t_n - st) = 0.$$

(*d*) Choosing m such that $|s_n - s| < \frac{1}{2}|s|$ if $n \geq m$, we see that

$$|s_n| > \tfrac{1}{2}|s| \qquad (n \geq m).$$

Given $\varepsilon > 0$, there is an integer $N > m$ such that $n \geq N$ implies

$$|s_n - s| < \tfrac{1}{2}|s|^2 \varepsilon.$$

Hence, for $n \geq N$,

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| = \left| \frac{s_n - s}{s_n s} \right| < \frac{2}{|s|^2} |s_n - s| < \varepsilon.$$

3.4 Theorem

(*a*) *Suppose* $\mathbf{x}_n \in R^k$ $(n = 1, 2, 3, \ldots)$ *and*

$$\mathbf{x}_n = (\alpha_{1,n}, \ldots, \alpha_{k,n}).$$

Then $\{\mathbf{x}_n\}$ *converges to* $\mathbf{x} = (\alpha_1, \ldots, \alpha_k)$ *if and only if*

(2) $$\lim_{n \to \infty} \alpha_{j,n} = \alpha_j \qquad (1 \leq j \leq k).$$

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(*b*) *Suppose* $\{\mathbf{x}_n\}, \{\mathbf{y}_n\}$ *are sequences in* R^k, $\{\beta_n\}$ *is a sequence of real numbers, and* $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{y}_n \to \mathbf{y}$, $\beta_n \to \beta$. *Then*

$$\lim_{n \to \infty} (\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y}, \qquad \lim_{n \to \infty} \mathbf{x}_n \cdot \mathbf{y}_n = \mathbf{x} \cdot \mathbf{y}, \qquad \lim_{n \to \infty} \beta_n \mathbf{x}_n = \beta \mathbf{x}.$$

Proof

(*a*) If $\mathbf{x}_n \to \mathbf{x}$, the inequalities

$$|\alpha_{j,n} - \alpha_j| \leq |\mathbf{x}_n - \mathbf{x}|,$$

which follow immediately from the definition of the norm in R^k, show that (2) holds.

Conversely, if (2) holds, then to each $\varepsilon > 0$ there corresponds an integer N such that $n \geq N$ implies

$$|\alpha_{j,n} - \alpha_j| < \frac{\varepsilon}{\sqrt{k}} \qquad (1 \leq j \leq k).$$

Hence $n \geq N$ implies

$$|\mathbf{x}_n - \mathbf{x}| = \left\{ \sum_{j=1}^{k} |\alpha_{j,n} - \alpha_j|^2 \right\}^{1/2} < \varepsilon,$$

so that $\mathbf{x}_n \to \mathbf{x}$. This proves (*a*).

Part (*b*) follows from (*a*) and Theorem 3.3.

SUBSEQUENCES

3.5 Definition Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\{p_{n_i}\}$ is called a *subsequence* of $\{p_n\}$. If $\{p_{n_i}\}$ converges, its limit is called a *subsequential limit* of $\{p_n\}$.

It is clear that $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p. We leave the details of the proof to the reader.

3.6 Theorem

(*a*) *If* $\{p_n\}$ *is a sequence in a compact metric space* X, *then some subsequence of* $\{p_n\}$ *converges to a point of* X.

(*b*) *Every bounded sequence in* R^k *contains a convergent subsequence.*

Proof

(*a*) Let E be the range of $\{p_n\}$. If E is finite then there is a $p \in E$ and a sequence $\{n_i\}$ with $n_1 < n_2 < n_3 < \cdots$, such that

$$p_{n_1} = p_{n_2} = \cdots = p.$$

The subsequence $\{p_{n_i}\}$ so obtained converges evidently to p.

If E is infinite, Theorem 2.37 shows that E has a limit point $p \in X$. Choose n_1 so that $d(p, p_{n_1}) < 1$. Having chosen $n_1, \ldots, n_{i-1}$, we see from Theorem 2.20 that there is an integer $n_i > n_{i-1}$ such that $d(p, p_{n_i}) < 1/i$. Then $\{p_{n_i}\}$ converges to p.

(*b*) This follows from (*a*), since Theorem 2.41 implies that every bounded subset of R^k lies in a compact subset of R^k.

3.7 Theorem *The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.*

Proof Let E^* be the set of all subsequential limits of $\{p_n\}$ and let q be a limit point of E^*. We have to show that $q \in E^*$.

Choose n_1 so that $p_{n_1} \neq q$. (If no such n_1 exists, then E^* has only one point, and there is nothing to prove.) Put $\delta = d(q, p_{n_1})$. Suppose $n_1, \ldots, n_{i-1}$ are chosen. Since q is a limit point of E^*, there is an $x \in E^*$ with $d(x, q) < 2^{-i}\delta$. Since $x \in E^*$, there is an $n_i > n_{i-1}$ such that $d(x, p_{n_i}) < 2^{-i}\delta$. Thus

$$d(q, p_{n_i}) \le 2^{1-i}\delta$$

for $i = 1, 2, 3, \ldots$. This says that $\{p_{n_i}\}$ converges to q. Hence $q \in E^*$.

CAUCHY SEQUENCES

3.8 Definition A sequence $\{p_n\}$ in a metric space X is said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there is an integer N such that $d(p_n, p_m) < \varepsilon$ if $n \ge N$ and $m \ge N$.

In our discussion of Cauchy sequences, as well as in other situations which will arise later, the following geometric concept will be useful.

3.9 Definition Let E be a nonempty subset of a metric space X, and let S be the set of all real numbers of the form $d(p, q)$, with $p \in E$ and $q \in E$. The sup of S is called the *diameter* of E.

If $\{p_n\}$ is a sequence in X and if E_N consists of the points $p_N, p_{N+1}, p_{N+2}, \ldots$, it is clear from the two preceding definitions that *$\{p_n\}$ is a Cauchy sequence if and only if*

$$\lim_{N\to\infty} \operatorname{diam} E_N = 0.$$

3.10 Theorem

(*a*) *If $\bar{E}$ is the closure of a set E in a metric space X, then*

$$\operatorname{diam} \bar{E} = \operatorname{diam} E.$$

(*b*) *If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \ldots)$ and if*

$$\lim_{n\to\infty} \operatorname{diam} K_n = 0,$$

then $\bigcap_1^\infty K_n$ consists of exactly one point.

Proof

(*a*) Since $E \subset \bar{E}$, it is clear that

$$\operatorname{diam} E \le \operatorname{diam} \bar{E}.$$

Fix $\varepsilon > 0$, and choose $p \in \bar{E}, q \in \bar{E}$. By the definition of $\bar{E}$, there are points p', q', in E such that $d(p, p') < \varepsilon$, $d(q, q') < \varepsilon$. Hence

$$d(p, q) \le d(p, p') + d(p', q') + d(q', q)$$
$$< 2\varepsilon + d(p', q') \le 2\varepsilon + \operatorname{diam} E.$$

It follows that

$$\operatorname{diam} \bar{E} \le 2\varepsilon + \operatorname{diam} E,$$

and since ε was arbitrary, (*a*) is proved.

(*b*) Put $K = \bigcap_1^\infty K_n$. By Theorem 2.36, K is not empty. If K contains more than one point, then $\operatorname{diam} K > 0$. But for each n, $K_n \supset K$, so that $\operatorname{diam} K_n \ge \operatorname{diam} K$. This contradicts the assumption that $\operatorname{diam} K_n \to 0$.

3.11 Theorem

(*a*) *In any metric space X, every convergent sequence is a Cauchy sequence.*

(*b*) *If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X.*

(*c*) *In R^k, every Cauchy sequence converges.*

Note: The difference between the definition of convergence and the definition of a Cauchy sequence is that the limit is explicitly involved in the former, but not in the latter. Thus Theorem 3.11(*b*) may enable us to decide whether or not a given sequence converges without knowledge of the limit to which it may converge.

The fact (contained in Theorem 3.11) that a sequence converges in R^k if and only if it is a Cauchy sequence is usually called the *Cauchy criterion* for convergence.

Proof

(*a*) If $p_n \to p$ and if $\varepsilon > 0$, there is an integer N such that $d(p, p_n) < \varepsilon$ for all $n \geq N$. Hence

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < 2\varepsilon$$

as soon as $n \geq N$ and $m \geq N$. Thus $\{p_n\}$ is a Cauchy sequence.

(*b*) Let $\{p_n\}$ be a Cauchy sequence in the compact space X. For $N = 1, 2, 3, \ldots$, let E_N be the set consisting of $p_N, p_{N+1}, p_{N+2}, \ldots$. Then

$$\lim_{N\to\infty} \operatorname{diam} \bar{E}_N = 0, \tag{3}$$

by Definition 3.9 and Theorem 3.10(*a*). Being a closed subset of the compact space X, each $\bar{E}_N$ is compact (Theorem 2.35). Also $E_N \supset E_{N+1}$, so that $\bar{E}_N \supset \bar{E}_{N+1}$.

Theorem 3.10(*b*) shows now that there is a unique $p \in X$ which lies in every $\bar{E}_N$.

Let $\varepsilon > 0$ be given. By (3) there is an integer N_0 such that diam $\bar{E}_N < \varepsilon$ if $N \geq N_0$. Since $p \in \bar{E}_N$, it follows that $d(p, q) < \varepsilon$ for every $q \in \bar{E}_N$, hence for every $q \in E_N$. In other words, $d(p, p_n) < \varepsilon$ if $n \geq N_0$. This says precisely that $p_n \to p$.

(*c*) Let $\{\mathbf{x}_n\}$ be a Cauchy sequence in R^k. Define E_N as in (*b*), with $\mathbf{x}_i$ in place of p_i. For some N, diam $E_N < 1$. The range of $\{\mathbf{x}_n\}$ is the union of E_N and the finite set $\{\mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\}$. Hence $\{\mathbf{x}_n\}$ is bounded. Since every bounded subset of R^k has compact closure in R^k (Theorem 2.41), (*c*) follows from (*b*).

3.12 Definition A metric space in which every Cauchy sequence converges is said to be *complete*.

Thus Theorem 3.11 says that *all compact metric spaces and all Euclidean spaces are complete*. Theorem 3.11 implies also that *every closed subset E of a complete metric space X is complete*. (Every Cauchy sequence in E is a Cauchy sequence in X, hence it converges to some $p \in X$, and actually $p \in E$ since E is closed.) An example of a metric space which is not complete is the space of all rational numbers, with $d(x, y) = |x - y|$.

Theorem 3.2(*c*) and example (*d*) of Definition 3.1 show that convergent sequences are bounded, but that bounded sequences in R^k need not converge. However, there is one important case in which convergence *is* equivalent to boundedness; this happens for monotonic sequences in R^1.

3.13 Definition A sequence $\{s_n\}$ of real numbers is said to be

(*a*) *monotonically increasing* if $s_n \leq s_{n+1}$ $(n = 1, 2, 3, \ldots)$;
(*b*) *monotonically decreasing* if $s_n \geq s_{n+1}$ $(n = 1, 2, 3, \ldots)$.

The class of monotonic sequences consists of the increasing and the decreasing sequences.

3.14 Theorem *Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.*

Proof Suppose $s_n \leq s_{n+1}$ (the proof is analogous in the other case). Let E be the range of $\{s_n\}$. If $\{s_n\}$ is bounded, let s be the least upper bound of E. Then

$$s_n \leq s \qquad (n = 1, 2, 3, \ldots).$$

For every $\varepsilon > 0$, there is an integer N such that

$$s - \varepsilon < s_N \leq s,$$

for otherwise $s - \varepsilon$ would be an upper bound of E. Since $\{s_n\}$ increases, $n \geq N$ therefore implies

$$s - \varepsilon < s_n \leq s,$$

which shows that $\{s_n\}$ converges (to s).

The converse follows from Theorem 3.2(*c*).

UPPER AND LOWER LIMITS

3.15 Definition Let $\{s_n\}$ be a sequence of real numbers with the following property: For every real M there is an integer N such that $n \geq N$ implies $s_n \geq M$. We then write

$$s_n \to +\infty.$$

Similarly, if for every real M there is an integer N such that $n \geq N$ implies $s_n \leq M$, we write

$$s_n \to -\infty.$$

It should be noted that we now use the symbol $\to$ (introduced in Definition 3.1) for certain types of divergent sequences, as well as for convergent sequences, but that the definitions of convergence and of limit, given in Definition 3.1, are in no way changed.

3.16 Definition Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers x (in the extended real number system) such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$. This set E contains all subsequential limits as defined in Definition 3.5, plus possibly the numbers $+\infty$, $-\infty$.

We now recall Definitions 1.8 and 1.23 and put

$$s^* = \sup E,$$
$$s_* = \inf E.$$

The numbers s^*, s_* are called the *upper* and *lower limits* of $\{s_n\}$; we use the notation

$$\limsup_{n\to\infty} s_n = s^*, \qquad \liminf_{n\to\infty} s_n = s_*.$$

3.17 Theorem *Let $\{s_n\}$ be a sequence of real numbers. Let E and s^* have the same meaning as in Definition 3.16. Then s^* has the following two properties:*

(a) $s^* \in E$.

(b) *If $x > s^*$, there is an integer N such that $n \ge N$ implies $s_n < x$.*

Moreover, s^* is the only number with the properties (a) and (b).

Of course, an analogous result is true for s_*.

Proof

(a) If $s^* = +\infty$, then E is not bounded above; hence $\{s_n\}$ is not bounded above, and there is a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \to +\infty$.

If s^* is real, then E is bounded above, and at least one subsequential limit exists, so that (a) follows from Theorems 3.7 and 2.28.

If $s^* = -\infty$, then E contains only one element, namely $-\infty$, and there is no subsequential limit. Hence, for any real M, $s_n > M$ for at most a finite number of values of n, so that $s_n \to -\infty$.

This establishes (a) in all cases.

(b) Suppose there is a number $x > s^*$ such that $s_n \ge x$ for infinitely many values of n. In that case, there is a number $y \in E$ such that $y \ge x > s^*$, contradicting the definition of s^*.

Thus s^* satisfies (a) and (b).

To show the uniqueness, suppose there are two numbers, p and q, which satisfy (a) and (b), and suppose $p < q$. Choose x such that $p < x < q$. Since p satisfies (b), we have $s_n < x$ for $n \ge N$. But then q cannot satisfy (a).

3.18 Examples

(a) Let $\{s_n\}$ be a sequence containing all rationals. Then every real number is a subsequential limit, and

$$\limsup_{n\to\infty} s_n = +\infty, \qquad \liminf_{n\to\infty} s_n = -\infty.$$

(b) Let $s_n = (-1^n)/[1 + (1/n)]$. Then

$$\limsup_{n\to\infty} s_n = 1, \qquad \liminf_{n\to\infty} s_n = -1.$$

(c) For a real-valued sequence $\{s_n\}$, $\lim_{n\to\infty} s_n = s$ if and only if

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s.$$

We close this section with a theorem which is useful, and whose proof is quite trivial:

3.19 Theorem *If $s_n \le t_n$ for $n \ge N$, where N is fixed, then*

$$\liminf_{n\to\infty} s_n \le \liminf_{n\to\infty} t_n,$$
$$\limsup_{n\to\infty} s_n \le \limsup_{n\to\infty} t_n.$$

SOME SPECIAL SEQUENCES

We shall now compute the limits of some sequences which occur frequently. The proofs will all be based on the following remark: If $0 \le x_n \le s_n$ for $n \ge N$, where N is some fixed number, and if $s_n \to 0$, then $x_n \to 0$.

3.20 Theorem

(a) *If $p > 0$, then* $\lim_{n\to\infty} \dfrac{1}{n^p} = 0$.

(b) *If $p > 0$, then* $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(d) *If $p > 0$ and α is real, then* $\lim_{n\to\infty} \dfrac{n^\alpha}{(1+p)^n} = 0$.

(e) *If $|x| < 1$, then* $\lim_{n\to\infty} x^n = 0$.

3.36 Remarks The ratio test is frequently easier to apply than the root test, since it is usually easier to compute ratios than nth roots. However, the root test has wider scope. More precisely: Whenever the ratio test shows convergence, the root test does too; whenever the root test is inconclusive, the ratio test is too. This is a consequence of Theorem 3.37, and is illustrated by the above examples.

Neither of the two tests is subtle with regard to divergence. Both deduce divergence from the fact that a_n does not tend to zero as $n \to \infty$.

3.37 Theorem *For any sequence $\{c_n\}$ of positive numbers,*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n},$$

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Proof We shall prove the second inequality; the proof of the first is quite similar. Put

$$\alpha = \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

If $\alpha = +\infty$, there is nothing to prove. If α is finite, choose $\beta > \alpha$. There is an integer N such that

$$\frac{c_{n+1}}{c_n} \le \beta$$

for $n \ge N$. In particular, for any $p > 0$,

$$c_{N+k+1} \le \beta c_{N+k} \qquad (k = 0, 1, \ldots, p-1).$$

Multiplying these inequalities, we obtain

$$c_{N+p} \le \beta^p c_N,$$

or

$$c_n \le c_N \beta^{-N} \cdot \beta^n \qquad (n \ge N).$$

Hence

$$\sqrt[n]{c_n} \le \sqrt[n]{c_N \beta^{-N}} \cdot \beta,$$

so that

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \beta, \tag{18}$$

by Theorem 3.20(*b*). Since (18) is true for every $\beta > \alpha$, we have

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \alpha.$$

POWER SERIES

3.38 Definition Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n \tag{19}$$

is called a *power series*. The numbers c_n are called the *coefficients* of the series; z is a complex number.

In general, the series will converge or diverge, depending on the choice of z. More specifically, with every power series there is associated a circle, the circle of convergence, such that (19) converges if z is in the interior of the circle and diverges if z is in the exterior (to cover all cases, we have to consider the plane as the interior of a circle of infinite radius, and a point as a circle of radius zero). The behavior on the circle of convergence is much more varied and cannot be described so simply.

3.39 Theorem *Given the power series $\Sigma c_n z^n$, put*

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \qquad R = \frac{1}{\alpha}.$$

(If $\alpha = 0$, $R = +\infty$; if $\alpha = +\infty$, $R = 0$.) Then $\Sigma c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

Proof Put $a_n = c_n z^n$, and apply the root test:

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{|z|}{R}.$$

Note: R is called the radius of convergence of $\Sigma c_n z^n$.

3.40 Examples

(*a*) The series $\Sigma n^n z^n$ has $R = 0$.

(*b*) The series $\sum \frac{z^n}{n!}$ has $R = +\infty$. (In this case the ratio test is easier to apply than the root test.)

3.36 Remarks The ratio test is frequently easier to apply than the root test, since it is usually easier to compute ratios than nth roots. However, the root test has wider scope. More precisely: Whenever the ratio test shows convergence, the root test does too; whenever the root test is inconclusive, the ratio test is too. This is a consequence of Theorem 3.37, and is illustrated by the above examples.

Neither of the two tests is subtle with regard to divergence. Both deduce divergence from the fact that a_n does not tend to zero as $n \to \infty$.

3.37 Theorem *For any sequence $\{c_n\}$ of positive numbers,*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n},$$

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Proof We shall prove the second inequality; the proof of the first is quite similar. Put

$$\alpha = \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

If $\alpha = +\infty$, there is nothing to prove. If α is finite, choose $\beta > \alpha$. There is an integer N such that

$$\frac{c_{n+1}}{c_n} \le \beta$$

for $n \ge N$. In particular, for any $p > 0$,

$$c_{N+k+1} \le \beta c_{N+k} \qquad (k = 0, 1, \ldots, p-1).$$

Multiplying these inequalities, we obtain

$$c_{N+p} \le \beta^p c_N,$$

or

$$c_n \le c_N \beta^{-N} \cdot \beta^n \qquad (n \ge N).$$

Hence

$$\sqrt[n]{c_n} \le \sqrt[n]{c_N \beta^{-N}} \cdot \beta,$$

so that

(18) $$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \beta,$$

by Theorem 3.20(*b*). Since (18) is true for every $\beta > \alpha$, we have

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \alpha.$$

POWER SERIES

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(19) $$\sum_{n=0}^{\infty} c_n z^n$$

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(*a*) The series $\Sigma n^n z^n$ has $R = 0$.

(*b*) The series $\sum \frac{z^n}{n!}$ has $R = +\infty$. (In this case the ratio test is easier to apply than the root test.)

(c) The series Σz^n has $R = 1$. If $|z| = 1$, the series diverges, since $\{z^n\}$ does not tend to 0 as $n \to \infty$.

(d) The series $\sum \frac{z^n}{n}$ has $R = 1$. It diverges if $z = 1$. It converges for all other z with $|z| = 1$. (The last assertion will be proved in Theorem 3.44.)

(e) The series $\sum \frac{z^n}{n^2}$ has $R = 1$. It converges for all z with $|z| = 1$, by the comparison test, since $|z^n/n^2| = 1/n^2$.

SUMMATION BY PARTS

3.41 Theorem *Given two sequences $\{a_n\}$, $\{b_n\}$, put*

$$A_n = \sum_{k=0}^{n} a_k$$

if $n \geq 0$; put $A_{-1} = 0$. Then, if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p. \tag{20}$$

Proof

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1},$$

and the last expression on the right is clearly equal to the right side of (20).

Formula (20), the so-called "partial summation formula," is useful in the investigation of series of the form $\Sigma a_n b_n$, particularly when $\{b_n\}$ is monotonic. We shall now give applications.

3.42 Theorem *Suppose*

(a) *the partial sums A_n of Σa_n form a bounded sequence;*
(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;
(c) $\lim b_n = 0$.

Then $\Sigma a_n b_n$ converges.

Proof Choose M such that $|A_n| \leq M$ for all n. Given $\varepsilon > 0$, there is an integer N such that $b_N \leq (\varepsilon/2M)$. For $N \leq p \leq q$, we have

$$\begin{aligned}\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\ &\leq M \left|\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p\right| \\ &= 2Mb_p \leq 2Mb_N \leq \varepsilon.\end{aligned}$$

Convergence now follows from the Cauchy criterion. We note that the first inequality in the above chain depends of course on the fact that $b_n - b_{n+1} \geq 0$.

3.43 Theorem *Suppose*

(a) $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$;
(b) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ $\quad (m = 1, 2, 3, \ldots)$;
(c) $\lim_{n\to\infty} c_n = 0$.

Then Σc_n converges.

Series for which (b) holds are called "alternating series"; the theorem was known to Leibnitz.

Proof Apply Theorem 3.42, with $a_n = (-1)^{n+1}$, $b_n = |c_n|$.

3.44 Theorem *Suppose the radius of convergence of $\Sigma c_n z^n$ is 1, and suppose $c_0 \geq c_1 \geq c_2 \geq \cdots$, $\lim_{n\to\infty} c_n = 0$. Then $\Sigma c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly at $z = 1$.*

Proof Put $a_n = z^n$, $b_n = c_n$. The hypotheses of Theorem 3.42 are then satisfied, since

$$|A_n| = \left|\sum_{m=0}^{n} z^m\right| = \left|\frac{1 - z^{n+1}}{1 - z}\right| \leq \frac{2}{|1 - z|},$$

if $|z| = 1$, $z \neq 1$.

ABSOLUTE CONVERGENCE

The series Σa_n is said to *converge absolutely* if the series $\Sigma |a_n|$ converges.

3.45 Theorem *If Σa_n converges absolutely, then Σa_n converges.*

Proof The assertion follows from the inequality

$$\left|\sum_{k=n}^{m} a_k\right| \le \sum_{k=n}^{m} |a_k|,$$

plus the Cauchy criterion.

3.46 Remarks For series of positive terms, absolute convergence is the same as convergence.

If Σa_n converges, but $\Sigma|a_n|$ diverges, we say that Σa_n converges *nonabsolutely*. For instance, the series

$$\sum \frac{(-1)^n}{n}$$

converges nonabsolutely (Theorem 3.43).

The comparison test, as well as the root and ratio tests, is really a test for absolute convergence, and therefore cannot give any information about nonabsolutely convergent series. Summation by parts can sometimes be used to handle the latter. In particular, power series converge absolutely in the interior of the circle of convergence.

We shall see that we may operate with absolutely convergent series very much as with finite sums. We may multiply them term by term and we may change the order in which the additions are carried out, without affecting the sum of the series. But for nonabsolutely convergent series this is no longer true, and more care has to be taken when dealing with them.

ADDITION AND MULTIPLICATION OF SERIES

3.47 Theorem *If* $\Sigma a_n = A$, *and* $\Sigma b_n = B$, *then* $\Sigma(a_n + b_n) = A + B$, *and* $\Sigma c a_n = cA$, *for any fixed c.*

Proof Let

$$A_n = \sum_{k=0}^{n} a_k, \qquad B_n = \sum_{k=0}^{n} b_k.$$

Then

$$A_n + B_n = \sum_{k=0}^{n} (a_k + b_k).$$

Since $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$, we see that

$$\lim_{n\to\infty} (A_n + B_n) = A + B.$$

The proof of the second assertion is even simpler.

Thus two convergent series may be added term by term, and the resulting series converges to the sum of the two series. The situation becomes more complicated when we consider multiplication of two series. To begin with, we have to define the product. This can be done in several ways; we shall consider the so-called "Cauchy product."

3.48 Definition Given Σa_n and Σb_n, we put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \qquad (n = 0, 1, 2, \ldots)$$

and call Σc_n the *product* of the two given series.

This definition may be motivated as follows. If we take two power series $\Sigma a_n z^n$ and $\Sigma b_n z^n$, multiply them term by term, and collect terms containing the same power of z, we get

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots)\\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \cdots\\ &= c_0 + c_1 z + c_2 z^2 + \cdots.\end{aligned}$$

Setting $z = 1$, we arrive at the above definition.

3.49 Example If

$$A_n = \sum_{k=0}^{n} a_k, \qquad B_n = \sum_{k=0}^{n} b_k, \qquad C_n = \sum_{k=0}^{n} c_k,$$

and $A_n \to A$, $B_n \to B$, then it is not at all clear that $\{C_n\}$ will converge to AB, since we do not have $C_n = A_n B_n$. The dependence of $\{C_n\}$ on $\{A_n\}$ and $\{B_n\}$ is quite a complicated one (see the proof of Theorem 3.50). We shall now show that the product of two convergent series may actually diverge.

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

converges (Theorem 3.43). We form the product of this series with itself and obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c_n = 1 &- \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{2}} + \frac{1}{\sqrt{3}}\right)\\ &- \left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{3}} + \frac{1}{\sqrt{4}}\right) + \cdots,\end{aligned}$$

so that

$$c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}}.$$

Since

$$(n-k+1)(k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \le \left(\frac{n}{2}+1\right)^2,$$

we have

$$|c_n| \ge \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2},$$

so that the condition $c_n \to 0$, which is necessary for the convergence of Σc_n, is not satisfied.

In view of the next theorem, due to Mertens, we note that we have here considered the product of two nonabsolutely convergent series.

3.50 Theorem *Suppose*

(a) $\sum_{n=0}^{\infty} a_n$ *converges absolutely,*

(b) $\sum_{n=0}^{\infty} a_n = A$,

(c) $\sum_{n=0}^{\infty} b_n = B$,

(d) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ $\quad (n = 0, 1, 2, \ldots)$.

Then

$$\sum_{n=0}^{\infty} c_n = AB.$$

That is, the product of two convergent series converges, and to the right value, if at least one of the two series converges absolutely.

Proof Put

$$A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{k=0}^{n} b_k, \quad C_n = \sum_{k=0}^{n} c_k, \quad \beta_n = B_n - B.$$

Then

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \\ &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + \cdots + a_n(B + \beta_0) \\ &= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0 \end{aligned}$$

Put

$$\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0.$$

We wish to show that $C_n \to AB$. Since $A_n B \to AB$, it suffices to show that

(21) $$\lim_{n\to\infty} \gamma_n = 0.$$

Put

$$\alpha = \sum_{n=0}^{\infty} |a_n|.$$

[It is here that we use (a).] Let $\varepsilon > 0$ be given. By (c), $\beta_n \to 0$. Hence we can choose N such that $|\beta_n| \le \varepsilon$ for $n \ge N$, in which case

$$\begin{aligned} |\gamma_n| &\le |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \cdots + \beta_n a_0| \\ &\le |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + \varepsilon\alpha. \end{aligned}$$

Keeping N fixed, and letting $n \to \infty$, we get

$$\limsup_{n\to\infty} |\gamma_n| \le \varepsilon\alpha,$$

since $a_k \to 0$ as $k \to \infty$. Since ε is arbitrary, (21) follows.

Another question which may be asked is whether the series Σc_n, if convergent, must have the sum AB. Abel showed that the answer is in the affirmative.

3.51 Theorem *If the series* Σa_n, Σb_n, Σc_n *converge to* A, B, C, *and* $c_n = a_0 b_n + \cdots + a_n b_0$, *then* $C = AB$.

Here no assumption is made concerning absolute convergence. We shall give a simple proof (which depends on the continuity of power series) after Theorem 8.2.

REARRANGEMENTS

3.52 Definition Let $\{k_n\}$, $n = 1, 2, 3, \ldots$, be a sequence in which every positive integer appears once and only once (that is, $\{k_n\}$ is a 1-1 function from J onto J, in the notation of Definition 2.2). Putting

$$a_n' = a_{k_n} \quad (n = 1, 2, 3, \ldots),$$

we say that $\Sigma a_n'$ is a *rearrangement* of Σa_n.

If $\{s_n\}$, $\{s_n'\}$ are the sequences of partial sums of Σa_n, $\Sigma a_n'$, it is easily seen that, in general, these two sequences consist of entirely different numbers. We are thus led to the problem of determining under what conditions all rearrangements of a convergent series will converge and whether the sums are necessarily the same.

3.53 Example Consider the convergent series

(22) $$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots$$

and one of its rearrangements

(23) $$1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \cdots$$

in which two positive terms are always followed by one negative. If s is the sum of (22), then

$$s < 1 - \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{5}{6}.$$

Since

$$\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > 0$$

for $k \geq 1$, we see that $s_3' < s_6' < s_9' < \cdots$, where s_n' is nth partial sum of (23). Hence

$$\limsup_{n\to\infty} s_n' > s_3' = \tfrac{5}{6},$$

so that (23) certainly does not converge to s [we leave it to the reader to verify that (23) does, however, converge].

This example illustrates the following theorem, due to Riemann.

3.54 Theorem *Let Σa_n be a series of real numbers which converges, but not absolutely. Suppose*

$$-\infty \leq \alpha \leq \beta \leq \infty.$$

Then there exists a rearrangement $\Sigma a_n'$ with partial sums s_n' such that

(24) $$\liminf_{n\to\infty} s_n' = \alpha, \qquad \limsup_{n\to\infty} s_n' = \beta.$$

Proof Let

$$p_n = \frac{|a_n| + a_n}{2}, \qquad q_n = \frac{|a_n| - a_n}{2} \qquad (n = 1, 2, 3, \ldots).$$

Then $p_n - q_n = a_n$, $p_n + q_n = |a_n|$, $p_n \geq 0$, $q_n \geq 0$. The series Σp_n, Σq_n must both diverge.

For if both were convergent, then

$$\Sigma(p_n + q_n) = \Sigma |a_n|$$

would converge, contrary to hypothesis. Since

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} (p_n - q_n) = \sum_{n=1}^{N} p_n - \sum_{n=1}^{N} q_n,$$

divergence of Σp_n and convergence of Σq_n (or vice versa) implies divergence of Σa_n, again contrary to hypothesis.

Now let $P_1, P_2, P_3, \ldots$ denote the nonnegative terms of Σa_n, in the order in which they occur, and let $Q_1, Q_2, Q_3, \ldots$ be the absolute values of the negative terms of Σa_n, also in their original order.

The series ΣP_n, ΣQ_n differ from Σp_n, Σq_n only by zero terms, and are therefore divergent.

We shall construct sequences $\{m_n\}$, $\{k_n\}$, such that the series

(25) $$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} + \cdots,$$

which clearly is a rearrangement of Σa_n, satisfies (24).

Choose real-valued sequences $\{\alpha_n\}$, $\{\beta_n\}$ such that $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\alpha_n < \beta_n$, $\beta_1 > 0$.

Let m_1, k_1 be the smallest integers such that

$$P_1 + \cdots + P_{m_1} > \beta_1,$$
$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} < \alpha_1;$$

let m_2, k_2 be the smallest integers such that

$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} > \beta_2,$$
$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} < \alpha_2;$$

and continue in this way. This is possible since ΣP_n and ΣQ_n diverge.

If x_n, y_n denote the partial sums of (25) whose last terms are P_{m_n}, $-Q_{k_n}$, then

$$|x_n - \beta_n| \leq P_{m_n}, \qquad |y_n - \alpha_n| \leq Q_{k_n}.$$

Since $P_n \to 0$ and $Q_n \to 0$ as $n \to \infty$, we see that $x_n \to \beta$, $y_n \to \alpha$.

Finally, it is clear that no number less than α or greater than β can be a subsequential limit of the partial sums of (25).

3.55 Theorem *If Σa_n is a series of complex numbers which converges absolutely, then every rearrangement of Σa_n converges, and they all converge to the same sum.*

Proof Let $\Sigma a'_n$ be a rearrangement, with partial sums s'_n. Given $\varepsilon > 0$, there exists an integer N such that $m \geq n \geq N$ implies

$$\sum_{i=n}^{m} |a_i| \leq \varepsilon. \tag{26}$$

Now choose p such that the integers $1, 2, \ldots, N$ are all contained in the set $k_1, k_2, \ldots, k_p$ (we use the notation of Definition 3.52). Then if $n > p$, the numbers $a_1, \ldots, a_N$ will cancel in the difference $s_n - s'_n$, so that $|s_n - s'_n| \leq \varepsilon$, by (26). Hence $\{s'_n\}$ converges to the same sum as $\{s_n\}$.

EXERCISES

1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?
2. Calculate $\lim_{n \to \infty} (\sqrt{n^2 + n} - n)$.
3. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \qquad (n = 1, 2, 3, \ldots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \ldots$.

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

5. For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that

$$\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

6. Investigate the behavior (convergence or divergence) of Σa_n if
   (a) $a_n = \sqrt{n+1} - \sqrt{n}$;
   (b) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$;
   (c) $a_n = (\sqrt[n]{n} - 1)^n$;
   (d) $a_n = \dfrac{1}{1 + z^n}$, for complex values of z.
7. Prove that the convergence of Σa_n implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n}$$

if $a_n \geq 0$.

10M) Mertens questi...

8. If Σa_n converges, and if $\{b_n\}$ is monotonic and bounded, verges.
9. Find the radius of convergence of each of the following p
   (a) $\sum n^3 z^n$,
   (b) $\sum \dfrac{2^n}{n!} z^n$,
   (c) $\sum \dfrac{2^n}{n^2} z^n$,
   (d) $\sum \dfrac{n^3}{3^n} z^n$.
10. Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.
11. Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and Σa_n diverges.
    (a) Prove that $\sum \dfrac{a_n}{1 + a_n}$ diverges.
    (b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \dfrac{a_n}{s_n}$ diverges.

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \dfrac{a_n}{s_n^2}$ converges.

(d) What can be said about

$$\sum \frac{a_n}{1 + na_n} \quad \text{and} \quad \sum \frac{a_n}{1 + n^2 a_n}?$$

12. Suppose $a_n > 0$ and Σa_n converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) Prove that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \dfrac{a_n}{r_n}$ diverges.

UNIT - II

Limits of Functions

4.1 Let X and Y be metric spaces. suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write,

$$f(x) \to q \text{ as } x \to p \quad (\text{or}) \quad \lim_{x \to p} f(x) = q.$$

(i) If there is a point $q \in Y$ with the following property:

For every $\epsilon > 0$ $\exists$ a $\delta > 0$ $\ni$: $d_Y(f(x), q) < \epsilon$.

For all points $x \in E$ for which $0 < d_X(x, p) < \delta$.

The symbols d_X & d_Y refer to the distances in X & Y respectively.

If X & Y are replaced by the real line, the complex plane or by some Euclidean space R^k the distances d_X, d_Y are of course replaced by absolute values or by appropriate norms.

4.2 **Theorem. 1**

Let X, Y be metric spaces. suppose $E \subset X$, f maps E into Y, and p is a limit point of E. Then, $\lim_{x \to p} f(x) = q$ iff $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E $\ni$: $p_n \neq p$, $\lim_{n \to \infty} p_n = p$.

Proof

Let us assume that

$$\lim_{x \to p} f(x) = q$$

Let us assume that for every sequence $\{P_n\}$ in E

$$P_n \neq P, \quad \lim_{n\to\infty} P_n = P$$

<u>To prove</u> $\lim_{n\to\infty} f(P_n) = q$

ie, $d_Y(f(P_n), q) < \epsilon$

Give that, $\lim_{x\to p} f(x) = q$

Given $\epsilon > 0$ $\exists$ $\delta > 0$ $\ni:$ $d_Y(f(x), q) < \epsilon$ if $x \in E$ & $0 < d_X(x, p) < \delta$

Also $\exists$ $N \ni:$ $n > N$ implies $0 < d_X(P_n, P) < \delta$

Thus $n > N$ we have $d_Y(f(P_n), q) < \epsilon$

$\therefore \lim_{n\to\infty} P_n = P$ if $P_n \neq P$

<u>Sufficient part</u> ($\Leftarrow$:)

Conversely assume that $\lim_{n\to\infty} f(P_n) = q$ for every sequence $\{P_n\}$ in E $\ni:$ $P_n \neq P$, $\lim_{n\to\infty} P_n = P$.

<u>T.o.P</u> $\lim_{x\to p} f(x) = q$

Let us assume that p is not a limit point of E. $\lim_{x\to p} f(x) \neq q$

Then $\exists$ some $\epsilon > 0$ $\ni:$ for every $\delta > 0$ $\exists$ a point $x \in E$ (depending on δ) for which $d_Y(f(x), q) \geq \epsilon$ but $0 < d_X(x, p) < \delta$.

Taking $\delta_n = \frac{1}{n}$ $(n = 1, 2, \ldots)$

Then we find a sequence $\{p_n\}$ in $E \ni$:

$\lim_{n\to\infty} p_n = p$, $p_n \neq p$. But $d_Y(f(p_n), q) > \epsilon$

$\Rightarrow \Leftarrow$ to $\lim_{n\to\infty} f(p_n) = q$

$\therefore$ our assumption that p is not a limit point in wrong.

$\therefore$ p is a limit point.

$\therefore \lim_{x\to p} f(x_p) = q$

4.3

Corollary.

If f has a limit at p, this limit is unique

This follows from theorem 3.2(b) & 4.2

Defn

Let f & g are two complex function which is defined on E. for each point x of E.

$f \to g$ defined us,

$$f+g = f(x) + g(x).$$

Similarly, the difference $f-g$, the product fg, and the quotient f/g of the two functions, with the understanding that the quotient is defined only at those points x of E at which $g(x) \neq 0$

If f assigns to each point x of E the same number c.

ie, $f = c$

Then f is said to be a constant function (or) simply constant.

If f & g are real functions and if

$f(x) \geq g(x)$ $\forall$ $x \in E$

Similarly, If f & g map E into R^k. Then we define $f+g$ & fg by:

$$(f+g)(x) = f(x) + g(x)$$

$$(f \cdot g)(x) = f(x) \cdot g(x). \ \&$$

If λ is a real number

$$(\lambda f)(x) = \lambda f(x).$$

4.4

Theorem. 2

Suppose $E \subset X$, a metric space, p is a limit point of E, f & g are complex functions on E, and.

$$\lim_{x \to p} f(x) = A, \quad \lim_{x \to p} g(x) = B.$$

Then, (a) $\lim_{x \to p} (f+g)(x) = A + B.$

(b) $\lim_{x \to p} (fg)(x) = AB.$

(c) $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ if $B \neq 0$

Ref

Remark: If f & g map E into R^K. Then (a) remains true, and (b) becomes (b').

$$\lim_{x \to p} (f \cdot g)(x) = A \cdot B. \quad (T\text{-}3.4)$$

Continuous Functions.

Defn :-

4.5 Suppose X & Y are metric spaces, $E \subset X$, $p \in E$ and f maps E into Y. Then f is said to be continuous at p if for every $\epsilon > 0$ $\exists$ a $\delta > 0$ $\ni:$ $d_Y(f(x), f(p)) < \epsilon$. $\forall$ points $x \in E$ for which $d_X(x, p) < \delta$.

If f is continuous at every point of E, then f is said to be continuous on E.

It should be noted that f has to be defined at the point p in order to be continuous at p.

If p is an isolated point of E. then every function f which has E.

Given $\epsilon > 0$ $\exists$ $\delta > 0$ so that the only point $x \in E$ for which $d_X(x, p) < \delta$ is $x = p$.

Then, $d_Y(f(x), f(p)) = 0 < \epsilon$

4.6 Theorem

p is a limit point of E. Then f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$

Proof:

This is clear if we compare definition 4.1 & 4.5

4.7 Theorem.

Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y, g maps the range of f, $f(E)$ into Z, and h is the mapping of E into Z defined by,

$$h(x) = g(f(x)), \quad x \in E.$$

If f is continuous at a point $p \in E$ & If g is continuous at the point $f(p)$, then h is continuous at p.

This function h is called the composition (or) the composite of f & g. The notation

$$h = g \circ f$$

is frequently used in this context.

Proof

Let g is continuous at $f(p)$.

Given $\epsilon > 0$ $\exists$ $\eta > 0$ $\ni$:

$d_Z(g(y), g(f(p))) < \epsilon$ if $d_Y(y, f(p)) < \eta$ & $y \in f(E)$.

Let f is continuous at p.

$\exists$ $\delta > 0$ $\ni$: $d_Y(f(x), f(p)) < \eta$ if $d_X(x, p) < \delta$ & $x \in E$.

$\therefore$ We have to show that h is continuous

ie, $d_Z(h(x), h(p)) < \epsilon$.

$\therefore d_Z(h(x), h(p)) = d_Z(g(f(x)), g(f(p)))$

$< \epsilon$.

$\therefore d_Z(h(x), h(p)) < \epsilon$ if $d_X(x, p) < \delta$ & $x \in E$

Thus h is continuous at p.

Theorem

A mapping f of a metric space X into a metric space Y is continuous on X iff $f^{-1}(V)$ is open in X for every open set V in Y.

Proof:- Neccessary part

Let us assume that f is continuous on X.

Neccessary part and V is an open set in Y.

$f^{-1}(V)$ is open in X is every point of $f^{-1}(V)$ is an interior point $f^{-1}(V)$.

Suppose $p \in X$ & $f(p) \in V$

Since, V is open $\exists\ \epsilon > 0 \ni: y \in V$ if $d_Y(f(p), y) < \epsilon$

$f(x) \in V$
$x \in f^{-1}(V)$

Since, f is continuous at p $\exists\ \delta > 0 \ni:$ $d_Y(f(x), f(p)) < \epsilon$ if $d_X(x, p) < \delta$.

Thus $x \in f^{-1}(V)$ as soon as $d_X(x, p) < \delta$.

$\therefore$ x is an interior point of $f^{-1}(V)$

$\because$ x is orbitrary every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$

$\therefore$ $f^{-1}(V)$ is open in X for every open set V in Y.

Sufficient part.

Conversely, suppose that $f^{-1}(V)$ is open in X for every open set V in Y.

T.P f is continuous on X.

Fix $p \in E$ & $\epsilon > 0$.

Let V be the set of all $y \in Y \ni:$ $d_y(y, f(p)) < \epsilon$.

$x \in f^{-1}(V)$ $\therefore f(x) \in V$

Then V is open.

Hence $f^{-1}(V)$ is open.

Hence $\exists\ \delta > 0 \ni: x \in f^{-1}(V)$ as soon as $d_x(p, x) < \delta$.

But if $x \in f^{-1}(V)$. Then $f(x) \in V$. So that $d_y(f(x), f(p)) < \epsilon \ \forall\ x \in E$ for which $d_x(x, p) < \delta$

$\therefore$ f is continuous.

Hence the theorem //

Corollary.

A mapping f of a metric space X into a metric space Y is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y.

Proof

This follows from the theorem since a set is closed iff its complement is open and since $f^{-1}(E^c) = [f^{-1}(E)]^c$ for every $E \subset Y$.

4.9 Theorem

Let f & g be complex continuous functions on a metric space X. Then $f+g$, fg & f/g are continuous on X.

In the last case, we must of course assume that $g(x) \neq 0 \ \forall\ x \in X$.

Th- 4.4 & 4.6

4.10 Theorem

(a) Let $f_1, f_2, \ldots, f_k$ be real functions on a metric space X, and Let f be the mapping of X into R^k defined by,

$$f(x) = \left(f_1(x), \ldots, f_k(x)\right), \quad x \in X$$

then f is continuous iff each of the functions $f_1, f_2, \ldots, f_k$ is continuous.

(b) If f & g are continuous mappings of X into R^k, then $f+g$ and $f \cdot g$ are continuous on X

The functions $f_1, \ldots, f_k$ are called the components of f. Note that $f+g$ is a mapping into R^k whereas $f \cdot g$ is a real function on X.

Proof: part (a) follows from the inequalities

$$|f_j(x) - f_j(y)| \leq |f(x) - f(y)|$$
$$= \left\{ \sum_{i=1}^{k} |f_i(x) - f_i(y)|^2 \right\}^{1/2},$$

for $j = 1, 2, \ldots K$

part (b) follows from (a) & T4.9

Continuity and Compactness.

A mapping f of a set E into R^k is said to be bounded if there is a real number M such that $|f(x)| \leq M \ \forall \ x \in E$

Theorem.

5m

Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

Proof:-

Let $\{V_\alpha\}$ be an open cover of $f(X)$.

Since f is continuous. Each of the sets $f^{-1}(V_\alpha)$ is open (T-4.8).

Since X is compact there are finitely many indices (Points), say $\alpha_1, \ldots, \alpha_n \ni$:

$X \subset f^{-1}(V_{\alpha_1}) \cup \ldots \cup f^{-1}(V_{\alpha_n})$ —— ①

Since $f(f^{-1}(E)) \subset E \ \forall \ E \subset Y$

$\therefore$ ① $\Rightarrow f(X) \subset V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}$

$\therefore f(x) \subset \bigcup_{i=1}^{n} V_{\alpha_i}$

$\therefore \{V_\alpha\}$ has a finite subcover.

$\therefore f(x)$ is compact.

4.15

Theorem.

UQ (X) 5m

If f is a continuous mapping of a compact metric space X into R^k, then $f(X)$ is closed and bounded. Thus f is bounded.

The result is particularly important when f is real

4.16 Theorem.

Suppose f is a continuous real function on a compact metric space X and

$$M = \sup_{p\in X} f(p), \quad m = \inf_{p\in X} f(p)$$

Then $\exists$ points $p, q \in X \ni: f(p) = M$ & $f(q) = m$.

The notation is above means that M is the least upper bound of the set of all numbers $f(p)$, where p ranges over X, and that m is the greatest lower bound of this set of numbers.

The conclusion may also be stated as follows:
$\exists$ points p & q in $X \ni: f(q) \leq f(x) \leq f(p)$ $\forall$ $x \in X$.

i.e), f attains its maximum (at p) and its minimum (at q).

Proof :-
$f(x)$ is a closed and bounded set of real numbers.

Hence $f(X)$ contains

$M = \sup f(X)$ & $m = \inf f(X)$. (By T-2.28)

4.17 Theorem.

Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by

$$f^{-1}(f(x)) = x, \quad x \in X$$

is a continuous mapping of Y onto X.

Proof :-

Uniformly Continuous function.

Let f be a mapping of a metric space X into a metric space Y. We say that f is uniformly continuous on X if for every $\epsilon>0$ $\exists$ $\delta>0$ $\ni$: $d_Y(f(p), f(q)) < \epsilon$. $\forall$ p & q in X for which $d_X(p,q) < \delta$.

$$\left[\begin{array}{l} \epsilon>0,\ \exists\ \delta>0 \ni: |f(x)-f(y)| < \epsilon \\ \forall\ |x-y| < \delta. \end{array}\right.$$

Theorem.

Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

Proof:

Given that f is continuous.

Given $\epsilon>0$ each point $p\in X$ a positive number $\phi(p) \ni: q\in X$, $d_X(p,q) < \phi(p)$

$$\Rightarrow d_Y(f(p), f(q)) < \epsilon/2 \quad \text{——} \ (1)$$

Let $J(p)$ be the set of all $q\in X$ for which $d_X(p,q) < \frac{1}{2}\phi(p)$.

$\therefore$ $p\in J(p)$. The collection of all sets $J(p)$ is an open cover of X.

$\because$ X is compact, there is a finite-set of points $p_1, \dots, p_n$ in $X \ni$:

$$X \subset J(p_1) \cup J(p_2) \cup \dots \cup J(p_n) \quad \text{——} \ (2)$$

We put $\delta = \frac{1}{2}\min[\phi(p_1), \dots, \phi(p_n)]$

Then $\delta > 0$

Now let p & q be points of X, $\ni$:

$d_X(p, q) < \delta$.

By ②, there is an integer m, $1 \le m \le n \ni$:

$p \in J(p_m)$

Hence $d_X(q, p_m) < \frac{1}{2}\phi(p_m)$.

and we have also,

$$d_X(p, p_m) \le d_X(p, q) + d_X(q, p_m)$$

$$\le \delta + \frac{1}{2}\phi(p_m)$$

$$\le \phi(p_m)$$

Finally, ① show that,

$$\therefore d_Y(f(p), f(q)) \le d_Y(f(p), f(p_m)) + d_Y(f(q), f(p_m)).$$

$$< \epsilon.$$

$\therefore d_Y(f(p), f(q)) < \epsilon$

$\therefore f$ is uniformly continuous on X.

<u>Theorem.</u>

Let E be a noncompact set in R^1. Then

(a) There exists a continuous function on E which is not bounded.

(b). There exists a continuous and bounded function on E which has no maximum

If in addition, E is bounded, then

(c) there exists a continuous function on E which is not uniformly continuous.

Proof:

Suppose we first assume that E is bounded.

So that there exists a limit point x_0 of E which is not a point of E.

Consider, $f(x) = \frac{1}{x - x_0}$, $x \in E$

This is continuous on E. but evidently unbounded.

$\therefore$ $f(x)$ is not uniformly continuous. (T-4.9)

Given $\epsilon > 0$ $\exists$ $\delta > 0$ be arbitrary. & choose a point $x \in E$ $\ni$: $|x - x_0| < \delta$.

Taking t close enough to x_0.

$|f(t) - f(x)| > \epsilon \Rightarrow |t - x| < \delta$.

$\therefore$ This is true for every $\delta > 0$

f is not uniformly continuous on E.

Secondly consider the function

$g(x) = \frac{1}{1 + (x - x_0)^2}$, $x \in E$.

is continuous on E. and is bounded

$\therefore$ $0 < g(x) < 1$.

It is clear that $\sup_{x \in E} g(x) = 1$. Where as $g(x) < 1$ $\forall$ $x \in E$.

Thus g has no maximum on E

Having proved the theorem for bounded sets E. Let us now suppose that E is unbounded.

Then $f(x) = x$ establishes (a), whereas

$$h(x) = \frac{x^2}{1+x^2}, \quad x \in E \text{ establishes (b).}$$

$\because \sup_{x \in E} h(x) = 1$ & hence $h(x) < 1 \ \forall \ x \in E$.

Assertion (c) would be false if boundedness were omitted from the hypothesis.

Let E be the set of all integers. Then every function defined on E will be uniformly continuous on E. If $\delta < 1$.

CONTINUITY AND CONNECTEDNESS :-

Theorem :- 4.22

5m

If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X. Then f(E) is connected.

Proof :-

Let us assume that f(E) is not connected

Then f(E) is the union of two non-empty separated sets.

$\therefore f(E) = A \cup B$, Where A & B are non empty separated subsets of Y.

put $G = E \cap f^{-1}(A)$, $H = E \cap f^{-1}(B)$.

Then $E = G \cup H$. & neither G nor H is empty.

$\because A \subset \bar{A}$ (The closure of A).

We have $G \subset f^{-1}(\bar{A})$ the latter set is closed.

$\because f$ is continuous.

Hence $\bar{G} \subset f^{-1}(\bar{A})$

$\Rightarrow f(\bar{G}) \subset \bar{A}$

$\because f(H) = B$ & $\bar{A} \cap B$ is empty.

$\bar{G} \cap H$ is empty

The same argument shows that $G \cap \bar{H}$ is empty.

Thus G & H are separated

$\therefore E$ is the union of two non-empty separated sets.

$\Rightarrow \Leftarrow$ E is connected.

Our assumption that $f(E)$ is not connected is wrong.

$\therefore f(E)$ is connected.

Theorem.

Let f be a continuous real function on the interval $[a,b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a,b)$ such that $f(x) = c$.

Proof:-

$[a,b]$ is connected (T-2.47)

Then by the previous then $A[a,b]$ is a connected subset of R'.

ie. Then theorem states that a continuous real function assumes all intermediate values on interval.

Discontinuities.

If x is a point in the domain of definition of the function f at which f is not continuous, we say that f is discontinuous at x. or that f has discontinuity of x.
We have to defined the right hand & Left hand limits of f at x. which we denote by $f(x+)$ & $f(x-)$ respectively.

Defn.

Let f be defined on (a,b). consider any point $x \ni: a \leq x < b$. we write $f(x+) = q$

If $f(t_n) \to q$ as $n \to \infty$ $\forall$ sequences $\{t_n\}$ in $(x,b) \ni: t_n \to x$

To obtain the definition of $f(x-)$ for $a < x \leq b$.

We restrict ourselves to sequences $\{t_n\}$ in (a,x).

It is clear that any point x of (a,b) $\lim\limits_{t \to x} f(t)$ exists iff $f(x+) = f(x-) = \lim\limits_{t \to x} f(t)$.

Defn:

Let f be defined on (a,b). If f is discontinuous at a point x and if $f(x+)$ & $f(x-)$ exist, then f is said to be have a discontinuity of the first kind (or) a simple discontinuity at x. otherwise the discontinuity is said to be of the second kind.

There are two ways in which a function can have a simple discontinuity either
$$f(x+) \neq f(x-) \quad (\text{or}) \quad f(x+) = f(x-) \neq f(x).$$

Monotonic Functions

Defn:-

Let f be real on (a,b). Then f is said to be monotonically increasing on (a,b). if $a<x<b \Rightarrow f(x) \leq f(y)$. If the last inequality is reversed we obtain the definition of a monotonically decreasing function.

Theorem.

Let f be monotonically increasing on (a,b). Then $f(x+)$ and $f(x-)$ exists at every point of x of (a,b). More precisely,
$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t).$$
Furthermore, if $a<x<y<b$, then
$$f(x+) \leq f(y-)$$

Analogous results evidently hold for monotonically decreasing functions.

Proof:-

Let us assume that,

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t). \quad \text{—— (A)}$$

Further more of $a<x<y<b$ then

$f(x+) \leq f(y-)$

By hypothesis the set of numbers $f(t)$, where $a<t<x$ is bounded above by the number $f(x)$ and therefore has a least upper bounded which we shall denote by A.

Evidently $A \leq f(x)$. We have to show that

$A = f(x-)$.

Let $\epsilon>0$ be given. It follows from the definition of A as a least upper bound that

$\exists\ \delta>0 \ni: a<x-\delta<x$ & $A-\epsilon < f(x-\delta) \leq A$. —— (1)

$\because f$ is monotonic. we have,

$f(x-\delta) \leq f(t) \leq A$ —— (2) $x-\delta<t<x$.

Hence $f(x-) = A$.

The second half of (A) is proved in preciously the same way

Next if $a<x<y<b$ we have from (A)

that $f(x+) = \inf\limits_{x<t<b} f(t) = \inf\limits_{x<t<y} f(t).$ —— (3)

The last equality is obtained by applying (A) to (a, y) in place of (a, b).

$|||^{ly}$, $$f(y-) = \sup_{a<t<y} f(t) = \sup_{x<t<y} f(t). \quad \text{--- (4)}$$

Comparing (3) & (4) we have:

$$f(x+) = f(y-)$$

Note :-

Monotonic function have no discontinuities of the second kind.

Theorem.

Let f be monotonic on (a,b). Then the set of points of (a,b) at which f is discontinuous is atmost countable.

Proof :-

Let us assume that f is monotonic increasing

Let E be the set of points at which f is discontinuous.

With every point x of E we associate a rational number $r(x) \ni : f(x-) < r(x) < f(x+)$

$\because x_1 < x_2 \Rightarrow f(x_1+) \leq f(x_2-)$ we see that $r(x_1) \neq r(x_2)$ if $x_1 \neq x_2$

Thus we have a 1-1 correspondence between the set E & a subset of the set of rational numbers.

$\therefore$ Then the set of points of (a,b) at which f is discontinuous is at most countable.

Infinite Limits and Limits at infinity.

Defn

For any real c, the set of real numbers $x \ni: x > c$ is called a neighbourhood of $+\infty$ and it is written $(c, +\infty)$

$|||^{ly}$ the set $(-\infty, c)$ is a neighbourhood of $-\infty$.

Defn

Let f be a real function defined on E. We say that $f(t) \to A$ as $t \to x$, where A & x are in the extended real number system, if for every neighbourhood U of A there is a neighbourhood V of $x \ni: V \cap E$ is not empty & $\ni: f(t) \in U \;\forall\; t \in V \cap E,\ t \neq x$.

Theorem.

Let f & g be defined on E. Suppose $f(t) \to A$, $g(t) \to B$ as $t \to x$.

Then, (a) $f(t) \to A' \Rightarrow A' = A$

(b) $(f+g)(t) \to A+B$

(c) $(fg)(t) \to AB$

(d) $(f/g)(t) \to A/B$.

provided the right numbers of (b), (c) & (d) are defined.

DIFFERENTIATION.

The Derivative of a Real Function.

Let f be defined on $[a,b]$. For any $x \in [a,b]$ form the quotient

$$\phi(t) = \frac{f(t)-f(x)}{t-x} \quad a<t<b, \; t \neq x \quad \text{①}$$

and define, $$f'(x) = \lim_{t \to x} \phi(t) \quad \text{②}.$$

We thus associate with the function f a function f' whose domain is the set of points x at which the limit ② exists. f' is called the derivative of f.

[If f' is defined at a point x, we say that f is differentiable at x. If f' is defined at every point of a set $E \subset [a,b]$, we say that f is differentiable on E]

If f is defined on a segment (a,b) and if $a<x<b$, then $f'(x)$ is defined by ① & ② as above. But $f'(a)$ & $f'(b)$ are not defined in this case.

Theorem.

Let f be defined on $[a,b]$. If f is differentiable at a point $x \in [a,b]$, then f is continuous at x.

Let f is differentiable at a point x

Top. f is continuous at x

ie, $\lim_{t \to x} f(t) = f(x)$ ——— ①.

Hence $f'(x)$ exists.

$\because$ f is differentiable.

$\therefore f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$, $t \neq x$, ~~$t - x \to f'(x) \cdot 0$~~

Now consider,

$\lim_{t \to x} f(t) - f(x) = f'(x)(t - x)$

$= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} (t - x)$

$= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \lim_{t \to x} (t - x)$

$= f'(x) \cdot 0$

$\lim_{t \to x} f(t) - f(x) = 0 \Rightarrow \lim_{t \to x} f(t) - \lim_{t \to x} f(x) = 0$

$\Rightarrow \lim_{t \to x} f(t) - f(x) = 0.$

$\therefore \lim_{t \to x} f(t) = f(x).$

Hence f is continuous at x.

Note:-

But the converse is not true. That is an continuous fun need not always be differentiable

Theorem.

Suppose f & g are defined on $[a,b]$ and are differentiable at a point $x \in [a,b]$. Then, $f+g$, fg, & f/g are differentiable at x, and

(a) $(f+g)'(x) = f'(x) + g'(x)$

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

(c) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$

In (c), we assume of course that $g(x) \neq 0$.

Proof:-

Given that f & g are defined on $[a,b]$ and we differentiable at $x \in [a,b]$. So $f'(x)$ & $g'(x)$ exists,

ie, $$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t-x}$$

$$g'(x) = \lim_{t \to x} \frac{g(t) - g(x)}{t-x}$$

(a). Give that $(f+g)$ is differentiable at x

$$(f+g)'(x) = f'(x) + g'(x).$$

Top:- Let us consider,

$$\lim_{t \to x} \frac{(f+g)(t) - (f+g)(x)}{t-x} = (f+g)'(x) \quad \text{①}$$

Now,

$$\lim_{t\to x} \frac{(f+g)(t)-(f+g)(x)}{t-x} = \lim_{t\to x} \frac{f(t)+g(t)-f(x)-g(x)}{t-x}$$

$$= \lim_{t\to x} \frac{[f(t)-f(x)]+[g(t)-g(x)]}{t-x}$$

$$= \lim_{t\to x} \frac{f(t)-f(x)}{t-x} + \lim_{t\to x} \frac{g(t)-g(x)}{t-x}$$

$$(f+g)'(x) = f'(x)+g'(x)$$

$$\therefore (f+g)'(x) = f'(x)+g'(x).$$

(b). Given that fg is differentiable at x.

<u>TOP</u>:- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

Let us consider

$$(fg)'(x) = \lim_{t\to x} \frac{(fg)(t)-(fg)(x)}{t-x}$$

$$= \lim_{t\to x} \frac{f(t)g(t)-f(x)g(x)}{t-x}$$

$$= \lim_{t\to x} \frac{f(t)g(t)-f(x)g(x)+g(t)f(x)-\{g(t)f(x)+f(x)g(x)-f(x)g(x)+f(t)g(x)-f(t)g(x)}{t-x}.$$

$$= \lim_{t \to x} \frac{f(x)[g(t)-g(x)] + g(x)[f(t)-f(x)] + [g(t)-g(x)][f(t)-f(x)]}{t-x}$$

$$= \lim_{t \to x} f(x) \frac{[g(t)-g(x)]}{t-x} + \lim_{t \to x} g(x) \frac{[f(t)-f(x)]}{t-x}$$

$$+ \lim_{t \to x} \frac{[g(t)-g(x)][f(t)-f(x)]}{t-x}$$

$$= f(x)g'(x) + g(x)f'(x) + 0.$$

$$= f(x)g'(x) + g(x)f'(x).$$

$\therefore$ $(fg)'(x) = f(x)g'(x) + g(x)f'(x).$

(c) Given that f/g is differentiable at x.

Top:- $$(f/g)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Let us consider,

$$(f/g)'(x) = \lim_{t \to x} \frac{(f/g)(t) - (f/g)(x)}{t-x}$$

$$= \lim_{t \to x} \frac{\frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}}{t-x}$$

$$= \lim_{t \to x} \frac{f(t)g(x) - f(x)g(t)}{g(x)g(t)(t-x)}$$

$$= \lim_{t \to x} \frac{f(t)g(x) - f(x)g(t) - f(x)g(x) + f(x)g(x)}{g(x)g(t)(t-x)}$$

$$= \lim_{t \to x} \frac{f(t)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(t)}{g(x)g(t)(t-x)}$$

$$= \lim_{t \to x} \frac{1}{g(x)g(t)} \left\{ \lim_{t \to x} \frac{g(x)[f(t)-f(x)]}{t-x} - \frac{f(x)[g(t)-g(x)]}{t-x} \right\}$$

$$(f/g)'(x) = \frac{1}{g^2(x)} \left\{ g(x)f'(x) - f(x)g'(x) \right\}.$$

$$\therefore (f/g)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Hence the proof //

Chain Rule

Theorem.

Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If

$$h(t) = g(f(t)), \quad a \le t \le b.$$

Then h is differentiable at x, and

$$h'(x) = g'(f(x))f'(x)$$

Proof :-

The function f & g exists such that the range of f is contained in the domain of g

$\Rightarrow$ g is a function of f, where f is a function of f.

g is the function of a function denoted by

$$h(t) = g\{f(t)\}. \quad \text{--- (1)}$$

Given that f is differentiable at a point x

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

Let us consider the function $u(t) \ni: u(t) \to 0$ as $t \to x$, where $t \in [a, b]$

$$f(t) - f(x) = (t - x)\left[f'(x) + u(t)\right]. \quad \text{--- (2)}$$

and g is differentiable at $f(x)$.

$$\therefore g'(f(x)) = \lim_{t \to x} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)}$$

Let $f(x) = y$ & $f(t) = s$ we have,

$$\therefore g'(y) = \lim_{s \to y} \frac{g(s) - g(y)}{s - y}. \quad \text{--- (3)}$$

$$g(s) - g(y) = (s - y)\{g'(y) + v(s)\},$$

Where $v(s) \to 0$ as $s \to y$.

Now,

$$h(t) - h(x) = g(f(t)) - g(f(x)) \quad \text{(by ①)}$$

$$= g(s) - g(y)$$

$$= (s-y)\{g'(y) + v(s)\}$$

$$= \{f(t) - f(x)\}\{g'(y) + v(s)\}$$

$$= (t-x)[f'(x) + u(t)][g'(y) + v(s)]$$

$$\frac{h(t) - h(x)}{t-x} = [g'(y) + v(s)][f'(x) + u(t)]$$

$\because$ f is continuous,

$$\lim_{t\to x} \frac{h(t)-h(x)}{t-x} = \lim_{t\to x} \lim_{s\to y} [g'(y) + v(s)][f'(x) + u(t)]$$

$$= \lim_{t\to x} [f'(x) + u(t)] \lim_{s\to y} [g'(y) + v(s)]$$

$$= f'(x) g'(y)$$

$$= f'(x) g'(f(x))$$

$$\therefore h'(x) = f'(x) g'(f(x)).$$ //

Mean Value Theorem.

Def^n.

Let f be a real function defined on a metric space X. We say that f has a local maximum at a point $p \in X$ if $\exists\ \delta > 0$ $\ni$: $f(q) \leq f(p)$ $\forall\ q \in X$ with $d(p, q) < \delta$.

Theorem.

Let f be defined on $[a, b]$. if f has a local maximum at a point $x \in (a, b)$ & if $f'(x)$ exists, then $f'(x) = 0$.

Proof :-

Given $f'(x)$ exists.

ie, $f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$

Choose δ as in the above definition.

So that, $a < x - \delta < x < x + \delta < b$.

If $x - \delta < t < x$, then

$$\frac{f(t) - f(x)}{t - x} \geq 0.$$

$$\therefore \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \geq 0$$

$$\therefore f'(x) \geq 0 \quad \text{—— (1)}.$$

If $x < t < x+\delta$. Then

$$\frac{f(t)-f(x)}{t-x} \leq 0.$$

$$\therefore \lim_{t \to x} \frac{f(t)-f(x)}{t-x} \leq 0$$

$\therefore f'(x) \leq 0$ —— (2)

From (1) & (2), we have

$$f'(x) = 0.$$

Hence the theorem //.

Generalised Mean Value Theorem

5m

Theorem.

If f & g are continuous real functions on $[a,b]$ which are differentiable in (a,b). Then there is a point $x \in (a,b)$ at which

$$[f(b)-f(a)]\, g'(x) = [g(b)-g(a)]\, f'(x)$$

$$\Rightarrow \frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Note that differentiability is not required at the endpoints.

Proof:-

Let us consider the function,

$$h(t) = [f(b)-f(a)]\, g(t) - [g(b)-g(a)]\, f(t),$$

$$a \leq t \leq b.$$

Then h is continuous on $[a,b]$.

$h=a$ h is differentiable in (a,b) and

$h(a) = f(b)g(a) - f(a)g(a) - g(b)f(a) + f(a)g(a)$

$$h(a) = f(b)g(a) - f(a)g(b)$$

$$= h(b)$$

To prove the theorem it is sufficient to show that $h'(x) = 0$ for some $x \in (a,b)$.

Case (i)

If h is a constant function:

Then, $h'(x) = 0 \;\; \forall \; x \in (a,b)$.

$$[f(b) - f(a)]\, g'(x) = [g(b) - g(a)]\, f'(x).$$

Case (ii)

If $h(t) > h(a) \;\; \forall \;\; t \in (a,b)$

Let x be a point on $[a,b]$ at which h attains its maximum. Then by the previous theorem.

Let f be defined on $[a,b]$ if f has a ~~real~~ local maximum at a point $x \in (a,b)$ and $f'(x)$ exists the $f'(x) = 0$.

$$h'(x) = 0.$$

$$\therefore [f(b) - f(a)]\, g'(x) = [g(b) - g(a)]\, f'(x).$$

case (iii)

If $h(t) < h(a)$ for same $t \in (a,b)$ the same argument applies if we choose for x a point on $[a,b]$, where h attains its minimum.

$\therefore h'(x) = 0$.

$\therefore [f(b) - f(a)]\, g'(x) = [g(b) - g(a)]\, f'(x)$.

Mean Value theorem.

If f is a real continuous function on $[a,b]$ which is differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$$f(b) - f(a) = (b-a) f'(x).$$

Proof:-

From the previous theorem we have

$$[f(b) - f(a)]\, g'(x) = [g(b) - g(a)]\, f'(x).$$

put $g(x) = x$.

$$\therefore f(b) - f(a) = (b-a) f'(x). //$$

Theorem.

suppose f is differentiable in (a,b).

(a) If $f'(x) \geq 0 \ \forall\, x \in (a,b)$, then f is monotonically increasing.

(b). If $f'(x) = 0 \ \forall\, x \in (a,b)$, then f is constant.

(c) If $f'(x) \leq 0 \ \forall\, x \in (a,b)$, then f is monotonically decreasing.

Proof :-

All conclusions can be read off from the equation.

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(x)$$

Which is valid for each pair of numbers x_1, x_2 in (a,b) for some x between x_1 & x_2

The continuity of derivatives :-

Theorem.

Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a,b) \ni: f'(x) = \lambda$.

A similar results holds of course if $f'(a) > f'(b)$.

Proof :-

Let us consider the function.

$$g(t) = f(t) - \lambda t$$

$$g'(t) = f'(t) - \lambda$$

$$g'(a) = f'(a) - \lambda$$

But given that $f'(a) < \lambda < f'(b)$

$$f'(a) < \lambda \Rightarrow f'(a) - \lambda < 0.$$

$$\Rightarrow g'(a) < 0$$

So that $g(t_1) < g(a) \ \forall \ t_1 \in (a,b)$.

$$g'(b) = f'(b) - \lambda$$

w.r.t $\quad f'(b) - \lambda > 0$

$$g'(b) > 0$$

So that $g(t_2) < g(b)$ for some $t_2 \in (a,b)$

Hence g attains its minimum on $[a,b]$ at some point $x \ni : a < x < b$

$\therefore g'(x) = 0$

Hence $f'(x) - \lambda = 0$

$\therefore f'(x) = \lambda$.

Corollary :-

If f is differentiable on $[a,b]$ then f' cannot have any simple discontinuities on $[a,b]$.

Indeterminate form

$\frac{0}{0}$, $0 \times \infty$, $\infty \times \infty$, $\frac{\infty}{\infty}$, $\infty - \infty$

L'Hospital's Rule :-

$\lim\limits_{x \to 0} \frac{x}{\tan x}$

Theorem :-

Suppose f & g are real and differentiable in (a,b), and $g'(x) \neq 0 \;\forall\; x \in (a,b)$, where $-\infty \le a < b \le +\infty$. Suppose,

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a,$$

$$f(x) \to 0 \;\&\; g(x) \to 0 \text{ as } x \to a$$

$$g(x) \to +\infty \text{ as } x \to a,$$

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a.$$

Proofs :-

Consider $-\infty \leq A \leq \infty$

choose a real number $q \ni: A < q$ &

choose $r \ni: A < r < q$ ——— (*).

Now, $\exists$ a point $c \in (a,b) \ni:$

$$a < x < c \Rightarrow \frac{f'(x)}{g'(x)} < r.$$

If $a < x < y < c$, then there is a point

$$t \in (x,y) \ni: \frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)} < r. \quad \text{——— ①}$$

Suppose $f(x) \to 0$ & $g(x) \to 0$ as $x \to a$ holds

Taking $x \to a$ sides of ① we have

$$\frac{f(y)}{g(y)} \leq r < q \quad , \quad a < y < c \quad \text{——— ②}$$

Next suppose $g(x) \to \infty$ as $x \to a$ holds.

keeping y fixed in ①. we can choose, a point $c_1 \in (a,y) \ni: g(x) > g(y)$ & $g(x) > 0$ if $a < x < c_1$

Multiplying ① by $\dfrac{g(x)-g(y)}{g(x)}$ we obtain

$$\frac{f(x)-f(y)}{\cancel{g(x)-g(y)}} \times \frac{\cancel{g(x)-g(y)}}{g(x)} < r\left[\frac{g(x)-g(y)}{g(x)}\right]$$

$$\frac{f(x)-f(y)}{g(x)} < r\left[\frac{g(x)-g(y)}{g(x)}\right]$$

$$\frac{f(x)}{g(x)} - \frac{f(y)}{g(x)} < r - r\frac{g(y)}{g(x)}$$

$$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}, \quad a < x < c_1 \longrightarrow (3)$$

Given that $g(x) \to \infty$ as $x \to a$ taking limit $x \to a$ on both sides of (*), (3) and the above shows that there is a point $c_2 \in (a, c_1)$.

$$\Rightarrow \frac{f(x)}{g(x)} < r < q$$

$$\Rightarrow \frac{f(x)}{g(x)} < q, \quad a < x < c_2 \longrightarrow (4)$$

Summing up (2) & (4) show that for any q subject only to the condition $A < q$ there is a point $c_2 \to \frac{f(x)}{g(x)} < q$ if $a < x < c_2$.

In the same manner if $-\infty \leq A \leq +\infty$ & p is chosen so that $p < A$, we can find a point $c_2 \ni: p < \frac{f(x)}{g(x)}, \quad a < x < c_3$

and $\frac{f(x)}{g(x)} \to A$ as $x \to a$ follows from there two statements.

$$p < \cancel{A} < \cancel{r} < q$$

Derivatives of Higher order

Defn:-

If f has a derivative f' on an interval and if f' is itself differentiable we denote the derivative of f' by f'' and call f'' the second derivative of f.
continuing in this manner, we obtain functions
$f, f', f'', f^{(3)}, \ldots f^{(n)}$,

Each of which is the derivative of the preceding one. $f^{(n)}$ is called the n^{th} derivative (or) the derivative of order n of f.

In order for $f^{(n)}(x)$ to exist at a point x. $f^{(n-1)}(t)$ must exist in a neighbourhood and $f^{(n-1)}$ must be differentiable at x.

Taylor's Theorem :-

Theorem.

Suppose f is a real function on $[a,b]$, n is a positive integer $f^{(n-1)}$ is continuous on $[a,b]$, $f^{n}(t)$ exists for every $t \in (a,b)$.
Let α, β be distinct points of $[a,b]$, and define,

$$p(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k \quad \text{——— (1)}$$

Then $\exists$ a point x between α & β $\ni$:

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n \quad \text{———} (2)$$

For $n=1$, this is just the mean value theorem. In general the theorem shows that f can be approximated by a polynomial of degree $n-1$ and that allows us to estimate the error, if we know bounds on $|f^{(n)}(x)|$.

Proof :-

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k \quad \text{———} (1)$$

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n \quad \text{———} (2)$$

Let M be the number defined by,

$$f(\beta) = P(\beta) + M(\beta-\alpha)^n \quad \text{———} (3)$$

& put; $g(t) = f(t) - P(t) - M(t-\alpha)^n$ ——— (4)

$(a \leq t \leq b)$.

We have to show that $n!M = f^{(n)}(x)$ for some x between hence α & β by (1) & (4).

$$g^n(t) = f^{(n)}(t) - n!M \quad \text{———} (5) \quad , \; a<t<b.$$

Hence the proof win be completed if we show that $g^n(x) = 0$ for some x between α & β.

$\because \beta^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, 1, 2, \ldots\ldots n-1$

have, $g(\alpha) = g'(\alpha) \cdots = g^{n-1}(\alpha) = 0$.

But our choice of M show that $g(\beta)=0$.

So that $g'(x)=0$ for some x between α & β by mean value theorem.

Since, $g'(\alpha)=0$ we can conclude similarly that $g''(x)=0$ for some x_2 between α & x_1.

proceeding in this way for n-steps we've that $g^n(x_n)=0$ for some, x_n between α & x_{n-1} is between α & β.

Hence the theorem //.

28.8.19

UNIT-III

THE RIEMANN-STIELTJES INTEGRAL

Definition and Existence of the integral

Definition (6.1)

Let $[a,b]$ be a given integral. By a partition P of $[a,b]$ we mean a finite set of points $x_0, x_1, \ldots, x_n$.

where $a = x_0 \le x_1 \le \ldots \le x_{n-1} = b$.

we write $\boxed{\Delta x_i = x_i - x_{i-1}}$ $(i = 1, 2, \ldots n)$

2m.

Now suppose f is a bdd real fn defined on $[a,b]$. Corresponding to each partition P of $[a,b]$

we put $M_i = \sup f(x)$ (LUB) $(x_{i-1} \le x \le x_i)$

and $m_i = \inf f(x)$ (GLB) $(x_{i-1} \le x \le x_i)$

$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$ and $L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$.

and finally

$$\overline{\int_a^b} f\,dx = \inf U(P,f) \quad ---- (1) \qquad \underline{\int_a^b} f\,dx = \sup L(P,f) \quad ---- (2)$$

where the inf and the sup are taken over all partitions P of $[a,b]$.

The left members of (1) and (2) are called the upper and lower Riemann integrals of f over $[a,b]$, resp.

2m

If the upper and lower integrals are equal, we say that f is Riemann integrable on $[a,b]$,

we write $f \in R$ (ie, R denotes the set of all Riemann integrable fns).

2m.

we denote the common value of eqn (1) and (2),

$$\int_a^b f\,dx \quad ---- (3)$$

$$(\text{or}) \quad \int_a^b f(x)\,dx \quad ---- (4).$$

This is the Riemann integral of f over $[a,b]$. Since f is bdd, there exist two numbers m and M such that

$m \le f(x) \le M \quad (a \le x \le b)$

Hence, for every P,

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

so that the numbers $L(P,f)$ and $U(P,f)$ form a bdd set.

Definition (6.2)

Let α be a monotonically increasing fn. on $[a,b]$. (Since $\alpha(a)$ and $\alpha(b)$ are finite, it follows that α is bdd on $[a,b]$). Corresponding to each partition P of $[a,b]$, we write $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$.

It is clear that $\Delta x_i \geq 0$.

2M For any real fn f which is bdd on $[a,b]$, we put $U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$ and

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$M_i = \sup f(x)$
$m_i = \inf f(x)$

Where M_i, m_i have the same meaning as in defn (6.1).

We define $\overline{\int_a^b} f\,d\alpha = \inf U(P,f,\alpha)$ ----- (1)

$$\underline{\int_a^b} f\,d\alpha = \sup L(P,f,\alpha) \quad \text{-----} (2)$$

the inf and sup again being taken over all partitions.

If the left members of (1) and (2) are equal, we ~~denote~~ their common value by $\int_a^b f\,d\alpha$ ---- (3)

or sometimes by $\int_a^b f(x)\,d\alpha(x)$ ----- (4)

This is the Riemann - Stieltjes integral (or simply the stieltjes integral) of f with respect to α, over $[a,b]$)

Definition (6.3) U.Q.

2M. We say that the partition P^* is a refinement of P if $P^* \supset P$ (ie, if every point of P is a point of P^*). Given two partitions P_1 and P_2 we say that P^* is their common refinement if $P^* = P_1 \cup P_2$.

Theorem (6.4)

5M

If P^* is a refinement of P, then

U.Q. $L(P,f,\alpha) \leq L(P^*,f,\alpha)$

$U(P^*,f,\alpha) \leq U(P,f,\alpha)$

Proof

To prove ~~U.C.~~ $L(P,f,\alpha) \leq L(P^*,f,\alpha)$

Suppose first that P^* contains just one point more than P.

Let this extra point be x^*.

Suppose $x_{i-1} \le x^* \le x_i$, where x_{i-1} and x_i are two consecutive points of P.

Put $w_1 = \inf f(x) \quad (x_{i-1} \le x \le x^*)$

$w_2 = \inf f(x) \quad (x^* \le x \le x_i)$

Clearly $w_1 \ge m_i$ and $w_2 \ge m_i$

where $m_i = \inf f(x) \quad (x_{i-1} \le x \le x_i)$

Hence,

$$L(P^*, f, \alpha) - L(P, f, \alpha) = w_1 [\alpha(x^*) - \alpha(x_{i-1})] + w_2 [\alpha(x_i) - \alpha(x^*)] - m_i [\alpha(x_i) - \alpha(x_{i-1})]$$

$$\Rightarrow L(P^*, f, \alpha) - L(P, f, \alpha) = (w_1 - m_i)[\alpha(x^*) - \alpha(x_{i-1})] + (w_2 - m_i)[\alpha(x_i) - \alpha(x^*)]$$

$$\Rightarrow L(P^*, f, \alpha) - L(P, f, \alpha) \ge 0.$$

If P^* $\therefore L(P^*, f, \alpha) \ge L(P, f, \alpha)$

$\Rightarrow L(P, f, \alpha) \le L(P^*, f, \alpha)$

M'ly $U(P^*, f, \alpha) \le U(P, f, \alpha)$.

Hence the proof.

Theorem (6.5) $$\underline{\int_a^b} f \, d\alpha \le \overline{\int_a^b} f \, d\alpha.$$

Proof Let P^* be the common refinement of two partitions P_1 and P_2.

By thm 6.4.

$$L(P_1, f, \alpha) \le L(P^*, f, \alpha) \le U(P^*, f, \alpha) \le U(P_2, f, \alpha)$$

Hence, $$L(P_1, f, \alpha) \le U(P_2, f, \alpha) \quad \text{------} ①$$

If P_2 is fixed and the sup is taken over all P_1.

Eqn (1) gives $$\underline{\int} f \, d\alpha \le U(P_2, f, \alpha) \quad \text{----} ②$$

The thm follows by taking the inf over all P_2 in eqn②

Hence the result.

5m.

<u>Thm (6.6)</u> $f \in R(\alpha)$ on $[a,b]$ iff for every $\varepsilon > 0$ there exists a partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \quad \text{------} \ (1)$$

<u>Proof:</u> For every P we have

$$L(P,f,\alpha) \le \underline{\int} f\,d\alpha \le \overline{\int} f\,d\alpha \le U(P,f,\alpha)$$

$$(1) \Rightarrow \quad 0 \le \overline{\int} f\,d\alpha - \underline{\int} f\,d\alpha < \varepsilon$$

Hence if eqn (1) can be satisfied for every $\varepsilon > 0$, we have $\overline{\int} f\,d\alpha = \underline{\int} f\,d\alpha$.

ie, $f \in R(\alpha)$.

Conversely, suppose $f \in R(\alpha)$, and let $\varepsilon > 0$ be given. Then there exist partitions P_1 and P_2 such that

$$U(P_2,f,\alpha) - \int f\,d\alpha < \varepsilon/2 \quad \text{------} \ (2)$$

$$\int f\,d\alpha - L(P_1,f,\alpha) < \varepsilon/2 \quad \text{------} \ (3)$$

we choose P to the common ~~refinement of P_1, P_2.~~

Then thm (6.4) together with (2) and (3) shows that

$$U(P,f,\alpha) \le U(P_2,f,\alpha) < \int f\,d\alpha + \frac{\varepsilon}{2} < L(P_1,f,\alpha) + \frac{\varepsilon}{2}$$
$$\le L(P,f,\alpha) + \varepsilon.$$

so that eqn (1) holds for this partition P.

ie, $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

Hence the theorem.

<u>Thm (6.7)</u>

(a) If $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$, for some P and some ε, then $U(P,f,\alpha) - L(P,f,\alpha)$ (with the same ε) for every refinement of P.

(b) If $U(P,f,\alpha) - L(P,f,\alpha)$, for $P = \{x_0, \dots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$,

then $\displaystyle\sum_{i=0}^{n} |f(s_i) - f(t_i)|\,\Delta\alpha_i < \varepsilon$

(c) If $f \in R(\alpha)$ and the hypotheses of (b) hold, then $\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \varepsilon$.

Proof

Thm 6.4 $\Rightarrow$ (a).

Under the assumptions made in (b), both $f(s_i)$ and $f(t_i)$ lie in $[m_i, M_i]$

so that $|f(s_i) - f(t_i)| \le M_i - m_i$.

Thus, $\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i \le U(P,f,\alpha) - L(P,f,\alpha)$

ie, $\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i < \varepsilon$. [∵ from thm (6.6)

which proves (b).

The obvious inequalities

$L(P,f,\alpha) \le \sum f(t_i)\Delta\alpha_i \le U(P,f,\alpha)$ ---- ①

and $L(P,f,\alpha) \le \int f\,d\alpha \le U(P,f,\alpha)$ ---- ②

From ① and ②

$\Rightarrow L(P,f,\alpha) \le \sum f(t_i)\Delta\alpha_i \le \int f\,d\alpha \le U(P,f,\alpha)$

ie, ~~$L(P,f,\alpha)$~~ $\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \varepsilon$

Prove (c).

Hence the theorem.

3.9.19

Thm (6.8) ⊛ 2m — The fn f is said to be cts at $x=a$ if given any $\varepsilon>0$, $\exists$ $\delta>0$ $\ni$ $|f(x)-f(a)|<\varepsilon$ whenever $|x-a|<\delta$

If f is continuous on $[a,b]$, then $f \in R(\alpha)$ on $[a,b]$.

2m. The fn $f: M_1 \to M_2$ is said to be uniformly cts on M_1 if $\varepsilon>0$ $\exists$ $\delta>0$ $\ni$ $|f(x)-f(y)|<\varepsilon$ whenever $|x-y|<\delta$

Proof: Let $\varepsilon>0$ be given. Choose $\eta>0$

$\ni$ $[\alpha(b)-\alpha(a)]\eta < \varepsilon$.

Since f is uniformly continuous on $[a,b]$ (thm 4.19)

$\exists$ a $\delta>0$ $\ni$ $|f(x) - f(t)| < \eta$ if $x \in [a,b]$, $t \in [a,b]$ and $|x-t| < \delta$ ----- ①

If P is any partition of $[a,b]$ such that $\Delta x_i < \delta$ for all i.

$① \Rightarrow \quad M_i - m_i \le \eta \quad (i = 1 \ldots n)$

$\therefore \; U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i$

$\le \eta \sum_{i=1}^{n} \Delta\alpha_i$

$= \eta \, [\alpha(b) - \alpha(a)]$

$\Rightarrow U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

ie, $f \in R(\alpha)$.

Hence the theorem.

Thm (6.9) V.I 5m. 5M (or) 2m

If f is monotonic on $[a, b]$, and if α is continuous on $[a, b]$, then $f \in R(\alpha)$. $\sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$

Proof

Let $\varepsilon > 0$ be given.

For any positive integer n, choose a partition such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n} \quad (i = 1 \ldots n)$$

This is possible since α is continuous (Thm 4.23)

we suppose that f is monotonically increasing.

Then $M_i = f(x_i)$, $m_i = f(x_{i-1})$. $(i = 1 \ldots n)$

So that

$$U(P, f, \alpha) - L(P, f, \alpha) = \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \cdot [f(b) - f(a)]$$

$\Rightarrow U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$. if n is taken large enough.

ie, $f \in R(\alpha)$.

Thm (6.11)

Suppose $f \in R(\alpha)$ on $[a, b]$, $m \le f \le M$, ϕ is continuous on $[m, M]$ and $h(x) = \phi(f(x))$ on $[a, b]$.

Then $h \in R(\alpha)$ on $[a, b]$.

Proof: Choose $\varepsilon > 0$, since ϕ is uniformly continuous on $[m, M]$ $\exists$ $\delta > 0$ $\ni$ $\delta < \varepsilon$

and $|\phi(s) - \phi(t)| < \varepsilon$ if $|s-t| \le \delta$ and $s, t \in [m, M]$

Since $f \in R(\alpha)$, there is a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ $\ni$ $U(P, f, \alpha) - L(P, f, \alpha) < \delta^2$ ------ ①

Let M_i, m_i have the same meaning as in defn 6.1

Let M_i^*, m_i^* be the analogous numbers for (h)

Divide the numbers $1, \ldots, n$ into two classes:

$i \in A$ if $M_i - m_i < \delta$, $i \in B$ if $M_i - m_i \ge \delta$.

For $i \in B$, $M_i^* - m_i^* \le 2K$, where $K = \sup|\phi(t)|$, $m \le t \le M$.

By eqn ① we have.

$$\delta \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B} (M_i - m_i)\Delta\alpha_i < \delta^2.$$

so that $\sum_{i \in B} \Delta\alpha_i < \delta$.

It follows that

$$U(P, h, \alpha) - L(P, h, \alpha) = \sum_{i \in A} (M_i^* - m_i^*)\Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*)\Delta\alpha_i$$

$$\le \varepsilon[\alpha(b) - \alpha(a)] + 2K\delta$$

$$U(P,h,\alpha) - L(P,h,\alpha) < \varepsilon[\alpha(b) - \alpha(a) + 2K].$$

Since ε was arbitrary.

Thm (6.6) $\Rightarrow$ $h \in R(\alpha)$.

Hence the theorem.

PROPERTIES OF THE INTEGRAL

V.I 8. 5m and 2m KH

Thm (6.13) If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a,b]$, then

(a) $fg \in R(\alpha)$

(b) $|f| \in R(\alpha)$ and $\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$.

U.Q

Proof If we take $\phi(t) = t^2$

Thm (6.11) shows that $f^2 \in R(\alpha)$ if $f \in R(\alpha)$

8. com or KR

The identity $4fg = (f+g)^2 - (f-g)^2$.

$fg \in R(\alpha)$

Completes the proof (a). $\rightarrow f^2 + g^2 + 2fg - [f^2 + g^2 - 2fg]$

$\Rightarrow f^2 + g^2 + 2fg - f^2 - g^2 + 2fg$

$\Rightarrow 4fg$

(b) If we take $\phi(t) = |t|$. thm (6.11) shows similarly that $|f| \in R(\alpha)$.

Choose $c = \pm 1$ so that $c\int f d\alpha \geq 0$

Then, $|\int f d\alpha| = c\int f d\alpha$

$= \int c f d\alpha$

ie, $|\int f d\alpha| \leq \int |f| d\alpha$.

Since $cf \leq |f|$.

Hence the result.

Definition (6.14)

The unit step function I is defined by

$$I(x) = \begin{cases} 0 & (x \leq 0), \\ 1 & (x > 0). \end{cases}$$

Thm (6.15) - 2M U.Q.

If $a < s < b$, f is bdd on $[a,b]$, f is continuous at s, and $\alpha(x) = I(x-s)$, then

$$\int_a^b f d\alpha = f(s)$$

Proof Consider partitions $P = \{x_0, x_1, x_2, x_3\}$, where $x_0 = a$, $x_1 = s < x_2 < x_3 = b$.

Then $U(P, f, \alpha) = M_2$ and $L(P, f, \alpha) = m_2$.

Since f is continuous at s, we see that M_2 and m_2 converge to $f(s)$ as $x_2 \to s$.

Thm (6.16) 10m . 10M

Suppose $c_n \geq 0$, for $1, 2, \ldots$ Σc_n converges, $\{s_n\}$ is a sequence of distinct points in (a,b) and

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n) \quad \text{---} (1)$$

10M Let f be continuous on $[a,b]$.

then, $$\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n).$$

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n) \longrightarrow (1)$$ (9)

Proof: The comparison test shows that the series (1) converges for every x.

Its sum $\alpha(x)$ is evidently monotonic and $\alpha(a) = 0$, $\alpha(b) = \sum c_n$. (Remark 4.31)

Let $\varepsilon > 0$ be given, and choose N so that

$$\sum_{n=N+1}^{\infty} c_n < \varepsilon.$$

Put $\alpha_1(x) = \sum_{n=1}^{N} c_n I(x - s_n)$ and

$$\alpha_2(x) = \sum_{N+1}^{\infty} c_n I(x - s_n).$$

Choose N

By thm 6.12 and 6.15,

$$\int_a^b f\,d\alpha_1 = \sum_{n=1}^{N} c_n f(s_n). \quad \text{------ (3)}$$

Since $\alpha_2(b) - \alpha_2(a) < \varepsilon$

ie, $\left|\int_a^b f\,d\alpha_2\right| \le M\varepsilon$ ---- (4)

~~Where $M = \sup|f(x)|$.~~ Since $\alpha = \alpha_1 + \alpha_2$

It follows from (3) and (4).

$$\left|\int_a^b f\,d\alpha_1 = \int_a^b f\,d\alpha_2\right| \le \sum_{n=1}^{N} c_n f(s_n) - M\varepsilon.$$

ie, $\left|\int_a^b f\,d\alpha - \sum_{n=1}^{N} c_n f(s_n)\right| \le M\varepsilon.$

If we let $N \to \infty$, then $\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$

Hence the proof.

V.I

Thm (6.17) (X) 10m vvv

Assume α is increases monotonically and $\alpha' \in R$ on $[a,b]$.

Let f be a bdd real fn on $[a,b]$.

Then $f \in R(\alpha)$ iff $f\alpha' \in R$.

10M

In that case

$$\int_a^b f\,d\alpha = \int_a^b f(x)\,\alpha'(x)\,dx. \quad \text{---- (1)}$$

$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ iff $f \in \mathcal{R}(\alpha)$

Proof:

Let $\varepsilon > 0$ be given and apply thm (6.6) to α'.

There is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$

$\ni$ $U(P,\alpha') - L(P,\alpha') < \varepsilon$ ------ (2)

The mean value thm furnishes points $t_i \in [x_{i-1}, x_i]$

$\ni$ $\boxed{\Delta\alpha_i = \alpha'(t_i)\Delta x_i}$ $\{ [f(b)-f(a)]g'(x) = [g(b)-g(a)]f'(x) \}$

for $i = 1, \ldots n$. If $s_i \in [x_{i-1}, x_i]$, $\{f(b)-f(a)\} = (b-a)f'(x)$

then $\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i < \varepsilon$ ----- (3)

By eqn (2) and thm 6.7(b).

Put $M = \sup|f(x)|$

Since $\sum_{i=1}^{n} f(s_i)\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i \Rightarrow \Delta\alpha_i = \alpha'(t_i)\Delta x_i$

It follows that from eqn (3)

$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i \leq M\varepsilon$ ---- (4)

In particular, $\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq U(P, f\alpha') + M\varepsilon$

for all choices of $s_i \in [x_{i-1}, x_i]$, so that

$U(P,f,\alpha) \leq U(P, f\alpha') + M\varepsilon$.

The same argument leads from (4)

$U(P, f, \alpha') \leq U(P, f, \alpha) + M\varepsilon$

Thus $|U(P,f,\alpha) - U(P, f\alpha')| \leq M\varepsilon$ ---- (5)

Now eqn (2) remains true if P is replaced by any refinement.

Hence eqn (5) also remains true.

We conclude that $\left| \int_a^{-b} f\,d\alpha - \int_a^{-b} f(x)\alpha'(x)\,dx \right| \leq M\varepsilon$.

But ε is arbitrary.

Hence, $\int_a^{-b} f\,d\alpha = \int_a^{-b} f(x)\alpha'(x)\,dx$ for any bdd f

Hence the theorem.

INTEGRATION AND DIFFERENTIATION

Thm (6.20)

Let $f \in R(\alpha)$ on $[a,b]$. For $a \le x \le b$.

Put $F(x) = \int_a^x f(t)dt$.

Then F is continuous on $[a,b]$; furthermore, if f is continuous at a point x_0 of $[a,b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$

Proof:

Since $f \in R$, f is bdd.

Suppose $|f(t)| \le M$, for $a \le t \le b$.

If $a \le x \le y \le b$, then

$|F(y) - F(x)| = \left| \int_x^y f(t)dt \right| \le M(y-x)$

Given $\varepsilon > 0$, we see that

$|F(y) - F(x)| < \varepsilon$ provided that $|y - x| < \varepsilon / M$

This proves continuity (and, in fact, uniform continuity) of F.)

Now suppose f is continuous at x_0.

Given $\varepsilon > 0$, choose $\delta > 0$

$\ni$ $|f(t) - f(x_0)| < \varepsilon$, if $|t - x_0| < \delta$ and $a \le t \le b$.

Hence, if

$x_0 - \delta < s \le x_0 \le t < x_0 + \delta$ and $a \le s < t \le b$

we have, by thm

$$\left| \frac{F(t) - F(s)}{t-s} - f(x_0) \right| = \left| \frac{1}{t-s} \int_s^t [f(u) - f(x_0)] du \right|$$

$$< \varepsilon$$

$\Delta x_i < \delta$

ie, $\left| \dfrac{F(t) - F(s)}{t-s} - f(x_0) \right| < \varepsilon$.

$\Sigma \Delta x_i$

$\therefore \left| \dfrac{F(t) - F(s)}{t-s} \right| = f(x_0)$

ie, $F'(x_0) = f(x_0)$.

Hence the theorem.

Thm (6.21) (The fundamental thm of calculus)

If $f \in R$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ $\ni F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof:

Let $\varepsilon > 0$ be a given. Choose a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ so that $U(P,f) - L(P,f) < \varepsilon$. ---- ①

The mean value thm furnishes points $t_i \in [x_{i-1}, x_i]$

$\ni$ $F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$, for $i = 1 \ldots n$.

Thus $\sum_{i=1}^{n} f(t_i)\Delta x_i = F(b) - F(a)$. ---- ②

WKT $\sum_{i=1}^{n} f(t_i)\Delta x_i \doteq \int_a^b f(x)\,dx$ ---- ③ Thm (6.7) c.

From eqns ② and ③.

$\Rightarrow \left| F(b) - F(a) - \int_a^b f(x)\,dx \right| < \varepsilon$.

Since for every $\varepsilon > 0$.

$\therefore \int_a^b f(x)\,dx = F(b) - F(a)$.

Hence the theorem.

Thm (6.22) (Integration by parts)

Suppose F and G are differentiable fns on $[a,b]$, $F' = f \in R$ and $G' = g \in R$. Then.

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx.$$

Proof:

Put $H(x) = F(x)G(x)$

Apply thm (6.21) to H and its derivative.

INTEGRATION OF VECTOR-VALUED FUNCTIONS

~~Thm~~ **Definition (6.23)**

Let $f_1, \ldots, f_k$ be a real fns on $[a,b]$ and let $f = (f_1, \cdots f_k)$ be the corresponding mapping of $[a,b]$ into R^k.

If α increases monotonically on $[a,b]$, to say that $f \in R(\alpha)$ means that $f_j \in R(\alpha)$, for $j = 1, 2, \ldots, k$.

If this is the case, we define

$$\int_a^b f \, d\alpha = \left(\int_a^b f_1 \, d\alpha, \int_a^b f_2 \, d\alpha, \ldots, \int_a^b f_k \, d\alpha \right)$$

In other words, $\int f \, d\alpha$ is the point in R^k whose jth Coordinate is $\int f_j \, d\alpha$.

RECTIFIABLE CURVES

Definition (6.26)

A Continuous mapping γ of -- interval $[a,b]$ into R^k is called a curve in R^k.

To emphasize the parameter interval $[a,b]$, we may also say that γ is a curve on $[a,b]$.

If γ is 1-1, γ is called an arc.

If $\gamma(a) = \gamma(b)$, γ is said to be a closed curve.

We define a curve to be a mapping, not a point set. Of course, with each curve γ in R^k there is associated a subset of R^m, namely the range of γ, but different curves may have the same range.

We associate to each partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ and to each curve γ on $[a,b]$ the number

$$\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$$

The ith term in this sum is the distance (in R^k) between the points $\gamma(x_{i-1})$ and $\gamma(x_i)$.

Hence $\Lambda(P, \gamma)$ is the lengths of a polynomial path with vertices at $\gamma(x_0), \gamma(x_1), \ldots, \gamma(x_n)$ in this order.

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As our partition becomes finer and finer, this polygon approaches the range of γ more and more closely. This makes it seem reasonable to define the length of γ as

$$\Lambda(\gamma) = \sup \Lambda(P, \gamma).$$

where the sup is taken over all partitions of [a, b].

If ~~...~~ $\Lambda(\gamma) < \infty$, we say that γ is rectifiable.

In certain cases, $\Lambda(\gamma)$ is give by a Riemann integral. We shall prove that this for continuously differentiable curves.

i.e, For curves γ whose derivative γ' is continuous.

V. I

Thm (6.27) ⓧ 10m 10M 10M.

If γ' is continuous on $[a,b]$, then γ is rectifiable, and $\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt.$ $\Lambda(\gamma)$ A = B A ≤ B B ≤ B

Proof: If $a \le x_{i-1} < x_i \le b$, then

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\, dt \right|$$

$$|\gamma(x_i) - \gamma(x_{i-1})| \le \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt$$

Consequently

Hence, $\Lambda(P, \gamma) \le \int_a^b |\gamma'(t)|\, dt$, for every partition P of [a, b].

Consequently, $\Lambda(\gamma) \le \int_a^b |\gamma'(t)| dt$ ------ Ⓐ

To prove the opposite inequality,

Let ~~...~~

i.e, $\int_a^b |\gamma'(t)|\, dt \le \Lambda(\gamma)$.

Let $\varepsilon > 0$ be given. Since γ' is uniformly continuous on $[a, b]$, there exists $\delta > 0$

Uniformly Conti $\cancel{|f(x)-f(y)| < \varepsilon \quad |x-y| < \delta}$

$\Rightarrow |\gamma'(s) - \gamma'(t)| < \varepsilon$ if $|s-t| < \delta$.

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$, with $\Delta x_i < \delta$ for all i. $\delta > 0$

If $x_{i-1} \le t \le x_i$, it follows that

$$|\gamma'(t)| \le |\gamma'(x_i)| + \varepsilon$$

Hence,

$$\int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt \le |\gamma'(x_i)|\,\Delta x_i + \varepsilon\,\Delta x_i$$

$$= \left| \int_{x_{i-1}}^{x_i} \left[(\gamma'(t) + \gamma'(x_i) - \gamma'(t) \right) dt \right| + \varepsilon\,\Delta x_i$$

$$\le \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\,dt \right| + \left| \int_{x_{i-1}}^{x_i} \left[\gamma'(x_i) - \gamma'(t) \right] dt \right| + \varepsilon\,\Delta x_i$$

$$\le |\gamma(x_i) - \gamma(x_{i-1})| + 2\varepsilon\,\Delta x_i.$$

If we add these inequalities, we obtain

$$\int_a^b |\gamma'(t)|\,dt \le \Lambda(P, \gamma) + 2\varepsilon\,(b-a)$$

$$\cancel{\le \Lambda(\gamma) + 2\varepsilon(b-a)}$$

Since ε was arbitrary.

$$\therefore \int_a^b |\gamma'(t)|\,dt \le \Lambda(\gamma) \quad \text{-----} \; (B)$$

From eqns (A) and (B),

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\,dt$$

Hence the theorem.

UNIT - 4

SEQUENCES AND SERIES OF FUNCTIONS

7.1 Defn :-

Suppose $\{f_n\}$, $n = 1, 2, 3 \ldots\ldots$, is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. Then we define a function f by

$$f(x) = \lim_{n \to \infty} f_n(x), \quad (x \in E) \longrightarrow ①$$

Hence, we say that $\{f_n\}$ converges on E and that f is the limit, or the limit function of $\{f_n\}$. Sometimes it may be descripted as "$\{f_n\}$ converges to f pointwise on E."

Defn :-

If the series $\Sigma f_n(x)$ converges for every $x \in E$, and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad (x \in E) \longrightarrow ②$$

Then, the function f is called the sum of the series Σf_n.

Main problem :-

To determine whether important properties of functions are preserved under the limit operations defined eqn ① & ② above.

i.e i) if the functions f_n are continuous, or differentiable, or integrable. Then, is the same true of the limit function f? (2)

ii) What are the relations between f_n' and f' or between the integrals of f_n and that of f?

iii) We know that if f is continuous at x then we have $\lim_{t \to x} f(t) = f(x)$.

The next question is whether the limit of the sequence of continuous function is continuous.

i.e) $$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

Note :-

We shall show in the following example that,

i) The limit processes cannot be interchanged in general.

ii) But under certain conditions, the order in the limit processes can be interchanged.

Ex :- 7.2

Consider the "double sequence"

$$S_{m,n} = \frac{m}{m+n} \quad \text{for } m = 1, 2, 3, \ldots\ldots, \; n = 1, 2, 3, \ldots\ldots$$

Then for every fixed n,

$$\lim_{m \to \infty} S_{m,n} = \lim_{m \to \infty} \frac{m}{m+n}$$

$$= \lim_{m\to\infty} \frac{1}{1+n/m} = 1 \qquad \left[\lim_{m\to\infty} \frac{n}{m} = \frac{n}{\infty} = 0\right]$$

(3)

$$\therefore \lim_{n\to\infty} \lim_{m\to\infty} S_{m,n} = \lim_{n\to\infty} 1 = 1 \longrightarrow (1)$$

For every fixed m,

$$\lim_{n\to\infty} S_{m,n} = \lim_{n\to\infty} \frac{m}{m+n} = 0 \Rightarrow \lim_{n\to\infty} \frac{1}{1+n/m} \Rightarrow \frac{1}{1+\frac{\infty}{1}} \Rightarrow \frac{1}{1+\frac{\infty}{0}} \Rightarrow \frac{1}{\frac{0+1}{0}} = \frac{0}{1} = 0$$

so that, $\lim_{m\to\infty} \lim_{n\to\infty} S_{m,n} = 0 \longrightarrow (2)$

Now, for every (1) & (2)

$$\lim_{n\to\infty} \lim_{m\to\infty} S_{m,n} \neq \lim_{m\to\infty} \lim_{n\to\infty} S_{m,n}$$

$\therefore$ The limit operations are not interchangeable.

Ex. 7.3

To show that a convergent series of continuous functions may have a discontinuous sum.

Sol:-

Let $f_n(x) = \dfrac{x^2}{(1+x^2)^n}$, $(x \text{ real}; n = 0, 1, 2, \ldots)$

consider $f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \dfrac{x^2}{(1+x^2)^n}$

Since $f_n(0) = 0$, we have $f(0) = 0$. For $x \neq 0$, an infinite geometric series with common ratio $\dfrac{1}{1+x^2}$

Its sum, $S_n = \dfrac{x^2}{1 - \frac{1}{1+x^2}}$ $\left[\text{For } a + ar + ar^2 + \ldots \quad S_n = \dfrac{a}{1-r}\right]$

$$= \frac{(1+x^2)\,x^2}{(1+x^2)-1} = 1+x^2$$

(4)

$$\therefore \quad f(x) = \begin{cases} 0 & , \text{ when } x=0 \\ 1+x^2 & , \text{ when } x \neq 0 \end{cases}$$

Thus, a convergent series of continuous functions may have a discontinuous sum.

Ex:- 7.4.

If an everywhere discontinuous limit function which is not Riemann-integrable.

For $m = 1, 2, 3, \ldots\ldots$ put $f_m(x) = \lim\limits_{n\to\infty} (\cos m!\,\pi x)^{2n}$

when $m!x$ is an integer, $f_m(x) = 1$.

For all other values of x, $f_m(x) = 0$

Sol :-

Let $f(x) = \lim\limits_{m\to\infty} f_m(x)$

when x is irrational, $f_m(x) = 0$, for every m.

Hence $f(x) = 0 \longrightarrow (1)$

when x is rational, say $x = p/q$ where p & q are integers.

$m!x$ is an integer if $m \geq q$.

$\therefore\ f(x) = 1 \longrightarrow (2)$ [by the earlier discussion]

From (1) & (2)

$$\lim_{m\to\infty} \lim_{n\to\infty} (\cos m!\,\pi x)^{2n} = \begin{cases} 0 & , \text{ when } x \text{ is irrational} \\ 1 & , \text{ when } x \text{ is rational} \end{cases}$$

Thus, we obtained an everywhere discontinuous limit function which is not Riemann-integrable.

X. Ex: 7.5

To show that the limit of the integral need not be equal to the integral of the limit.

Let $f_n(x) = n^2 x(1-x^2)^n$, $(0 \le x \le 1,\ n=1,2,3,\ldots\ldots)$

For $0 < x \le 1$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} n^2 x(1-x^2)^n = 0 \longrightarrow ①$

Since $f_n(0)=0$, we see that

$$\lim_{n\to\infty} f_n(x) = 0 \qquad (0 \le x \le 1)$$

Now, $\int_0^1 x(1-x^2)^n\, dx = \int_0^{\pi/2} \cos^{2n+1}\theta \,.\, d(\cos\theta)$

$$= \left[\frac{\cos^{2n+2}\theta}{2n+2}\right]_0^{\pi/2} = 0 - \frac{1}{2n+2}$$

$$= \frac{-1}{2n+2}$$

Hence, $\int_0^1 f_n(x)\, dx = n^2 \int_0^1 x(1-x^2)^n\, dx = \frac{-n^2}{2n+2}$

$$\therefore \lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \lim_{n\to\infty} \frac{n^2}{2n+2} = \infty \longrightarrow ②$$

If $f_n(x) = nx(1-x^2)^n$ [replacing n^2 by n]

then, $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\, dx = \lim\limits_{n\to\infty} \frac{n}{2n+2}$

$$= \lim_{n\to\infty} \frac{n}{n(2+\frac{2}{n})} = \lim_{n\to\infty} \frac{1}{2+2/n}$$

$$= 1/2$$

Again, $\int_0^1 \left\{\lim\limits_{n\to\infty} f_n(x)\right\} dx = 0 \longrightarrow ③$ using ①

From (2) & (3) we get that the limit of the integral need not be equal to the integral of the limit even if both are finite.

(b)

7.7 Defn:-

UNIFORM CONVERGENCE

A sequence of functions $\{f_n\}$ $n = 1, 2, 3 \ldots$ is said to converge uniformly on a set E to a function f if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \varepsilon, \quad \forall x \in E.$$

Note:-

It is obvious that every uniformly convergent sequence is pointwise convergent.

Concept (i):-

If $\{f_n\}$ converges pointwise on E, then there exists a function f such that, for every $\varepsilon > 0$ and for every $x \in E$, there is an integer N, depending on ε and on x such that $|f_n(x) - f(x)| \leq \varepsilon$

Concept (ii):-

If $\{f_n\}$ converges uniformly on E, it is possible for each $\varepsilon > 0$, to find one integer N such that $n \geq N$

$$\Rightarrow |f_n(x) - f(x)| \leq \varepsilon, \quad \forall x \in E.$$

Defn:-

The series $\sum f_n(x)$ is said to converge uniformly on E, if the sequence $\{S_n\}$ of partial sums defined by $\sum_{i=1}^{n} f_i(x) = S_n(x)$ converges uniformly on E.

Theorem :- 7.8

10M **Cauchy criterion for uniform convergence:**

The sequence of functions $\{f_n\}$ defined on a set E, converges uniformly if and only if for every $\varepsilon > 0$, there exists an integer N such that $m \geq N$, $n \geq N$, $x \in E$ implies $|f_n(x) - f_m(x)| \leq \varepsilon$

Proof :- N.P

$\Rightarrow$:- Suppose $\{f_n\}$ converges uniformly on E and let f be the limit function.

Then ~~th~~ by defn,

$\exists$ an integer N $\ni$ $n \geq N$

$x \in E \Rightarrow |f_n(x) - f(x)| \leq \varepsilon/2$

$|||^{ly}$ $|f_m(x) - f(x)| \leq \varepsilon/2$

$$\therefore |f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)|$$

$$\leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$$

$$\leq \varepsilon/2 + \varepsilon/2$$

$$\leq \varepsilon$$

Hence, $|f_n(x) - f_m(x)| \leq \varepsilon$, for $n, m \geq N$

$\Leftarrow$:-

Conversely, suppose the cauchy conditions holds.

i.e) suppose that for every $\varepsilon > 0$, $\exists$ an integer N $\ni$ $m \geq N$, $n \geq N$, $x \in E \Rightarrow |f_n(x) - f_m(x)| \leq \varepsilon$

Proof :- S.P To Prove

$\{f_n\}$ converges uniformly on E. The sequence $\{f_n\}$ converges for every x, to a limit (say) $f(x)$.

$\therefore \{f_n\}$ converges on E to $f(x)$.

To proof :-

The convergence is uniform.

(8)

Let $\varepsilon > 0$ be given.

choose $N \ni |f_n(x) - f_m(x)| \leq \varepsilon$ is true.

Fix n and allow $m \to \infty$

since $f_m(x) \to f(x)$ as $m \to \infty$, we have

$$|f_n(x) - f(x)| \leq \varepsilon \quad \text{[from ①]}$$

For every $n \geq N$ & every $x \in E$

$\therefore$ The convergence is uniform. (by defn).

7.9 Theorem :-

Suppose $\lim\limits_{n\to\infty} f_n(x) = f(x)$, $(x \in E)$

put $M_n = \sup\limits_{x \in E} |f_n(x) - f(x)|$

Then f_n converges to f uniformly on E if and only if $M_n \to 0$ as $n \to \infty$

Proof :-

$\Rightarrow$:- To prove $M_n \to 0$ as $n \to \infty$

Since $f_n \to f$ uniformly on E,

$$|f_n - f| \leq \varepsilon \text{ for } n \geq N \text{ \& for each } x \in E.$$

$$\therefore \sup_{x \in E} |f_n - f| \leq \varepsilon \quad \text{i.e) } M_n \leq \varepsilon$$

since ε is arbitary, $M_n \to 0$ as $n \to \infty$.

$\Leftarrow$:-

Conversely, let $M_n \to 0$ as $n \to \infty$

$\therefore M_n \leq \varepsilon$, since $\varepsilon > 0$ is arbitary

i.e) $\sup\limits_{x \in E} |f_n(x) - f(x)| \leq \varepsilon$ for $n \geq N$ & each $x \in E$

$\Rightarrow |f_n - f| \leq \varepsilon$ for $n \geq N$ & each $x \in E$

$\Rightarrow$ $f_n \to f$ uniformly on E.

T.10 Theorem:- WEIERSTRASS TEST 5M

Suppose $\{f_n\}$ is a sequence of functions defined on E and suppose $|f_n(x)| \leq M_n$ $(x \in E, n = 1, 2, 3, \ldots)$. Then Σf_n converges uniformly on E if ΣM_n converges.

(The converse of the theorem not true)

Proof :-

If ΣM_n converges, then, for arbitary $\varepsilon > 0$,

$$\left| \sum_{i=1}^{m} f_i(x) - \sum_{K=1}^{n} f_K(x) \right| = \sum_{i=K}^{m} M_i \leq \varepsilon \quad (x \in E)$$

$$\therefore |f_n(x)| \leq M_n$$

when m and n are large enough.

Hence by Cauchy criterion for uniform convergence, Σf_n converges uniformly on E.

23.9.2019

T.11 Theorem:- UNIFORM CONVERGENCE AND CONTINUITY

Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E and suppose that.

$\lim\limits_{t \to x} f_n(t) = A_n$ $(n = 1, 2, 3, \ldots)$. Then $\{A_n\}$ converges

and $\lim\limits_{t \to x} f(t) = \lim\limits_{n \to \infty} A_n$.

Proof :- Let $\varepsilon > 0$ be given.

since, f_n converges to f uniformly on E.

By defn, $\exists$ an integer $N \ni n \geq N,\ m \geq N,\ t \in E$

$$\Rightarrow |f_n(t) - f_m(t)| \leq \varepsilon \longrightarrow (1)$$

Allow $t \to x$ in (1)

$$\therefore (1) \Rightarrow |f_n(x) - f_m(x)| \leq \varepsilon$$

ie, $|A_n - A_m| \leq \varepsilon$ for $n \geq N,\ m \geq N$

$\therefore \{A_n\}$ is a Cauchy sequence & converges to (say) A.

Now, $|f(t) - A| = |f(t) - f_n(t) + f_n(t) - A_n + A_n - A|$

$$\leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \longrightarrow (2)$$

First choose $n \ni |f(t) - f_n(t)| \leq \varepsilon/3,\ \forall\, t \in E \longrightarrow (3)$

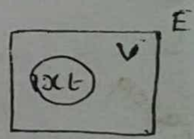
(This is possible by uniform convergence)

Since $\{A_n\}$ converges to A, we have

$$|A_n - A| \leq \varepsilon/3 \longrightarrow (4)$$

For this n, choose a neighbourhood V of x,

$$\ni |f_n(t) - A_n| \leq \varepsilon/3 \longrightarrow (5)$$



if $t \in V \cap E,\ t \neq x$

using (3), (4), (5) in (2), we've

$$|f(t) - A| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

given
$\therefore \lim_{t \to x} f_n(t) = A_n$
$\lim_{n \to \infty} f_n(t) = f(t)$

ie) $|f(t) - A| \leq \varepsilon$ when $t \in V \cap E,\ t \neq x$

$$\Rightarrow \lim_{t \to x} f(t) = A = \lim_{n \to \infty} A_n$$

$$\Rightarrow \lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

7.12 Theorem :- (Corollary Thm 7.11) 5M

If $\{f_n\}$ is a sequence of continuous functions on a set E and if f_n cgs to f uniformly on E then f is continuous ~~of~~ on E. (11)

Proof :-

Given $\{f_n\}$ converges to f uniformly on E.

$\therefore$ By defn, for every $\varepsilon > 0$, $\exists$ an integer N

$\ni n \geq N \Rightarrow |f_n(x) - f(x)| \leq \varepsilon/3 \longrightarrow (1)$, $\forall x \in E$

$\therefore |f_n(x_0) - f(x_0)| \leq \varepsilon/3 \longrightarrow (2)$, $\forall x_0 \in E$

Also given $\{f_n\}$ is continuous at (say) $x = x_0$.

$\therefore$ for every $\varepsilon > 0$ $\exists$ a $\delta > 0$ $\ni |x - x_0| < \delta$

$\Rightarrow |f_n(x) - f_n(x_0)| \leq \varepsilon/3 \longrightarrow (3)$

To proof :-

f is continuous.

Now, $|f(x) - f(x_0)| = |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)|$

$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$

$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ [using (1), (2), (3)]

ie) $|f(x) - f(x_0)| \leq \varepsilon$ for $|x - x_0| < \delta$

Hence f is continuous.

Note :-

i) The converse of the above theorem (is corollary) is not true in general.

ie) a sequence of continuous function may ~~coverage~~ converge to a continuous function but the convergence need not be uniform.

ii) The convergence may be uniform under certain conditions, which is the following thm 7.13.

Cor 7.13 Theorem :-

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Suppose K is compact and

a) $\{f_n\}$ is a sequence of continuous functions on K.

b) $\{f_n\}$ converges pointwise to a continuous function f on K.

c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$

Then $\{f_n\}$ converges to f uniformly on K.

Proof :-

Put $g_n = f_n - f \longrightarrow ①$

Since $\{f_n\}$ is continuous & f is also continuous.

$\therefore f_n - f$ is continuous.

Hence g_n is continuous.

Given $\{f_n\}$ converges to the continuous function f on K.

$\therefore f_n - f$ ie) $g_n \to 0$ pointwise on K. $\longrightarrow ②$

Since $f_n(x) \geq f_{n+1}(x)$, $f_n(x) - f(x) \geq f_{n+1}(x) - f(x)$

$\Rightarrow \quad g_n \geq g_{n+1} \quad$ [using ①]

To proof :-

$g_n \to 0$ uniformly on K. | ie) $f_n - f \to 0$ uniformly $\Rightarrow f_n \to f$ uniformly.

Let $\varepsilon > 0$ be given.

Let K_n be the set of all x in $K \ni g_n(x) \geq \varepsilon \longrightarrow ③$

since g_n is continuous, & K_n is closed.

Since $g_n \geq g_{n+1}$, $K_n \supset K_{n+1}$ [$\because K_n$ is a subset of K thm 2.35]

Fix $x \in K$.

since $g_n(x) \to 0$, $x \notin K_n$ if n is sufficiently large.

This is true for every n

$\therefore x \notin \cap K_n$

$\therefore \cap K_n = \phi$

using (3) if $x \in K_n$

then $g_n(x) \geq \varepsilon > 0$

ie) $g_n(x) \geq 0 \Rightarrow g_n \nrightarrow 0$

(13)

$\therefore K_N$ is empty for some N.

$\therefore 0 \leq g_n(x) < \varepsilon$ for all $x \in K$ and for all $n \geq N$.

Hence, the proof.

$\Rightarrow \Leftarrow$ to (3)

our assumption that $g_n(x) \geq \varepsilon$ is false

Hence, we've $0 \leq g_n(x) < \varepsilon$ & $\forall n \geq N$

$\therefore \{f_n\} \leq g_s$ to f uniformly on K.

Note:-

Compactness is an important condition required to prove the above theorem. For example, if

$$f_n(x) = \frac{1}{nx+1} \quad (0 < x < 1 ; n = 1, 2, 3 \ldots\ldots)$$

Then $f_n(x) \to 0$ monotonically in $(0,1)$, but the convergence is not uniform.

7.14 Defn:-

Let X be a metric space and $\mathscr{C}(X)$ denote the set of all complex - values, continuous, bounded functions with domain X.

With each function $f \in \mathscr{C}(X)$ associate its supremum norm given by,

$$\|f\| = \sup_{x \in X} |f(x)|$$

Since, f is assumed to be bounded $\|f\|$ is finite.

To show:-

with the above norm :

$\mathscr{C}(X)$ is a metric space. Define Let $f, g \in \mathscr{C}(X)$ and define the distance between f & g to $\|f-g\|$

ie) $d(f,g) = \|f-g\|$

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i) $d(f,g) = \|f-g\| = \sup\limits_{x \in X} |f - g(x)| \geq 0$

& $d(f,g) = 0$ only if $\sup\limits_{x \in X} |f(x) - g(x)| = 0$

ie) if $|f(x) - g(x)| = 0$ ie) if $f = g$

ii) $d(f,g) = \|f-g\| = \|g-f\| = d(g,f)$

iii) If $h = f+g$, then $|h(x)| = |f(x) + g(x)|$

$$\leq |f(x)| + |g(x)|$$

$$\leq \sup |f(x)| + \sup |g(x)| = \|f\| + \|g\|$$

$\therefore |f+g| \leq \|f\| + \|g\|$

Taking supremum

$$\|f+g\| \leq \|f\| + \|g\|$$

Hence, $\mathscr{C}(X)$ is a metric space.

Note:

i) Using the above defn we concentrate the thm 7.9 as follows. " A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathscr{C}(X)$ if and only if $f_n \to f$ uniformly on X.

ii) Accordingly, closed subsets of $\mathscr{C}(X)$ are sometimes called uniformly closed, the closure of a set ~~$A \in \mathscr{C}(X)$~~ $A \subset \mathscr{C}(X)$ is called its uniform closure and so on.

7.15 Theorem :- (X). 5 M. Sculpt c

The above metric makes $\mathscr{C}(X)$ into a complete metric space.

[ie) $\mathscr{C}(X)$ with the metric defined in sec 7.14 is a complete metric space] 15

Proof :-

W.K.T $\mathscr{C}(X)$ is a metric space with the metric

7.14 → $\|f\| = \sup_{x \in X} |f(x)|$ or each $f \in \mathscr{C}(X)$

To proof :-

$\mathscr{C}(X)$ is the ~~metric~~ complete metric space.

ie) To prove every cauchy sequence in $\mathscr{C}(X)$ is convergent. ← (complete M.S)

Let $\{f_n\}$ be a cauchy sequence in $\mathscr{C}(X)$

∴ By defn, to each $\varepsilon > 0$ ∃ an integer N

∋ $\|f_n - f_m\| < \varepsilon$ if $n, m \geq N$ [$N \to$ +ve integer]

Since $\mathscr{C}(X)$ is the set of all complex valued continuous and bounded functions by cauchy criterion of uniform convergence (Thm 7.8)

$\{f_n\}$ converges to f with domain X, f is continuous and f is bounded. (Thm 7.12)

Since, there is an n ∋ $|f(x) - f_n(x)| < 1$, $\forall x \in X$ and f_n is bounded. → $|f(x)| \leq k$ if $k \geq 0$

Thus $f \in \mathscr{C}(X)$ and since $\{f_n\}$ converges to f uniformly on X,

we've $\|f - f_n\|$ converges to 0 as $n \to \infty$

Hence $\{f_n\}$ converges in $\mathscr{C}(X)$

Hence the theorem.

UNIFORM CONVERGENCE AND INTEGRATION

7.16 Theorem :- 5M

U.Q Nov 2017.

Let α be monotonically increasing on $[a,b]$. suppose $f_n \in R(\alpha)$ on $[a,b]$, for $n = 1,2,3 \ldots$, and suppose $f_n \to f$ uniformly on $[a,b]$. Then $f \in R(\alpha)$ on $[a,b]$ and

$$\int_a^b f \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha. \quad (16)$$

(The existence of the limit is part of the conclusion.)

Proof :-

It is sufficient to prove the result for the real function f_n.

put $\varepsilon_n = \sup |f_n(x) - f(x)| \longrightarrow (1)$

The supremum is taken over $a \le x \le b$.

$$\left.\begin{aligned} &\text{From (1)}, \quad |f(x) - f_n(x)| \le \varepsilon_n \\ &\text{ie)} \quad -\varepsilon_n \le f(x) - f_n(x) \le \varepsilon_n \end{aligned}\right\} \longrightarrow (*)$$

$|x| \le \infty$

$-\infty \le x \le \infty$

$\therefore f_n(x) - \varepsilon_n \le f(x) \le f_n(x) + \varepsilon_n \longrightarrow (2)$

using the defn of upper and lower integral of f. [Defn 6.2]

we've

$$\int_a^b (f_n - \varepsilon_n)\,d\alpha \le \underline{\int} f\,d\alpha \le \overline{\int} f\,d\alpha \le \int_a^b (f_n + \varepsilon_n)\,d\alpha$$

$$\therefore 0 \le \overline{\int} f\,d\alpha - \underline{\int} f\,d\alpha \le \int_a^b (f_n + \varepsilon_n)\,d\alpha - \int_a^b (f_n - \varepsilon_n)\,d\alpha$$

$$\text{ie)} \quad 0 \le \overline{\int} f\,d\alpha - \underline{\int} f\,d\alpha \le 2\varepsilon_n \int_a^b d\alpha$$

$\int_a^b dx = [x]_a^b = b-a$

$$\text{ie)} \quad 0 \le \overline{\int} f\,d\alpha - \underline{\int} f\,d\alpha \le 2\varepsilon_n [\alpha[b] - \alpha[a]] \longrightarrow (3)$$

Given f_n converges to f on $[a,b]$ uniformly.

$\therefore$ By Thm 7.9 $\varepsilon_n \to 0$ as $n \to \infty$

$$(3) \Rightarrow \quad 0 \le \overline{\int} f\,d\alpha - \underline{\int} f\,d\alpha \le 0$$

$$\Rightarrow \overline{\int} f \cdot d\alpha = \underline{\int} f\, d\alpha$$

Hence. $f \in R(\alpha)$

Also from ② $\int_a^b f_n\, d\alpha - \varepsilon_n \int_a^b d\alpha \le \int_a^b f \cdot d\alpha \le \int_a^b f_n d\alpha + \varepsilon_n \int_a^b d\alpha$

ie) $-\varepsilon_n \int_a^b d\alpha \le \int_a^b f \cdot d\alpha - \int_a^b f_n \cdot d\alpha \le \varepsilon_n \int_a^b d\alpha$ (17)

$$\Rightarrow \left| \int_a^b f \cdot d\alpha - \int_a^b f_n\, d\alpha \right| \le \varepsilon_n \int_a^b d\alpha \quad [\text{From } ⊛] \quad \left[\because f \in R(\alpha),\ \int f \cdot d\alpha = \overline{\int} f \cdot d\alpha = \underline{\int} f \cdot d\alpha\right]$$

$$= \varepsilon_n [\alpha(b) - \alpha(a)]$$

since $\varepsilon_n \to 0$ as $n \to \infty$, we've

$$\int_a^b f \cdot d\alpha = \lim_{n\to\infty} \int_a^b f_n\, d\alpha$$

Corollary:-

If $f_n \in R(\alpha)$ on $[a,b]$ and if $f(x) = \sum_{n=1}^{\infty} f_n(x)$ $(a \le x \le b)$ is converging uniformly on $[a,b]$ then

$$\int_a^b f \cdot d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \cdot d\alpha$$

ie) The series may be integrated term by term.

UNIFORM CONVERGENCE AND DIFFERENTIATION

7.17 Theorem:- EX. 10m 10M

Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$ to a function f and

$$f'(x) = \lim_{n\to\infty} f_n'(x) \quad (a \le x \le b)$$

Proof :-

Given $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$.

Hence by Cauchy criterion of convergence,

given $\varepsilon > 0$ choose $N \ni n \geq N,\ m \geq N$ (18)

$$\Rightarrow |f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2} \longrightarrow ①$$

Also given $\{f_n'\}$ converges uniformly on $[a,b]$.

$$\therefore |f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)} \quad (a \leq t \leq b) \longrightarrow ②$$

Applying mean value theorem to the function $f_n - f_m$, we've $\frac{f(b) - f(a)}{b-a} = f'(x)$

$$\frac{|f_n(x) - f_m(x) - \{f_n(t) - f_m(t)\}|}{|x-t|} = |f_n'(t) - f_m'(t)|$$

for any x, t on $[a,b]$ if $n, m \geq N$.

ie) $$|f_n(x) - f_m(x) - \{f_n(t) - f_m(t)\}| \leq \frac{|x-t|\,\varepsilon}{2(b-a)} \quad [\text{using } ②]$$

$$\leq \frac{\varepsilon}{2} \text{ for any } x, t \text{ on } [a,b] \text{ if } n, m \geq N \longrightarrow ③$$

Now,

$$|f_n(x) - f_m(x)| = |f_n(x) - f_n(x_0) + f_n(x_0) - f_m(x_0) + f_m(x_0) - f_m(x)|$$

$$\leq |f_n(x) - f_m(x_0) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad [\text{using } ①, ③]$$

$$\leq \varepsilon$$

ie) $|f_n(x) - f_m(x)| < \varepsilon \quad (a \leq x \leq b,\ n \geq N,\ m \geq N)$

so that $\{f_n\}$ converges uniformly on $[a,b]$.

Let $f(x) = \lim_{n\to\infty} f_n(x) \quad (a \le x \le b) \longrightarrow (4)$

Now, fix a point x on $[a,b]$ and define (19)

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t-x}, \quad \phi(t) = \frac{f(t)-f(x)}{t-x}$$

for $a \le t \le b$, $t \ne x$. Then

$$\lim_{t\to x} \phi_n(t) = f_n'(x), \quad n = 1,2,3 \ldots \longrightarrow (5)$$

From (3)

$$\left| \frac{f_n(t) - f_n(x)}{t-x} - \frac{f_m(t) - f_m(x)}{t-x} \right| \le \frac{\varepsilon}{2(b-a)}$$

ie) $\left| \phi_n(t) - \phi_m(t) \right| \le \dfrac{\varepsilon}{2(b-a)}$, $n \ge N, m \ge N$ [using (4)]

$\therefore \{\phi_n\}$ converges uniformly for $t \ne x$

Since $\{f_n\}$ converges to f,

$$\lim_{n\to\infty} \phi_n(t) = \phi(t) \longrightarrow (6) \quad \text{[using (4)]}$$

uniformly for $a \le t \le b$, $t \ne x$.

By a thm on uniform convergence & continuity,

From (5) & (6)

$$\Rightarrow \lim_{t\to x} \phi(t) = \lim_{n\to\infty} f_n'(x) \quad \text{[Thm 7.11]}$$

ie) $f'(x) = \lim_{n\to\infty} f_n'(x)$ [using (5)]

Hence, the theorem.

—

7.18 Theorem :-

There exists a real continuous function on the real line which is ~~nowhere~~ not differentiable.

Proof :-

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Define $\varphi(x) = |x| \quad (-1 \le x \le 1)$

Extend the above defn of $\varphi(x)$ to all real x

$\ni \varphi(x+2) = \varphi(x)$

Then for all s and t,

$$|\varphi(s) - \varphi(t)| = \big| |s| - |t| \big| \le |s-t|$$

In particular, the function φ is continuous on R'.

Define $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x) \longrightarrow (1)$

since $0 \le \varphi \le 1$, by thm 7.10, the series (1) converges uniformly on R'.

Also f is continuous on R' [By thm 7.12]

Now fix a real number x and a positive integer m.

Put $\delta_m = \pm \frac{1}{2}(4^{-m})$ where the sign is chosen that no integer lies between $4^m x$ and $4^m(x+\delta_m)$.

This can be done, since $4^m |\delta_m| = \frac{1}{2}$

Define $\gamma_n = \dfrac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m} \longrightarrow (2)$

when $n \ge m$, $4^n \delta_m = 4^n (\pm \frac{1}{2}(4^{-m}))$

$= \pm \frac{1}{2}(4^{n-m})$

$=$ a multiple of 2.

$\therefore 4^n \delta_m$ is an even integer.

Hence $\varphi(4^n(x+\delta_m)) - \varphi(4^n x) = 0$ | $\because$ both the fns are even & no integer lies between them.

using this in (2), $\gamma_n = 0$

when $0 \le n \le m$, since

$|Q(s) - Q(t)| \le |s-t|$ $\Rightarrow$ (2) $\Rightarrow$ $|\gamma_n| =$ whole mod (2)

$$|\gamma_n| = \frac{|Q(4^n(x+\delta_m)) - Q(4^n x)|}{|\delta_m|}$$

$$\le \frac{|4^n(x+\delta_m) - 4^n x|}{|\delta_m|} = \frac{|4^n x + 4^n \delta_m - 4^n x|}{|\delta_m|} =$$

Hence $|\gamma_n| \le 4^n$.

since $|\gamma_m| \le 4^m$, we've

From (1) $\Rightarrow$
$$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| = \left|\frac{\sum_{n=0}^{m} \left(\frac{3}{4}\right)^n Q(4^n(x+\delta_m)) - \sum_{n=0}^{m}\left(\frac{3}{4}\right)^n Q(4^n x)}{\delta_m}\right|$$

$$= \sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \left|\frac{Q(4^n(x+\delta_m)) - Q(4^n x)}{\delta_m}\right|$$

$$= \sum_{0}^{m}\left(\frac{3}{4}\right)^n \gamma_n$$

$$= \left(\frac{3}{4}\right)^0 \gamma_0 + \left(\frac{3}{4}\right)^1 \gamma_1 + \left(\frac{3}{4}\right)^2 \gamma_2 + \dots \left(\frac{3}{4}\right)^m \gamma_m$$

$$\ge \frac{3^m}{4^m}\gamma_m - \sum_{0}^{m-1}\frac{3^n}{4^n}\gamma_n \qquad \frac{3^m}{4^m}\gamma_m - \left[\left(\frac{3}{4}\right)^0\gamma_0 + \left(\frac{3}{4}\right)^1\gamma_1 + \dots + \left(\frac{3}{4}\right)^{m-1}\gamma_{m-1}\right]$$

$$\ge \frac{3^m}{4^m}4^m - \sum_{0}^{m-1}\frac{3^n}{4^n}4^n$$

$$= 3^m - [1 + 3 + 3^2 + \dots + 3^{m-1}]$$

$$= 3^m - \frac{3^m - 1}{3-1} = 3^m - \left(\frac{3^m-1}{2}\right)$$

$$= \frac{3^m+1}{2} \qquad \text{G.P } [1 + ar + ar^2 + \dots + ar^{n-1}] = \frac{a(r^n-1)}{r-1}$$

$$\left(\frac{2\cdot 3^m - 3^m + 1}{2}\right)$$

As $m \to \infty$, $\delta_m = \frac{1}{2}\left(\frac{1}{4^m}\right) \to 0$

$$\therefore \lim_{\delta_m \to 0}\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| \ge \lim_{m\to\infty}\frac{3^m+1}{2} \to \infty$$

Thus, the limit does not exist and hence f is not differentiable.

Hence, the theorem.

EQUICONTINUOUS FAMILIES OF FUNCTIONS

7.19 Defn:-

Let $\{f_n\}$ be a sequence of functions defined on a set E. we say that $\{f_n\}$ is pointwise bounded on E if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$, ie) if $\exists$ a finite valued function ϕ defined on E such that

$$|f_n(x)| < \phi(x) \qquad (x \in E, n = 1, 2, 3 \ldots\ldots)$$

we say that $\{f_n\}$ is uniformly bounded on E if $\exists$ a number M $\ni$

$$|f_n(x)| < M \qquad (x \in E, n = 1, 2, \ldots\ldots)$$

if $\{f_n\}$ is pointwise bounded on E and E_1 is a countable subset of E, it is always possible to find a sub sequence $\{f_{n_k}\}$ $\ni$ $\{f_{n_k}(x)\}$ converges for every $x \in E_1$.

If $\{f_n\}$ is a uniformly bounded sequence of continuous functions on a compact set E, there need not exist a subsequence which converges pointwise on E.

Ex. 7.20

Every convergent sequence contain a uniformly convergent sub sequence.

Let $f_n(x) = \sin nx \quad (0 \le x \le 2\pi,\ n = 1, 2, 3 \ldots\ldots)$

Suppose, there exists a sequence $\{n_k\}$ $\ni$ $\{\sin n_k x\}$ converges, for every $x \in [0, 2\pi]$

In that case, we must have

$$\lim_{k\to\infty} (\sin n_k x - \sin n_{k+1} x) = 0 \qquad (0 \le x \le 2\pi)$$

Hence $\lim_{k\to\infty} (\sin n_k x - \sin n_{k+1} x)^2 = 0 \qquad (0 \le x \le 2\pi)$

By Lebesgue's theorem concerning integration of boundedly convergent sequences.

$$\lim_{k\to\infty} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 . dx = 0$$

But a simple calculation shows that

$$\int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 . dx = 2\pi$$

which contradicts.

7.21 Ex :-

Every convergent sequence need not contain a uniformly convergent sub sequence.

Let $f_n(x) = \dfrac{x^2}{x^2 + (1-nx)^2} \qquad (0 \le x \le 1, n = 1, 2, \dots)$

Then $|f_n(x)| \le 1$ so that $\{f_n\}$ is uniformly bounded on $[0,1]$. Also,

$$\lim_{n\to\infty} f_n(x) = 0, \qquad 0 \le x \le 1$$

Let $\{\frac{1}{n}\}$ be a subsequence of $f_n(n)$.

But $f_n(\frac{1}{n}) = 1, \quad (n = 1, 2, 3, \dots)$

so that no subsequence can converge uniformly on $[0,1]$.

7.22 Defn :- U.Q. X.

A family F of complex functions f defined on a set E in a metric space X is said to be equicontinuous on E if for every $\varepsilon > 0$ $\exists$ a $\delta > 0$

such that

$$|f(x) - f(y)| < \varepsilon$$

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2M whenever $d(x,y) < \delta$, $x \in E$, $y \in E$ and $f \in F$. Hence, d denotes the metric of X.

It is clear that every member of an equicontinuous family is uniformly continuous.

X. 7.23 Theorem :- 5M

If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

Proof :-

Let $\{x_i\}$, $i = 1, 2, 3 \ldots\ldots$ be the points of E, arranged in a sequence. since $\{f_n(x_1)\}$ is bounded, there exists a subsequence $\{f_{1,k}\}$ such that $\{f_{1,k}(x_1)\}$ converges as $k \to \infty$.

Let us now consider sequences $S_1, S_2, \ldots\ldots$ which we represent by the array,

$$S_1 : f_{1,1} \quad f_{1,2} \quad f_{1,3} \ldots\ldots$$

$$S_2 : f_{2,1} \quad f_{2,2} \quad f_{2,3} \ldots\ldots$$

$$S_3 : f_{3,1} \quad f_{3,2} \quad f_{3,3} \ldots\ldots$$

$$\ldots\ldots$$

and which have the following properties:

a) S_n is a subsequence of S_{n-1} for $n = 2, 3, 4 \ldots\ldots$

ie) $S_n \subset S_{n-1}$

b) $\{f_{n,k}(x_n)\}$ converges as $k \to \infty$ (the boundness of

$\{f_n(x_n)\}$ makes it possible to choose S_n in this way.) (25)

c) The order in which the functions appear in the same in each sequence. ie) If one function precedes in S_1, another in every S_n, until one or the other is deleted.

Hence, when going from one row in the above array to the next below, functions may move to the left but never to the right.

Now, we consider the diagonal of the array,

$S : f_{1,1} \quad f_{2,2} \quad f_{3,3} \ldots\ldots$

By (c), the sequence S(except possibly its first n-1 terms) is a subsequence of S_n, for $n = 1, 2, 3 \ldots\ldots$ Hence, (b) implies that $\{f_{n,n}(x_i)\}$ converges as $n \to \infty$ for every $x_i \in E$.

7.24 Theorem :- (V) 5M V.Q.

5M

If K is a compact metric space, if $f_n \in C(K)$ for $n = 1, 2, 3, \ldots\ldots$ and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.

Proof :-

Let K is a compact metric space. $f_n \in C(K)$ and $\{f_n\}$ converges uniformly.

A m.s M is said be compact if M both Complete & totally bdd. Totally bdd: A m.s (M, d) is totally bdd if every seq in M contains a cauchy subsequence.

To proof:- $\{f_n\}$ is equicontinuous on K.

ie) for every $\varepsilon > 0$ $\exists$ $\delta > 0$ $\ni$ $|f_n(x) - f_n(y)| < \varepsilon$ whenever $d(x, y) < \delta$, $x, y \in K$, $f_n \in C(K)$.

Given $\{f_n\}$ converges uniformly, given $\varepsilon > 0$ there is an integer N $\ni$

$\|f_n - f_N\| < \varepsilon$, $n > N$ | defn of uniformly converges.

Uni cts ⇒ cts
cts ⇏ uni cts
cts ⇒ uni cts if f compact

Since continuous functions are uniformly continuous on compact sets, there is a $\delta > 0$ $\ni |f_i(x) - f_i(y)| < \varepsilon$

(26)

if $1 \le i \le N$ and $d(x,y) < \delta$ $\ni$

$$|f_n(x) - f_n(y)| = |f_n(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f_n(y)|$$
$$\le |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)|$$
$$< \varepsilon + \varepsilon + \varepsilon$$
$$< 3\varepsilon = \varepsilon$$

$\therefore |f_n(x) - f_n(y)| < \varepsilon$

$\therefore \{f_n\}$ is equicontinuous on K.

7.25 Theorem :-

10M

If K is compact, if $f_n \in C(K)$ for $n = 1, 2, 3, \ldots$ and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then

10M

a) $\{f_n\}$ is uniformly bounded on K.

b) $\{f_n\}$ contains a uniformly convergent subsequence.

Proof :-

To prove (a) :-

Let $\varepsilon > 0$ be given and choose $\delta > 0$ by defn (7.22)

$|f_n(x) - f_n(y)| < \varepsilon$, $\forall n$, $\ni d(x,y) < \delta$

Since K is compact, there are finitely many points $p_1, p_2, \ldots p_r$ in K such that to every $x \in K$ corresponds atleast one p_i with $d(x, p_i) < \delta$.

Since, $\{f_n\}$ is pointwise bounded,

$\exists M_i < \infty \ni |f_n(p_i)| < M_i$ for all n.

If $M = \max(M_1, M_2, \dots\dots M_r)$, then $|f(x)| < M+\varepsilon$

for every $x \in K$.

This proves (a).

A subset A of a M.S (M,d) is said to be dense if $\bar{A} = M$

$\bar{A} = M$

To prove (b) :-

Let E be a countable dense subset of K.

Thm 7.23 shows that $\{f_n\}$ has a subsequence $\{f_{n_i}\}$ such that $\{f_{n_i}(x)\}$ converges for every $x \in E$.

put $f_{n_i} = g_i$ [to simplify the notation]

we shall prove that $\{g_i\}$ converges uniformly on K.

Let $\varepsilon > 0$, pick $\delta > 0$.

Let $V(x, \delta)$ be the set of all $y \in K$ with $d(x,y) < \delta$

Since E is dense in K and K is compact, there are finitely many points $x_1, x_2, \dots\dots x_m$ in E

$\ni K \subset V(x_1, \delta) \cup \dots\dots \cup V(x_m, \delta) \longrightarrow ①$

since $\{g_i(x)\}$ converges for every $x \in E$, there is an integer $N \ni |g_i(x_s) - g_j(x_s)| < \varepsilon \longrightarrow ②$

whenever $i \geq N$, $j \geq N$, $1 \leq s \leq m$.

If $x \in K$, ① shows that $x \in V(x_s, \delta)$ for some s,

so that $|g_i(x) - g_i(x_s)| < \varepsilon$ for every i.

If $i \geq N$ & $j \geq N$, it follows from ②

$$|g_i(x) - g_j(x)| = |g_i(x) - g_i(x_s) + g_i(x_s) + g_j(x_s) - g_j(x_s) - g_j(x)|$$

$$\leq |g_i(x) - g_i(x_s)| + |g_i(x_s) - g_j(x_s)| + |g_j(x_s) - g_j(x)|$$

$$< 3\varepsilon$$

$\therefore \{g_i(x)\}$ converges uniformly.

ie) $\{f_{n_i}(x)\}$ converges uniformly.

$\therefore \{f_n\}$ contains a uniformly convergence sequence.

2018, 2017

V.Q. 10M

7.26 Theorem :-

If f is a continuous complex function on $[a,b]$, there exists a sequence of polynomials P_n $\ni$ $\lim_{n\to\infty} P_n(x) = f(x)$ uniformly on $[a,b]$. If f is real then P_n may be taken real.

Proof :-

We may assume, without loss of generality that $[a,b] = [0,1]$. We may also assume that $f(0) = f(1) = 0$. For if the theorem is proved for this case consider,

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \quad (0 \le x \le 1)$$

Here, $g(0) = g(1) = 0$, and if g can be obtained as the limit of a uniformly convergent sequence of polynomials, it is clear that the same is true for f, since $f-g$ is a polynomial.

Furthermore, we define $f(x)$ to be zero for x outside $[0,1]$. Then f is uniformly continuous on the whole line. We put

$$Q_n(x) = c_n(1-x^2)^n \quad (n = 1,2,3,\dots\dots) \quad \text{----- (1)}$$

where c_n is chosen so that

$$\int_{-1}^{1} Q_n(x)\,dx = 1 \quad (n = 1,2,3,\dots\dots) \quad \text{----- (*)}$$

We need some information about the order of magnitude of c_n. Since,

$\cancel{\int_{-1}^{1} Q_n(x)\,dx = 1 \quad (n=1,2,3,\dots)}$

$\ge 2\int_0^{1/\sqrt{n}} (1-nx^2)\,dx$

$\ge 2\left(x - \frac{nx^3}{3}\right)_0^{1/\sqrt{n}}$

$\ge \frac{2}{3}\left(3x - nx^3\right)_0^{1/\sqrt{n}}$

$\ge \frac{2}{3}\left[\frac{3}{\sqrt{n}} - n\left(\frac{1}{\sqrt{n}}\right)^3\right] - 0$

$\ge \frac{2}{3}\left\{\frac{3}{\sqrt{n}} - n\frac{1}{n\sqrt{n}}\right\}$

$\ge \frac{2}{3}\left[\frac{2}{\sqrt{n}}\right]$

$\ge \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}$

$$\int_{-1}^{1}(1-x^2)^n dx = 2\int_0^1 (1-x^2)^n dx$$

$$\geq 2\int_0^{1/\sqrt{n}} (1-x^2)^n dx$$

$$\geq 2\int_0^{1/\sqrt{n}} (1-nx^2) dx$$

$$= \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}$$

(29)

it follows from ① that, $c_n < \sqrt{n} \longrightarrow ②$

The inequality $(1-x^2)^n \geq 1-nx^2$ which we used above is easily shown to be true by considering the function

$$(1-x^2)^n - 1 + nx^2$$

which is zero at $x=0$ and whose derivative is positive in $(0,1)$.

For any $\delta > 0$, ② implies

$$Q_n(x) \leq \sqrt{n}(1-\delta^2)^n \quad (\delta \leq |x| \leq 1) \longrightarrow ③ \quad [\because \text{eqn } ①$$

so that $Q_n \to 0$ uniformly in $\delta \leq |x| \leq 1$

Now let,

$$P_n(x) = \int_{-1}^{1} f(x+t)\, Q_n(t).dt, \quad 0 \leq x \leq 1.$$

Our assumptions about f show, by a simple change of variable, that

$$P_n(x) = \int_{-x}^{1-x} f(x+t).\, Q_n(t).dt = \int_0^1 f(t).\, Q_n(t-x)dt,$$

when $x+t=0$, $t=-x$; $x+t=1$, $t=1-x$; so that $x=1, t=-1$; $x=0, t=1$

and the last integral is clearly a polynomial in x. Thus $\{P_n\}$ is a sequence of polynomials, which are real if f is real.

Given $\varepsilon > 0$, we choose $\delta > 0$ such that $|y-x| < \delta$

$$\Rightarrow |f(y) - f(x)| < \frac{\varepsilon}{2}.$$

Let $M = \sup|f(x)|$. Using ① & ③ and the fact that $Q_n(x) \geq 0$, we see that for $0 \leq x \leq 1$,

$$
\begin{aligned}
|P_n(x) - f(x)| &= \left| \int_{-1}^{1} [f(x+t) - f(x)]\, Q_n(t)\,dt \right| \\
&\le \int_{-1}^{1} |f(x+t) - f(x)|\, Q_n(t)\,dt \qquad (30) \\
&\le 2M \int_{-1}^{-\delta} Q_n(t)\,dt + \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t)\,dt + 2M \int_{\delta}^{1} Q_n(t)\,dt \\
&\le 4M\sqrt{n}\,(1-\delta^2)^n + \frac{\varepsilon}{2} \qquad \{\because \text{using eqn } ③ \,\&\, ④ \\
&< \varepsilon
\end{aligned}
$$

$\le 4M \int_{-1}^{1} Q_n(t)\,dt + \frac{\varepsilon}{2}$

$|P_n(x) - f(x)|$

for all large enough n.

Hence, the theorem.

7.27 Corollary :-

For every interval $[-a,a]$ there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on $[-a,a]$.

Proof :-

By the previous theorem, there exists a sequence $\{P_n^*\}$ of real polynomials which converges to $|x|$ uniformly on $[-a,a]$.

In particular $P_n^*(0) \to 0$ as $n \to \infty$

The polynomials $P_n(x) = P_n^*(x) - P_n^*(0)$, $n = 1, 2, 3, \dots$ have desired properties.

7.28 Defn :-

A family A of complex functions defined on a set E is said to be an algebra if i) $f+g \in A$ ii) $fg \in A$ and iii) $cf \in A$ for all $f \in A$, $g \in A$ and for all complex constants c.

ie) if A is closed under addition, multiplication & scalar multiplication.

Consider A has algebra of real functions. in this case.

iii) only required to hold for all real c

If A has the property that $f \in A$ whenever $f_n \in A$ $(n=1,2,\dots)$ and $f_n \to f$ uniformly on E, then A is said to be uniformly closed.

Let B be the set of all functions which are limits of uniformly convergent sequences of members of A. Then B is called the uniform closure of A.

7.29 Theorem:- U.Q. x

Let B be the uniform closure of an algebra A of bounded functions. Then B is a uniformly closed algebra.

Proof:-

If $f \in B$ and $g \in B$, there exist uniformly convergent sequences $\{f_n\}, \{g_n\}$ such that

$$f_n \to f,\ g_n \to g \text{ and } f_n, g_n \in A.$$

Since we are dealing with bounded functions, it is easy to show that,

$$f_n + g_n \to f+g, \quad f_n g_n \to fg \quad cf_n \to cf$$

where c is any constant, the convergence being uniform in each case.

Hence, $f+g \in B$, $fg \in B$, and $cf \in B$, so that B is an algebra.

$\therefore$ B is uniformly closed. [By Thm 2.27]

7.30 Defn:-

Let A be a family of functions on a set E. Then A is said to separate points on E if to every pair of distincts point $x_1, x_2 \in E$ there corresponds a function $f \in A \ni f(x_1) \neq f(x_2)$

If to each $x \in E$ there corresponds a function $g \in A$ such that $g(x) \neq 0$, we say that A vanishes at no point of E.

7.31 Theorem:-

Suppose A is an algebra of functions on a set E, A separates points on E, and A vanishes at no point of E. Suppose x_1, x_2 are distinct points of E and c_1, c_2 are constants (real if A is a real algebra). Then A contains a function f such that

$$f(x_1) = c_1, \quad f(x_2) = c_2$$

Proof:-

The assumptions show that A contains functions g, h and k such that

$$g(x_1) \neq g(x_2), \quad h(x_1) \neq 0, \quad k(x_2) \neq 0$$

put $u = gk - g(x_1)k$, $\quad v = gh - g(x_2)h$

Then $u \in A$, $v \in A$, $u(x_1) = v(x_2) = 0$, $u(x_2) \neq 0$ and $v(x_1) \neq 0$.

$$\therefore f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$$

has the desired properties.

Thm 7.33 :-

Suppose A is a self-~~dis~~adjoint algebra of complex continuous functions on a compact set K, A separates points on K and A vanishes at no point of K. Then the uniform closure B of A consists of all complex continuous functions on K. In other words, A is dense $C(K)$.

Proof :-

Let A_R be the set of all real function on K which belong to A. (33)

If $f \in A$ and $f = u + iv$ with u, v real, then $2u = f + \bar{f}$ and since A is self-adjoint, we see that $u \in A_R$

If $x_1 \neq x_2$ $\exists$ $f \in A$ $\ni$ $f(x_1) = 1$, $f(x_2) = 0$

Hence, $0 = u(x_2) \neq u(x_1) = 1$, which shows that A_R separates points on K.

If $x \in K$, then $g(x) \neq 0$, for some $g \in A$ and there is a complex number λ such that $\lambda g(x) > 0$

if $f = \lambda g$, $f = u + iv$, it follows that $u(x) > 0$.

Hence, A_R vanishes at no point of K.

Thus A_R satisfies the hypotheses of thm 7.32.

It follows that every real continuous function on K. lies in the uniform closure of A_R, hence lies in B.

If f is a complex continuous functions on k, $f = u + iv$, then $u \in B$, $v \in B$, hence $f \in B$.

This completes the proof.