CORE COURSE III

ORDINARY DIFFERENTIAL EQUATIONS

Objectives

1. To give an in-depth knowledge of differential equations and their applications.
2. To study the existence, uniqueness, stability behavior of the solutions of the ODE

UNIT I

The general solution of the homogeneous equation– he use of one known solution to find another – The method of variation of parameters – Power Series solutions. A review of power series– Series solutions of first order equations – Second order linear equations; Ordinary points.

UNIT II

Regular Singular Points – Gauss's hypergeometric equation – The Point at infinity - Legendre Polynomials – Bessel functions – Properties of Legendre Polynomials and Bessel functions.

UNIT III

Linear Systems of First Order Equations – Homogeneous Equations with Constant Coefficients – The Existence and Uniqueness of Solutions of Initial Value Problem for First Order Ordinary Differential Equations – The Method of Solutions of Successive Approximations and Picard's Theorem.

UNIT IV

Oscillation Theory and Boundary value problems – Qualitative Properties of Solutions – Sturm Comparison Theorems – Eigenvalues,Eigenfunctions and the Vibrating String.

UNIT V

Nonlinear equations: Autonomous Systems; the phase plane and its phenomena – Types of critical points; Stability – critical points and stability for linear systems – Stability by Liapunov's direct method – Simple critical points of nonlinear systems.

TEXT BOOKS

G.F. Simmons, Differential Equations with Applications and Historical Notes, TMH, New Delhi, 1984.

UNIT – I	Chapter 3: Sections 15, 16, 19 and Chapter 5: Sections 25 to 27
UNIT – II	Chapter 5 : Sections 28 to 31 and Chapter 6: Sections 32 to 35
UNIT – III	Chapter 7: Sections 37, 38 and Chapter 11: Sections 55, 56
UNIT – IV	Chapter 4: Sections 22 to 24
UNIT – V	Chapter 8: Sections 42 to 44

REFERENCES

1. W.T. Reid, Ordinary Differential Equations, John Wiley & Sons, New York, 1971.
2. E.A. Coddington and N. Levinson, Theory of Ordinary Differential Equaitons, McGraw Hill Publishing Company, New York, 1955.

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3/9/19

Unit - 1

Second order linear eqns - Defn

The general second order linear differential equation is $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + Q(x)y = R(x)$ (or)

$$y'' + p(x)y' + Q(x)y = R(x) \quad \text{①}$$

where $p(x)$, $Q(x)$ & $R(x)$ are functions of 'x' alone.

The existence and uniqueness theorem (proof not include in Syllabus)

St: If $p(x)$, $Q(x)$ & $R(x)$ are continuous functions on a $[a,b]$. If x_0 is any point in the $[a,b]$ and if y_0 & y_0' are any numbers whatever the eqn ① has one & only one solution $y(x)$ on the interval ∋ $y(x_0) = y_0$ and $y'(x_0) = y_0'$.

Note:

i.e., the soln of given a differential eqn over all of $[a,b]$ is completely determined by its value & the value of its derivative at a single point.

6/9/10

Theorem 1: The general soln of the homogeneous eqns

(X) 10 marks. Nov-04,03

If $y_1(x)$ & $y_2(x)$ are linearly independent solutions of homogeneous equations $y'' + p(x)y' + Q(x)y = 0$ → ① on $[a, b]$. Then $c_1 y_1(x) + c_2 y_2(x)$ — ② is the general soln of ① on $[a,b]$.

In the sense that every solution of eqn ① of this interval can be obtained from eqn ② by suitable choice of the arbitrary constant c_1, c_2

proof :

Before proving the above theorem, let us prove 2 lemmas.

(X) 5 marks

Lemma (i) : If $y_1(x)$ and $y_2(x)$ are any two solutions of eqn ① on $[a, b]$. Then their wronskian eqn

$W = W(y_1, y_2)$ is either identically '0' or never '0' on $[a, b]$

proof of lemma : $W = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= y_1 y_2' - y_2 y_1'$$

Let the wronskian not vanish identically

$$W = y_1 y_2' - y_2 y_1'$$

$$W' = y_1 y_2'' + y_1' y_2' - [y_2 y_1'' + y_2' y_1']$$

$$W' = y_1 y_2'' - y_2 y_1'' \longrightarrow Ⓐ$$

put $\underline{y = y_1}$ in ① eqn $\rightarrow y'' + p(x) y' + Q(x) y = 0$

$$y_1'' + p(x) y_1' + Q(x) y_1 = 0 \longrightarrow ③$$

Since y_1 & y_2 are soln of y.

$\underline{y = y_2}$ in ① eqn

$$y_2'' + p(x) y_2' + Q(x) y_2 = 0 \longrightarrow ④.$$

③ y_2 - ④ y_1 $\Rightarrow$

$$y_1'' y_2 + p(x) y_1' y_2 + Q(x) y_1 y_2 - [y_2'' y_1 + p(x) y_2' y_1 + Q(x) y_2 y_1] = 0$$

$$y_1'' y_2 - y_2'' y_1 + p(x)[y_1' y_2 + y_2' y_1] = 0$$

$$-W' - p(x) W = 0 \quad [\text{from } Ⓐ]$$

$$W' + p(x) W = 0$$

$$\frac{dW}{dx} = -p(x) W$$

$$\frac{dW}{W} = -p(x) dx.$$

$\int$ing on both sides,

$$\log W = -\int p(x)\,dx + c \Rightarrow \log\left(\frac{W}{c}\right) = -\int p\,dx$$

$$W = ce^{-\int p\,dx}$$

R.H.S never becomes zero as the exponential factor never becomes zero.

Lemma (ii):

④ 5 marks

If $y_1(x)$ & $y_2(x)$ are two solns of eqn ① on $[a, b]$, then they are linearly dependent on this interval iff their wronskian is

$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ is identically zero.

If part: $(\Rightarrow)$ Necessary condition

Let y_1 & y_2 ~~are~~ be linearly ~~in~~dependent

We have to prove $y_1 y_2' - y_2 y_1' = 0$ $W(y_1, y_2) =$

If $y_1 = 0$ (or) $y_2 = 0$ then the result is clear.

$\therefore$ Without loss of generality, let us assume neither is identically zero. We have,

one is constant multiple of another

$y_1(x) = k\, y_2(x)$ $\quad y_2 = cy_1$

$y_2' = cy_1'$

$y_1'(x) = k y_2'(x)$

$y_2 = c y_1$

$y_2' = c y_1'$

$\frac{y_1}{y_2} = \frac{y_1'}{y_2'} = k$

$w(y_1, y_2) = y_1 y_2' - y_2 y_1' = k y_2 y_2' - k y_2' y_2 = 0$

$w(y_1, y_2) = y_1 y_2' - y_2 y_1'$

$= y_1 c y_1' - c y_1 y_1'$

$= 0$ [using previous result]

$y_1 y_2' - y_2 y_1' = 0$

Hence proved.

Only if part : ($\Leftarrow$) Sufficient condition / To prove linearly dependent

(i.e) Wronskian is identically zero

Let us assume $y_1 y_2' - y_2 y_1' = 0$. If y_1 is identically zero on $[a,b]$ then, $y_1(x) = 0$

$= 0 \cdot y_2(x)$

To prove: [illegible]

$\therefore y_1(x)$ & $y_2(x)$ are linearly dependent

Let us assume that $y_1(x)$ does not vanish identically on $[a,b]$.

$\therefore$ By continuity $\exists$ a subinterval $[c,d]$ of the interval $[a,b]$: y_1 does not vanish at all of these subinterval.

Since the Wronskian is identically zero on $[a,b]$.

We can divide it by y_1^2, we get.

$$\frac{y_1 y_2' - y_2 y_1'}{y_1^2} = 0$$

$$d\left(\frac{y_2}{y_1}\right) = 0$$

$$\frac{y_2}{y_1} = c$$

$$y_2 = c y_1$$

$$y_2(x) = c\, y_1(x) \quad \forall \; x \text{ in } [a, b]$$

Because, $y_2(x) = c\,y_1(x)$, we see that $y_2(x)$ & $c\,y_1(x)$ have the equal derivatives.

∴ By uniqueness theorem $y_2(x) = c\,y_1(x)$ throughout the interval $[a, b]$

proof of main theorem:

Let $y(x)$ be any solution of eqn ① on [a,

We have to show that constants c_1 & c_2 can be found out

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad \forall \; x \text{ in } [a, b]$$

By uniqueness theorem, the solns ① of the given differential eqn overall of the $[a, b]$ is completely determined by its value and the value of the derivative at a single point. Because $y(x)$ and $c_1 y_1(x)$ ~~&~~ + $c_2 y_2(x)$ are both solns of the given differential eqn on $[a, b]$.

It is enough ~~if~~ to we show that for some point x_0 in the interval $[a, b]$.

c_1 & c_2 can be found out

$y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0)$ and

$y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0)$

i.e, for the above system solvable for c_1 & c_2.

The condition is $y_1 y_2' - y_2 y_1' \neq 0$ $\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \Rightarrow$

this is defined by $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ → this is known as the wronskian of (y_1, y_2)

i.e, we have to prove $y_1 y_2' - y_2 y_1' \neq 0$

By lemma (ii) $y_1(x)$ & $y_2(x)$ are linearly independent, the wronskian is not identically zero and by lemma (i), the wronskian is never zero.

WKT If $y_1(x)$ & $y_2(x)$ are l.i. [illegible] $c_1y_1(x)+c_2y_2(x)$ is the general soln of $y''+$ [illegible]

Problems

1) Show that, $y = c_1 \sin x + c_2 \cos x$ is the general soln of $y'' + y = 0$, on any ~~integral~~ interval and find the particular soln for which $y(0) = 2, y'(0) = 3$

Soln:

- Let $y_1(x) = \sin x$

$y_2(x) = \cos x$

$\frac{y_1}{y_2} = \tan x \neq$ not a constant

∴ The solns are linearly independent

[∴ General soln of homogeneous eqn is $c_1y_1(x) + c_2y_2(x)$]

Then they are the solns of $y'' + y = 0$

(or)

Now let us prove that $y_1(x)$ & $y_2(x)$ are linearly independent.

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$$

$$= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= -\sin^2 x - \cos^2 x = -1 \neq 0$$

∴ Given that $y = c_1 \sin x + c_2 \cos x$ is the general soln of $y'' + y = 0$.

Given that, $y(0) = 2$ & $y'(0) = 3$

$y = 2,\ x = 0$; $y' = 3,\ x' = 0.$

$y = c_1 \sin x + c_2 \cos x$; $y' = c_1 \cos x - c_2 \sin x$

$2 = c_1 \sin 0 + c_2 \cos 0$; $3 = c_1 \cos 0 - c_2 \sin 0$

$\boxed{c_2 = 2}$; $\boxed{c_1 = 3}$

$\therefore$ The particular soln is $y = 3 \sin x + 2 \cos x$

2) Show that $\sin x$, $\cos x$ are independent solns

(X-) 2 marks of $y'' + y = 0$ on any interval.

Proof: Let $y_1(x) = \sin x$

$y_2(x) = \cos x$

Then they are the solns of $y'' + y = 0$.

T.p $y_1(x)$ & $y_2(x)$ are linearly independent.

Using wronskian,

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0.$$

$$= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$= -\sin^2 x - \cos^2 x = -1 \neq 0.$

$\therefore$ $\sin x$, $\cos x$ are independent solns of $y'' + y = 0$.

(7) 2marks Papers

5) Show that e^x, e^{-x} are linearly independent solns of $y'' - y = 0$. on any interval.

Another method

proof:

Let $y_1(x) = e^x$

$y_2(x) = e^{-x}$

$\frac{y_1}{y_2} = \frac{e^x}{e^{-x}} = e^{2x} \neq 0 \Rightarrow$ The soln is L.I

$y_1'' - y_1 = 0$

$e^x - e^x = 0$

$\therefore y_1(x)$ is a soln

Illy $y_2(x)$ "

Then they are the solns of $y'' - y = 0$.

(or)

T.P $y_1(x)$ & $y_2(x)$ are linearly independent.

Using wronskian,

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$$

$$= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$= -e^{x-x} - e^{x-x}$$

$$= -1 - 1$$

$$= -2 \neq 0$$

$\therefore e^x, e^{-x}$ are solns of $y'' - y = 0$

4) Show that $y = c_1 x + c_2 x^2$ is the general soln of $x^2 y'' - 2xy' + 2y = 0$ on any $\overset{\text{IV}}{\text{integral}}$ not containing zero $\overset{\&}{}$ find the particular soln for which $y(1) = 3$, $y'(1) = 5$.

<u>proof</u>: Let $y_1(x) = x$

$$y_2(x) = \cancel{e^x} x^2$$

<u>T.P</u> $y_1(x)$ & $y_2(x)$ are linearly independent using wronskian,

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

$$= 2x^2 - x^2$$

$$= x^2 \neq 0.$$

$\therefore \overset{y_1(x)}{x}, \overset{y_2(x)}{x^2}$ are solns of $x^2 y'' - 2xy' + 2y = 0$.

Given that $\underbrace{y(1) = 3}_{}$ & $\underbrace{y'(1) = 5}_{}$.

$\underline{y = 3,\ x = 1}$ $\qquad$ $\underline{y' = 5,\ x' = 1}$

x^2 $\quad y = c_1 x + c_2 x^2$ $\qquad$ $y' = c_1 + 2c_2 x$

$8 = c_1 + c_2$ $\quad$ & $\qquad$ $5 = c_1 + 2c_2$.

$\quad \llcorner ①$ $\qquad\qquad \llcorner ②$

① - ②
$$\begin{aligned} c_1 + c_2 &= 3 \\ c_1 + 2c_2 &= 5 \\ \hline -c_2 &= -2 \end{aligned}$$

$$\boxed{c_2 = 2}$$

From ①
$$\boxed{c_1 = 1.}$$

∴ The particular soln is ~~$y = 2x + 1x^2$~~ $y = x + 2x^2$

5) Show that $y = c_1 e^{2x} + c_2 x e^{2x}$ is the general soln of $y'' - 4y' + 4y = 0$ on any interval.

Proof.

$$y_1(x) = e^{2x}$$
$$y_2(x) = xe^{2x}$$

T.P $y_1(x)$ & $y_2(x)$ are linearly independent.

using wronskian $W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$

$$= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix}$$

$$= 2xe^{4x} + e^{4x} - 2xe^{4x}$$

$$= e^{4x} + xe^{4x}$$

$$= e^{4x} (\cos x) \neq 0.$$

T.P $y'' - 4y' + 4y = 0$ | $y_1 = e^{2x}$, $y_1' = 2e^{2x}$

$y_1'' = 4e^{2x}$.

$$4e^{2x} - 8e^{2x} + 4e^{2x} = 0$$

$$8e^{2x} - 8e^{2x} = 0$$

$\therefore$ $y(x)$ is a soln.

III^{ly} $y_2(x)$ is also a soln.

$\therefore$ $y_1(x)$ & $y_2(x)$ are general solns of

$y'' - 4y' + 4y = 0$.

14/9/10

Given a known solution to find another

Bookwork: Given a soln find another of the 2nd order homogeneous eqn

Let $y'' + P(x)y' + Q(x)y = 0$ ——①

Given $\boxed{y_1(x)}$ is one soln of eqn ①

Let $\boxed{y_2(x) = Vy_1(x)}$ be another soln of eqn ①.

where $V = V(x)$

$$y_2' = vy_1' + v'y_1$$

$$y_2'' = vy_1'' + v'y_1' + v'y_1' + v''y_1$$

$$y_2'' = vy_1'' + 2v'y_1' + v''y_1$$

$y_2'' + P(x)y_2' + Q(x)y_2 = 0$ [$\because$ y_2 is the soln eqn ①]

$$vy_1'' + 2v'y_1' + v''y_1 + P(vy_1' + v'y_1) + Q(vy_1) = 0$$

$vy_1'' + 2v'y_1' + v''y_1 + pvy_1' + pv'y_1 + Qvy_1 = 0$

$v(y_1'' + py_1' + Qy_1) + v'(2y_1' + py_1) + v''y_1 = 0$

$v'(2y_1' + py_1) + v''y_1 = 0$ [$\because y_1$ is the soln of eqn ①]

$v''y_1 = -v'(2y_1' + py_1)$

$y'' + py' + Qy = 0$

$\frac{v''}{v'} = -\frac{2y_1'}{y_1} - p$

$\int \frac{f'(x)}{f(x)} dx = \log f(x)$

$\int$ing on both sides

$\log v' = -2\log y_1 - \int pdx = -2\log y + \log e^{-\int pdx}$

$= \log y^{-2}$

$\log(y^{-2} e^{-\int pdx})$

$\log v' = (\log y_1)^{-2} - \int pdx$.

$v' = (y_1)^{-2} e^{-\int pdx}$

Again $\int$ing on both sides,

$$\boxed{v = \int \frac{1}{y_1^2} e^{-\int pdx} dx}$$

To show that y_1 & y_2 are linearly independen

$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

$= y_1(v'y_1 + vy_1') - vy_1 y_1'$

$= v'y_1^2 = e^{-\int pdx} \neq 0$.

$\therefore y_1$ & y_2 are L.I.

Problems

2 mark

① Prove $y_1 = x^2$ is a soln of $x^2y'' + xy' - 4y = 0$ and find the 2^{nd} soln.

proof: 2^{nd} soln $y_2 = vy_1$

Given $y_1 = x^2$.

$x^2y''+xy'-4y=0$

$$y''+\frac{1}{x}y'-\frac{4}{x^2}y=0$$

W.K.T $$V=\int\frac{1}{y^2}e^{-\int p\,dx}.dx$$

$$V=\int\frac{1}{x^4}e^{-\int\frac{1}{x}dx}dx$$

$$=\int\frac{1}{x^4}e^{-\log x}dx$$

$$=\int\frac{1}{x^4}e^{\log x^{-1}}dx$$

$$V=\int\frac{1}{x^5}dx=\frac{x^{-4}}{-4}$$

$$y_2=Vy_1$$

$$y_2=-\frac{1}{4x^4}x^2$$

$$\boxed{y_2=-\frac{1}{4x^2}}$$

2marks

(X) 2) Find the general solution for $(1-x^2)y''-2xy'+2y=0$ given $y_1=x$ as one solution.

Soln: Let $y_2(x)=Vy_1(x)$ be the 2^{nd} solution, where

$$V=\int\frac{1}{y_1^2}e^{-\int p\,dx}dx$$

Given $(1-x^2)y''-2xy'+2y=0$

$$\Rightarrow y''-\frac{2x}{1-x^2}y'+\frac{2}{1-x^2}y=0.$$

$$\therefore \quad V = \int \frac{1}{x^2} e^{-\int \frac{-2x}{1-x^2}} dx$$

$$= \int \frac{1}{x^2} e^{-\log(1-x^2)} dx$$

$$= \int \frac{1}{x^2} e^{\log(1-x^2)^{-1}} dx$$

$$= \int \left(\frac{1}{x^2} \cdot \frac{1}{1-x^2}\right) dx$$

$$\frac{1}{x^2} + \frac{1}{1-x^2}$$

$$= \int \frac{dx}{x^2(1-x^2)} = \int \left(\frac{1}{x^2} + \frac{1}{1-x^2}\right) dx$$

$$= \int \frac{1}{x^2} dx + \int \frac{1}{1-x^2} dx$$

$$= \frac{x^{-2+1}}{-2+1} + \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) + c \Bigg/ \int \frac{dx}{a^2-x^2} = \frac{1}{2}\log\left(\frac{a+x}{a-x}\right)$$

$$V = -\frac{1}{x} + \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) + c$$

$$\therefore y_2(x) = V y_1(x)$$

$$= \left[-\frac{1}{x} + \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\right] x$$

$$y_2(x) = \frac{x}{2}\log\left(\frac{1+x}{1-x}\right) - 1$$

W.K.T General soln is $y = c_1 y_1(x) + c_2 y_2(x)$

$$\Rightarrow \quad y = c_1 x + c_2 \left[\frac{x}{2}\log\left(\frac{1+x}{1-x}\right) - 1\right]$$

operator.
more.
5mark

Q. Find the general soln of $x^2y'' + xy' - y = 0$ if $y_1 = x$ is known.

3) Find the general solution of $x^2y''+xy'+(x^2-\frac{1}{4})y=0$ given that $y_1=(x^{-1/2}\sin x)$ as one solution.

↳ $\frac{\sin x}{\sqrt{x}}$

Soln: Given $x^2y''+xy'+(x^2-\frac{1}{4})y=0$

$$y''+\frac{1}{x}y'+\frac{(x^2-\frac{1}{4})}{x^2}y=0$$

Let $y_2(x)=vy_1(x)$ be the 2nd soln where

$$v=\int\frac{1}{y_1^2}e^{-\int p\,dx}dx$$

$$v=\int\frac{1}{(x^{-\frac{1}{2}}\sin x)^2}e^{-\int\frac{1}{x}dx}dx=\int\left(\frac{x^{1/2}}{\sin x}\right)^2e^{\log x^{-1}}dx$$

$$=\int\frac{x}{\sin^2x}\frac{1}{x}dx=\int\operatorname{cosec}^2x\,dx$$

$$v=-\cot x+c$$

Now, $y_2(x)=vy_1(x)\Rightarrow-\cot x\cdot\frac{\sin x}{\sqrt{x}}$

$$=-\frac{\cos x}{\sin x}\cdot\frac{\sin x}{\sqrt{x}}$$

$$\boxed{y_2(x)=-\frac{\cos x}{\sqrt{x}}}$$

∴ General soln is $y=c_1\frac{\sin x}{\sqrt{x}}-c_2\frac{\cos x}{\sqrt{x}}$ $[\because y=c_1y_1(x)+c_2y_2(x)]$

4) Find the general solution of $y'' - \frac{x}{x-1} y' + \frac{1}{x-1} y = 0$

Soln :

Given $y'' - \frac{x}{x-1} y' + \frac{1}{x-1} y = 0$ — ①

Let $y_1 = x$ be one soln of ①

Let $y_2 = Vy_1$ be another soln, where $V = \int \frac{1}{y_1^2} e^{-\int P dx} dx$.

$$V = \int \frac{1}{x^2} e^{\int \frac{x}{x-1} dx} dx$$

$$= \int \frac{1}{x^2} e^{\int \frac{x-1+1}{x-1} dx} dx$$

$$= \int \frac{1}{x^2} e^{\left(\int \frac{x-1}{x-1} dx + \int \frac{1}{x-1} dx\right)} dx$$

$$= \int \frac{1}{x^2} e^{\int dx + \int \frac{dx}{x-1}} dx$$

$$= \int \frac{1}{x^2} e^{x} e^{\log(x-1)} dx$$

$$= \int \frac{1}{x^2} e^{x} (x-1) dx$$

$$= \int \frac{e^x}{x} dx - \int \frac{e^x}{x^2} dx$$

$$= \int x^{-1} e^x dx - \int x^{-2} e^x dx$$

$\int u dv = uv - \int v du$

$u = x^{-1}$

$du = -\frac{1}{x^2}$

$$= \frac{e^x}{x} + \cancel{\int \frac{e^x}{x^2} dx} - \cancel{\int \frac{e^x}{x^2} dx} = \frac{e^x}{x}$$

$\therefore y_2 = V y_1(x) \Rightarrow \frac{e^x}{x} \cdot x = e^x \Rightarrow \boxed{y_2 = e^x}$

$\therefore$ The general soln is $\boxed{y = c_1 x + c_2 e^x}$

5) Find the general soln of .

a) $y'' + y = 0$, $y_1 = \sin x \Rightarrow y = c_1 \sin x + c_2 \cos x$

b) $y'' - y = 0$, $y_1 = e^x \Rightarrow y = c_1 e^x + c_2 e^{-x}$

c) $xy'' + 3y' = 0$, $y_1 = 1 \Rightarrow y = c_1 + c_2 x^{-2}$.

d) $xy'' - (2x+1)y' + (x+1)y = 0$, $y_1 = e^x \Rightarrow y = e^x(c_1 + x^2 c_2)$

☆ 10 marks

THE METHOD OF VARIATION OF PARAMETERS.

We are finding a particular soln of a non-homogenous eqn $y'' + p(x)y' + Q(x)y = R(x)$ where the co-efficients of $p(x)$ & $Q(x)$ are constants & $R(x)$ has a particular simple form.

Nov.01 10 marks

Bookwork : The particular soln by the method of variation of parameters for the non-homogenous differential eqn $y'' + p(x)y' + Q(x)y = R(x)$.

Soln :

$$y'' + p(x)y' + Q(x)y = R(x) \quad \text{—} ①$$

Let $y_1(x)$ and $y_2(x)$ be the soln of the homogenous eqn

$$y'' + p(x)y' + Q(x)y = 0 \quad \text{—} ②$$

$\underline{y = v_1 y_1 + v_2 y_2}$ be a particular soln of ①

[General soln $y = c_1 y_1 + c_2 y_2$]

$$y' = v_1 y_1' + v_1' y_1 + v_2 y_2' + v_2' y_2$$

$$= v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2$$

Let $v_1' y_1 + v_2' y_2 = 0$ — ③

$\Rightarrow \quad y' = v_1 y_1' + v_2 y_2' \qquad$ [by ③]

Again, $\quad y'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'$

Substitute y' & y'' in ①

① $\Rightarrow \quad y'' + P(x) y' + Q(x) y = R(x)$

$$v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2' + P[v_1 y_1' + v_2 y_2']$$
$$+ Q(v_1 y_1 + v_2 y_2) = R(x)$$

$$v_1 (\underbrace{y_1'' + P y_1' + Q y_1}_{(0)}) + v_2 (\underbrace{y_2'' + P y_2' + Q y_2}_{(0)}) + v_1' y_1' + v_2' y_2' = R$$

$$v_1' y_1' + v_2' y_2' = R(\cancel{x}) \qquad \text{[by (2)]}$$

└ ④

from ③ $\quad \underset{a_1 x \; + \; b_1 y \; + \; c_1}{v_1' y_1 + v_2' y_2} = 0$

from ④ $\quad \underset{a_2 x \; + \; b_2 y \; + \; c_2}{v_1' y_1' + v_2' y_2' - R(x)} = 0$

$$\frac{v_1'}{y_2(-R(x)) - y_2'(0)} = \frac{-v_2'}{y_1(-R(x)) - y_1'(0)} = \frac{1}{y_1 y_2' - y_1' y_2}$$

$a_1 x + b_1 y + c_1 = 0$

$a_2 x + b_2 y + c_2 = 0$

$$\Rightarrow \frac{x}{b_1 c_2 - b_2 c_1} = \frac{-y}{a_1 c_2 - a_2 c_1} = \frac{1}{a_1 b_2 - a_2 b_1}$$

$v_1' \quad v_2' \quad 1$

$y_2 \quad 0 \quad y_1 \quad y_2$

$y_2' \quad -R(x) \quad y_1' \quad y_2'$

$$\frac{v_1'}{-y_2 R(x)} = \frac{-v_2'}{-y_1 R(x)} = \frac{1}{y_1 y_2' - y_1' y_2}$$

$$\therefore v_1' = \frac{-y_2 R(x)}{W(y_1, y_2)} \quad \text{and} \quad v_2' = \frac{y_1 R(x)}{W(y_1, y_2)}$$

$$\boxed{\therefore v_1 = \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx} \quad \text{and} \quad \boxed{v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx}$$

where $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = (y_1 y_2' - y_1' y_2)$

$\therefore$ The particular soln is $\underline{\underline{y = y_1 \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 R(x)}{W(y_1, y_2)} dx}}$.

Problems.

5 marks

(5) marks) Find the particular soln for $y''+y = \operatorname{cosec} x$

Soln:

Auxillary eqn is $(D^2+1)y=0$

$D=\pm i$

Given eqn can be written as $(D^2+1)y = \operatorname{cosec} x$

Auxillary eqn is $m^2+1=0$

$m = \pm i$

General soln is $y = A\cos x + B\sin x$

Complex roots.

$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x$

i.e., $y_1(x) = \cos x$ & $y_2(x) = \sin x$ — ①

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$\therefore$ The particular soln is $y = v_1 y_1 + v_2 y_2$ ~~where~~

where $v_1 = \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx$ & $v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)}$

$$\therefore v_1 = \int \frac{-\sin x \operatorname{cosec} x}{1} dx \quad \& \quad v_2 = \int \frac{\cos x \operatorname{cosec} x}{1} dx.$$

$$v_1 = -\int dx \quad \& \quad v_2 = \int \cot x \, dx$$

$$V_1 = -x \quad \& \quad v_2 = \log \sin x \quad \text{—②}$$

$\int \cot x \, dx = \log \sin x$ or $\log \operatorname{cosec} x$

① & ② gives

∴ The particular soln is $\underline{y = -x \cos x + \sin x \log \sin x}$.

2) Find the particular soln of $y'' - y' - 6y = e^{-x}$ by the method of variation of parameters.

5 marks

Soln: Given eqn $\Rightarrow (D^2 - D - 6) y = 0$

-3
$-6 < 2$
-1

A.E $\quad m^2 - m - 6 = 0$

$m_1 = +3 \ \& \ m_2 = -2$

Roots are distinct

$y = Ae^{m_1 x} + Be^{m_2 x}$

General soln is $y = Ae^{3x} + Be^{-2x}$

$y_1(x) = e^{3x} \ \& \ y_2(x) = e^{-2x}$ —①

$$W(y_1, y_2) = \begin{vmatrix} e^{3x} & e^{-2x} \\ 3e^{3x} & -2e^{-2x} \end{vmatrix} = -2e^{x} - 3e^{x} = -5e^{x}$$

Now, the particular soln is $y = v_1 y_1 + v_2 y_2$

$$\text{where } v_1 = \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx \quad \& \quad v_2 = \int \frac{+y_1 R(x)}{W(y_1, y_2)} dx.$$

$v_1 = \int \frac{-e^{-2x}e^{-x}}{-5e^{x}}\,dx$; $v_2 = \int \frac{e^{3x}e^{-x}}{-5e^{x}}\,dx$

$v_1 = \frac{1}{5}\int e^{-4x}\,dx$; $v_2 = \frac{-1}{5}\int e^{x}\,dx$

$v_1 = \frac{e^{-4x}}{-20}$; $v_2 = \frac{-e^{x}}{5}$

└ ②

from ① & ②

The particular soln is $y = \frac{e^{3x}e^{-4x}}{-20} + e^{-2x}\left(\frac{-e^{x}}{5}\right)$

$= \frac{e^{-x}}{-20} - \frac{e^{-x}}{5} = \frac{e^{-x}+4e^{-x}}{+20}$

$= \frac{-5e^{-x}}{20}$

∴ Particular soln is $\underline{y = -\frac{1}{4}e^{-x}}$

⑧ mark

Find the particular soln of $y'' + 4y = \tan 2x$.

Soln: Given eqn => $(D^2+4)y = \tan 2x$

A.E $m^2+4=0$

$m = \pm 2i$

∴ The general soln is $y = e^{0x}[A\cos 2x + B\sin 2x]$

=> $y_1(x) = \cos 2x$ & $y_2(x) = \sin 2x$

└ ①

$$W(y_1, y_2) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x \quad = 2(1)$$

Now,

$$= 2$$

The particular soln is $y = v_1 y_1 + v_2 y_2$

where $v_1 = \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx$ & $v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$.

$$v_1 = \int -\frac{\sin 2x \tan 2x}{2} dx \;;\; v_2 = \int \frac{\cos 2x \cdot \tan 2x}{2} dx.$$

$$v_1 = -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx \;;\; v_2 = \frac{1}{2}\int \sin 2x \, dx$$

$$v_1 = -\frac{1}{2}\left[\int \frac{1-\cos^2 2x}{\cos 2x} dx\right] \;;\; v_2 = \frac{1}{2}\int \sin 2x \, dx.$$

$$v_1 = -\frac{1}{2}\left[\int \frac{dx}{\cos 2x} - \int \cos 2x \, dx\right] \;;\; v_2 = \frac{1}{2}\left[\frac{-\cos 2x}{2}\right]$$

$$v_1 = -\frac{1}{2}\left[\int \sec 2x \, dx - \left(\frac{\sin 2x}{2}\right)\right. \;;\; v_2 = -\frac{\cos 2x}{4}$$

$$v_1 = -\frac{1}{2}\left[\frac{1}{2}\log(\sec 2x + \tan 2x) - \frac{\sin 2x}{2}\right] \;;\; v_2 = -\frac{\cos 2x}{4}$$

$$\Rightarrow v_1 = \frac{1}{4}\sin 2x - \frac{1}{4}\log(\sec 2x + \tan 2x) \;;\; v_2 = -\frac{\cos 2x}{4}$$

① & ② gives

∴ The particular soln is $y = \cos 2x\left[\frac{1}{4}\sin 2x - \frac{1}{4}\log(\sec 2x + \tan 2x)\right]$

$$+ \sin 2x\left(-\frac{\cos 2x}{4}\right)$$

$$y = \frac{\sin 2x\,\cancel{\cos 2x}}{4} - \frac{\cos 2x}{4}\log(\sec 2x + \tan 2x) - \frac{\sin 2x\,\cancel{\cos 2x}}{4}$$

$$\therefore y = -\frac{\cos 2x}{4}\log(\sec 2x + \tan 2x)$$

5 marks

④ Find the particular soln of $y'' + 2y' + y = e^{-x}\log x$.

Soln: Given eqn ⇒ $(D^2 + 2D + 1)y = e^{-x}\log x$

A.E $m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m = -1, -1$

Roots are equal

$y = (Ax + B)e^{mx}$

∴ The general soln is $y = (Ax + B)e^{-x}$

⇒ $y_1(x) = xe^{-x}$ & $y_2(x) = e^{-x}$ —— ①

$$W(y_1, y_2) = \begin{vmatrix} xe^{-x} & e^{-x} \\ e^{-x}(1-x) & -e^{-x} \end{vmatrix}$$

$d(xe^{-x}) = e^{-x} - xe^{-x}$

$= e^{-x}(1-x)$

$$= -xe^{-2x} - e^{-x}(e^{-x} - xe^{-x})$$

$$= -xe^{-2x} - e^{-2x} + xe^{-2x}$$

$$W(y_1, y_2) = -e^{-2x}.$$

∴ The particular soln is $y = v_1 y_1 + v_2 y_2$

where $v_1 = \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx$, $v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$

$$\therefore v_1 = \int \frac{-e^{-x} e^{-x} \log x}{-e^{-2x}} dx \; ; \; v_2 = \int \frac{x e^{-x} e^{-x} \log x}{-e^{-2x}} dx$$

$$v_1 = \int \log x \, dx \; ; \; v_2 = -\int x \log x \, dx$$

$\int u\,dv = uv - \int v\,du$

$u = \log x$, $dv = dx$

$du = \frac{1}{x} dx$, $v = x$

$\int u\,dv = uv - \int v\,du$

$u = \log x$, $dv = x\,dx$

$du = \frac{1}{x} dx$, $v = x^2/2$

$$v_1 = x \log x - \int x \frac{1}{x} dx \; ; \; v_2 = -\left[\frac{x^2}{2} \log x - \int \frac{x^2}{2} \frac{1}{x} dx \right]$$

$$v_1 = x \log x - x \; ; \; v_2 = -\frac{x^2}{2} \log x + \frac{x^2}{4}$$

V,17

②

∴ The particular soln is

$$y = x e^{-x} [x \log x - x] + e^{-x} \left[-\frac{x^2}{2} \log x + \frac{x^2}{4} \right]$$

$$= x^2 e^{-x} \log x - x^2 e^{-x} - \frac{x^2}{2} \log x \, e^{-x} + \frac{x^2}{4} e^{-x}$$

$$y = \frac{x^2}{2} e^{-x} \log x - \frac{3}{4} e^{-x}$$

5) Find the particular soln of $y'' - 2y' - 3y = 64xe^{-x}$

Soln. Given eqn $\Rightarrow (D^2 - 2D - 3)y = 0$

A.E $\quad m^2 - 2m - 3 = 0$

$m_1 = 3, \quad m_2 = -1$

$\therefore$ The general soln is $Ae^{3x} + Be^{-x}$

$\Rightarrow \quad y_1(x) = e^{3x} \quad \& \quad y_2(x) = e^{-x}$ ①

$$W(y_1, y_2) = \begin{vmatrix} e^{3x} & e^{-x} \\ 3e^{3x} & -e^{-x} \end{vmatrix} = -e^{2x} - 3e^{2x} = -4e^{2x}$$

The particular soln is $y = v_1 y_1 + v_2 y_2$

where $v_1 = \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx \quad \& \quad v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$

$v_1 = \int \frac{-e^{-x} 64xe^{-x}}{-4e^{2x}} dx \quad \& \quad v_2 = \int \frac{e^{3x} 64xe^{-x}}{-4e^{2x}} dx$

$v_1 = \int 16xe^{-4x} dx \quad \& \quad v_2 = -\int 16x \, dx$

$v_1 = 16 \int xe^{-4x} dx \quad \& \quad v_2 = -16 \int x \, dx$

$v_1 = 16 \left[\frac{-xe^{-4x}}{4} + \int \frac{e^{-4x}}{4} dx \right] \quad \& \quad v_2 = -16 \left[x^2/2 \right]$

$v_1 = -4xe^{-4x} - e^{-4x} \quad \& \quad v_2 = -8x^2$

$v_1 = -e^{-4x}(4x+1) \quad \& \quad v_2 = -8x^2$

∴ The particular soln is $y = -e^{-4x}(4x+1)e^{3x} + (-8x^2e^{-x})$

$$= -e^{-x}(4x+1) - 8e^{-x}x^2$$

$$\underline{\underline{y = -e^{-x}\left[8x^2+4x+1\right]}}$$

6) Find the particular soln of $y''+2y'+5y = e^{-x}\sec 2x$.

Soln:

Given eqn $\Rightarrow (D^2+2D+5)y = 0$

A.E $\quad m^2+2m+5=0$

$$m = \frac{-2 \pm \sqrt{4-4(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$m = -1 \pm 2i$$

∴ The general soln is $y = e^{-x}\left[A\cos 2x + B\sin 2x\right]$

$\Rightarrow y_1(x) = e^{-x}\cos 2x \ \& \ y_2(x) = e^{-x}\sin 2x$ ①

$$W(y_1,y_2) = \begin{vmatrix} e^{-x}\cos 2x & e^{-x}\sin 2x \\ -e^{-x}[2\sin 2x + \cos 2x] & e^{-x}[2\cos 2x - \sin 2x] \end{vmatrix}$$

$$= 2e^{-2x}\cos^2 2x - e^{-2x}\cancel{\cos 2x\sin 2x} + 2e^{-2x}\sin^2 2x + e^{-2x}\cancel{\sin 2x\cos 2x}$$

$W(y_1,y_2) = 2e^{-2x}$

∴ The particular soln is $y = v_1y_1 + v_2y_2$

where $v_1 = \int \frac{-y_2R(x)}{W(y_1,y_2)}dx \ \& \ v_2 = \int \frac{y_1R(x)}{W(y_1,y_2)}dx$

Scanned with CamScanner

$$v_1 = \int \frac{-e^{-x}\sin 2x \; e^{-x}\sec 2x}{2e^{-2x}}\,dx \;;\quad v_2 = \int \frac{e^{-x}\cos 2x \; e^{-x}\sec 2x}{2e^{-2x}}\,dx.$$

$$v_1 = -\frac{1}{2}\int \tan 2x\,dx \;;\quad v_2 = \frac{1}{2}\int dx$$

~~$v_1 = -\frac{1}{2}\left[\frac{\sec^2 2x}{2}\right.$~~

$$v_1 = -\frac{1}{2}\left[-\frac{1}{2}\log(\cos 2x)\right] \;;\quad v_2 = x/2$$

$$v_1 = \frac{1}{4}\log(\cos 2x) \;;\quad v_2 = x/2$$

└ ②

from ① & ②

The particular soln is $y = \frac{1}{4}\log(\cos 2x)\,e^{-x}\cos 2x + \frac{x}{2}\,e^{-x}\sin 2x$

$$y = \frac{1}{2}x e^{-x}\sin 2x + \frac{1}{4}e^{-x}\cos 2x \log(\cos 2x)$$

a/c.

⑦ solve $y'' - y' - 2y = 4x^2$ ~~using the~~ method of variation of parameters.

POWER SERIES SOLUTIONS:

$y = -2(x^2 - x + 3/2)$ ↙

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots.$$

is called a power series in x. This series is said to converge at a point x if the limit

$\lim_{m\to\infty} \sum_{n=0}^{m} a_n x^n$ exists and in this case the sum of the series is the value of this limit

If $\mathop{Lt}_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ exists, then we call this as the radius of convergence of power series and it is denoted by R. ie, $R=\mathop{Lt}_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

Series solutions of first order eqns.

5 marks 1) Find the power series soln for $y'=y$.

Soln: $y'=y \quad -① \Rightarrow \underline{y'-y=0}$

Let $y=\sum_{n=0}^{\infty} a_n x^n$ be the power series soln of ①

$$y'=\sum_{n=1}^{\infty} n\,a_n x^{n-1}$$

$$=\sum_{n+1=1}^{\infty}(n+1)\,a_{n+1}x^{n}$$

$$y'=\sum_{n=0}^{\infty}(n+1)\,a_{n+1}x^{n}$$

① $\Rightarrow$ $y'-y=0 \Rightarrow \sum_{n=0}^{\infty}(n+1)\,a_{n+1}x^n-\sum_{n=0}^{\infty}a_n x^n$

$$\sum_{n=0}^{\infty}\left[(n+1)\,a_{n+1}-a_n\right]x^n=0$$

(or) $\sum_{n=0}^{\infty}(n+1)\,a_{n+1}x^n=\sum_{n=0}^{\infty}a_n x^n.$

Equating co-eff of x^n on both sides,

$$(n+1)a_{n+1} = a_n$$

$$a_{n+1} = \frac{a_n}{n+1}$$

Put

$n=0$, $a_1 = \frac{a_0}{1}$

$n=1$ $a_2 = \frac{a_1}{2} = \frac{a_0}{2}$

$n=2$ $a_3 = \frac{a_2}{3} = \frac{a_0}{6}$

∴ The power ~~series~~ soln is $y = a_0 + a_1 x + a_2 x^2 + \ldots$

$$y = a_0 + a_0 x + \frac{a_0}{2}x^2 + \frac{a_0}{6}x^3 + \ldots$$

$$= a_0\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right]$$

$$\underline{\underline{y = a_0 e^x}}$$

Find the diff eqn satisfied by the fn $y = (1+x)^p$ & then solve this eqn by power series.

2) Express $y = (1+x)^p$ in the power series form (or)

Find the power series soln for $y'(1+x) = py$, $y(0) = 1$

Soln : Given $y = (1+x)^p$ — ①

$$y' = p(1+x)^{p-1} = p(1+x)^p \cdot (1+x)^{-1}$$

$$y'(1+x) = p(1+x)^p$$

$$y'(1+x) = py \quad — ②$$

(or)

$y' + xy' - py = 0$

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be the soln of ①

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n+1=1}^{\infty} (n+1) a_{n+1} x^{n+1-1}$$

$$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1)a_{n+1} x^n + \cdots$$

$$xy' = a_1 x + 2a_2 x^2 + \cdots + n a_n x^n + \cdots$$

$$\Rightarrow xy' = \sum_{n=0}^{\infty} n a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$xy' = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n$$

$$py = p a_0 + p a_1 x + p a_2 x^2 + \cdots + p a_n x^n + \cdots$$

$$py \sum_{n=0}^{\infty} p a_n x^n$$

② $\Rightarrow$ $y' + xy' - py = 0$

$$\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} x^n + n a_n x^n - p a_n x^n \right] = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} + n a_n - p a_n \right] x^n = 0.$$

Equating the coefficient of x^n we get,

$$(n+1) a_{n+1} = p a_n - n a_n$$

$$(n+1) a_{n+1} = (p-n) a_n$$

$$a_{n+1} = \frac{p-n}{n+1} a_n$$

Put $n=0$, $a_1 = Pa_0$

$n=1$, $a_2 = \frac{p-1}{2} a_1 = \frac{p(p-1)}{2} a_0$

$n=2$, $a_3 = \frac{p-2}{3} a_2 = \frac{p(p-1)(p-2)}{6} a_0$

. . . . . .

$\therefore$ The power series soln is $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$y = a_0 + x\,pa_0 + x^2 \frac{p(p-1)}{2} a_0 + x^3 \frac{p(p-1)(p-2)}{6} a_0 + \cdots$$

$$y = a_0 \left[1 + px + \frac{p(p-1)x^2}{2!} + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots \right]$$

Given $y(0) = 1 \Rightarrow \boxed{a_0 = 1}$ $y = \overrightarrow{1 + px + \frac{p(p-1)}{2!} x^2} + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots$ $\boxed{y = (1+x)^p}$

② Find the power series soln for $y' = 2xy$ ~~and~~ verify your answer by solving the eqn directly.

$\downarrow$ validancy $|x|<1$

Soln: Let $y' = 2xy$ —①

Let $y = \sum_0^\infty a_n x^n$ be the soln of eqn ①

W.K.T $y' = \sum_{n=0}^\infty (n+1) a_{n+1} x^n$

$2\sum_0^\infty a_n x^{n+1}$

$2\sum_{n-1=0}^\infty a_n x^n = 2\sum_{n=1}^\infty a_{n-1} x^n$

$n \to n-1$

① $\Rightarrow$ $y' = 2xy$

$$\sum_{n=0}^\infty [(n+1) a_{n+1} x^n = 2x \sum_{n=0}^\infty a_n x^n = 2\sum_{n=0}^\infty a_n x$$

a_{n-1}

$$\sum_{n=0}^\infty \left[(n+1) a_{n+1} - 2a_{n-1}\right] x^n = 0.$$

Equating the coeff of x^n,

$$(n+1)a_{n+1} = 2a_{n-1}$$

$$a_{n+1} = \frac{2a_{n-1}}{n+1}$$

n=0, $a_1 = \frac{2a_{-1}}{1} = 0$

n=1, $a_2 = \frac{2a_0}{2} = a_0$

n=2, $a_3 = \frac{2a_1}{3} = 2\cdot 0 = 0$

. . . .

∴ The power series soln is $y = a_0 + a_1 x + a_2 x^2 + \cdots$

$$= a_0 + 0 + a_0 x^2 + 0 + \frac{a_0}{2} x^4 + \cdots$$

$$= a_0 \left[1 + x^2 + \frac{(x^2)^2}{2!} + \cdots \right]$$

$$y = a_0 e^{x^2}$$

Verification:

Given $\frac{dy}{dx} = 2xy$

$$\frac{dy}{2y} = x\,dx$$

$$\frac{1}{2}\log y = \frac{x^2}{2} + c$$

$$y = ce^{x^2}$$

2 mark

① Find the power series soln for $y'+y=\overset{0}{\text{[illegible]}}$

Soln: $y'+y=\overset{0}{\text{[illegible]}}$ — ①

Let $y=\sum_{n=0}^{\infty} a_n x^n$ be the soln of ①

W.K.T $y'=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$

$① \Rightarrow \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = -\sum_{n=0}^{\infty} a_n x^n \quad [\because y'=-y]$ $\overset{0}{}$

Put n=0 & equate the constant terms $\Rightarrow (0+1)a_1+a_0=1$

$\Rightarrow a_1+a_0=1$

$\boxed{a_1=1-a_0}$

Equating the coeff of x^n.

$\boxed{a_1=-a_0}$

$(n+1)a_{n+1}=-a_n$

$a_{n+1}=\frac{-a_n}{n+1}$ $\quad n=0 \quad a_1=\frac{-a_0}{1}$

n=0, $a_1 \neq 1-1a_0$

$a_1=-a_0$

$a_2=\frac{-a_1}{n+1}=\frac{a_0}{2!}$

$a_3=\frac{-a_2}{3}=\frac{-a_0}{6}$

n=1, $a_2=-\frac{a_1}{2}=\frac{a_0-1}{2}$

n=2, $a_3=-\frac{a_2}{3}=\frac{1-a_0}{6}$

n=3, $a_4=-\frac{a_3}{4}=\frac{a_0-1}{24}$

. . . .

The power series soln is $y=a_0+a_1x+a_2x^2+\cdots$

$y=a_0+(1-a_0)x+\frac{(1-a_0)}{2!}x^2+\frac{(1-a_0)}{3!}x^3+\frac{(1-a_0)}{4!}x^4+\cdots$

$=1+a_0-1+(1-a_0)x-\frac{(1-a_0)}{2!}x^2+\cdots$ [Adding & subtracting '1']

$$y = 1 + (a_0 - 1)\left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right]$$

$$y = 1 + (a_0 - 1)e^{-x}$$

$\frac{dy}{dx} = 1 - y$

Power series solns for 2nd order linear eqn.

Define ordinary ~~pt of a 2nd order linear~~ diff eqn.

2 marks. **Ordinary point & Singular point**

Consider the general homogenous 2nd order linear eqn as

$$y'' + p(x)y' + Q(x)y = 0.$$

We say that x_0 is an ordinary point of the above eqn, if $p(x)$ & $Q(x)$ can be expanded by means of the power series in the neighbourhood of the point x_0.

Otherwise the point x_0 is called a singular point.

Ex: $y'' + p(x)y' + Q(x)y = 0.$

$(1-x^2)y'' + 2xy' - 2y = 0$

i.e. $y'' + \frac{2x}{1-x^2}y' - \frac{2}{1-x^2}y = 0$

$x = 0$ is an ordinary point. since,

when $x = 0$, $p(x) = \frac{2x}{1-x^2} = 0$ & $Q(x) = -\frac{2}{1-x^2} = -2$

$\therefore x = 0$ is an ordinary point

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when $x=1$, $P(x)=\infty$ & $Q(x)=\infty$

$x=-1$, $P(x)=\infty$ & $Q(x)=\infty$.

$\therefore$ $x=\pm 1$ are singular points.

Problems:

1) Find the general soln of $y''+xy'+y=0$ in the form $y=a_0y_1(x)+a_1y_2(x)$ where $y_1(x)$ & $y_2(x)$ are power series.

Soln: $y''+xy'+y=0$ — ①

Let $y=\sum_{n=0}^{\infty} a_n x^n$ be the soln of ①

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n+2=2}^{\infty}(n+2)(n+2-1)a_{n+2}x^{n+2-2}$$

$$y''=\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n$$

$$xy'=\sum_{n=0}^{\infty} n a_n x^n \qquad \left[xy'= a_1x+2a_2x^2+\cdots+na_nx^n+\cdots\right]$$

① $\Rightarrow$ $$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n+\sum_{n=0}^{\infty} n a_n x^n+\sum_{n=0}^{\infty} a_n x^n=0.$$

~~$\sum_{n\geq 0}$~~ $$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+na_n+a_n\right]x^n=0.$$

Equating the co-efficient of x^n, we got

$$(n+1)(n+2)\,a_{n+2} = -a_n n - a_n$$

$$(n+1)(n+2)\,a_{n+2} = -a_n(n+1)$$

$$a_{n+2} = \frac{-a_n\cancel{(n+1)}}{\cancel{(n+1)}(n+2)}$$

$$a_{n+2} = \frac{-a_n}{n+2}$$

Put $n=0$, $a_2 = \frac{-a_0}{2}$

$n=1$, $a_3 = \frac{-a_1}{3}$

$n=2$, $a_4 = \frac{-a_2}{4} = \frac{a_0}{2\cdot4}$

$n=3$, $a_5 = \frac{-a_3}{5} = \frac{a_1}{3\cdot5}$

$n=4$, $a_6 = \frac{-a_4}{6} = \frac{-a_0}{2\cdot4\cdot6}$

$n=5$, $a_7 = \frac{-a_5}{7} = \frac{-a_1}{3\cdot5\cdot7}$

Now, the power series soln is $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$

$$y = a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{3}x^3 + \frac{a_0}{2\cdot4}x^4 + \frac{a_1}{3\cdot5}x^5 - \frac{a_0}{2\cdot4\cdot6}x^6 - \frac{a_1}{3\cdot5\cdot7}x^7 + \cdots$$

$$y = a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{2\cdot4} - \frac{x^6}{2\cdot4\cdot6} + \cdots\right] + a_1\left[x - \frac{x^3}{3} + \frac{x^5}{3\cdot5} - \frac{x^7}{3\cdot5\cdot7} + \cdots\right]$$

$$y = \underline{a_0y_1(x) + a_1y_2(x)}$$

where $y_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2\cdot4} - \frac{x^6}{2\cdot4\cdot6} + \cdots$

$y_2(x) = x - \frac{x^3}{3} + \frac{x^5}{3\cdot5} - \frac{x^7}{3\cdot5\cdot7} + \cdots$

2) Find the general soln of $(1+x^2)y''+2xy'-2y=0$ in terms of power series of x.

Soln: $(1+x^2)y''+2xy'-2y=0$ —①

Let $y=\sum_{n=0}^{\infty} a_n x^n$ be the soln of ①.

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$xy'=\sum_{n=0}^{\infty} n a_n x^n$$

$$y''=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

W.K.T $y''=\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n$ [by previous pbm]

$$x^2y''=\sum_{n=2}^{\infty} n(n-1)a_n x^n \quad \left(\sum_{n+2=2}^{\infty}\right)(n+2)(n+2-1)a_{n+}$$

$$=\sum_{n=0}^{\infty} n(n-1)a_n x^n$$

from ① $\quad y''+x^2y''\cancel{+2xy'-}2y=0$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+n(n-1)a_n+2na_n-2a_n\right]x^n=0$$

Equating the coeff of x^n

$$(n+1)(n+2)a_{n+2}=-\left[n(n-1)+2(n-1)\right]a_n$$

$$a_{n+2}=-\left[\frac{n(n-1)+2(n-1)}{(n+1)(n+2)}\right]a_n$$

$$=-\left[\frac{(n-1)(n+\cancel{2})}{(n+1)(n+\cancel{2})}\right]a_n$$

$$a_{n+2} = -\frac{(n-1)}{(n+1)} a_n$$

Put $n=0$, $a_2 = a_0$ | $n=3$, $a_5 = \frac{2a_3}{4} = 0$

$n=1$, $a_3 = 0$ | $n=4$, $a_6 = \frac{3a_4}{5} = \frac{a_0}{5}$

$n=2$, $a_4 = \frac{a_2}{3}$ | $n=5$, $a_7 = \frac{4a_5}{6} = 0$.

$\therefore$ The power series soln is $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$$y = a_0 + a_1 x + a_0 x^2 + 0 - \frac{a_0}{3} x^4 + 0 + \frac{a_0}{5} x^6 + 0 + \dots$$

$$y = a_0 \left[1 + x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \dots\right] + a_1 x$$

$$= a_0 \left[1 + x\left(x - \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)\right] + a_1 x$$

$\tan^{-1} x$ $\quad$ $-\tan^{-1} x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

$$\therefore y = a_0 \left(1 + x \sin x\right) + a_1 x.$$

3) Find the soln in the power series form for $y'' + y' - xy = 0$. Given that $y_1(0) = 1$, $y_1'(0) = 0$, $y_2(0) = 0$, $y_2'(0) = 1$.

Soln: $y'' + y' - xy = 0$ — ①

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be the soln of ①

W.K.T $y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

W.k.T $y'' = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, x^n$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n-1=0}^{\infty} a_{n-1} x^n$$

$$xy = \sum_{0}^{\infty} a_{n-1} x^n$$

∴ from ①

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} + (n+1) a_{n+1} - a_{n-1} \right] x^n = 0.$$

$$(n+1)(n+2)\, a_{n+2} = a_{n-1} - (n+1) a_{n+1}$$

$$a_{n+2} = \frac{a_{n-1} - (n+1) a_{n+1}}{(n+1)(n+2)}$$

$a_5 = \frac{-4a_1 + a_0}{5}$

put

$n=0,\ a_2 = \frac{a_{-1} - a_1}{1\cdot 2} = \frac{-a_1}{2!}$

$n=1,\ a_3 = \frac{a_0 - 2a_2}{2\cdot 3} = \frac{a_0 + a_1}{2\cdot 3}$

$n=2\ a_4 = \frac{a_1 - 3a_3}{3\cdot 4} = \frac{a_1 - a_0 - a_1}{2\cdot 3\cdot 4}$

$a_4 = \frac{-a_0}{4!}$ $\frac{a_1 - a_0}{4!}$

$-3(a_0+a_1)$ $\frac{-3a_0 - 3a_1}{2\cdot 3\cdot 3\cdot 4}$ $\frac{a_1 - 3a_0 - 3a_1}{2\cdot 3\cdot 3\cdot 4}$

$n=3,\ a_5 = \frac{a_2 - 4a_4}{4\cdot 5}$

$= \frac{-\frac{a_1}{2} \overset{-}{⊕} 4 \frac{a_0}{4!}}{4\cdot 5}$

$a_5 = \frac{-a_1 + a_0}{5!}$

The power series soln is $y = a_0 + a_1 x + a_2 x^2 + \cdots$

$$y = a_0 + a_1 x - \frac{a_1}{2!} x^2 + \left(\frac{a_0 + a_1}{3!}\right) x^3 + \frac{a_0}{4!} x^4 + \frac{a_0 - a_1}{5!}\Big) x^5 - \cdots$$

$$= a_0 \left[1 + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots \right] + a_1 \left[x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots \right]$$

where $y_1(x) = a_0\left[1 + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots\right]$ &

$$y_2(x) = a_1\left[x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots\right]$$

Given $y_1(0) = 1$

$y_1(0) = a_0$ i.e, $\boxed{a_0 = 1}$

Given $y_2(0) = 0$

$y_2(0) = a_1\left[0 - \frac{0}{2} - \cdots\right]$

$\boxed{y_2(0) = 0}$

Given $y_1'(0) = 0$

$y_1'(x) = a_0\left[\frac{3x^2}{3!} - \frac{4x^3}{4!} + \frac{5x^4}{5!} \cdots\right]$

$\boxed{y_1'(0) = 0}$

Given $y_2'(0) = 1$

$y_2'(x) = a_1\left[1 - \frac{2x}{2!} + \frac{3x^2}{3!} \cdots\right]$

$\boxed{y_2'(0) = a_1} \Rightarrow a_1 = 1$

to more Repeated

4) Find the power series soln for the Legendre's eqn $(1-x^2)y'' - 2xy' + p(p+1)y = 0$, where p is a constant. / Solve Legendre's eqn $(1-x^2)y'' - 2xy' + p(p+1)y = 0$ where p is a constant [Note: Did not mention power

(or) obtain the formal soln of Legendre's eqn

Soln: $(1-x^2)y'' - 2xy' + p(p+1)y = 0$ — ①

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be the soln of ①

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n+1=1}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

w.r.t $y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$

$$x^2 y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n = \sum_{n=0}^{\infty} n(n-1) a_n x^n$$

$$xy' = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n.$$

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① =) $y'' - x^2y'' - 2xy' + p(p+1)y$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2na_n + p(p+1)a_n\right]x^n = 0$$

Equating the coeff of x^n, we get

$$(n+1)(n+2)a_{n+2} = n(n-1)a_n + 2na_n + p(p+1)a_n$$

$$a_{n+2} = a_n\frac{[n(n-1)+2n + p(p+1)]}{(n+1)(n+2)}$$

n^2-n+2n

n^2+n

$n(n+1)-$

$$= a_n\frac{[n(n+1) - p(p+1)]}{(n+1)(n+2)}$$

$$= \frac{a_n[n^2+n-p^2-p]}{(n+1)(n+2)}$$ / $n^2-p^2+n-p = (n-p)[n+p+1]$

$$a_{n+2} = \frac{(n-p)(n+p+1)}{(n+1)(n+2)}a_n$$ => This is called recursion formula. This gives a_n in terms of a_0 or a_1

put $n=0$, $a_2 = \frac{a_0(-p)(p+1)}{1\cdot 2}$

$n=1$, $a_3 = \frac{a_1(1-p)(p+2)}{2\cdot 3}$

$n=2$, $a_4 = \frac{a_2(2-p)(3+p)}{3\cdot 4}$

$$= \frac{-p(p+1)(2-p)(3+p)}{1\cdot 2\cdot 3\cdot 4}a_0$$

$n=3$, $a_5 = \frac{a_3(3-p)(4+p)}{4\cdot 5}$

$$= \frac{a_1(1-p)(p+2)(3-p)(4+p)}{2\cdot 3\cdot 4\cdot 5}$$

$$= \frac{(p-1)(p-3)(p+2)(p+4)}{5!}a_1$$

∴ The power series soln $y = a_0 + a_1 x + a_2 x^2 + \cdots$

$$y = a_0 + a_1 x + \frac{(-p)(p+1)}{2!} a_0 x^2 + \frac{(1-p)(p+2)}{3!} a_1 x^3 +$$

$$\frac{p(p-2)(p+1)(p+3)}{4!} a_0 x^4 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!} a_1 x^5 + \cdots$$

$$y = a_0 \left[1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p-2)(p+1)(p+3)}{4!} x^4 - \frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!} x^6 \cdots \right.$$

$$+ a_1 \left[x - \frac{(p-1)(p+2)}{3!} x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!} x^5 \right.$$

$$\left. - \frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!} x^7 + \cdots \right]$$

mod State chebyshev's eqn

$L(x) =$

25) Find the 2 linearly independent power series solns for the chebyshev's eqn $(1-x^2)y'' - xy' + p^2 y = 0$ where p is a constant.

Soln: $(1-x^2)y'' - xy' + p^2 y = 0$ — ①

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be the soln of ①

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad xy' = \sum_{n=1}^{\infty} n a_n x^n$$

$$\therefore xy' = \sum_{n=0}^{\infty} n a_n x^n$$

W.K.T $y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$

$$x^2 y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n = \sum_{n=0}^{\infty} n(n-1) a_n x^n \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

from ①

$$y'' - x^2 y'' - xy' + p^2 y = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} - n(n-1)a_n - na_n + p^2 a_n \right] x^n = 0$$

Equating the coeff of x^n

$$(n+1)(n+2)\, a_{n+2} = \left[n(n-1+1) + p^2\right] a_n$$

$$a_{n+2} = \frac{n^2 - p^2}{(n+1)(n+2)}\, a_n$$

$$\therefore a_{n+2} = \frac{(n-p)(n+p)}{(n+1)(n+2)}\, a_n$$

Put

$n=0$, $a_2 = \dfrac{-p^2}{1 \cdot 2} a_0$

$n=1$, $a_3 = \dfrac{(1+p)(1-p)}{2 \cdot 3} a_1$

$n=2$, $a_4 = \dfrac{(2+p)(2-p)}{3 \cdot 4} a_2 = \dfrac{p^2 (p-2)(p+2)}{4!} a_0$

$n=3$, $a_5 = \dfrac{(3+p)(3-p)}{4 \cdot 5} a_3 = \dfrac{(p-1)(p-3)(p+1)(p+3)}{5!} a_1$

. . . .

$\therefore$ The power series soln is $y = a_0 + a_1 x + a_2 x^2 + \cdots$

$$y = a_0 + a_1 x - \frac{p^2}{2!} a_0 x^2 - \frac{(p-1)(p+1)}{3!} a_1 x^3 + \frac{p^2 (p-2)(p+2)}{4!} a_0 x^4 + \frac{(p-1)(p-3)(p+1)(p+3)}{5!} a_1 x^5 + \cdots$$

$$y = a_0 \left[1 - \frac{p^2}{2!} x^2 + \frac{p^2 (p-2)(p+2)}{4!} x^4 - \frac{p^2 (p-2)(p-4)(p+2)(p+4)}{6!} x^6 + \cdots \right]$$

$$+ a_1 \left[x - \frac{(p-1)(p+1)}{3!} x^3 + \frac{(p-1)(p-3)(p+1)(p+3)}{5!} x^5 - \cdots \right]$$

and 2nd is a polynomial. If p is odd the 2nd series terminates & the first series is a polynomial. In each case the other series is a infinite series.

6) Find the 2 linearly independent power series soln for the Hermites eqn $y'' - 2xy' + 2py = 0$.

<u>Soln</u>: $y'' - 2xy' + 2py = 0$ —①

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be the soln of ①

W.K.T $xy' = \sum_{n=0}^{\infty} n a_n x^n$ and

$$y'' = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

from ①
$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} - 2n a_n + 2p a_n \right] = 0.$$

Equating the coeff of x^n

$$(n+1)(n+2) a_{n+2} = \underline{2(n-p) a_n}$$

$$a_{n+2} = \frac{2(n-p)}{(n+1)(n+2)} a_n$$

Put

$n=0$, $a_2 = \frac{-2p}{1\cdot 2} a_0$

$n=1$, $a_3 = \frac{2(1-p)}{2\cdot 3} a_1 = \frac{-2(p-1)}{3!} a_1$

$n=2$, $a_4 = \frac{2(2-p)}{3\cdot 4} a_2 = \frac{2^2 p(p-2)}{4!} a_0$

$n=3$, $a_5 = \frac{2(3-p)}{4\cdot 5} a_3 = \frac{2^2 (p-1)(p-3)}{5!} a_1$

∴ The power series soln is $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$y = a_0 + a_1 x - \frac{2p}{2!} a_0 x^2 - \frac{2(p-1)}{3!} a_1 x^3 + \frac{2^2 p(p-2)}{4!} a_0 x^4 + \frac{2^2 (p-1)(p-3)}{5!} a_1 x^5$$

$+ -$

$$y = a_0\left[1 - \frac{2p}{2!}x^2 + \underbrace{\frac{2^2 p(p-2)}{4!}}x^4 - \underbrace{\frac{2^3 p(p-2)(p-4)}{6!}}x^6 + \cdots\right] +$$

$$a_1\left[x - \underbrace{\frac{2(p-1)}{3!}}x^3 + \underbrace{\frac{2^2 (p-1)(p-3)}{5!}}x^5 - \underbrace{\frac{2^3 (p-1)(p-3)(p-5)}{7!}}x^7 \cdots\right]$$

2) S.T for the soln of the eqn $y'' + (p + \frac{1}{2} - \frac{1}{4}x^2)y = 0$. where p is constant and the coefficients are related by the 3 term recursion formula.

Soln :

$$(n+1)(n+2)a_{n+2} + (p + \tfrac{1}{2})a_n - \frac{1}{4}a_{n-2} = 0.$$

$$y'' + (p + \tfrac{1}{2} - \frac{1}{4}x^2)y = 0 \quad \text{—①}$$

$$\Rightarrow y'' + (p + \tfrac{1}{2})y - \tfrac{1}{4}x^2 y = 0$$

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be the soln of ①

$$x^2 y = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=0}^{\infty} a_{n-2}\, x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}\, x^n.$$

∴ from ① $y'' + (p + \tfrac{1}{2})y - \tfrac{1}{4}x^2 y = 0$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2} + (p + \tfrac{1}{2})a_n - \tfrac{1}{4}a_{n-2}\right]x^n = 0$$

i.e, $(n+1)(n+2)a_{n+2} + (p + \tfrac{1}{2})a_n - \frac{1}{4}a_{n-2} = 0.$

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Unit-2

2 marks

Ordinary Point, singular points

The point x_0 is called an ordinary point of $y''+P(x)y'+Q(x)y=0$, if $P(x)$ & $Q(x)$ are analytic at x_0.

[$P(x)$ & $Q(x)$ are said to be analytic at x_0, if $P(x)$ & $Q(x)$ can be expressed in the form of a power series in the neighbourhood of x_0].

Otherwise x_0 is said to be a singular point

Regular and Irregular singular points

A singular point x_0 of $y''+p(x)y'+Q(x)y=0$ is said to be regular, if $(x-x_0)p(x)$ and $(x-x_0)^2 Q(x)$ are analytic and Otherwise irregular.

regular singular
$(x-x_0)\,p(x) = 0$ or constant
$(x-x_0)^2\,Q(x) = 0$ is cons

Problems

5 mark

1) Classify the singular points for $(1-x^2)y''-2xy'+p(p+1)y=0$

Soln: Given $(1-x^2)y''-2xy'+p(p+1)y=0$ —①

$$① \Rightarrow \quad y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0$$

$x=1$ & $x=-1$ are singular points.

$$\left[\because \lim_{x\to\pm1}\frac{-2x}{1-x^2}=\infty \text{ and } \lim_{x\to\pm1}\frac{p(p+1)}{1-x^2}=\infty\right].$$

At $x=1$, $\lim\limits_{x\to 1} (x-1)\dfrac{(-2x)}{(1+x)(1-x)} = \lim\limits_{x\to 1} \dfrac{2x}{1+x}$

$= \lim\limits_{x\to 1} 2x(1+x)^{-1} \Rightarrow \lim\limits_{x\to 1} 2x\left[1-x+x^2-x^3+\dots\right]$

$= \lim\limits_{x\to 1} 2(1)\left[1-1+1-1+\dots\right]$

$= 0$

$\lim\limits_{x\to 1} (x-1)^2 \dfrac{p(p+1)}{(1+x)(1-x)} = \dfrac{p(p+1)}{1+x}(1-x) = p(p+1)(1-x)\left[1-x+x^2\dots\right]$

$= 0.$

At $x=-1$ $\lim\limits_{x\to -1} (x+1)\dfrac{(-2x)}{(1+x)(1-x)} = -2x[1-x]^{-1} = -2x[1+x+x^2+\dots] = -2(-1)(1-1+1\dots] = 0.$

$|||^{ly}$ $x = -1$ is also a regular singular point.

2 marks

Q. Locate and classify the singular point for

$x^3(x-1)y'' - 2(x-1)y' + 3xy = 0.$ — ①

Soln.

① $\Rightarrow y'' - \dfrac{2(x-1)}{x^3(x-1)}y' + \dfrac{3x}{x^3(x-1)}y = 0$

$\Rightarrow y'' - \dfrac{2}{x^3}y' + \dfrac{3}{x^2(x-1)}y = 0$

$x=0$ and $x=1$ are singular points.

At $x=0$, $\lim\limits_{x\to 0}(x-0)p(x) = \lim\limits_{x\to 0} x\left[\dfrac{-2}{x^3}\right] = \lim\limits_{x\to 0}\dfrac{-2}{x^2} = \infty$

$\therefore x=0$ is an irregular singular point.

At $x=1$, $\lim\limits_{x\to 1}(x-1)p(x) = \lim\limits_{x\to 1}(x-1)\left[\dfrac{-2}{x^3}\right] = 0$

$$\underset{x\to1}{Lt}\ (x-1)^2 Q(x) = \underset{x\to1}{Lt}\ (x-1)^{\cancel{2}}\left[\frac{3x}{x^3(x-\cancel{1})}\right] = 0$$

∴ $x=1$ is a regular singular point.

3ϕ) Locate & classify the singular points for

$$x^2(x^2-1)^2y'' - x(1-x)y' + 2y = 0 \quad ①$$

soln:

$$① \Rightarrow \ y'' - \frac{x(1-x)}{x^2(x^2-1)^2}y' + \frac{2}{x^2(x^2-1)^2}y = 0.$$

$$\Rightarrow \ y'' - \frac{(1-x)}{x(x^2-1)^2}y' + \frac{2}{x^2(x^2-1)^2}y = 0$$

∴ $x=0$, $x=\pm1$ are singular points.

At $x=0$,

$$\underset{x\to0}{Lt}\ (x-0)P(x) = \underset{x\to0}{Lt}\ x\left(\frac{-(1-x)}{x(x^2-1)^2}\right) = -1$$

$$\underset{x\to0}{Lt}\ (x-0)^2 Q(x) = \underset{x\to0}{Lt}\ x^2\left[+\left(\frac{2}{x^2(x^2-1)^2}\right)\right] = 2$$

$x=0$ is a regular singular point.

At $x=1$

$$\underset{x\to1}{Lt}\ (x-1)P(x) = \underset{x\to1}{Lt}\ x-1\left[\frac{-(1-x)}{x(x^2-1)^2}\right] = \underset{x\to1}{Lt}\frac{(x-\cancel{1})^2}{x(x+1)^2(x-\cancel{1})}$$

$$= \underset{x\to1}{Lt}\ \frac{-1}{x(x+1)^2} = \frac{1}{4}$$

$$\lim_{x \to 1} (x-1)^2 Q(x) = \lim_{x \to 1} (x-1)^2 \left[\frac{2}{x^2 (x^2-1)^2} \right] = \lim_{x \to 1} \cancel{(x-1)^2} \left[\frac{2}{x^2 (x+1)^2 \cancel{(x-1)^2}} \right]$$

$$= \lim_{x \to 1} \frac{2}{x^2 (x+1)^2} = \frac{1}{2} .$$

$\therefore$ x = 1 is a regular singular point.

At x = -1

$$\lim_{x \to -1} (x+1) P(x) = \lim_{x \to -1} \cancel{(x+1)} \left[\frac{-(1-x)}{x (x+1)^{\cancel{2}} (x-1)^2} \right]$$

$$= \lim_{x \to -1} \frac{\cancel{x-1}}{x(x+1)(x-1)^{\cancel{2}}} = \lim_{x \to -1} \frac{1}{x(x+1)(x-1)} = \infty$$

$$\lim_{x \to -1} (x+1)^2 Q(x) = \lim_{x \to -1} \cancel{(x+1)^2} \left[\frac{2}{x^2 \cancel{(x+1)^2} (x-1)^2} \right]$$

$$= \lim_{x \to -1} \left[\frac{2}{x^2 \, \text{[illegible]} \, (x-1)^2} \right] = \frac{2}{4} = \frac{1}{2}$$

x = -1 is an irregular singular point

Solns are

x = 0, x = 1 are regular singular points, as -1, 2, 1/4 & 1/2 can be expressed in forms of power series

But x = -1 is an irregular singular point as ∞ cannot be expressed in forms of power series.

4) Locate and classify the singular points for

$(3x+1)\,xy'' - (x+1)\,y' + 2y = 0 \quad \llcorner ①$

Soln :

$$① \Rightarrow y'' - \frac{(x+1)}{x(3x+1)}\,y' + \frac{2}{x(3x+1)}\,y = 0$$

$\therefore x = 0$ and $x = \frac{-1}{3}$ are singular points.

At $x=0$,

$$\lim_{x\to 0} (x-0)\,P(x) = \lim_{x\to 0} \cancel{x}\left[\frac{-(x+1)}{\cancel{x}(3x+1)}\right] = -1$$

$$\lim_{x\to 0} (x-0)^2\,Q(x) = \lim_{x\to 0} x^{\cancel{2}}\left[\frac{2}{\cancel{x}(3x+1)}\right] = 0$$

At $x = -1/3$

$$\lim_{x\to -1/3} (x+1/3)\,P(x) = \left(\frac{\cancel{3x+1}}{3}\right)\left[\frac{-(x+1)}{x(\cancel{3x+1})}\right] = \frac{-(-1/3+1)}{\cancel{3}(-1/\cancel{3})}$$

$$= 2/3$$

$$\lim_{x\to -1/3} (x+1/3)^2\,Q(x) = \lim_{x\to -1/3}\left(\frac{3x+1}{3}\right)^{\cancel{2}}\left[\frac{2}{x(\cancel{3x+1})}\right] = \lim_{x\to -1/3} \frac{2(3x+1)}{9x} = 0$$

$\therefore x=0,\ x=-1/3$ are regular singular points as $-1, 0, 2/3$ & 0 can be expressed in terms of power series.

Determine the nature of the point $x=0$ for each of the following eqns.

1) $y'' + (\sin x)\, y = 0$.

Soln: At $x=0$.

$\lim_{x\to 0} p(x) = 0$ and $\lim_{x\to 0} Q(x) = \lim_{x\to 0} \sin x = 0$.

$\therefore$ $x=0$ is an ordinary point as 0 can be expressed in terms of power series.

2) $xy'' + (\sin x)\, y = 0$. ①

Soln:

$$① \Rightarrow y'' + \frac{\sin x}{x}\, y = 0$$

$$\left[\because \sin x = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right) \quad \frac{\sin x}{x} = \frac{\cancel{x}\left(1 - x^2/5 + x^4/5 - \dots\right)}{\cancel{x}}\right]$$

At $x=0$

$\lim_{x\to 0} p(x) = 0$ and $\lim_{x\to 0} Q(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1$

$\therefore$ $x=0$ is an ordinary point as 1 can be expressed in terms of power series.

3) $x^2 y'' + \sin x\, y = 0$ ①

$$\left[\because \frac{\sin x}{x^2} = \frac{\cancel{x}\left(1 - x^2/5 + x^4/5 - \dots\right)}{x^2}\right.$$

Soln ① $\Rightarrow$ $y'' + \frac{\sin x}{x^2}\, y = 0$

At $x=0$ $\lim_{x\to 0} p(x) = 0$ and $\lim_{x\to 0} Q(x) = \lim_{x\to 0} \frac{\sin x}{x^2} = \infty$

∴ $x=0$ is not an ordinary point as ∞ cannot be expressed in terms of power series

At $x=0$

$\underset{x\to 0}{Lt}\ (x-0)\,P(x)=0$ and $\underset{x\to 0}{Lt}\ (x-0)^2\,Q(x)$

$$= \underset{x\to 0}{Lt}\ x^2\,\frac{\sin x}{x^2} = \underset{x\to 0}{Lt}\ \sin x = 0.$$

∴ $x=0$ is the regular singular point as 0 can be expressed in terms of power series.

4) $x^3y''+\sin x\, y=0$ — ①

soln:

① ⇒ $y''+\dfrac{\sin x}{x^3}\,y=0.$

At $x=0$ $\underset{x\to 0}{Lt}\ P(x)=0$ & $\underset{x\to 0}{Lt}\ Q(x)=\underset{x\to 0}{Lt}\ \dfrac{\sin x}{x^3}=\underset{x\to 0}{Lt}\ \dfrac{\sin x}{x}\cdot\dfrac{1}{x^2}$

$$= 1\cdot\frac{1}{0}=\infty.$$

∴ $x=0$ is not an ordinary point as ∞ cannot be expressed in terms of power series.

At $x=0$,

$\underset{x\to 0}{Lt}\ (x-0)\,P(x)=0$ & $\underset{x\to 0}{Lt}\ (x-0)^2\,Q(x)=\underset{x\to 0}{Lt}\ x^2\,\dfrac{\sin x}{x^3}$

$$=\underset{x\to 0}{Lt}\ \frac{\sin x}{x}=1$$

∴ $x=0$ is a regular singular pt as 0 & 1 can be expressed in terms of power series.

5) $x^4 y'' + \sin x\, y = 0$
└①

Soln:

$$① \Rightarrow y'' + \frac{\sin x}{x^4} y = 0$$

At $x=0$, $\lim_{x\to 0} P(x) = 0$ & $\lim_{x\to 0} Q(x) = \lim_{x\to 0} \frac{\sin x}{x^4} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{x^3}$

$$= 1 \cdot \frac{1}{0} = \infty$$

∴ $x=0$ is not an ordinary point as ∞ cannot be expressed in terms of power series.

At $x=0$, $\lim_{x\to 0} (x-0) P(x) = 0$

$$\lim_{x\to 0} (x-0)^2 Q(x) = \lim_{x\to 0} \cancel{x^2} \frac{\sin x}{x^{\cancel{4}}} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{x} = 1 \cdot \frac{1}{0}$$
$$= \infty$$

∴ $x = 0$ is an irregular singular point as ∞ cannot be expressed in terms of power series.

Frobenius series solution for a given differential equation:

In this case we take the soln in the form $y = x^m (a_0 + a_1 x + a_2 x^2 + \cdots)$ where $a_0 \neq 0$.

5 mark

Problems

(5) Find the indicial eqn of $2x^2y''+x(2x+1)y'-y=0$.

1) Find a Frobenius series soln for

$$2x^2y''+x(2x+1)y'-y=0 \quad \text{—①}$$

Soln:

$$① \Rightarrow y''+\frac{x(2x+1)}{2x^2}y'-\frac{y}{2x^2}=0$$

i.e.,
$$y''+\frac{x+1/2}{x}y'-\frac{1/2}{x^2}=0 \quad \text{— ②}$$

Let the soln be $y=x^m(a_0+a_1x+a_2x^2+\cdots)$

$x^{m-2}[a_0m+a_1(m+1)x+a_2(m+2)x^2+\cdots]$

$$y=a_0x^m+a_1x^{m+1}+a_2x^{m+2}+\cdots$$

$\frac{y}{x^2}=a_0x^{m-2}+a_1x^{m-1}+a_2x^m$

$$y'=a_0mx^{m-1}+a_1(m+1)x^m+a_2(m+2)x^{m+1}+\cdots$$

$\frac{y'}{x}=a_0mx^{m-2}+a_1(m+1)x^{m-1}+a_2(m+2)x^m+$

$$y''=a_0m(m-1)x^{m-2}+a_1m(m+1)x^{m-1}+a_2(m+1)(m+2)x^m+\cdots$$

when $a_0\neq0$.

Substitute y, y' & y'' in ② & taking x^{m-2} as a factor

$$x^{m-2}\Big\{[a_0m(m-1)+a_1m(m+1)x+a_2(m+1)(m+2)x^2+\cdots]$$
$$+(x+1/2)[a_0m+a_1(m+1)x+a_2(m+2)x^2+\cdots]$$
$$-\frac{1}{2}[a_0+a_1x+a_2x^2+\cdots]\Big\}=0$$

(ie) $\{\ \}=0$.

Equating constant, $a_0 m(m-1) + \frac{1}{2} a_0 m - \frac{1}{2} a_0 = 0$ — ③

Equating coeff of x, $a_1\left[m(m+1) + \frac{1}{2}(m+1) - \frac{1}{2}\right] + a_0 m = 0$ — ④

Equating coeff of x^2, $a_2\left[(m+1)(m+2) + \frac{1}{2}(m+2) - \frac{1}{2}\right] + a_1(m+1) = 0$ — ⑤

from ③ $a_0\left[m(m-1) + \frac{1}{2}m - \frac{1}{2}\right] = 0$

$$\therefore m(m-1) + \frac{1}{2}m - \frac{1}{2} = 0 \quad [\because a_0 \neq 0]$$

ie. $(m-1)\left[m + \frac{1}{2}\right] = 0 \Rightarrow \boxed{m = 1 \text{ or } m = -1/2}$

For m=1.

④ $\Rightarrow a_1\left[2 + \frac{1}{2}(2) - \frac{1}{2}\right] + a_0 = 0$

$$\frac{5a_1}{2} + a_0 = 0$$

$$\boxed{a_1 = \frac{-2}{5} a_0}$$

⑤ $\Rightarrow a_2\left[(2)(3) + \frac{1}{2}(3) - \frac{1}{2}\right] + 2a_1 = 0$

$$a_2[6+1] = -2a_1$$

$$a_2 = \frac{4}{5}\frac{a_0}{7}$$

$$\boxed{\therefore a_2 = \frac{4}{35} a_0}$$

For m= −1/2

④ $\Rightarrow a_1\left[\frac{-1}{2}\cdot\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{2}\right] - \frac{a_0}{2} = 0$

$$a_1\left[-\frac{1}{4} + \frac{1}{4} - \frac{1}{2}\right] = \frac{a_0}{2}$$

$$\boxed{a_1 = -a_0}$$

⑤ $\Rightarrow a_2\left[\frac{1}{2}\cdot 3/2 + 1/2 \cdot 3/2 - 1/2\right] + 1/2\, a_1 = 0$

$$a_2\left[3/4 + 3/4 - 1/2\right] + 1/2\, a_1 = 0$$

$$a_2 = -\frac{a_1}{2}$$

$$\boxed{a_2 = \frac{a_0}{2}}$$

∴ The Frobenius soln is $y = x^1\left[1 - 2/5\, x + 4/35\, x^2 - \dots\right]$

and

$$y = x^{-1/2}\left[1 - x + 1/2\, x^2 - \dots\right]$$

Eqn ① is called indicial eqn of the given differential eqn.

It is of the form $m(m-1)+mP_0+q_0=0$ where P_0 is the constant in $x\,p(x)$ and q_0 is the constant in $x^2 Q(x)$.

$\underset{x\to0}{Lt}\,(x-0)p(x) = \underset{x\to0}{Lt}\,\frac{\cancel{x}(x+x)}{\cancel{x}} = \frac{1}{2}$

$\underset{x\to0}{Lt}\,(x-0)^2Q(x) = \underset{x\to0}{Lt}\,x^2\frac{(-1/2)}{x^2} = -\frac{1}{2}$.

2) Find the indicial eqn and its roots for

$x^3y'' + (\cos 2x - 1)y' + 2xy = 0.$ → ①

Soln:

$m(m-1)+mP_0+q_0=0$ Indicial eqn →

① ⇒ $y'' + \frac{\cos 2x - 1}{x^3}y' + \frac{2x}{x^3}y = 0$

ie, $y'' + \frac{\cos 2x-1}{x^3}y' + \frac{2}{x^2}y = 0$

$m(m-1)-2m+2=0$

Roots, $m_1=2,\ m_2=1$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x = 1-2\sin^2 x$

$\cos 2x = 1-2\sin^2$

$= -2[x$

$x=0$ is the singular point

At $x=0$; $\underset{x\to0}{Lt}\,(x-0)p(x) = \underset{x\to0}{Lt}\,x\left[\frac{\cos 2x-1}{x^3}\right]$

$= \underset{x\to0}{Lt}\,\frac{\cos 2x-1}{x^2} = \underset{x\to0}{Lt}\,\frac{-2\sin^2 x}{x^2}$

$= -2\,\underset{x\to0}{Lt}\,\frac{\sin^2 x}{x^2} = 1$

-2

$\sin^2 x = \left(x - \frac{x^3}{3} + \frac{x^5}{5}\ldots\right)^2$

$= x^2$

∴ $P_0 = xP$

$\underset{x\to0}{Lt}\,(x-0)^2Q(x) = \underset{x\to0}{Lt}\,x^2\left[\frac{2}{x^2}\right] = 2$

∴ $x=0$ is the regular singular point.

Eqn ① is called the indicial eqn & its roots are 2 & 1

3) Find the Indicial eqn and its roots for

$4x^2y'' + (2x^4 - 5x)y' + (3x^2+2)y = 0$. L①

Soln :

$$① \Rightarrow y'' + \frac{2x^4-5x}{4x^2}y' + \frac{3x^2+2}{4x^2}y = 0$$

$$y'' + \frac{\frac{1}{2}x^3 - 5/4}{x}y' + \frac{3/4\,x^2 + 1/2}{x^2}y = 0$$

The indicial eqn is $m(m-1) + mP_0 + q_0 = 0$ where

$P_0 = $ constant in $xp(x)$ and $q_0 = $ constant in $x^2Q(x)$.

$m(m-1) + m(9/4)$

$\therefore m(m-1) + m(-5/4) + 1/2 = 0$ ✓ I. Eqn

constant in $xp(x) = x\left[\frac{\frac{1}{2}x^3 - \boxed{5/4}}{x}\right]$ considering

$m^2 - m - 5/4\,m + 1/2 = 0$

constant in $x^2Q(x) = x^2\left[\frac{3/4\,x^2+1/2}{x^2}\right]$

$\Rightarrow 4m^2 - 4m - 5m + 2 = 0$

$4m^2 - 9m + 2 = 0$

$(m - 8/4)(m - 1/4) = 0$

$m = 2, 1/4$

$\therefore$ The roots are 2 and $1/4$. ✓

4) Verify that the origin is the regular singular point for $4xy'' + 2y' + y = 0$ and find 2 independent Frobenius series soln. L①

Soln ① $\Rightarrow$ $y'' + \frac{2}{4x}y' + \frac{y}{4x} = 0$.

$$\Rightarrow y'' + \frac{1}{2x}y' + \frac{1}{4x}y = 0 \quad \text{—②}$$

At $x=0$, $\underset{x\to 0}{Lt}\,(x-0)\,p(x) = \underset{x\to 0}{Lt}\; x \cdot \frac{1}{2x} = \frac{1}{2}$

$$\underset{x\to 0}{Lt}\,(x-0)^2\,Q(x) = \underset{x\to 0}{Lt}\; x^2 \cdot \frac{1}{4x} = 0$$

$\therefore x=0$ is a regular singular point.

Let the soln be $y = x^m(a_0 + a_1x + a_2x^2 + \cdots)$

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots$$

$$y' = a_0(m)x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \cdots$$

$$y'' = a_0m(m-1)x^{m-2} + a_1(m+1)m\,x^{m-1} + a_2(m+2)(m+1)x^m + \cdot$$

when $a_0 \neq 0$

taking x^{m-2} as a factor & sub in ②, we get

$$② \Rightarrow y'' + \frac{1}{2x}y' + \frac{1}{4x}y = 0$$

$$\Rightarrow x^{m-2}\Big\{\big[a_0m(m-1) + a_1(m+1)(m)x + a_2(m+2)(m+1)x^2 + \cdots\big] + \frac{1}{2}\big[a_0m + a_1(m+1)x + a_2(m+2)x^2 + \cdots\big] + \frac{x}{4}\big[a_0 + a_1x + a_2x^2 + \cdots\big]\Big\} = 0$$

Equating the constant, $a_0\left[m(m-1) + \frac{1}{2}m\right] = 0$ —③

Equating the coeff of x, $a_1\left[(m+1)m + \frac{1}{2}(m+1)\right] + \frac{a_0}{4} = 0$ —④

Equating the coeff of x^2, $a_2\left[(m+2)(m+1) + \frac{1}{2}(m+2)\right] + \frac{a_1}{4} = 0$ —⑤

From ③ $m^2 - m + \frac{1}{2}m = 0 \Rightarrow m(m-1+\frac{1}{2}) = 0$

$m(m-\frac{1}{2}) = 0$ ie. $m=0$ or $m=\frac{1}{2}$

for $m=0$

from ④ $\Rightarrow a_1\left[0+\frac{1}{2}\right] + \frac{a_0}{4} = 0$

$\Rightarrow \frac{a_1}{2} + \frac{a_0}{4} = 0$

$a_1 = -\frac{2a_0}{4}$

$$\boxed{a_1 = -\frac{a_0}{2}}$$

from ⑤ $\Rightarrow a_2\left[2+1\right] + \left(\frac{-a_0/2}{4}\right) = 0$

$3a_2 = \frac{a_0}{8} \Rightarrow a_2 = \frac{a_0}{24}$

$$\boxed{a_2 = \frac{a_0}{4!}}$$

for $m = \frac{1}{2}$

from ④ $\Rightarrow a_1\left[(\frac{1}{2}+1)\frac{1}{2} + \frac{1}{2}(\frac{1}{2}+1)\right] + \frac{a_0}{4} = 0$

$a_1\left[\frac{3}{2}\cdot\frac{1}{2} + \frac{3}{2}\cdot\frac{1}{2}\right] + \frac{a_0}{4} = 0$

$a_1\frac{3}{2} + \frac{a_0}{4} = 0$

$$\boxed{a_1 = -\frac{a_0}{3!}}$$

from ⑤ $\Rightarrow a_2\left[(\frac{1}{2}+2)(\frac{1}{2}+1) + \frac{1}{2}(\frac{1}{2}+2)\right] + \left(\frac{-a_0}{3!\cdot 4}\right) = 0$

$a_2\left[\frac{5}{2}\cdot\frac{3}{2} + \frac{1}{2}\cdot\frac{5}{2}\right] - \frac{a_0}{3!4} = 0$

$\Rightarrow a_2\left(\frac{20}{4}\right) = \frac{a_0}{4!}$

$$\boxed{a_2 = \frac{a_0}{5!}}$$

∴ The Frobenius series soln is

$y_1(x) = x^0\left[1 - \frac{x}{2!} + \frac{x^2}{4!} - \dots\right]$; $y_2(x) = x^{1/2}\left[1 - \frac{x}{3!} + \frac{x^2}{5!} + \dots\right]$

$y_1(x) = \left[1 - \frac{x}{2!} + \frac{x^2}{4!} \dots\right]$; $y_2(x) = x^{1/2}\left[1 - \frac{x}{3!} + \frac{x^2}{5!} + \dots\right]$

⑤ Find the 2 independent Frobenius series soln for $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$ ---- ①

Soln: ① => $y'' + \frac{x}{x^2} y' + \frac{x^2 - \frac{1}{4}}{x^2} y = 0$

=> $y'' + \frac{1}{x} y' + \frac{x^2 - \frac{1}{4}}{x^2} y = 0$ ---- ②

Let the soln be $y = x^m (a_0 + a_1 x + a_2 x^2 + \dots)$

$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \dots$

$y' = a_0 m x^{m-1} + a_1 (m+1) x^m + a_2 (m+2) x^{m+1} + \dots$

$y'' = a_0 m(m-1) x^{m-2} + a_1 (m+1)(m) x^{m-1} + a_2 (m+2)(m+1) x^m + \dots$

when $a_0 \neq 0$

taking x^{m-2} as a factor & sub in ①

$$x^{m-2} \{ [a_0 (m)(m-1) + a_1 (m+1)(m) x + a_2 (m+2)(m+1) x^2 + \dots] + [a_0 (m) + a_1 (m+1) x + a_2 (m+2) x^2 + \dots] + (x^2 - 1/4) [a_0 + a_1 x + a_2 x^2 + \dots] \} = 0$$

Equating the ~~coefficient~~ constants, $a_0 [m(m-1) + m - \frac{1}{4}] = 0$ ---- ③

Equating the coeff of x, $a_1 [(m+1)m + (m+1) - \frac{1}{4}] = 0$ ---- ④

Equating the coeff of x^2, $a_2 [(m+2)(m+1) + (m+2) - \frac{1}{4}] + a_0 = 0$ ---- ⑤

∴ from ③ => $m^2 - m + m - 1/4 = 0$ => $m^2 - 1/4 = 0$

$\boxed{m = \pm 1/2}$

For $m = \frac{1}{2}$

from ④ $\Rightarrow a_1\left[\left(\frac{1}{2}+1\right)\frac{1}{2}+\left(\frac{1}{2}+1\right)\cdot -\frac{1}{4}\right]=0$

$a_1\left[\frac{3}{2}\cdot\frac{1}{2}+\frac{3}{2}\cdot -\frac{1}{4}\right]=0 \Rightarrow a_1\left[\frac{3}{4}+\frac{3}{8}\right]=0$

$a_1(2)=0 \Rightarrow \boxed{a_1=0}$

from ⑤ $\Rightarrow a_2\left[\left(\frac{1}{2}+2\right)\left(\frac{1}{2}+1\right)+\left(\frac{1}{2}+2\right)\cdot -\frac{1}{4}\right]+a_0=0$

$\Rightarrow a_2\left[\frac{5}{2}\cdot\frac{3}{2}+\frac{5}{2}\cdot -\frac{1}{4}\right]+a_0=0$

$\Rightarrow a_2\left[\frac{15}{4}+\frac{9}{4}\right]+a_0=0$

$a_2\left(\frac{24}{4}\right)=-a_0$

$\boxed{a_2=-\frac{a_0}{3!}}$

For $m = -\frac{1}{2}$

from ④ $\Rightarrow a_1\left[\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}+1\right)\cdot -\frac{1}{4}\right]=0$

$a_1\left[\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)+\left(\frac{1}{2}\right)\cdot -\frac{1}{4}\right]=0 \quad a_1\left[\frac{-2}{4}+\frac{1}{2}\right]=0$

$\boxed{a_1=0}$

from ⑤ $\Rightarrow a_2\left[\left(-\frac{1}{2}+2\right)\left(-\frac{1}{2}+1\right)+\left(-\frac{1}{2}+2\right)\cdot -\frac{1}{4}\right]+a_0=0$

$a_2\left[\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{3}{2}\right)\cdot -\frac{1}{4}\right]+a_0=0$

$a_2\left[\frac{3}{4}-\frac{1}{4}+\frac{3}{2}\right]+a_0=0$

$\Rightarrow a_2\left[\frac{1}{2}+\frac{3}{2}\right]+a_0=0 \Rightarrow 2a_2=-a_0$

$\boxed{a_2=-\frac{a_0}{2!}}$

$y_1(x)=x^{1/2}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right]$; $y_2(x)=x^{-1/2}\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right]$

$y_1(x)=x^{1/2}\sin x$ $\qquad$ $y_2(x)=x^{-1/2}(\cos x)$

mark

6) Show that the eqn $x^2y''+xy'+(x^2-1)y=0 \;\llcorner①$ has only one Frobenius series soln.

Soln: ① $y''+\frac{x}{x^2}y'+\frac{(x^2-1)}{x^2}y=0$

$\Rightarrow y''+\frac{1}{x}y'+\frac{(x^2-1)}{x^2}y=0$ —②

Let the soln be $y=x^m(a_0+a_1x+a_2x^2+\cdots)$

$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots$

$y' = a_0 m x^{m-1} + a_1 (m+1) x^m + a_2 (m+2) x^{m+1} + \cdots$

$y'' = a_0 m(m-1) x^{m-2} + a_1 m(m+1) x^{m-1} + a_2 (m+1)(m+2) x^m + \cdots$

when $a_0 \neq 0$

taking x^{m-2} as a factor & sub in ①

$$x^{m-2} \Big\{ [a_0 (m)(m-1) + a_1 (m+1)(m) x + a_2 (m+2)(m+1) x^2 + \cdots] + [a_0 (m) + a_1 (m+1) x + a_2 (m+2) x^2 + \cdots] + (x^2-1)[a_0 + a_1 x + a_2 x^2 + \cdots] \Big\} = 0$$

Equating the constant, $a_0[m(m-1) + m - 1] = 0$ — ③

Equating the coeff of x, $a_1[(m+1)m + (m+1)] = 0$ — ④

Equating the coeff of x^2, $a_2[(m+2)(m+1) + (m+2) - 1] + a_0 = 0$ ⑤

from ③ $\Rightarrow a_0[m^2 - m + m - 1] = 0 \Rightarrow m^2 - 1 = 0$

$$\boxed{m = \pm 1}$$

for m=1

from ④ $a_1[2+2] = 0$

$$\boxed{a_1 = 0}$$

from ⑤ $\Rightarrow a_2[3 \cdot 2 + 3 - 1] + a_0 = 0$

$8a_2 = -a_0$

$$\boxed{a_2 = -\frac{a_0}{8}}$$

for m=-1

from ④ $\Rightarrow a_1 (0) = 0$

$$\boxed{a_1 = 0}$$

from ⑤ $\Rightarrow a_2[1-1] + a_0 = 0$

$$\boxed{a_2 = 0}$$

$$y_1(x) = x\left[1 - \frac{x^2}{8} + \cdots\right]$$

There is only one soln for Farberius series.

7) S.T the eqn $x^2y'' + xy' + x^2y = 0$ ── ① has only one root for indicial eqn & P.T the Frobenius series soln is $y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$

Soln: ① ⇒ $y'' + \frac{x}{x^2}y' + \frac{x^2}{x^2}y = 0$ (i.e) $y'' + \frac{1}{x}y' + \frac{x^2}{x^2}y = 0$ ── ②

At $x=0$,

$\lim_{x\to 0}(x-0)P(x) = \lim_{x\to 0} x \cdot \frac{1}{x} = 1$ and $\lim_{x\to 0}(x-0)^2 Q(x) = \lim_{x\to 0} \frac{x^2}{x^2} |=1$

∴ Eqn ② is called the indicial eqn and 1 is the only root.

Let the soln Be $y = x^m(a_0 + a_1x + a_2x^2 + \cdots)$

$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots$

$y' = a_0mx^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \cdots$

$y'' = a_0m(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + a_2(m+2)(m+1)x^m + \cdots$

when $a_0 \neq 0$

taking x^{m-2} as a factor & sub in ①

$$x^{m-2}\{[a_0 m(m-1) + a_1 (m+1)(m) x + a_2 (m+2)(m+1)x^2 + \dots]$$

$$+ [a_0(m) + a_1 (m+1) x + a_2 (m+2) x^2 + \dots] +$$

$$x^2 [a_0 + a_1 x + a_2 x^2 + \dots]\} y = 0$$

Equating the constant,

$a_0[m(m-1)+m] = 0$ — ③

Equating the coeff of x, $a_1[(m+1)(m) + (m+1)] = 0$ — ④

Equating the coeff of x^2, $a_2[(m+2)(m+1) + (m+2)] + a_0 = 0$ — ⑤

∴ from ③ $m^2 - m + m = 0$ i.e, $m^2 = 0$

$\Rightarrow \boxed{m=0}$

for $m=0$

④ $\Rightarrow a_1[0+1] = 0$

$a_1 = 0$

$a_2[2+2] + a_0 = 0$

$4a_2 = -a_0$

$a_2 = \frac{-a_0}{4}$

$y_1(x) = x^m[a_0 + a_1 x + a_2 x^2$

$= x^0 [1 - \frac{x^2}{8}$

2marks

(*) Find the indicial eqn of $x^2y'' - 3xy' + (4x+4)y = 0$

Ans: $y = x^2(1 - 4x + 4x^2 +$

Gauss Hyper Geometric equation.

The differential equation

$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$ where a, b, c are constants is called Gauss Hyper Geometric eqn.

Note:

The roots of the indicial eqn are called exponents.

To find the general soln of the Hyper geometric eqn near the singular point $x=0$.

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

$$y'' + \frac{c-(a+b+1)x}{x(1-x)}y' - \frac{ab}{x(1-x)}y = 0$$

Here, $p(x) = \frac{c-(a+b+1)x}{x(1-x)}$ and $Q(x) = \frac{-ab}{x(1-x)}$

The singular points are $x=0$ & $x=1$.

At $x=0$, $\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x \frac{c-(a+b+1)x}{x(1-x)} = \lim_{x\to 0} [c-(a+b+1)x](1-x)^{-1}$

$$= \lim_{x\to 0} [c-(a+b+1)x][1+x+x^2+\cdots]$$

$$= c$$

$$\lim_{x\to 0} x^2 Q(x) = \lim_{x\to 0} x^2\left[\frac{-ab}{x(1-x)}\right] = \lim_{x\to 0}\left[-x ab(1+x+x^2+\cdots)\right]$$

$$= \cancel{1}\ 0$$

$\therefore$ $x=0$ is a regular singular point.

III^{ly} $x=1$ is also a regular singular point.

In the neighbourhood of the point $x=0$

The Indicial eqn is $m(m-1) + m p_0 + q_0 = 0$

i.e, $m(m-1) + mc = 0 \Rightarrow m^2 - m + \overset{mc}{\cancel{m+}}\, c = 0$

i.e, $m^2 - m(1-c) = 0$ $\quad m[m-(1-c)] = 0$

i.e. $m=0$ (or) $m = 1-c$

for $m=0$

The soln is $y = x^m(a_0 + a_1 x + a_2 x^2 + \cdots)$

$y = x^0(a_0 + a_1 x + a_2 x^2 + \cdots)$

$$\boxed{a_{n+1} = \frac{(a+n)(b+n)}{(n+1)(c+n)}\, a_n}$$

Take $a_0 = 1$

$n=0$, $a_1 = \frac{ab}{1\cdot c} = \frac{ab}{1!c}$

$n=1$, $a_2 = \frac{(a+1)(b+1)}{2!(c+1)} a_1$

$= \frac{a(a+1)b(b+1)}{2!\,c(c+1)}$

$n=2$, $a_3 = \frac{(a+2)(b+2)}{3!(c+2)} a_2$

$= \frac{a(a+1)(a+2)\,b(b+1)(b+2)}{3!\,c(c+1)(c+2)}$

∴ from ①

$$y = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)\, b(b+1)(b+2)\cdots(b+n-1)}{n!\; c(c+1)(c+2)\cdots(c+n-1)} \cdot x^n$$

i.e, $y = F(a, b, c, x)$ where

$$F(a,b,c,x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)\, b(b+1)(b+2)\cdots(b+n-1)}{n!\; c(c+1)(c+2)\cdots(c+n-1)} \cdot x^n$$

Note:

$$F(a,b,c,x) = F(b,a,c,x)$$

for $m = 1-c$,

The soln is $y = x^m(a_0 + a_1 x + a_2 x^2 + \cdots)$

i.e, $y = x^{1-c} z$, where $z = a_0 + a_1 x + a_2 x^2 + \cdots$

Find y' and y'' & sub in the H.G eqn

$$x(1-x)z'' + [z - c - \{(a-c+1) + (b-c+1) + 1\}x]z' - (a-c+1)(b-c+1)z = 0$$

∴ The soln is $z = F(a-c+1,\, b-c+1,\, 2-c,\, x)$

∴ The required soln is $y = x^{1-c} F[a-c+1,\, b-c+1,\, 2-c,\, x]$

provided c is not an integer.

Hyper Geometric eqn is

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

The general soln near $x=0$ is

$$y = c_1 F(a, b, c, x) + c_2 F(a-c+1, b-c+1, 2-c, x) x^{1-c}.$$

The general soln near $x=1$ is

$$y = c_1 F(a, b, a+b-c+1, 1-x) + c_2 F(c-b, c-a, c-a-b+1, 1-x)(1-$$

Problems

1) Prove $(1+x)^p = F(-p, b, b, -x)$

Soln:

$$F(a, b, c; x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{n!\, c(c+1)(c+2)\cdots(c+n-1)}$$

$$F(-p, b, b, -x) = 1 + \sum_{n=1}^{\infty} \frac{-p(-p+1)(-p+2)\cdots(-p+n-1)\, b(b+1)\cdots(b+}{n!\, b(b+1)(b+2)\cdots(b+n-1)}$$

$$= 1 + \frac{(-p)}{1!}(-x) + \frac{(-p)(-p+1)}{2!}(-x)^2 + \cdots$$

$$= 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots$$

~~$= (1 + pc_1 x + pc_2 x^2$~~

$$= 1 + pc_1 + pc_2 x^2 + \cdots \quad = (1+x)^p.$$

2) Prove $\log(1+x) = x\,F(1,1,2,-x)$

Soln

$$F(a,b,c,x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)\,b(b+1)\cdots(b+n-1)}{n!\,c(c+1)(c+2)\cdots(c+n-1)} x^n$$

$$\therefore x\,F(1,1,2,-x) = x\left[1 + \sum_{n=1}^{\infty} \frac{1(2)(3)\cdots(n)\,1(2)(3)\cdots(n)}{n!\,(2)(3)4\cdots(n+1)}(-x)^n\right]$$

$$= x\left[1 + \sum_{n=1}^{\infty} \frac{(-x)^n}{n+1}\right]$$

$$= x\left[1 + \frac{-x}{2} + \frac{x^2}{3} + \frac{-x^3}{4} + \cdots\right.$$

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right]$$

$$= \log(1+x)$$

3) Prove $\sin^{-1} x = x\,F\left(\frac{1}{2}, \frac{1}{2}, 3/2, x^2\right)$

Soln:

$$F(a,b,c,x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)\,b(b+1)(b+2)\cdots(b+n-1)}{n!\,c(c+1)(c+2)\cdots(c+n-1)} x^n$$

$$F(1/2, 1/2, 3/2, x^2) = 1 + \sum_{n=1}^{\infty} \frac{\left[1/2 \cdot 3/2 \cdot 5/2 \cdots (n-1/2)\right]\left[1/2 \cdot 3/2 \cdots (n-1/2)\right]}{n!\; 3/2 \cdot 5/2 \cdot 7/2 \cdots (n+1/2)} x^{2n}$$

4. Prove $\tan^{-1}(x) = x F\left(\frac{1}{2}, 1, \frac{3}{2}, -x^2\right)$

Soln:

$$F(a,b,c,x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{n!\, c(c+1)(c+2)\cdots(c+n-1)} x^n$$

$$\therefore F\left(\frac{1}{2}, 1, \frac{3}{2}, -x^2\right) = 1 + \sum_{n=1}^{\infty} \frac{\left[\frac{1}{2}\cdot\cancel{\frac{3}{2}}\cdot\cancel{\frac{5}{2}}\cdots\left(n-\frac{1}{2}\right)\right]\cancel{\left[1\cdot2\cdot3\cdots n\right]}}{\cancel{n!}\;\frac{3}{2}\cdot\frac{5}{2}\cdots\left(n+\frac{1}{2}\right)} (-x^2)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{-x^{2n}}{2\left(n+\frac{1}{2}\right)}$$

$$= 1 + \frac{-x^2}{2\cdot\frac{3}{2}} + \frac{-x^4}{2\cdot\frac{5}{2}} + \frac{-x^6}{2\cdot\frac{7}{2}} + \cdots$$

$$= 1 - \frac{x^2}{3} - \frac{x^4}{5} - \frac{x^6}{7} - \cdots$$

$$\therefore x F\left(\frac{1}{2}, 1, \frac{3}{2}, -x^2\right) = x - \frac{x^3}{3} - \frac{x^5}{5} - \frac{x^7}{7} \cdots$$

(X.) 2marks.

⑤ prove $e^x = \underset{b\to\infty}{Lt} F(a, b, a, x/b)$

Soln

$$F(a,b,c,x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{n!\, c(c+1)(c+2)\cdots(c+n-1)} x^n$$

$$\therefore F(a,b,a,x/b) = 1 + \sum_{n=1}^{\infty} \frac{\cancel{a}\cancel{(a+1)}(a+2)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{n!\, \cancel{a}\cancel{(a+1)}(a+2)\cdots(a+n-1)} (x/b)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\cancel{b^n}}{n!}\left[(1+1/b)(1+2/b)\cdots\left(1+\frac{n-1}{b}\right)\right]\frac{x^n}{\cancel{b^n}}$$

$$\underset{b\to\infty}{Lt} F(a,b,a,x/b) = \underset{b\to\infty}{Lt}\left[1 + \sum_{n=1}^{\infty} (1+1/b)(1+2/b)\cdots\left(1+\frac{n-1}{b}\right)\right]\cdot\frac{x^n}{n!}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\therefore \lim_{b\to\infty} F(a, b, a, x/b) = e^x .$$

⑥ prove $\sin x = x\left[\lim_{a\to\infty} F\left(a, a, 3/2, -x^2/4a^2\right)\right]$

$$F(a,b,c,x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)\,b(b+1)\cdots(b+n-1)}{n!\; c(c+1)(c+2)\cdots(c+n-1)}\, x^n$$

$$F\left(a,a,3/2,\frac{-x^2}{4a^2}\right) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)\,a(a+1)\cdots(a+n-1)}{n!\; 3/2\cdot 5/2\cdots(n+1/2)} \left(\frac{-x^2}{4a^2}\right)$$

$$= 1 + \frac{a\cdot a}{1!\,3/2}\left(\frac{-x^2}{4a^2}\right) + \frac{a(a+1)(a+1)a}{2!\,3/2\cdot 5/2}\left(\frac{-x^2}{4a^2}\right)^2 + \cdots$$

$$= 1 - \frac{x^2}{1\cdot 2\cdot 3} + \frac{(a+1)^2}{2\cdot 3\cdot 4\cdot 5}\cdot x^4 + \cdots$$

$$= 1 - \frac{x^2}{3!} + \frac{(a+1)^2}{5!}x^4 + \cdots$$

$$\lim_{a\to\infty} F\left(a,a,3/2,-x^2/4a^2\right) = \lim_{a\to\infty}\left[1 - \frac{x^2}{3!} + \frac{(a+1)^2}{5!}x^4 + \cdots\right]$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

$$x \lim_{a\to\infty} F\left(a,a,3/2,-x^2/4a^2\right) = x\left[1 - x^2/3! + \frac{x^4}{5!} - \cdots\right] = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \sin x .$$

⑦ Prove $\cos x = \underset{a\to\infty}{Lt}\ F\left(a, a, \frac{1}{2}, -\frac{x^2}{4a^2}\right)$

Soln:

$$F(a,b,c,x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{n!\, c(c+1)(c+2)\cdots(c+n-1)}\, x^n$$

$$F\left(a, a, \tfrac{1}{2}, -\tfrac{x^2}{4a^2}\right) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)\, a(a+1)\cdots(a+n-1)}{n!\, \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\left(n-\frac{1}{2}\right)} \left(\frac{-x^2}{4a}\right.$$

$$= 1 + \frac{a\cdot a}{1!\,\frac{1}{2}}\left(\frac{-x^2}{4a^2}\right) + \frac{a(a+1)(a)(a+1)}{2!\,\frac{1}{2}\cdot\frac{3}{2}}\left(\frac{-x^2}{4a^2}\right)^2 + \cdots$$

$$= 1 - \frac{x^2}{1!\,2} + \frac{(a+1)^2}{2!\,1\cdot3\cdot4}\, x^4 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{(a+1)^2}{4!}\, x^4 + \cdots$$

$$\underset{a\to\infty}{Lt}\ F\left(a, a, \frac{1}{2}, \frac{-x^2}{4a^2}\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$= \underline{\underline{\cos x}}.$$

For the hyper geometric series the 3 regular singular points are

i) $x=0$ with exp 0 & $1-c$

ii) $x=1$ with exp 0 & $c-a-b$

iii) $x=\infty$ with exp a & b.

Problems.

1) Find the general soln of $x(1-x)y'' + (3/2 - 2x)y' + 2y = 0$ be a singular point at $x = 0$.

Soln :

Hyper Geometric eqn is $x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$

The given eqn is $x(1-x)y'' + (\frac{3}{2} - 2x)y' + 2y = 0$

$\therefore c = 3/2, \quad a+b+1 = 2, \quad ab = -2$

$\Rightarrow b = -2/a$

$\therefore \quad a - \frac{2}{a} + 1 = 2 \Rightarrow a^2 - 2 + a = 2a \Rightarrow a^2 - a - 2 = 0$

$a(a-1) = 2 \quad$ i.e, $\underline{\underline{a = 2, -1}}$

$b = -2/2 = -1 \quad (or) \quad b = \frac{-2}{-1} = 2$

i.e, $\boxed{\begin{matrix} a = 2, -1 \\ b = -1, 2 \end{matrix}}$

$\frac{4-3+2}{2} \quad -2-3+2 \quad 3/2$

$F(a, b, c, x) = F(b, a, c, x)$

The soln near x is

$y = c_1 F(a, b, c, x) + c_2 x^{1-c} F(a-c+1, b-c+1, 2-c, x)$

$\therefore y = c_1 F(2, -1, 3/2, x) + c_2 x^{1-3/2} F(2 - 3/2 + 1, -1 - 3/2 + 1, 2 - 3/2, x)$

~~$F(2, -1, 3/2, x) = 1 + \sum_{n=1}^{\infty} (2)(3)(4)\ldots(n+1)(-1)$~~

$F(2, -1, 3/2, x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)(a+2)\ldots(a+n-1)b(b+1)\ldots(b+n-1)}{n!\,(c)(c+1)\ldots(c+n-1)} x^n$

$$= 1 + \sum_{n=1}^{\infty} \frac{(n+1)(n-2)}{n!\,(n+1/2)} x^n$$

$$= 1 + \frac{2(-1)}{1!\,3/2} x + \frac{3(0)}{2!\,5/2} x^2 + \cdots$$

$$= 1 - \frac{4}{3} x$$

$$y = \left(1 - \tfrac{4}{3} x\right) c_1 + x^{-1} F\left(3/2, -3/2, 1/2, x\right) c_2$$

To solve $(x-A)(x-B)y'' + (C+Dx)y' + Ey = 0$ where $y' = \frac{dy}{dx}$ putting $t = \frac{x-A}{B-A}$ we get $t(1-t)y'' + (F+Gt)y' + Hy = 0$

where $y' = \frac{dy}{dt}$ and $F = -\left[\frac{C+DA}{B-A}\right]$, $G = -D$, $H = -E$

Soln:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{B-A} \cdot \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\left(\frac{dt}{dx}\right) = \frac{d}{dt}\left(\frac{1}{B-A}\frac{dy}{dt}\right) \cdot \frac{1}{B-A}$$

$$= \frac{1}{(B-A)^2} \frac{d^2y}{dt^2}$$

$$(t-1) = \frac{x-A}{B-A} - 1 = \frac{x-A-B+A}{B-A} = \frac{x-B}{B-A}$$

$$\boxed{\therefore t-1 = \frac{x-B}{B-A}}$$

Sub in ①

$$t(B-A)(t-1)(B-A)\frac{1}{(B-A)^2}y''+\left[C+D\{(t-1)(B-A)+B\}\right]\frac{y'}{B-A}+Ey=0$$

$$\left[\because\ x-A=t(B-A)\ \&\ x-B=(t-1)(B-A)\right]$$

i.e. $t(t-1)y''+\left[C+D\{tB-B-tA+A+B\}\right]\frac{y'}{B-A}+Ey=0$

i.e, $t(t-1)y''+\left[C+D\{t(B-A)+A\}\right]\frac{y'}{B-A}+Ey=0$

i.e, $t(t-1)y''+\left[\frac{C+DA}{B-A}+Dt\frac{(B-A)}{B-A}\right]y'+Ey=0$

ie $t(1-t)y''+\left[\frac{-(C+DA)}{B-A}-Dt\right]y'-Ey=0$

i.e. $\underline{\underline{t(1-t)y''+\left[F+Gt\right]y'+Hy=0}}$, where $F=-\left[\frac{C+DA}{B-A}\right]$, $G=-D$, $H>E$

) Find the general soln of $(x^2-x-6)y''+(5+3x)y'+y=0$ near the singular point $x=3$.

Soln: H.G.E is $x(1-x)y''+\left[c-(a+b+1)x\right]y'-aby=0$

The given eqn is $(x^2-x-6)y''+(5+3x)y'+y=0$

ie, $(x-3)(x+2)y''+(5+3x)y'+y=0$

i.e. $(x-A)(x-B)y'+(C+Dx)y'+Ey=0$.

put $t=\frac{x-A}{B-A}$, i.e, $t=\frac{x-3}{-2-5}$ $\Rightarrow$ $\boxed{t=\frac{x-3}{-5}}$

∴ The transformed eqn is $t(1-t)y''+(F+Gt)y'+Hy=0$

where $F=-\left[\frac{C+DA}{B-A}\right]$, $G=-D$, $H=-E$

∴ $F=-\left[\frac{5+3\cdot3}{-2-3}\right]=-\left[\frac{5+9}{-5}\right]=\frac{14}{5}$, $G=-3$, $H=-1$

i.e., $t(1-t)y''+\left(\frac{14}{5}-3t\right)y'-y=0$

∴ $c=\frac{14}{5}$, $a+b+1=3$, $ab=1$

$b=1/a$

$\Rightarrow a+1/a+1=3 \Rightarrow a^2+a-3a=-1 \Rightarrow a^2-2a+1=0$

$(a-1)^2=0 \Rightarrow \boxed{a=1 \Rightarrow b=1}$

Now $t=\frac{x-3}{-5}$

∴ $x=3$ corresponds to $t=0$

∴ The soln is $y=c_1F(a,b,c,x)+c_2x^{1-c}F(a-c+1,b-c+1,2-c,x)$

$\therefore y=c_1F\left(1,1,\frac{14}{5},\frac{x-3}{-5}\right)+c_2\left(\frac{x-3}{-5}\right)^{1-\frac{14}{5}}F\left(1-\frac{14}{5}+1,1-\frac{14}{5}+1,2-\frac{14}{5},\frac{x-}{-5}\right.$

$y=c_1F\left(1,1,\frac{14}{5},\frac{x-3}{-5}\right)+c_2\left(\frac{x-3}{-5}\right)^{\frac{-9}{5}}F\left(\frac{-4}{5},\frac{-4}{5},\frac{-4}{5},\frac{x-3}{-5}\right)$.

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Note: If we take in $t = \frac{x-A}{B-A}$, for $A=-2$ & $B=3$ then

$t = \frac{x+2}{5}$ then $x=3$ corresponds to $t=1$.

∴ we've to write the formula.

$$y = C_1 F(a, b, a+b-c+1, 1-x) + C_2 (1-x)^{c-a-b} F(c-b, c-a, c-a-b+1, 1-x)$$

2) Find the general soln for $(2x^2+2x)y''+(1+5x)y'+y=0$ near $x=0$.

Soln: H.G.E is $x(1-x)y''+[c-(a+b+1)x]y'-aby=0$

The given eqn is $(2x^2+2x)y''+(1+5x)y'+y=0$

ie, $2x(x+1)y''+(1+5x)y'+y=0$

$\Rightarrow x(x+1)y''+\left(\frac{1+5x}{2}\right)y'+\frac{1}{2}y=0$

i.e, $(x-A)(x-B)y''+(C+Dx)y'+Ey=0$

put $t = \frac{x-A}{B-A}$ (i.e.,) $t = \frac{x-0}{-1-0} = -x$

∴ The transformed eqn is $t(1-t)y''+(F+Gt)y'+Hy=0$

where $F = \left[\frac{C+DA}{B-A}\right]$, $G=-D$, $H=-E$

$$\therefore F = \left[\frac{1/2+5/2\cdot 0}{-1-0}\right] = -\left[-1/2\right] = 1/2$$

$\Rightarrow G = -5/2$, $H = -1/2$.

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ie $t(1-t)y''+\left(\frac{1}{2}-\frac{5}{2}t\right)y'-\frac{1}{2}y=0$

$c=\frac{1}{2}$, $a+b+1=\frac{5}{2}$, $ab=\frac{1}{2}$

$b=\frac{1}{2}a$

$\therefore a+\frac{1}{2a}+1=\frac{5}{2} \Rightarrow 2a^2+2a-5a+1=0 \Rightarrow 2a^2-3a+1=0$

$\Rightarrow 2a(a-1)-1(a-1)=0 \Rightarrow (2a-1)(a-1)=0$

$$\boxed{a=\frac{1}{2},1 \quad \& \quad b=1,\frac{1}{2}}$$

Now $t=-x$

$\therefore x=0$ corresponds to $t=0$

$\therefore$ The soln is $y=c_1F(a,b,c,x)+c_2x^{1-c}F(a-c+1, b-c+1, 2-c, x)$

$y=c_1F\left(\frac{1}{2},1,\frac{1}{2},-x\right)+c_2(-x)^{1-\frac{1}{2}}F\left(\frac{1}{2}-\frac{1}{2}+1, 1-\frac{1}{2}+1, 2-\frac{1}{2},-x\right.$

$\therefore y=c_1F\left(\frac{1}{2},1,\frac{1}{2},-x\right)+c_2(-x)^{\frac{1}{2}}F\left(1,\frac{3}{2},\frac{3}{2},-x\right)$

3) Find the general soln for $(x^2-1)y''+(5x+4)y'+4y=0$ near $x=-1$

Soln: H.G.E is $x(1-x)y''+[c-(a+b+1)x]y'-aby=0$

The given eqn is $(x^2-1)y''+(5x+4)y'+4y=0$

i.e., $(x+1)(x-1)y''+(5x+4)y'+4y=0$

i.e, $(x-A)(x-B)y''+(C+Dx)y'+Ey=0$

put $t = \frac{x-A}{B-A} = \frac{x-(-1)}{1-(-1)} = \frac{x+1}{2}$

∴ The transformed eqn is $t(1-t)y'' + (F+Gt)y' + Hy = 0$

where $F = -\left[\frac{C+DA}{B-A}\right]$, $G = -D$, $H = -E$

$\therefore F = \left[\frac{4+5(-1)}{1-(-1)}\right] = -\left[\frac{4-5}{1+1}\right] = -\left[\frac{-1}{2}\right] = 1/2$, $G = -5$, $H = -4$.

ie. $t(1-t)y'' + (1/2 - 5t)y' - 4y = 0$

$c = 1/2$, $a+b+1 = 5$, $ab = 4 \Rightarrow b = 4/a$

$\therefore a + 4/a + 1 = 5 \Rightarrow a^2 + 4 + a - 5a = 0 \Rightarrow a^2 - 4a + 4 = 0$

$(a-2)^2 = 0 \Rightarrow \boxed{a = 2 \ \& \ b = 2}$

Now $t = \frac{x+1}{2}$

∴ $x = -1$ corresponds to $t = 0$

∴ The soln is $y = c_1 F(a,b,c,x) + c_2 x^{1-c} F(a-c+1, b-c+1, 2-c, x)$

$$y = c_1 F\left(2, 2, 1/2, \frac{x+1}{2}\right) + c_2 \left(\frac{x+1}{2}\right)^{1-1/2} F\left(2-1/2+1, 2-1/2+1, 2-1/2, \frac{x+1}{2}\right)$$

$$y = c_1 F\left(2, 2, 1/2, \frac{x+1}{2}\right) + c_2 \left(\frac{x+1}{2}\right)^{1/2} F\left(5/2, 5/2, 3/2, \frac{x+1}{2}\right)$$

5 marks

S.T the general soln for $(1-x^2)y''-xy'+p^2y=0$ near $x=1$ is $y=c_1F(p,-p,\frac{1}{2},\frac{1-x}{2})+c_2\left(\frac{1-x}{2}\right)^{\frac{1}{2}}F(p+\frac{1}{2},-p+\frac{1}{2},\frac{3}{2},\frac{1-}{2}$

Soln: H.G.E is $x(1-x)y''+[c-(a+b+1)x]y'-aby=0$

The given eqn is $(1-x^2)y''-xy'+p^2y=0$

(i.e.,) $(1+x)(1-x)y''-xy'+p^2y=0$

$\Rightarrow (x-1)(x+1)y''+xy'-p^2y=0$

ie $(x-A)(x-B)y''+(C+Dx)y'+Ey=0$

Put $t=\frac{x-A}{B-A}=\frac{x-1}{-1-1}=\frac{x-1}{-2}$

$\therefore$ The transformed eqn is $t(1-t)y''+(F+Gt)y'+Hy=0$

where $F=-\left[\frac{C+DA}{B-A}\right]$, $G=-D$, $H=-E$

$\therefore F=-\left[\frac{0+1(1)}{-1-1}\right]=-\left[-\frac{1}{2}\right]=\frac{1}{2}$, $G=-1$, $H=-(-p^2)=p^2$

ie, $t(1-t)y''+(\frac{1}{2}-t)y'+p^2y=0$

$c=\frac{1}{2}$, $a+b+1=1$, $ab=-p^2$

$b=-p^2/a$

$a-\frac{p^2}{a}+1=1 \Rightarrow a^2=p^2 \Rightarrow \boxed{a=p \;\&\; b=-p}$

Now $t=\frac{x-1}{-2}$

$\therefore$ $x=1$ corresponds to $t=0$

$\therefore$ The soln is $y=c_1F(a,b,c,x)+c_2x^{1-c}F(a-c+1,b-c+1,2-c,x)$

$\therefore y=c_1F(p,-p,\frac{1}{2},\frac{x-1}{-2})+c_2\left(\frac{x-1}{-2}\right)^{1-\frac{1}{2}}F(p-\frac{1}{2}+1,-p-\frac{1}{2}+1,2-\frac{1}{2},\frac{x-1}{2})$

$\therefore y=c_1F(p,-p,\frac{1}{2},\frac{1-x}{2})+c_2\left(\frac{1-x}{2}\right)^{\frac{1}{2}}F(p+\frac{1}{2},-p+\frac{1}{2},\frac{3}{2},\frac{1-x}{2})$

5 marks.

(*) 2 mark Define Legendre polynomial of degree n.

Legendre polynomials (chapter-6)

(*) 10 mark.

Consider the Legendre's eqn $(1-x^2)y''-2xy'+n(n+1)y=0$ where n is a non-negative integer. Then the Legendre's polynomial is defined as $F(-n, n+1; 1; \frac{1-x}{2})$ and denoted by $P_n(x)$

i.e, $$P_n(x) = 1+\frac{(-n)(n+1)}{1!\cdot 1}\left(\frac{1-x}{2}\right)+\frac{(-n)(-n+1)(n+1)(n+2)}{(2!)^2}\left(\frac{1-x}{2}\right)^2+\cdots$$

$$= 1+\frac{n(n+1)}{(1!)^2}\left(\frac{x-1}{2}\right)+\frac{n(n-1)(n+1)(n+2)}{(2!)^2}\left(\frac{x-1}{2}\right)^2+\cdots+\frac{(2n)!}{(n!)^2}\left(\frac{x-1}{2}\right)^n$$

$$= 1+\frac{n(n+1)}{(1!)^2}\cdot\frac{x-1}{2}+\frac{n(n-1)(n+1)(n+2)}{(2!)^2}\left(\frac{x-1}{2}\right)^2+\cdots+\frac{[n(n-1)\cdots n(n-0)][(n+1)(n+2)\cdots(n+n)]\left(\frac{x-1}{2}\right)^n}{(n!)^2}$$ → ①

Already we've proved the soln for the Legendre's eqn as $y=y_1(x)$ & $y=y_2(x)$ where

$$y_1(x) = a_0\left[1-\frac{p(p+1)}{2!}x^2+\frac{p(p-2)(p+1)(p+3)}{4!}x^4+\cdots\right] \text{ and}$$

$$y_2(x) = a_1\left[x-\frac{(p-1)(p+2)}{3!}x^3+\frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5+\cdots\right]$$

$\therefore P_n(x)$ is of the form

$$P_n(x) = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \cdots$$ [according as n is
or odd

$$\llcorner (2)$$

We've already proved the recursion formula

$$a_{n+2} = \frac{-(p-n)(p+n+1)}{(n+1)(n+2)} a_n$$

Replace p by n and n by k-2

$$a_k = -\frac{(n-k+2)(n+k-2+1)}{(k-2+1)(k-2+2)} a_{k-2}$$

i.e, $$a_k = \frac{-(n-k+2)(n+k-1)}{k(k-1)} a_{k-2}$$

In other words;

$$a_{k-2} = \frac{k(k-1)}{-(n-k+2)(n+k-1)} a_k$$

Now put k=n	put k=n-2
$\therefore a_{n-2} = \frac{-n(n-1)}{2(2n-1)} a_n$	$a_{n-4} = \frac{-(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{n(n-2)(n-1)(n-3)}{2.4(2n-1)(2n-}$

And from ① the coefficient of x^n in $P_n(x) = \frac{(2n)!}{(n!)^2 2^n} = a_n$ [from

Then substituting in ②

$$P_n(x) = \frac{(2n)!}{(n!)^2 2^n}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2^2\, 2!\,(2n-1)(2n-3)}x^{n-4} - \cdots \right.$$

$$\left. + \frac{(-1)^k n(n-1)(n-2)\cdots[n-(2k-1)]}{2^k k!\,(2n-1)(2n-3)\cdots[2n-(2k-1)]}x^{n-2k} + \cdots\right] \quad \text{③}$$

Now take.

$$n(n-1)(n-2)\,[n-(2k-1)] = \frac{n(n-1)(n-2)\cdots(n-2k+1)(n-2k)!}{(n-2k)!} = \frac{n!}{(n-2k)!}$$

$$(2n-1)(2n-3)\cdots[2n-(2k-1)] = \frac{2n(2n-1)(2n-2)\cdots[2n-(2k-1)]}{2n(2n-2)\cdots[2n-(2k-2)]}$$

$$= \frac{2n(2n-1)(2n-2)\cdots[2n-(2k-1)][2n-2k]!}{2^k n(n-1)\cdots[n-(k-1)][2n-2k]!}$$

$$= \frac{(2n)!\,(n-k)!}{2^k n(n-1)\cdots[n-(k-1)][n-k]!\,(2n-2k)!}$$

$$= \frac{(2n)!\,(n-k)!}{2^k n!\,(2n-2k)!}$$

$$\therefore P_n(x) = \frac{(2n)!}{(n!)^2 2^n}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \cdots + \frac{(-1)^k n!}{2^k k!(n-2k)!}\right.$$ $2^k n!(2n-2k)!$

$$\left.\frac{(-1)^k n!}{2^k k!\,(n-2k)!}\,\frac{2^k n!\,(2n-2k)!}{(2n)!\,(n-k)!}\,x^{n-2k} + \cdots\right]$$

∴ co-efficient of $x^{n-2k} = \dfrac{(-1)^k (2n-2k)!\, \cancel{(n!)^2}}{2^n k! (n-2k)! (n-k)!}$

i.e, $$\boxed{P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n k! (n-k)! (n-2k)!} x^{n-2k}}$$

marks

where $\left[\frac{n}{2}\right]$ denotes the greatest integer $\leq \frac{n}{2}$.

⑧. Rodrigue's formula for Legendre's Equation.

10 marks

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n k! (n-k)! (n-2k)!} x^{n-2k}$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2^n k! (n-k)!} \frac{d^n}{dx^n} x^{2n-2k}$$

$$\Big[\because \frac{d^n}{dx^n} x^{2n-2k} = (2n-2k)(2n-2k-1) \cdots (2n-2k-(n-1)) x^{2n-2k-n}$$

$$= (2n-2k)(2n-2k-1) \cdots (n-2k+1) x^{n-2k}$$

$$= \frac{(2n-2k) \cdots (n-2k+1)(n-2k)!}{(n-2k)!} x^{n-2k}$$

$$= \frac{(2n-2k)!}{(n-2k)!} x^{n-2k} \Big].$$

$$\therefore P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-k)!} (x^2)^{n-k} (-1)^k$$

$$\boxed{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n}$$

This is called Rodgrigue's formula.

where $P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$

$\vdots$

Problems

1) Prove $P_n(1) = 1$ and $P_n(-1) = (-1)^n$. [Given that the function on the left side of $\frac{1}{\sqrt{1-2xt+t^2}} = P_0(x)+P_1(x)t+\dots+P_n(x)t^n+\dots$ is called the generating function of the Legendre polynomials]

~~Soln~~

Soln W.K.T $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$

put $x=1$

$$L.H.S = \frac{1}{\sqrt{1-2t+t^2}} = \frac{1}{\sqrt{(1-t)^2}} = \frac{1}{1-t} = 1+t+t^2+\dots$$

$$\therefore 1+t+t^2+\dots+t^n+\dots = P_0(1)+P_1(1)t+\dots+P_n(1)t^n+\dots$$

Equating the coefficient of t^n on both sides

$$\boxed{1 = P_n(1)}$$

III^{ly} Put $x = -1$

$$\text{L.H.S} = \frac{1}{\sqrt{1+2t+t^2}} = \frac{1}{\sqrt{(1+t)^2}} = \frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots$$

$$\therefore 1 - t + t^2 - t^3 + \ldots + (-1)^n t^n = P_0(-1) - P_1(-1) + P_2(-1)t^2 - + P_n(-1)$$

Equating the coefficient of t^n on both sides

$$\boxed{(-1)^n = P_n(-1)}$$

2) Prove $P_{2n+1}(0) = 0$ and $P_{2n}(0) = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n \cdot n!}$ Using the relation $\frac{1}{\sqrt{1-2xt+t^2}} = P_0(x) + P_1(x)t + P_2(x)t^2 + \ldots + P_n(x)t$

Soln: W.K.T $\Rightarrow$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

Put $x = 0$,

$$\text{L.H.S} = \frac{1}{\sqrt{1+t^2}} = (1+t^2)^{-1/2} = 1 - \frac{1}{2}t^2 + \frac{1/2 \cdot 3/2}{2!}(t^2)^3 + \ldots$$

$$+ \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n n!}(t^2)^{n+1} + \ldots$$

$$1 - \frac{t^2}{2} + \frac{1 \cdot 3}{2^2 2!}(t^2)^2 + \ldots + \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n n!}(t^2)^{n+1} + \ldots = P_0(0) + P_1(0)t^2 + \ldots \rightarrow$$

$$P_{2n}(0)t^{2n} + P_{2n+1}(0)t^{2n+1} +$$

Equating coeff of t^{2n+1} on both sides

$$\boxed{0 = P_{2n+1}(0)} \quad \left[\because \text{There is no } (t^{2n+1})^{th} \text{ term}\right]$$

Equating the co-eff of t^{2n} terms on both sides

$$\boxed{\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^n n!} = P_n(0)}$$

marks.

5 marks

Orthogonal properties of the Legendre's polynomials.

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases}$$

prove that $P_0(x), P_1(x), P_2(x) \ldots P_n(x) \ldots$ is a sequence of orthogonal fns in the interval $-1 \leq x \leq 1$

proof: $$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{1}{2^n n!} \int_{-1}^{1} P_m(x) \frac{d^n}{dx^n} (x^2-1)^n dx$$

$$= \frac{1}{2^n n!} \left\{ \left[\frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n P_m(x) \right] - \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \cdot P_m'(x) dx \right\}$$

$$= \frac{(-1)^{1}}{2^n n!} \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n P_m'(x) dx \qquad \because \left[(x^2-1)^{n-1} 2x\right]_{-1}^{1} = 0$$

$$(x^2-1)^{n-1} \cdot x(n-1)(x^2-1)^{n-2} = 0$$

$$= \frac{(-1)^n}{2^n n!} \int_{-1}^{1} (x^2-1)^n P_m^{(n)}(x)\, dx$$

when $m<n$, $P_m^n(x)=0$

$$\boxed{\therefore P_m^n(x)=0 \text{ when } m\neq n}$$

$P_m^n(x)=\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n dx$, if $m<n$

$P_m^n(x)=\frac{d^{m+1}}{dx^{m+n}}(x^2-1)^m$, if $m>$

When $m=n$,

$$\int_{-1}^{1} P_n(x)P_n(x)\,dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1}(x^2-1)^n P_n^n(x)\,dx$$

$$=\frac{(-1)^{2n}(2n)!}{2^n n!\,2^n n!}\int_{-1}^{1}(1-x^2)^n dx$$

$$=\frac{(2n)!}{2^{2n}(n!)^2}\cdot 2\int_{0}^{1}(1-x^2)^n dx$$

$$\int_0^1 (1-x^2)^n dx = \int_0^{\pi/2}\cos^{2n+1}\theta\,d\theta = \frac{2n(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3}$$

$$=\frac{2^n n(n-1)\cdots 1\cdot 2n(2n-2)\cdots 2}{(2n+1)(2n)(2n-1)(2n-2)\cdots 3\cdot 2}$$

$$=\frac{2^{2n}(n!)^2}{(2n+1)!}$$

$$\int_{-1}^{1}P_n(x)P_n(x)\,dx = \frac{(2n)!}{2^{2n}(n!)^2}\cdot\frac{2^{2n}(n!)^2}{(2n+1)!} = \frac{2}{2n+1}$$

c) $$\boxed{\int_{-1}^{1}P_n^2(x)\,dx = \frac{2}{2n+1}}$$

to more.

prove $\int_{-1}^{1}(P_n')^2 dx = n(n+1)$.

1) Show that any polynomial p(x) of degree k has an expansion of the form $p(x) = \sum_{n=0}^{k} a_n P_n(x)$.

Soln: $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$

$\boxed{1 = P_0(x)}$, $\boxed{x = P_1(x)}$, $x^2 = \frac{2}{3} P_2(x) + \frac{1}{3} \Rightarrow x^2 = \frac{2}{3} P_2(x) + \frac{P_0}{3}(x)$

$x^3 = \frac{2P_3(x) + 3x}{5} \Rightarrow x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)$

Let p(x) be a polynomial of degree 3

Then $p(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$

$= b_0 P_0 x + b_1 P_1(x) + b_2 \left[\frac{1}{3} P_0(x) + \frac{2}{3} P_2(x)\right] +$

$b_3 \left[\frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)\right]$

$= P_0(x)\left[b_0 + \frac{b_2}{3}\right] + P_1(x)\left[b_1 + \frac{3}{5} b_3\right] + P_2(x)\left[\frac{2}{3} b_2\right]$

$+ P_3(x)\left[\frac{2}{5} b_3\right]$

$$= a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x)$$

$$= \sum_{k=0}^{3} a_k P_k(x)$$

$|||^{ly}$ a polynomial $p(x)$ of degree n can be writt

$$p(x) = \sum_{k=0}^{n} a_k P_k(x)$$

Legendre's series :

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \text{ where } a_n = (n+\tfrac{1}{2}) \int_{-1}^{1} f(x) P_n(x) dx$$

Proof :

Let $f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots + a_n P_n(x) + \cdots$ ——①

Multiplying both sides of ① by $P_n(x)$ & $\int$ing b/w -1 to 1

$$\int_{-1}^{1} f(x) P_n(x) dx = \int_{-1}^{1} a_n P_n^2(x) dx = \frac{2a_n}{2n+1}$$

$$\therefore a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx.$$

∴ Eqn. ① is called Legendre's. series.

5 marks

Problems.

1) Find the first 3 terms of the Legendre's series

where $f(x) = \begin{cases} 0 & \text{if } -1 \le x < 0 \\ x & \text{if } 0 \le x \le 1 \end{cases}$

Soln :

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$$

Here

$$a_n = \frac{2n+1}{2} \int_{-1}^{0} f(x) P_n(x) dx = 0 \quad [\because f(x) = 0 \text{ if } -1 \le x < 0]$$

$$\therefore \quad a_n = \frac{2n+1}{2} \int_{0}^{1} f(x) P_n(x) dx$$

Put n=0.

$$a_0 = \frac{1}{2} \int_{0}^{1} f(x) P_0(x) dx = \frac{1}{2} \int_{0}^{1} x(1) dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{4}$$

Put n=1

$$a_1 = \frac{3}{2} \int_{0}^{1} x \cdot P_1(x) dx = \frac{3}{2} \int_{0}^{1} x(x) dx = \frac{3}{2}\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{2}$$

Put n=2

$$a_2 = \frac{5}{2} \int_{0}^{1} x \cdot P_2(x) dx = \frac{5}{2} \int_{0}^{1} x\left[\frac{1}{2} 3x^2 - \frac{1}{2}\right] dx = \frac{5}{4} \int_{0}^{1} (3x^3 - x) dx$$

$$= \frac{5}{4}\left[\frac{3}{4} - \frac{1}{2}\right] = \frac{5}{16}$$

$$f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$$

$$\therefore f(x) = \frac{1}{4} P_0(x) + \frac{1}{2} P_1(x) + \frac{5}{16} P_2(x) //.$$

Q. Find the first 3 terms of Legendre's series where $f(x) = e^x$.

Soln:

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)\,P_n(x)\,dx = \frac{2n+1}{2}\int_{-1}^{1} e^x P_n(x)\,dx$$

Put $n=0$

$$a_0 = \frac{1}{2}\int_{-1}^{1} e^x P_0(x)\,dx = \frac{1}{2}\int_{-1}^{1} e^x \cdot 1\,dx = \frac{1}{2}\left[e^x\right]_{-1}^{1} = \frac{1}{2}\left[e - e^{-1}\right]$$

Put $n=1$

$$a_1 = \frac{3}{2}\int_{-1}^{1} e^x P_1(x)\,dx = \frac{3}{2}\int_{-1}^{1} x e^x\,dx$$

$$= \frac{3}{2}\left\{\left[xe^x\right]_{-1}^{1} - \int_{-1}^{1} e^x\,dx\right\} = \frac{3}{2}\left[(e+e^{-1}) - (e-e^{-1})\right]$$

$$= \frac{3}{2}\cdot 2e^{-1} = 3e^{-1}$$

Put $n=2$

$$a_2 = \frac{5}{2}\int_{-1}^{1} e^x P_2(x)\,dx = \frac{5}{2}\int_{-1}^{1} e^x\left[\frac{1}{2}(3x^2-1)\right]dx$$

$$= \frac{5}{4}\int_{-1}^{1} (3x^2e^x - e^x)\,dx$$

$$= \frac{5}{4}\left[3\int_{-1}^{1} x^2 e^x\,dx - \int_{-1}^{1} e^x\,dx\right]$$

$$= \frac{5}{4}\left[3\left\{(x^2e^x)_{-1}^{1} - 2\int_{-1}^{1} e^x x\,dx\right\} - \left[e^x\right]_{-1}^{1}\right]$$

$$= \frac{5}{4}\left[3\left\{(e-e^{-1}) - 2(2e^{-1})\right\} - e + e^{-1}\right]$$

$$= \frac{5}{4}\left[3e - 3e^{-1} - 12e^{-1} - e + e^{-1}\right]$$

$$= \frac{5}{4}\left[2e - 14e^{-1}\right]$$

$$= \frac{2.5}{4}\left[e - 7e^{-1}\right] = \frac{1}{2}\left[5e - 35e^{-1}\right]$$

$$f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$$

$$= \frac{1}{2}(e - e^{-1}) P_0(x) + 3e^{-1} P_1(x) + \frac{1}{2}(5e - 35e^{-1}) P_2(x).$$

Least square Approximation.

$$I = \int_{-1}^{1} \left[f(x) - P(x)\right]^2 dx \quad \because P(x) \text{ is a polynomial of degree } n$$

$$P(x) = b_0 P_0(x) + b_1 P_1(x) + \cdots + b_n P_n(x) = \sum_{k=0}^{n} b_k P_k(x)$$

We can write $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$

where $a_k = (k + \frac{1}{2}) \int_{-1}^{1} f(x) P_k(x)\, dx.$

$$I = \int_{-1}^{1} \left[f(x) - \sum_{k=0}^{n} b_k P_k(x)\right]^2 dx = \int_{-1}^{1} f(x)^2 dx + \sum_{k=0}^{n} bk^2 \int_{-1}^{1} P_k^2(x)\, dx$$

$$- \sum_{k=0}^{n} 2b_k \int f(x) P_k(x)\, dx.$$

$$= \int_{-1}^{1} f(x)^2 dx + \sum_{k=0}^{n} bk^2 \frac{2}{2k+1} - \sum_{k=0}^{n} 2b_k \frac{2a_k}{2k+1}$$

$$= \int_{-1}^{1} f(x)^2 dx = \sum_{k=0}^{n} \frac{2}{2k+1}(b_k - a_k)^2 - \sum_{k=0}^{n} a_k^2 \cdot \frac{2}{2k+1}$$

I is minimum when $b_k = a_k$.

Bessel Functions. (or) $J_p(x)$ Bessel's function

Bessel fcn

$$x^2 y'' + xy' + (x^2 - p^2) y = 0 \quad — ①$$

$y = x^m \sum a_n x^n$ be the soln.

$m(m-1) + mP_0 + q_0 = 0$

$x\,P(x) = x \cdot \frac{1}{x} = 1 \Rightarrow \boxed{\therefore P_0 = 1}$

$x^2 Q(x) = \frac{x^2 (x^2 - p^2)}{x^2} = x^2 - p^2 \quad \boxed{\therefore q_0 = -p^2}$

$\Rightarrow x^2 = p^2$

$m(m-1) + m - p^2 = 0 \Rightarrow m^2 = p^2 \Rightarrow \boxed{m = \pm P}$

when $m = P$,

$$\therefore y = x^P \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+P}$$

$$y' = \sum_{n=1}^{\infty} (n+p)\, a_n\, x^{n+p-1}$$

$$y'' = \sum_{n=2}^{\infty} (n+p)(n+p-1)\, a_n\, x^{n+p}$$

$$x^2 y'' = \sum_{n=0}^{\infty} (n+p)(n+p-1)\, a_n\, x^{n+p}$$

$$xy' = \sum_{n=0}^{\infty} (n+p)\, a_n\, x^{n+p}$$

$$x^2 y = \sum_{n=0}^{\infty} a_n x^{n+p+2} = \sum_{n=0}^{\infty} a_{n-2}\, x^{n+p}$$

$$-p^2 y = \sum_{n=0}^{\infty} -p^2 a_n x^{n+p}$$

substituting these in ① & collect the coeff of x^{n+p}

$$\sum_{n=0}^{\infty} x^{n+p}\left[(n+p)(n+p-1)a_n + (n+p)a_n + a_{n-2} - p^2 a_n\right] = 0$$

(ie) $$\sum_{n=0}^{\infty} x^{n+p}\left[\{a_n(n+p)(n+p-1+1)\} + a_{n-2} - p^2 a_n\right] = 0$$

(ie) $$\sum_{n=0}^{\infty} x^{n+p}\left[a_n(n+p)^2 + a_{n-2} - p^2 a_n\right] = 0$$

(ie) $$\sum_{n=0}^{\infty} x^{n+p}\left[n(n+2p)\,a_n + a_{n-2}\right] = 0$$

$$a_n = \frac{-a_{n-2}}{n(n+2p)}$$

Assume $a_0 \neq 0$

put $\underline{n=1}$, $a_1 = 0$

$\underline{n=2}$, $a_2 = \frac{-a_0}{2(2p+2)} = \frac{-a_0}{2^2(p+1)}$

$\underline{n=3}$, $a_3 = 0$

$\underline{n=4}$, $a_4 = \frac{-a_2}{4(2p+4)} = \frac{a_0}{2\cdot 4(2p+2)(2p+4)} = \frac{a_0}{2^4 \cdot 2!\,(p+1)(p+2)}$

$\underline{n=5}$, $a_5 = 0$

$n=6$, $a_6 = \frac{a_4}{6(2p+6)} = \frac{a_0}{2\cdot 4\cdot 6(2p+2)(2p+4)(2p+6)} = \frac{a_0}{2^6 \cdot 3!\,(p+1)(p+2)(p+3)}$

$$\therefore y = a_0 x^p \left[1 - \frac{x^2}{2^2 (p+1)} + \frac{x^4}{2^4 2!(p+1)(p+2)} - \frac{x^6}{2^6 . 3!(p+1)(p+2)(p+3)} + \dots + \frac{(-1)^n x^{2n}}{2^{2n} n!(p+1)(p+2)\dots(p+n)} \right]$$

$$J_p(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p}}{2^{2n} n!(p+1)(p+2)\dots(p+n)}$$

Putting $a_0 = \frac{1}{2^p p!}$

$$\boxed{J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n!(n+p)!}}$$

$J_p(x)$ is known as Bessel function of first kind of order p.

when $m = -p$

$y_2 = J_{-p}(x)$ is the 2^{nd} soln.

$\therefore$ The general soln is $y = C_1 J_p(x) + C_2 J_{-p}(x)$

But $J_{-p}(x)$ is not an independent soln.

$\because J_{-p}(x) = (-1)^p J_p(x)$ where p is an integer.

when p is an integer say $m \geq 0$.

$$J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n-m}}{n!(n-m)!} = \sum_{n=m}^{\infty} \frac{(-1)^n (x/2)^{2n-m}}{n!(n-m)!}$$ $\left[\because\right.$ for $n=0,1,2.m$ the same va

put n = n+m

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+m} (x/2)^{2n+m}}{(m+n)!\, n!} = (-1)^m \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+m}}{n!\,(n+m)!} = (-1)^m J_m(x)$$

when p is not an integer we get the 2nd independent soln,

$Y_p(x)$ given by

$$\boxed{Y_p(x) = \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi}}$$

when p is not an integer

$Y_p(x)$ is known as the Bessel function of the 2nd kind

Problems.

$\frac{d}{dx}\left[x J_1(x)\right] = x J_0(x)$ - Prove.

soln: $J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\,(n+p)!}$

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+1}}{n!\,(n+1)!} \Rightarrow x J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n+1}\, n!\,(n+1)!}$$

$$\frac{d}{dx}\left[x J_1(x)\right] = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2)\, x^{2n+1}}{2^{2n+1}\, n!\,(n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2(n+1)\, x^{2n+1}}{2^{2n+1}\, n!\,(n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)\, x^{2n+1}}{2^{2n}\, n!\, n!\,(n+1)} = x \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\, n!}$$

$$= x J_0(x)$$

Hence proved.

2) Prove $J_0'(x) = -J_1(x)$.

Gamma Function

$$\boxed{\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt, \quad p > 0}$$

2 mark

i) $\Gamma(p+1) = p!$

$$\Gamma(p+1) = \int_0^\infty t^p e^{-t}\, dt = \left[-e^{-t} t^p\right]_0^\infty + \int_0^\infty e^{-t} p\, t^{p-1} dt$$

$$= 0 + p\Gamma_p$$

$$= p\Gamma_p = p(p-1)\Gamma_{p-1}$$

$$= p(p-1)\cdots 1\,\Gamma_1 = p!$$

$$\left[\because \Gamma_1 = \int_0^\infty e^{-t} dt = \left[-e^{-t}\right]_0^\infty = 1\right] \Rightarrow \boxed{\Gamma_1 = 1}$$

ii) $\Gamma_{1/2} = \sqrt{\pi}$

$$\Gamma_{1/2} = \int_0^\infty t^{-1/2} e^{-t}\, dt$$

put $t = x^2$

$$\therefore \Gamma_{1/2} = \int_0^\infty \frac{1}{x} e^{-x^2} 2x\,dx = 2\int_0^\infty e^{-x^2}\, dx = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

iii) $(n+1/2)! = \frac{(2n+1)!}{2^{2n+1}\, n!}\sqrt{\pi}$

W.K.T $\Gamma_{p+1} = p\,\Gamma_p$ (or) $\Gamma_{p+1} = p!$

$$\Gamma_{n+3/2} = \Gamma(n+1/2+1) = (n+1/2)!$$

But $\Gamma_{n+3/2} = n+1/2\,\Gamma_{n+1/2} = n+1/2\,(n-1/2)\,\Gamma_{n-1/2}$

$$= (n+1/2)(n-1/2)(n-3/2)\,\Gamma_{n-3/2}$$

$$= (n+\tfrac{1}{2})(n-\tfrac{1}{2})(n-\tfrac{3}{2})\cdots\tfrac{1}{2}\,\Gamma_{1/2}$$

$$= \frac{2n+1}{2}\cdot\frac{2n-1}{2}\cdot\frac{(2n-3)}{2}\cdots\frac{1}{2}\sqrt{\pi}$$

$$= \frac{(2n+1)(2n)(2n-1)(2n-3)\cdots 3\cdot 2\cdot 1}{2^{n+1}\, 2n(2n-2)(2n-4)\cdots 2}\sqrt{\pi}$$

$$= \frac{(2n+1)!}{2^{2n+1}\, n!}\sqrt{\pi}$$

iv) $(n-\frac{1}{2})! = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi}$

$\Gamma n+\frac{1}{2} = (n-\frac{1}{2})!$

$\Gamma n-\frac{1}{2} = n-\frac{1}{2}\,\Gamma(n-\frac{1}{2}) = (n-\frac{1}{2})(n-\frac{3}{2})\,\Gamma(n-\frac{3}{2})$

$= (n-\frac{1}{2})(n-\frac{3}{2})\cdots\frac{1}{2}\,\Gamma\frac{1}{2}$

$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{1}{2}\sqrt{\pi}$

$= \frac{2n(2n-1)(2n-2)(2n-3)\cdots3\cdot2\cdot1}{2^n\,2n(2n-2)\cdots2}\sqrt{\pi}$

$= \frac{2n!\sqrt{\pi}}{2^n\,2^n(n-1)n\cdots1} = \frac{2n!\sqrt{\pi}}{2^{2n}n!}$

5 marks.

1) prove $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ → 2 marks.

Soln: $J_p(x) = \sum_{n=0}^{\infty}\frac{(-1)^n (x/2)^{2n+p}}{n!(n+p)!}$

$J_{1/2}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n (x/2)^{2n+1/2}}{n!(n+1/2)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1/2}}{2^{2n+1/2}\,n!}\cdot\frac{2^{2n+1}\,n!}{(2n+1)!\sqrt{\pi}}$

$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1/2}\,2^{1/2}}{(2n+1)!\sqrt{\pi}} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{x}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$= \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] = \sqrt{\frac{2}{\pi x}}\sin x.$

5 marks.

Prove $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ → 2 marks

Soln: $J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p}}{n!\,(n+p)!}$.

$$J_{-1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n-1/2}}{n!\,(n-1/2)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1/2}}{2^{2n-1/2}} \cdot \frac{2^{2n} \cdot n!}{(2n)!\,\sqrt{\pi}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1/2}\, 2^{1/2}}{(2n)!\,\sqrt{\pi}} = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$$

$$= \sqrt{\frac{2}{\pi x}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}} \cos x .$$

Properties of Bessel functions.

5 mark

(i) $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$ (or) $\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$. → 2 marks

(ii) $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{p} J_{p+1}(x)$. → 2 marks

To prove (i): $\frac{d}{dx}\left[x^p J_p(x)\right] = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2p}}{2^{2n+p}\, n!\,(n+p)!}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, 2(n+p)\, x^{2n+2p-1}}{2^{2n+p}\, n!\,(n+p)(n+p-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^p (x/2)^{2n+p-1}}{n!\,(n+p-1)!}$$

$$= x^p \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p-1}}{n!\,(n+p-1)!}$$

$$= x^p J_{p-1}(x)$$

To prove (ii) :

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+p} n!(n+p)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2n (x)^{2n-1}}{2^{2n+p} n!\ (n+p)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n x^{-p} x^{p} x^{2n-1}}{2^{2n+p-1} (n-1)!\ (n+p)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{-p} (x/2)^{2n}}{(n-1)!\ (n+p}$$

$$= x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+p-1}}{(n-1)!\ (n+p)!} = x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (x/2)^{2n+p+1}}{n!\ (n+p+1)!} [\because$$

$$= -x^{-p} J_{p+1}(x)$$

$$= -x^{-p} J_{p+1}(x).$$

iii) $\int x^p J_{p-1}(x)\,dx = x^p J_p(x) + c.$

iv) $\int x^{-p} J_{p+1}(x)\,dx = -x^{-p} J_p(x) + c$

marks.

Ⓧ v) $2J_p'(x) = J_{p-1}(x) - J_{p+1}(x)$

vi) $\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$

from (i) $x^p J_p'(x) + px^{p-1} J_p(x) = x^p J_{p-1}(x)$

(i.e) $J_p'(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$ —③

from (ii) $x^{-p} J_p'(x) - px^{-p-1} J_p(x) = -x^{-p} J_{p+1}(x)$

(ie) $J_p'(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x) \longrightarrow$ ④

③ + ④ gives $2J_p'(x) = J_{p-1}(x) - J_{p+1}(x) \longrightarrow$ property no(v)

③ - ④ gives $2\frac{p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x) \longrightarrow$ property no(vi)

5mark

Q. Orthogonal properties of Bessel functions.

Bessel fns.

St

$$\int_0^1 x J_p(\lambda_m x) J_p(\lambda_n x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{1}{2} J_{p+1}(\lambda_n)^2 & \text{if } m = n \end{cases}$$

where λ_m & λ_n are positive zeros of $J_p(x)$.

Proof: ∴ $y = J_p(x)$ is a soln of $x^2y'' + xy' + (x^2 - p^2)y = 0$ ⌞①

$y = J_p(ax)$ is a soln of $x^2y'' + xy' + (a^2x^2 - p^2)y = 0$ ⌞②

Put $z = ax$ then $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = ay'$, where $y' = \frac{dy}{dz}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(ay') = \frac{d}{dz}(ay')\frac{dz}{dx} = ay''$$

Sub in ② $\frac{z^2}{a^2} \cdot a^2y'' + \frac{z}{a} ay' + (z^2 - p^2)y = 0$

$z^2y'' + zy' + (z^2 - p^2)y = 0$ which is true by ①

uy = $J_p(z)$ is soln of

(ie) $u = J_p(ax)$ is a soln of $x^2u'' + xu' + (a^2x^2 - p^2)u = 0$

$u = J_p(ax)$ is a soln of $u'' + \frac{u'}{x} + \left(a^2 - \frac{p^2}{x^2}\right)u = 0$ ⌞③

111'y

$v = J_p(bx)$ is a soln of $v'' + \frac{v'}{x} + \left(b^2 - \frac{p^2}{x^2}\right) v = 0$ ← ④

③ × v - ④ × u gives $(u''v - v''u) + \frac{1}{x}(u'v - v'u) + (a^2-b^2)u$

ie $x(u''v - v''u) + (-u'v - v'u) + (a^2-b^2)\, uvx = 0$

$$\frac{d}{dx}\left[x(u'v - v'u)\right] + (a^2-b^2)\, uvx = 0$$

$(a^2-b^2)\int_0^1 uvx\,dx = \left[-x(u'v' - v'u)\right]_0^1$ where $u = u(x)$ & $v = v$

$$= -1\left[u'(1)v(1) - v'(1)u(1)\right] - 0$$

$$= -\left[aJ_p'(a)\,J_p(b) - bJ_p'(b)\,J_p(a)\right]$$

$$\therefore \int_0^1 x\,u(x)\,v(x)\,dx = \frac{-\left[aJ_p'(a)\,J_p(b) - bJ_p'(b)\,J_p(a)\right]}{a^2-b^2}$$

If $a = \lambda_m$ and $b = \lambda_n$ are distinct +ve zeros of $J_p(x)$ where $J_p(a) = 0$ & $J_p(b) = 0$ then $J_p(\lambda_m) = J_p(\lambda_n)$

$$\therefore \int_0^1 x\,J_p(\lambda_m x)\,J_p(\lambda_n x)\,dx = \frac{-\left[aJ_p'(a)\,J_p(\lambda_n) - bJ_p'(b)\,J_p(\lambda_m)\right]}{a^2-b^2}$$

$= 0$ when $m \neq n$,,

Case (2): when $m = n$,

③ × $2x^2u'$ gives ⇒ $2x^2u'u'' + 2xu'^2 + 2a^2x^2u'u - 2p^2u'u = 0$

$$\frac{d}{dx}(x^2u'^2) + \frac{d}{dx}(a^2x^2u^2) - 2a^2xu^2 - \frac{d}{dx}(p^2u^2) = 0$$

(ie) $2a^2xu^2 = \frac{d}{dx}(x^2u'^2 + a^2x^2u^2 - p^2u^2)$

(ie) $2a^2\int_0^1 xu^2dx = \left[x^2u'^2 + a^2x^2u^2 - p^2u^2\right]_0^1$

$= 1.a^2J_p'^2(a) + a^2J_p^2(a) - p^2J_p^2(a) + 0 + 0 + p^2J_p^2(0)$

$= a^2J_p'^2(a)$ [$\because J_p(a)=0$ being a zero of $J_p(x)$ & $J_p(0)=0$]

$$\int_0^1 xJ_p^2(\lambda_n x)\,dx = \frac{1}{2}J_p'(a)^2$$

$$\frac{d}{dx}(x^{-p}J_p(x)) = -x^{-p}J_{p+1}(x)$$

(ie) $J_p'(x) - \frac{p}{x} = -J_{p+1}(x)$

(ie) $J_p'(a) - \frac{p}{a}J_p(a) = -J_{p+1}(a)$ (ie) $J_p'(a) = -J_{p+1}(a)$

$$\boxed{\therefore \int_0^1 xJ_p^2(\lambda_n x)\,dx = \frac{1}{2}J_{p+1}^2(a)}$$

Bessel series:

If $f(x)$ is defined in the interval $0\le x\le 1$ and λ_n are +ve zero of $J_p(x)$, $p\geq 0$ then

$$f(x) = \sum_{n=}^{\infty} a_nJ_p(\lambda_n x) = a_1J_p(\lambda_1 x) + a_2J_p(\lambda_2 x) + \dots + a_nJ_p(\lambda_n x)$$

where $a_n = \frac{2}{J_{p+1}^2(\lambda_n)}\int_0^1 x f(x) J_p(\lambda_n x)\,dx$

Multiplying both sides by $x J_p(\lambda_n x)$ & $\int$ing b/w 0,

$$\therefore \int_0^1 x f(x). J_p(\lambda_n x) dx = a_n \int_0^1 x J_p^2(\lambda_n x) dx$$

$$= a_n \frac{1}{2} J_{p+1}^2(\lambda_n)$$

$$\boxed{\therefore a_n = \frac{2}{J_{p+1}^2(\lambda_n)} \int_0^1 x f(x) J_p(\lambda_n x) dx}$$

Problems

1) Compute the Bessel series for the functions $f(x)=1$ for the interval $0 \le x \le 1$ in terms of the functions $(\lambda_n x)$ where λ_n's are the +ve zeros of J_0

Soln: $\int x^p J_{p-1}(x) dx = x^p J_p(x) + c$

The Bessel series for $f(x)$ is Given by $\int x J_0(x) dx = x J_1(x)+$

$$f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x) \text{ where } a_n = \frac{2}{J_{p+1}^2(\lambda_n)} \int_0^1 x f(x) J_p(\lambda_n x$$

Given $J_p(\lambda_n x) = J_0(\lambda_n x)$

$$a_n = \frac{2}{J_1^2(\lambda_n)} \int_0^1 x.1.J_0(\lambda_n x) dx = \frac{2}{J_1^2(\lambda_n)} \left[\frac{1}{\lambda_n} x J_1(\lambda_n x)\right]_0^1$$

$$= \frac{2}{J_1^2(\lambda_n)} \cdot \frac{1}{\lambda_n} [1. J_1(\lambda_n) - 0] = \frac{2}{\lambda_n J_1(\lambda_n)}$$

$$\therefore 1 = \sum_{n=0}^{\infty} \frac{2}{\lambda_n J_1(\lambda_n)} J_0(\lambda_n x) //.$$

If $f(x) = \begin{cases} 1, & 0 \leq x \leq 1/2 \\ 1/2, & x = 1/2 \\ 0, & 1/2 < x \leq 1 \end{cases}$ S.T $\sum_{n=1}^{\infty} \frac{J_1(\lambda_n/2)}{\lambda_n J_1(\lambda_n)^2} J_0(\lambda_n x) = f(x)$

Bessel Expansion theorem [Proof not include].

If $f(x)$ & $f'(x)$ have atmost a finite number of jump discontinuities on the interval $0 \leq x \leq 1$ & if $0 < x < 1$ then the Bessel series $f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x)$ converges to $f(x)$ when x is a paint of continuity of the function & converges to $\frac{1}{2}[f(x-) + f(x+)]$ when x is a point of discontinuity.

Problems

1) prove $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right)$

Soln: W.K.T $\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$

put $p = \frac{1}{2}$

$$\frac{1}{x} J_{1/2}(x) = J_{-1/2}(x) + J_{3/2}(x) \quad (\text{ie}) \quad J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$$

$$J_{3/2}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}} \left[\frac{\sin x}{x} - \cos x\right].$$

2) Prove $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x}{x^2} - \frac{3\cos x}{x} - \sin x\right]$

Soln: W.K.T $\frac{2p}{x} J_p(x) = J_{p-1}(x) + J_{p+1}(x)$

Put $p = 3/2$

$\frac{3}{x} J_{3/2}(x) = J_{1/2}(x) + J_{5/2}(x)$ (ie) $J_{5/2}(x) = \frac{3}{x} J_{3/2}(x) - J_{1/2}$

$$J_{5/2}(x) = \frac{3}{x}\left[\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)\right] - \sqrt{\frac{2}{\pi x}}\sin x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x}{x^2} - \frac{3\cos x}{x} - \sin x\right].$$

—x—

3) ||ly $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[-\frac{\cos x}{x} - \sin x\right]$

$$J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3\cos x}{x^2} + \frac{3\sin x}{x} - \cos x\right].$$

Unit-3

Linear Systems

Let $a_i(t)$, $b_i(t)$, $f_i(t)$, $i = 1, 2$ are continuous on $[a, b]$ then,

→ non-homogeneous

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y + f_1(t) \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y + f_2(t) \end{aligned}\right\} \quad - ①$$

ated
marks

$$\left[\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y \end{aligned}\right.$$

Ex. If $f_1(t)$ and $f_2(t)$ are identically zero then ① is called the homogeneous linear system. Otherwise it is called the Non-homogeneous linear system.

The above system has a soln of the form

$x = x(t)$ &

$y = y(t)$

Ex: Consider the linear system $\frac{dx}{dt} = 4x - y$ & $\frac{dy}{dt} = 2x + y$.

This has a soln $\begin{cases} x = e^{3t} \\ y = e^{3t} \end{cases}$ and $\begin{cases} x = e^{2x} \\ y = 2e^{2t} \end{cases}$

Theorem-1: Existence and Uniqueness theorem for th linear system of eqn

IMP.
2mark

If $a_i(t)$, $b_i(t)$ and $f_i(t)$, $i=1,2$ are continuous on $[a,b]$ and if t_0 is any point of the interval $[a,$ and if x_0 and y_0 are any numbers, then

$$\frac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t)$$

$$\frac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t)$$

has one and only one solution

$$x = x(t)$$
$$y = y(t) \quad \text{on } [a,b] \ni x(t_0)=x_0 \ \& \ y(t_0)=$$

Theorem-2:

If the homogeneous system $\frac{dx}{dt} = a_1(t)x + b_1(t)y$

$$\frac{dy}{dt} = a_2(t)x + b_2(t)y$$

has 2 solns $\begin{matrix} x = x_1(t) \\ y = y_1(t) \end{matrix}$ and $\begin{matrix} x = x_2(t) \\ y = y_2(t) \end{matrix}$

on $[a,b]$, then

$$x = c_1x_1(t) + c_2x_2(t)$$
$$y = c_1y_1(t) + c_2y_2(t)$$

is also a soln on $[a,b]$ for any constants c_1 and c

Theorem-3 :

If the solns $x = x_1(t), y = y_1(t)$ and $x = x_2(t), y = y_2(t)$

of the homogeneous system
$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y \end{aligned}\right\} \to ①$$

has a wronskian $W(t)$ does not vanish on $[a,b]$ then
$$\left.\begin{aligned} x &= c_1 x_1(t) + c_2 x_2(t) \\ y &= c_1 y_1(t) + c_2 y_2(t) \end{aligned}\right\} \to ②$$ is the general solution.

proof: By uniqueness thm ② will be the general soln of ①, if ∃ a point t_0 in $[a,b]$: $x(t_0) = x_0, y(t_0) = y_0$

(i.e.,) In otherwords the system
$$c_1 x_1(t_0) + c_2 x_2(t_0) = x_0$$
$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0$$ is solvable for c_1 & c_2

for each t_0.

By elementary theory of determinants this is possible if $W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix}$ does not vanish on the closed interval $[a,b]$.

Theorem-4 :

5 marks

If $W(t)$ is the wronskian of the two solutions $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$

In simple $x_1(t) = x_2(t) = x(t)$ & $y_1(t) = y_2(t) = y(t)$

of the homogeneous system $\frac{dx}{dt} = a_1(t)x + b_1(t)y$

$$\frac{dy}{dt} = a_2(t)x + b_2(t)y,$$

then $W(t)$ is either identically zero or nowhere zero on $[a,b]$.

Proof: Let $W(t)$ be not identically zero

$$W(t) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1$$

$$\frac{dW}{dt} = x_1\frac{dy_2}{dt} + y_2\frac{dx_1}{dt} - x_2\frac{dy_1}{dt} - y_1\frac{dx_2}{dt}$$

$$= x_1(\cancel{a_2x_2} + b_2y_2) + y_2(a_1x_1 + \cancel{b_1y_1}) - x_2(\cancel{a_2x_1} + b_2y_1)$$

$$- y_1(a_1x_2 + \cancel{b_1y_2})$$

$$= x_1y_2(b_2 + a_1) - x_2y_1(b_2 + a_1)$$

$$= (b_2 + a_1)(x_1y_2 - x_2y_1)$$

$$\therefore \frac{dW}{dt} = [a_1(t) + b_2(t)]\,W(t)$$

Integrating on both sides,

$$\int \frac{dW}{W} = \int [a_1(t) + b_2(t)]\, dt$$

$$\Rightarrow \log W = \int [a_1(t) + b_2(t)]\, dt$$

i.e., $W = c e^{\int [a_1(t) + b_2(t)] dt}$ where c is some constants

∵ The exponential factor never vanishes, the wronskian never vanishes.

Theorem-5:

If the two solns $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ of the homogeneous system

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1(t)x + b_1(t)y \\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y \end{aligned}\right\} \to ①$$

are linearly independent on $[a,b]$, then

$$\left.\begin{aligned} x &= c_1 x_1(t) + c_2 x_2(t) \\ y &= c_1 y_1(t) + c_2 y_2(t) \end{aligned}\right\} \to ②$$

is the general soln of ① on $[a,b]$.

Proof:

We've to prove the 2 solns in ① are linearly independent if ② is not the general soln.

(ie) T.P the solns are linearly dep if $W(t) = 0$ [by thm.3]

(ie) T.P the solns are linearly dep if $W(t)$ is identically zero $\Rightarrow W(t) = 0$ [by thm 4]

If part :

Let the solns in ③ are linearly dependent.

$\therefore x_1 = k x_2(t)$ & $y_1 = k y_2(t)$ for some constant k

$$\therefore W(t) = \begin{vmatrix} kx_2 & x_2 \\ ky_2 & y_2 \end{vmatrix} = kx_2y_2 - ky_2x_2 = 0$$

$$\therefore W(t) = 0$$

Only if part :

Let $W(t)$ be identically zero.

We shall prove the solns are linearly dependent.

Let t_0 be a fixed point in $[a, b]$

Consider the system $c_1x_1(t_0) + c_2x_2(t_0) = 0$ and

$$c_1y_1(t_0) + c_2y_2(t_0) = 0$$

$\because W(t) = 0$ we've a soln c_1 & c_2 in which these numbers are not both zero.

Hence the soln of the homogeneous system is given by

$$\left.\begin{aligned} x &= c_1x_1(t) + c_2x_2(t) \\ y &= c_1y_1(t) + c_2y_2(t) \end{aligned}\right\} \rightarrow ②$$

equals the trivial soln at t_0.

It now follows from the uniqueness part of uniqueness thm that ② must equal the trivial soln throughout the interval $[a,b]$

$$\text{i.e., } c_1 x_1(t) + c_2 x_2(t) = 0 \text{ \& }$$
$$c_1 y_1(t) + c_2 y_2(t) = 0$$

is true throughout the interval where atleast one of c_i, $i=1,2$ is not equal to zero.

Hence the solns are linearly dependent.

Theorem-6 :

If the 2 solns $x = x_1(t)$, $y = y_1(t)$ and $x = x_2(t)$, $y = y_2(t)$ of the homogeneous system $\frac{dx}{dt} = a_1(t)x + b_1(t)y$, $\frac{dy}{dt} = a_2(t)x + b_2(t)y$ are linearly independent on $[a,b]$ and if $x = x_p(t)$, $y = y_p(t)$ is a particular soln of $\frac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t)$ and $\frac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t)$ on this interval then $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$ & $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$ is the general soln of the homogeneous system on the $[a,b]$.

Proof : Let $x = x(t)$, $y = y(t)$ be an arbitrary soln of the non-homogenous system.

$$\left.\begin{aligned}\therefore \frac{dx}{dt} &= a_1(t)x + b_1(t)y + f_1(t)\\ \frac{dy}{dt} &= a_2(t)x + b_2(t)y + f_2(t)\end{aligned}\right\} - ①$$

$\because x = x_p(t)$ & $y = y_p(t)$ is a particular soln of the non-homogeneous system

$$\left.\begin{aligned}\frac{dx_p}{dt} &= a_1(t)x_p + b_1(t)y_p + f_1(t)\\ \frac{dy_p}{dt} &= a_2(t)x_p + b_2(t)y_p + f_2(t)\end{aligned}\right\}$$

① - ② gives,

$$\frac{d(x-x_p)}{dt} = a_1(t)(x-x_p) + b_1(t)(y-y_p)$$

$$\frac{d(y-y_p)}{dt} = a_2(t)(x-x_p) + b_2(t)(y-y_p)$$

$\therefore \left.\begin{aligned}x-x_p\\ y-y_p\end{aligned}\right\} - ③$ are the solns of non-homogeneous sys

Given $x = x_1(t)$ & $x = x_2(t)$,
$y = y_1(t)$ & $y = y_2(t)$ are linearly independ

$\therefore$ By a known thm [thm 5] $x = c_1 x_1(t) + c_2 x_2(t)$
$y = c_1 y_1(t) + c_2 y_2(t)$ is

general soln of the homogeneous system.

$\therefore$ from ③ & ④ $x(t) - x_p(t) = c_1 x_1(t) + c_2 x_2(t)$ and

$$y(t) - y_p(t) = c_1 y_1(t) + c_2 y_2(t)$$

$\therefore x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

$y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$ are solns of the non. homogeneous system.

Problems

1) Show that $x = e^{4t}$, $y = e^{4t}$ & $x = e^{-2t}$, $y = -e^{-2t}$ are the solns of $\frac{dx}{dt} = x+3y$ and $\frac{dy}{dt} = 3x+y$. Find the particular soln of the system ① for which $x(0)=5$ & $y(0)=1$.

Soln : Given $x = e^{4t}$ $x = e^{-2t}$
$y = e^{4t}$ $y = -e^{-2t}$

$$\left.\begin{aligned} \frac{dx}{dt} &= x+3y \\ \frac{dy}{dt} &= 3x+y \end{aligned}\right\} - ①$$

Substituting these in ①

$$4e^{4t} = e^{4t} + 3e^{4t} \Rightarrow 4e^{4t}$$
$$4e^{4t} = 3e^{4t} + e^{4t} \Rightarrow 4e^{4t}$$

$\therefore$ ① is satisfied.

When $x = e^{-2t}$, $y = -e^{-2t}$

$$-2e^{-2t} = e^{-2t} - 3e^{-2t} = -2e^{-2t}$$
$$2e^{-2t} = 3e^{-2t} - e^{-2t} = 2e^{-2t}$$

$\therefore$ It satisfies ①

$\Rightarrow$ They are solns of ①.

To prove: linearly independent

The particular soln is $x = c_1x_1 + c_2x_2$

$y = c_1y_1 + c_2y_2$

$\therefore$ Given $x(0) = 5$ & $y(0) = 1$

$\Rightarrow 5 = c_1e^{4t} + c_2e^{-2t} \Rightarrow 5 = c_1 + c_2$

$1 = c_1e^{4t} + c_2(-e^{-2t}) \Rightarrow 1 = c_1 - c_2$

Consider

$$\begin{array}{l} 5 = c_1 + c_2 \\ 1 = c_1 - c_2 \\ \hline 6 = 2c_1 \end{array}$$

$$\boxed{c_1 = 3}$$

$5 = 3 + c_2 \Rightarrow \boxed{c_2 = 2}$

$\therefore \left.\begin{array}{l} x = 3e^{4t} + 2e^{-2t} \\ y = 3e^{4t} - 2e^{-2t} \end{array}\right\}$ is the general soln.

2) Replace the d.e $y'' - x^2y' - xy = 0$ by an equivalen system of 1^{st} order eqns.

Soln : Let $\frac{dy}{dx} = z$ — ①

$\Rightarrow y'' = z'$

$\Rightarrow z' - x^2z - xy = 0$

i.e, $\frac{dz}{dx} = x^2z + xy$ — ②

3) Replace the d.e $y''' - y'' - x^2 (y')^2$.

Soln: Let $\frac{dy}{dx} = z$ — ① and $\frac{dz}{dx} = w$ — ②

$\frac{d^2y}{dx^2} = z' = w$

$\frac{d^3y}{dx^3} = z'' = w'$

$\therefore \; w' - w - x^2 z^2 = 0$

i.e., $\frac{dw}{dx} = w + x^2 z^2$ — ③

Homogeneous linear system with constant coefficient:

$\frac{dx}{dt} = a_1 x + b_1 y$ — ① $\quad \frac{dy}{dt} = a_2 x + b_2 y$ — ②

Let $\left.\begin{array}{l} x = Ae^{mt} \\ y = Be^{mt} \end{array}\right\}$ be the soln.

Then from ① $\; mAe^{mt} = a_1 Ae^{mt} + b_1 Be^{mt}$

from ② $\; mBe^{mt} = a_2 Ae^{mt} + b_2 Be^{mt}$

i.e., $(a_1 - m) A + b_1 B = 0$ — ③

$a_2 A + (b_2 - m) B = 0$ — ④

③ & ④ will have non-trivial soln if $\begin{vmatrix} a_1 - m & b_1 \\ a_2 & b_2 - m \end{vmatrix} = 0$

i.e., $(a_1 - m)(b_2 - m) - a_2 b_1 = 0 \Rightarrow a_1 b_2 - b_2 m - a_1 m + m^2 - a_2 b_1 = 0$

i.e, $m^2 - (a_1 + b_2) m + a_1 b_2 - a_2 b_1 = 0$ — ⑤

This being a quadratic in m has 2 values for m.

Also ⑤ is the A.E of linear system.

Type (1): When the values of m are distinct §

Substitute the values of m in ③ & ④ we get a simple non-trivial soln for A & B.

Problems

Imp

X.

3 marks.

Repeated.

1) Find the general soln of $\frac{dx}{dt} = x+y$, $\frac{dy}{dt} = 4x-2y$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ └① └②.

X.

10 marks.

Soln: Let $\left.\begin{array}{l} x = Ae^{mt} \\ y = Be^{mt} \end{array}\right\}$ - ⑨ / A.E is obtained from $(a_1-m)A + b_1B=0$ & $a_2A+(b_2-m)$

i.e, $(1-m)A+B=0$ i.e, $(1-m)A+B=0$ — ④

$4A+(-2-m)B=0 \Rightarrow 4A-(2+m)B=0$ — ⑤

$$\therefore \begin{vmatrix} 1-m & 1 \\ 4 & -(2+m) \end{vmatrix} = 0$$

$\Rightarrow -(1-m)(2+m)-4=0 \Rightarrow (1-m)(2+m)+4=0$

$2+m-2m-m^2+4=0 \Rightarrow m^2+m-6=0$

$\Rightarrow m=-3, 2$

When m = -3

④ ⇒ $4A+B=0$

⑤ ⇒ $4A+B=0$

$\therefore 4A+B=0$

When $A=1$, $B=-4$

When m = 2

④ ⇒ $-A+B=0$

⑤ ⇒ $4A-4B=0$

$\therefore A-B=0$

When $A=1$, $B=1$.

∴ from ③ the soln is

A=1, B=-4

m=-3 $x_1 = e^{-3t}$

$y = -4e^{-3t}$

from ③ the soln is

A=1, B=1

m=2 $x_2 = e^{2t}$

$y_2 = e^{2t}$

∴ The general soln is $x = c_1 x_1 + c_2 x_2$ & $y = c_1 y_1 + c_2 y_2$

$$\therefore x = c_1 e^{-3t} + c_2 e^{2t}$$

$$y = -4c_1 e^{-3t} + c_2 e^{2t}$$

2) Find the general soln of $\frac{dx}{dt} = -3x + 4y$ ⌊①, $\frac{dy}{dt} = -2x + 3y$ ⌊②

marks.

Soln: Let $\left.\begin{aligned} x &= Ae^{mt} \\ y &= Be^{mt} \end{aligned}\right\} - ③$

A·E is obtained from $(a_1 - m)A + b_1 B = 0$ & $a_2 A + (b_2 - m)B = 0$

i.e, $(-3-m)A + 4B = 0$ i.e., $-(3+m)A + 4B = 0$ — ④

$-2A + (3-m)B = 0$ $-2A + (3-m)B = 0$ — ⑤

$$\therefore \begin{vmatrix} -(3+m) & 4 \\ -2 & (3-m) \end{vmatrix} = 0 \Rightarrow (3+m)(m-3) + 8 = 0$$

$$\Rightarrow 3m + m^2 - 3m - 9 + 8 = 0$$

$$m^2 - 1 = 0$$

$$\boxed{m = \pm 1}$$

when m=1

④ ⇒ $-4A + 4B = 0$ A=B

⑤ ⇒ $-2A + 2B = 0$

∴ $A - B = 0$

when A=1, B=1

from ③ the soln is $x = e^t$

$y = e^t$

when m=-1

④ ⇒ $-2A + 4B = 0$

⑤ ⇒ $-2A + 4B = 0$

∴ $A - 2B = 0$

when B=1, A=2

∴ from ③ the soln is $x = 2e^{-t}$

$y = e^{-t}$

A=2B

$\frac{A}{2} = B$

∴ The general soln is $x = c_1 x_1 + c_2 x_2$ and

$y = c_1 y_1 + c_2 y_2$

$\therefore x = c_1 e^{t} + 2c_2 e^{-t}$

$y = c_1 e^{t} + c_2 e^{-t}.$

Type 2: When the A.E has equal roots.

(i.e.,) m_1 & m_2 have the same value, one set of so is given by $x = Ae^{mt}$, $y = Be^{mt}$. The next set of soln is given by $x = (A_1 + A_2 t)e^{mt}$, $y = (B_1 + B_2 t)e^{mt}$ where A_1, A_2, B_1 & B_2 are foundout by substituti the 2^{nd} soln in the given d.e.

5 marks ① Repeated

(Ar) 10 marks.

Find the general soln of $\frac{dx}{dt} = 3x - 4y$ & $\frac{dy}{dt} = x - y$

$\quad\quad$ ① $\quad\quad$ ②

Soln:

The A.E is obtained from $(a_1 - m)A + b_1 B = 0$ &

$a_2 A + (b_2 - m)B = 0$

Let
$$\left.\begin{aligned} x &= Ae^{mt} \\ y &= Be^{mt} \end{aligned}\right\} - ③$$

$(3 - m)A - 4B = 0$ — ④

$A - (1+m)B = 0$ — ⑤

$$\therefore \begin{vmatrix} 3-m & -4 \\ 1 & -(1+m) \end{vmatrix} = 0$$

$\Rightarrow -(1+m)(3-m)+4=0 \Rightarrow -3 \mp 3m + m + m^2 + 4 = 0$

$\Rightarrow m^2 - 2m + 1 = 0$

$m = 1, 1$

When $m = 1$,

④ $\Rightarrow 2A - 4B = 0 \Rightarrow$ i.e, $A - 2B = 0$

⑤ $\Rightarrow A - 2B = 0$

when $B = 1$, $A = 2$

From ③ the soln is $x = 2e^t$

$y = e^t$

The next soln is given by $x = (A_1 + A_2 t)e^{mt}$

$y = (B_1 + B_2 t)e^{mt}$ } — ⑥

Sub this in ① & ②

① $\Rightarrow m(A_1 + A_2 t)e^{mt} + A_2 e^{mt} = 3(A_1 + A_2 t)e^{mt} - 4(B_1 + B_2 t)e^{mt}$

$\Rightarrow m(A_1 + A_2 t) - 3(A_1 + A_2 t) + 4(B_1 + B_2 t) + A_2 = 0$

$(A_1 + A_2 t)(m-3) + 4(B_1 + B_2 t) + A_2 = 0$

when $m = 1$

$-2(A_1 + A_2 t) + 4(B_1 + B_2 t) + A_2 = 0$

Equating the coeff of t, $-2A_2 + 4B_2 = 0 \Rightarrow A_2 - 2B_2 = 0$

Equating the coeff of constants, $-2A_1 + 4B_1 + A_2 = 0 \Rightarrow 2A_1 - 4B_1 - A_2 = 0$

② $\Rightarrow m(B_1 + B_2 t)e^{mt} + B_2 e^{mt} = (A_1 + A_2 t)e^{mt} - (B_1 + B_2 t)e^{mt}$

$\Rightarrow m(B_1 + B_2 t) + (B_1 + B_2 t) + B_2 - (A_1 + A_2 t) = 0$

$(B_1 + B_2 t)(m+1) - (A_1 + A_2 t) + B_2 = 0$

when $m = 1$,

$2(B_1 + B_2 t) - (A_1 + A_2 t) + B_2 = 0$

Equating the coeff of t, $2B_2 - A_2 = 0$ i.e, $A_2 - 2B_2 = 0$

Equating the coeff of constants, $2B_1 - A_1 + B_2 = 0$

i.e $2B_1 - A_1 + B_2 = 0$.

$\therefore$ $A_2 - 2B_2 = 0$ — ⑦ $A_2 = 2B_2 \Rightarrow \frac{A_2}{2} = B_2$

$2A_1 - 4B_1 - A_2 = 0$ — ⑧

$2B_1 - A_1 + B_2 = 0$ — ⑨

From ⑦, when $\boxed{B_2 = 1, \ A_2 = 2}$

From ⑧, $2A_1 - 4B_1 = 2 \Rightarrow A_1 - 2B_1 = 1$ ~~$A_1 - 2B_1 - 1 = 0$~~

From ⑨ $2B_1 - A_1 = -1 \Rightarrow A_1 - 2B_1 = 1$ ~~$A_1 - 2B_1 - 1 = 0$~~

when $A_1 = 1$, $B_1 = 0$

Now, from ⑥

$$\left.\begin{aligned} x &= (A_1 + A_2 t)e^{mt} \\ y &= (B_1 + B_2 t)e^{mt} \end{aligned}\right\} - ⑥$$

$\Rightarrow$ $x = (1+2t)e^{t}$

$y = te^{t}$

$\therefore$ The general soln is $x = 2c_1 e^{t} + c_2(1+2t)e^{t}$

$y = c_1 e^{t} + c_2 t e^{t}$.

$$\left[\begin{aligned} \because\ x &= c_1 x_1 + c_2 x_2 \\ y &= c_1 y_1 + c_2 y_2 \end{aligned}\right]$$

2) Find the general soln of $\frac{dx}{dt} = 5x+4y$ ↳① ; $\frac{dy}{dt} = -x+y$ ↳②

<u>soln</u> : Let $\left.\begin{aligned} x &= Ae^{mt} \\ y &= Be^{mt} \end{aligned}\right\} - ③$

The A.E is obtained from $(a_1 - m)A + b_1 B = 0$

$a_2 A + (b_2 - m)B = 0$

$(5-m)A + 4B = 0$ - ④

$-A + (1-m)B = 0$ - ⑤

$$\begin{vmatrix} 5-m & 4 \\ -1 & (1-m) \end{vmatrix} = 0 \Rightarrow (5-m)(1-m)+4=0 \Rightarrow 5-m-5m+m^2+4=0$$

$m^2 - 6m + 9 = 0$

$m = 3, 3$

Sub m in ④ & ⑤

④ $\Rightarrow 2A + 4B = 0 \Rightarrow A + 2B = 0$

⑤ $\Rightarrow -A - 2B = 0 \Rightarrow A + 2B = 0$

when $B = 1$, $A = -2$

∴ from ③ one set of soln is $x = -2e^{3t}$

$y = e^{3t}$

The next set of soln is given by $\left.\begin{aligned} x &= (A_1 + A_2 t)e^{mt} \\ y &= (B_1 + B_2 t)e^{mt} \end{aligned}\right\} - ⑥$

Sub. this ① & ②

① $\Rightarrow m(A_1+A_2t)e^{mt} + A_2e^{mt} = 5(A_1+A_2t)e^{mt} + 4(B_1+B_2t)e^{m}$

$\Rightarrow (m-5)(A_1+A_2t) - 4(B_1+B_2t) + A_2 = 0$

when $m=3$,

$-2(A_1+A_2t) - 4(B_1+B_2t) + A_2 = 0$

Equating coeff of t, $-2A_2 - 4B_2 = 0$ (i.e.,) $A_2 + 2B_2 = 0$

Equating coeff of constants, $-2A_1 - 4B_1 + A_2 = 0$ (ie) $2A_1 + 4B_1 - A_2 = 0$

② $\Rightarrow m(B_1+B_2t)e^{mt} + B_2e^{mt} = -(A_1+A_2t)e^{mt} + (B_1+B_2t)e^{mt}$

$\Rightarrow (m-1)(B_1+B_2t) + (A_1+A_2t) + B_2 = 0$

when $m=3$,

$2(B_1+B_2t) + (A_1+A_2t) + B_2 = 0$

Equating coeff of t, $2B_2 + A_2 = 0 \Rightarrow$ i.e, $A_2 + 2B_2 = 0$

Equating coeff of constants, $2B_1 + A_1 + B_2 = 0$ (ie) $A_1 + 2B_1 + B_2 = 0$

Now, $A_2 + 2B_2 = 0$ — ⑦

$2A_1 + 4B_1 - A_2 = 0$ — ⑧

$A_1 + 2B_1 + B_2 = 0$ — ⑨

from ⑦ $\Rightarrow$ when $\underline{B_2 = 1, A_2 = -2}$

from ⑧ $\Rightarrow \therefore 2A_1 + 4B_1 = -2$ (ie) $A_1 + 2B_1 = -2$

from ⑨ $\Rightarrow A_1 + 2B_1 = -1$ $A_1 = -2B$

$\frac{A_1}{7} = \frac{B_1}{-1}$

when $\underline{A_1 = 1}$, $2 + B_1 = 0 \Rightarrow$ i.e, $\underline{B_1 = -1}$

from ⑥ $x_2 = (1-2t)e^{3t}$

$y_2 = (-1+t)e^{3t}$

∴ The general soln is $x = -2c_1e^{3t} + c_2(1-2t)e^{3t}$

$y = c_1e^{3t} - (1-t)c_2e^{3t}.$

Type-3 : If m_1 & m_2 are distinct complex numbers of the form $a \pm ib$.

Here the 2 solns are given by

$x = e^{at}(A_1\cos bt - A_2\sin bt)$ and $x = e^{at}(A_1\sin bt + A_2\cos bt)$

$y = e^{at}(B_1\cos bt - B_2\sin bt)$ $y = e^{at}(B_1\sin bt + B_2\cos bt)$

5 marks

$e^{\alpha x}[A\cos\beta x + B\sin\beta x]$

Problems.

(1) Find the general soln of $\frac{dx}{dt} = 4x - 2y$, $\frac{dy}{dt} = 5x + 2y$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ↳① $\quad\quad\quad$ ↳②

soln: Let $\left.\begin{array}{l} x = Ae^{mt} \\ y = Be^{mt} \end{array}\right\} - ③$

The A·E is obtained from $(a_1 - m)A + b_1B = 0$

$a_2A + (b_2 - m)B = 0$

$(4-m)A - 2B = 0$ — ④

$5A + (2-m)B = 0$ — ⑤

$$\begin{vmatrix} 4-m & -2 \\ 5 & (2-m) \end{vmatrix} = 0$$

$(4-m)(2-m) + 10 \Rightarrow 8 - 2m - 4m + m^2 + 10 = 0$

$m^2 - 6m + 18 = 0$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \Rightarrow \frac{-(-6) \pm \sqrt{36-4(18)}}{2(1)}$$

$$\frac{6 \pm \sqrt{36-72}}{2} = \frac{6 \pm \sqrt{-36}}{2} = \frac{6 \pm i6}{2}$$

$$\boxed{m = 3 \pm i3}$$

$\therefore m_1 = 3+i3$ & $m_2 = 3-i3$

$$\left.\begin{array}{l} \therefore x = e^{3t}(A_1 \cos 3t - A_2 \sin 3t) \\ y = e^{3t}(B_1 \cos 3t - B_2 \sin 3t \end{array}\right\} \text{-⑥} \quad \& \quad \begin{array}{l} x = e^{3t}(A_1 \sin 3t + A_2 \cos 3 \\ y = e^{3t}(B_1 \overset{\sin 3t}{\cos 3t} + B_2 \overset{\cos 3}{\sin 3} \end{array}$$

└ ⑦

<u>Sub ⑥ in ① & ②</u>

$$① \Rightarrow e^{3t}\left[-3A_1 \sin 3t - 3A_2 \cos 3t\right] + 3e^{3t}\left[A_1 \cos 3t - A_2 \sin 3t\right]$$

$$= 4e^{3t}\left[A_1 \cos 3t - A_2 \sin 3t\right] - 2e^{3t}\left[B_1 \cos 3t - B_2 \sin\right.$$

Equating the coeff of ~~constant~~ cos 3t, $-3A_2 + 3A_1 = 4A_1 + 2B_1$

$$-3A_2 + 3A_1 - 4A_1 + 2B_1 = 0$$

$$3A_2 + A_1 - 2B_1 = 0 \quad \text{—⑧}$$

Equating the coeff of sin 3t, $-3A_1 - 3A_2 = -4A_2 + 2B_2$

$$-3A_1 - 3A_2 + 4A_2 - 2B_2 = 0 \quad \Rightarrow \quad -3A_1 + A_2 - 2B_2 = 0.$$

$$3A_1 - A_2 + 2B_2 = 0 \quad \text{—⑨} \qquad 3A_1 - A_2 + 2B_2$$

$$② \Rightarrow e^{3t}\left[-3B_1 \sin 3t - 3B_2 \cos 3t\right] + 3e^{3t}\left[B_1 \cos 3t - B_2 \sin 3t\right]$$

$$= 5e^{3t}\left[A_1 \cos 3t - A_2 \sin 3t\right] + 2e^{3t}\left[B_1 \cos 3t - B_2 \sin\right.$$

Equating the coeff of cos 3t, $-3B_2 + 3B_1 = 5A_1 + 2B_1$

i.e. $5A_1 + 3B_2 - B_1 = 0$ — ⑩

Equating the coeff of sin 3t $-3B_1 - 3B_2 = -5A_2 - 2B_2$

$5A_2 - 3B_1 - B_2 = 0$ — ⑪

put $\underline{B_1 = k_1 \;\&\; B_2 = k_2}$

$3k_1 \qquad = -k_2 \qquad \frac{k_1}{1} = \frac{-k_2}{3}$

$\therefore$ from ⑩ $5A_1 = k_1 - 3k_2$

$A_1 = \frac{k_1 - 3k_2}{5}$

from ⑪ $5A_2 = 3k_1 + k_2$

$A_2 = \frac{3k_1 + k_2}{5}$

$k_1 = -3k_2$

$k_1 = 3k_2$

put $k_1 = \underline{1}$ & $k_2 = -3$

$\boxed{\therefore A_1 = 2 \;\&\; A_2 = 0}$

Sub the values of k_1 & k_2 in B_1 & $B_2 \Rightarrow$ $\boxed{B_1 = 1, \; B_2 = -3}$

from ⑥ $\Rightarrow$ $x = e^{3t}(2\cos 3t)$

$y = e^{3t}[\cos 3t + 3\sin 3t]$

from ⑦ $\Rightarrow$ $x = e^{3t}[2\sin 3t]$

$y = e^{3t}[\cos 3t - 3\sin 3t]$

$\therefore$ The general soln is $x = 2c_1 e^{3t}\cos 3t + 2c_2 e^{3t}\sin 3t$

$y = c_1 e^{3t}(\cos 3t + 3\sin 3t) + c_2 e^{3t}(\cos 3t - 3\sin 3t)$

29) Ⓒ marks

Find the general soln of $\frac{dx}{dt} = x-2y$, $\frac{dy}{dt} = 4x+5y$

$\llcorner ①$ $\qquad \llcorner ②$

Soln :

The A.E is obtained from $(a_1 - m)A + b_1 B = 0$

$a_2 A + (b_2 - m)B = 0$

$\therefore (1-m)A - 2B = 0$ — ③

$4A + (5-m)B = 0$ — ④

$$\begin{vmatrix} 1-m & -2 \\ 4 & 5-m \end{vmatrix} = 0 \Rightarrow (1-m)(5-m) + 8 = 0 \Rightarrow 5 - 5m - m + m^2 + 8 = 0$$

$$m^2 - 6m + 13 = 0$$

$$m = \underbrace{\frac{-b \pm \sqrt{b^2-4ac}}{2a}} \Rightarrow \underbrace{\frac{-(-6) \pm \sqrt{36 - 4(13)}}{2(1)}} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2}$$

$$\boxed{m = 3 \pm 2i}$$ i.e, $m_1 = 3+2i$ &

$m_2 = 3-2i$

$\therefore x = e^{3t}(A_1 \cos 2t - A_2 \sin 2t)$

$y = e^{3t}(B_1 \cos 2t - B_2 \sin 2t)$ } & $\llcorner ⑤$

$x = e^{3t}(A_1 \sin 2t + A_2 \cos 2t)$

$y = e^{3t}(B_1 \cos 2t + B_2 \sin 2t$

$\llcorner ⑥$

Sub ⑤ in ① & ②

$① \Rightarrow e^{3t}\left[-2A_1 \sin 2t - 2A_2 \cos 2t\right] + 3e^{3t}\left[A_1 \cos 2t - A_2 \sin 2t\right.$

$= e^{3t}\left[A_1 \cos 2t - A_2 \sin 2t\right] - 2e^{3t}\left[B_1 \cos 2t - B_2\right.$

Equating the coeff of cos 2t, $-2A_2 + 3A_1 = A_1 - 2B_1$

i.e., $A_1 - A_2 + B_1 = 0$ — ⑦

Equating the coeff of sin 2t, $-2A_1 - 3A_2 = -A_2 + 2B_2$

i.e., $A_1 + A_2 + B_2 = 0$ — ⑧

② ⇒ $e^{3t}[-2B_1 \sin 2t - 2B_2 \cos 2t] + 3e^{3t}[B_1 \cos 2t - B_2 \sin 2t] =$

$4e^{3t}[A_1 \cos 2t - A_2 \sin 2t] + 5e^{3t}[B_1 \cos 2t - B_2 \sin 2t]$

Equating the coeff of cos 2t, $-2B_2 + 3B_1 = 4A_1 + 5B_1$

i.e., $2A_1 + B_1 + B_2 = 0$ — ⑨

Equating the coeff of sin 2t, $-2B_1 - 3B_2 = -4A_2 - 5B_2$

i.e., $2A_2 + B_2 - B_1 = 0$ — ⑩

put $B_1 = k_1$ & $B_2 = k_2$

∴ from ⑨ $2A_1 + k_1 + k_2 = 0$

$2A_1 = -(k_1 + k_2)$

$A_1 = -\dfrac{(k_1 + k_2)}{2}$

from ⑩ $2A_2 = k_1 - k_2$

$A_2 = \dfrac{k_1 - k_2}{2}$

put $k_1 = 1$, and $k_2 = 1$

∴ $A_1 = -1$ & $A_2 = 0$

Sub the values of k_1 & k_2 in B

∴ $B_1 = 1$ & $B_2 = 1$

from ⑤ $x = e^{3t}[-\cos 2t]$

$y = e^{3t}[\cos 2t - \sin 2t]$

from ⑥ $x = e^{3t}[-\sin 2t]$

$y = e^{3t}[\cos 2t + \sin 2t]$

$\therefore$ The general soln is $x = -c_1 e^{3t}\cos 2t - c_2 e^{3t}\sin 2t$

$$y = c_1 e^{3t}[\cos 2t - \sin 2t] + c_2 e^{3t}[\cos 2t + \sin 2$$

To solve the non-homogeneous linear system by method of variation of Parameters:

$$\frac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t) \text{ and}$$

$$\frac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t)$$

Let $\left.\begin{array}{l} x = x_1(t) \\ y = y_1(t) \end{array}\right\}$ & $\left.\begin{array}{l} x = x_2(t) \\ y = y_2(t) \end{array}\right\}$ be the soln of homogeneous system.

Then $\left.\begin{array}{l} x = v_1(t)x_1(t) + v_2(t)x_2(t) \\ y = v_1(t)y_1(t) + v_2(t)y_2(t) \end{array}\right\}$ is the particul

soln of the non-homogeneous system where v_1 & v_2

$v_1' x_1 + v_2' x_2 = f_1$

$v_1' y_1 + v_2' y_2 = f_2$.

1) Find the particular soln of $\frac{dx}{dt} = x + y - 5t + 2$ —①

$$\frac{dy}{dt} = 4x - 2y - 8t - 8 \quad —②.$$

Soln :

To discuss the nature of the point at ∞ of the hyper geometric series

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

put $t = \frac{1}{x}$ when $x \to \infty,\ t \to 0$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \left(\frac{-1}{x^2}\right)\frac{dy}{dt} = -t^2\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left[-t^2\frac{dy}{dt}\right]\left[\frac{-1}{x^2}\right]$$

$$= \left[-t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt}\right](-t^2)$$

$$\therefore \frac{1}{t}\left(1-\frac{1}{t}\right)(-t^2)\left[-t^2y'' - 2ty'\right] + \left[c - \frac{(a+b+1)}{t}\right](-t^2)y' - ab$$

$$\frac{t-1}{t^2}\,t^2\left[t^2y'' + 2ty'\right] - \left[ct - (a+b+1)\right]ty' - aby = 0$$

i.e, $$y'' + \left[\frac{t(2-c) + (a+b-1)}{t(t-1)}\right]y' - \frac{ab}{t^2(t-1)}\,y = 0$$

$$\therefore (t-0)\,P(x) = -(a+b+1)$$

$$(t-0)^2\,Q(x) = ab.$$

$\therefore t = 0$ is a regular singular point.

i.e, $x = \infty$ is a regular singular point.

$$m(m-1)+mp_0+q_0=0$$

when p_0 is the constant in $tp(t)$ and

q_0 is the constant in $t^2Q(t)$

$$tp(t) = -\left[t(a-c)+(a+b-1)\right]\left[1+t+t^2+\cdots\right] = -(a+b-1)$$

$$t^2q(t) = ab$$

$\therefore$ The indicial is $m(m-1)-m(a+b-1)+ab=0$

$$m^2-m(a+b)+ab=0$$

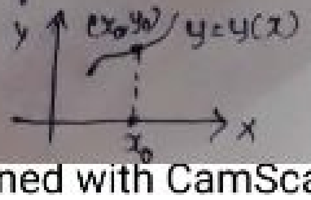
$$\text{i.e., } (m-a)(m-b)=0.$$

The method of successive approximation

To solve the initial value problem $y'=f(x,y)$, $y(x_0)=y_0$ where $f(x,y)$ is an arbitrary function defined & continuous in some neighbourhood of the points x_0 & y_0.

The geometrical meaning of initial value problem:

We have to derive a method for constructing the function $y=y(x)$ [which is soln of the initial value problem & represents a curve] whose graph passes through the point (x_0,y_0) satisfying the differential equation $y'=f(x,y)$ in some neighbourhood of the point x_0



The integral eqn of the initial value problem

$$\frac{dy}{dx} = f(x,y)$$

$$dy = f(x,y(x))\,dx$$

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(t, y(t))\,dt$$

$$y = y(x)$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(t, y(t))\,dt$$

i.e, $$\boxed{y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt}$$

5 marks

Ⓧ Picard's method of successive approximation

First we take a rough approximation $y_0(x) = y_0$

Now the integral eqn is $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$

Then we take the next approximation $y_1(x)$ defined b

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\,dt$$ where $y_0 \Rightarrow y(t) = y_0(t) = y_0$

Then the next better approximation is given by

$$y_2(x) = \int_{x_0}^{x} f(t, y_1(t))\,dt + y_0$$

$$y_3(x) = \int_{x_0}^{x} f(t, y_2(t))\,dt + y_0$$

$$\vdots$$

$$y_n(x) = \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt + y_0$$

Problems

1) Solve by method of successive approximation $y' = y$, $y(0) = 1$. compare the result with exact soln.

(x) more

Soln:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\,dt = 1 + \int_{0}^{x} f(t, 1)\,dt$$

$$= 1 + \int_{0}^{x} dt = 1 + x$$

Exact soln

$$\frac{dy}{dx} = y \Rightarrow \frac{dy}{y} = dx$$

$$\log y = x + c$$

$$y = e^{x+c}$$

$$y_1(x) = 1 + x$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1)\,dt = 1 + \int_{0}^{x} f(t, 1+t)\,dt = 1 + \int_{0}^{x} (1+t)\,dt \left(1 + t + \frac{t^2}{2}\right)_0^x$$

$$= 1 + x + \frac{x^2}{2}$$

$$y_3(x) = y_0 + \int_{x_0}^{x} f(t, y_2)\,dt = 1 + \int_{0}^{x} f\left(t, 1 + t + \frac{t^2}{2}\right)dt = 1 + \int_{0}^{x}\left(1 + t + \frac{t^2}{2}\right)dt$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$$

$$\vdots$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1})\,dt = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = e^x.$$

2mark $y_1(x), y_2(x)$

(X) Solve by method of successive approximation

5mark $y' = x+y$, $y(0) = 1$.

$y(0) \Rightarrow y_0 = 1$ & $x_0 =$

Soln $$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\,dt = 1 + \int_{0}^{x} f(t, 1)\,dt = 1 + \int_{0}^{x} (t+1)\,dt = 1+$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1)\,dt = 1 + \int_{0}^{x} f(t, 1+t+\frac{t^2}{2})\,dt = 1 + \int_{0}^{x} (2t+1+\frac{t^2}{2})\,dt$$

$$= 1 + x + x^2 + \frac{x^3}{3!}$$

$$y_3(x) = y_0 + \int_{x_0}^{x} f(t, y_2)\,dt = 1 + \int_{0}^{x} f(t, 1+t+t^2+\frac{t^3}{3!})\,dt$$

$$= 1 + \int_{0}^{x} (1+2t+t^2+\frac{t^3}{3!})\,dt = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4!}$$

⋮

$$y_n(x) = 1 + x + 2\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right] + \frac{x^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} y_n(x) = 1 + x + 2\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right] + 0 = 1 + x + 2\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} +\right.$$

$$= 1 + x + 2\left[e^x - 1 - x\right]$$

Exact soln $\therefore y_n(x) = 2e^x - 1 - x$ //

$$\frac{dy}{dx} - y = x$$

$$\boxed{I.F \cdot e^{-\int dx} = e^{-x}}$$

Integrating fac

$$\frac{dy}{dx} + Py = Q$$

$$ye^{\int Pdx} = \int e^{\int Pdx} Qd$$

The soln is $ye^{-x} = \int xe^{-x}$

$ye^{-x} = (-e^{-x} \cdot x) - e^{-x} + c$

$1 = 0 - 1 + c \quad [y_0 = 1, x_0 = 0]$

$\boxed{c = 2}$

$ye^{-x} + xe^{-x} + e^{-x} = 2$

$e^{-x}[x + y + 1] = 2$

$x + y + 1 = 2e^{x}$

$y = 2e^{x} - 1 - x$.

$\int u\,dv = uv - \int v\,du$

$u = x, \quad dv = e^{-x}$

$du = dx, \quad v = -e^{-x}$

$= (xe^{-x}] - \int -e^{-x} d$

$= -xe^{-x} - e^{-x}$

2 marks $[y_2(x), y_3(x)]$.

marks.

3) Find the exact solution of the initial value problem $y' = y^2$, $y(0) = 1$. Apply picards method of successive approximation and compare the result with the exact solution.

Soln: G.T $y' = y^2$, $y(0) = 1$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$

$$y_1(x) = y_0 + \int_0^x f(t, 1)\,dt = 1 + \int_0^x dt = 1 + x$$

$$y_2(x) = y_0 + \int_0^x f(t, y_1)\,dt = y_0 + \int_0^x f(t, 1+t)\,dt$$

$$= 1 + \int_0^x (1+t)^2\,dt = 1 + \int_0^x (1 + 2t + t^2)\,dt$$

$$y_2(x) = 1 + x + x^2 + \frac{x^3}{3}$$

$$y_3(x) = y_0 + \int_0^x f(t, y_2)\,dt = 1 + \int_0^x f\left(t, 1+t+t^2+\frac{t^3}{3}\right)dt$$

$$= 1 + \int_0^x \left(1+t+t^2+\frac{t^3}{3}\right)^2 dt$$

$$y_3(x) = 1+x+x^2+x^3+\frac{2}{3}x^4+\frac{x^5}{3}+\frac{x^6}{9}+\frac{x^7}{63}$$

⋮

$$y_n(x) = 1+x+x^2+x^3+x^4+\cdots$$

$$\therefore y_n(x) = (1-x)^{-1}$$

Exact soln

Given that $\frac{dy}{dx} = y^2$.

$$\int \frac{dy}{y^2} = \int dx$$

$$\frac{-1}{y} = x + c$$

Given $y(0)=1 \Rightarrow y=1$ & $x=0$

$\therefore c = -1$

$\Rightarrow \frac{-1}{y} = x-1$

$$y = \frac{-1}{x-1}$$

$$y = \frac{1}{1-x} \quad (\text{or}) \quad y = (1-x)^{-1}$$

~~y~~ i.e. $y = 1+x+x^2+x^3+\cdots$ when $|x|<1$.

(8.) 5 marks Find the equivalent integral eqn to the initial value pblm $y' = 2x(1+y)$; $y(0)$

2 mark

2) Find the exact soln of the initial value problem $y' = 2x(1+y)$, $y(0)=0$ calculate $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$ & compare the result with exact soln.

Soln :

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$

$$y_1(x) = y_0 + \int_0^x f(t, y_0)\,dt = 0 + \int_0^x f(t, 0)\,dt$$

$$= \int_0^x 2t\,dt = \left[\frac{2t^2}{2}\right]_0^x = x^2$$

$$y_2(x) = y_0 + \int_0^x f(t, y_1)\,dt = 0 + \int_0^x f(t, t^2)\,dt = \int_0^x 2t(1+t^2)\,dt$$

$$= \left[\frac{2t^2}{2} + \frac{2t^4}{4}\right]_0^x = x^2 + \frac{x^4}{2}$$

$$y_3(x) = y_0 + \int_0^x f(t, y_2)\,dt = 0 + \int_0^x f\left(t, t^2 + \frac{t^4}{2}\right)dt$$

$$= \int_0^x 2t\left(1 + t^2 + \frac{t^4}{2}\right)dt = \left[\frac{2t^2}{2} + \frac{2t^4}{4} + \frac{t^6}{6}\right]_0^6$$

$$y_3(x) = x^2 + \frac{x^4}{2} + \frac{x^6}{3!}$$

$$y_4(x) = y_0 + \int_0^x f(t, y_3)\,dt = 0 + \int_0^x f\left(t, t^2 + \frac{t^4}{2} + \frac{t^6}{3!}\right)dt$$

$$= \int_0^x 2t\left(1 + \frac{t^2}{1} + \frac{t^4}{2} + \frac{t^6}{6}\right)dt$$

$$= \left[t^2 + \frac{2t^4}{4} + \frac{t^6}{6} + \frac{t^8}{3.8} \right]_0^x$$

$$= x^2 + \frac{x^4}{4} + \frac{x^6}{6} + \frac{x^8}{4!}$$

$$y_4(x) = x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!}$$

⋮

$$y_n(x) = x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots + \frac{(x^2)^n}{n!}$$

When $n \to \infty$

$$y_n(x) = x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots + \cdots$$

$$= 1 + x^2 + \frac{(x^2)^2}{2!} + \cdots + \cdots - 1 = e^{x^2} - 1$$

$$y_n(x) = \underline{e^{x^2} - 1}.$$

Exact soln

Consider $\frac{dy}{dx} = 2x(1+y)$

$$\Rightarrow \int \frac{dy}{1+y} = \int 2x\,dx$$

$$\Rightarrow \log(1+y) = x^2 + c$$

Given $y(0)=0 \Rightarrow x=0, y=0 \Rightarrow \boxed{c=0}$

$$1+y = e^{x^2}$$

$$\underline{y = e^{x^2} - 1}.$$

(x) 10 marks.

5) Find the soln of $y' = x+y$, $y_0(x) = e^x$, $y(0) = 1$. ㉟

soln:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$

put $x=0$ i.e., $y_0(0) = 1$

$$y_1(t) = y_0 + \int_0^x f(t, y_0)\,dt = 1 + \int_0^x f(t, e^t)\,dt$$

$$= 1 + \int_0^x (t + e^t)\,dt = 1 + \frac{x^2}{2} + e^x - 1 = \frac{x^2}{2} + e^x$$

$$y_2(x) = y_0 + \int_0^x f(t, y_1)\,dt = 1 + \int_0^x f\left(t, \frac{t^2}{2} + e^t\right)dt = 1 + \int_0^x \left(t + \frac{t^2}{2} + e^t\right)dt$$

$$= 1 + \frac{x^2}{2} + \frac{x^3}{3!} + e^x - 1 = \frac{x^2}{2} + \frac{x^3}{3!} + e^x$$

$$y_3(x) = y_0 + \int_0^x f(t, y_2)\,dt = 1 + \int_0^x f\left(t, \frac{t^2}{2} + \frac{t^3}{3} + e^t\right)dt = 1 + \int_0^x \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + e^t\right)dt$$

$$= 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + e^x - 1 = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + e^x$$

$$y_n(x) = \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!} + e^x$$

$$\lim_{n \to \infty} y_n(x) = \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - (1+x) + e^x.$$

$$= e^x - 1 - x + e^x = 2e^x - 1 - x$$

Exact soln

$$\frac{dy}{dx} + Py = Q$$

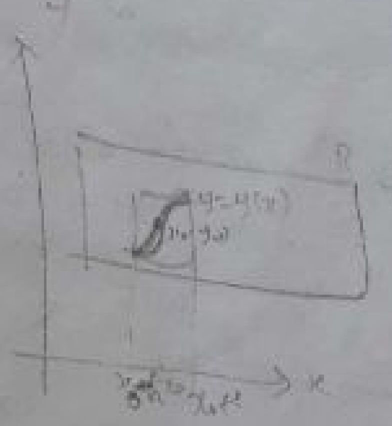
$$ye^{\int P dx} = \int e^{\int P dx} Q dx$$

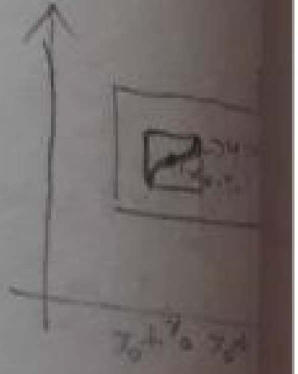
$$\frac{dy}{dx} - y = x$$

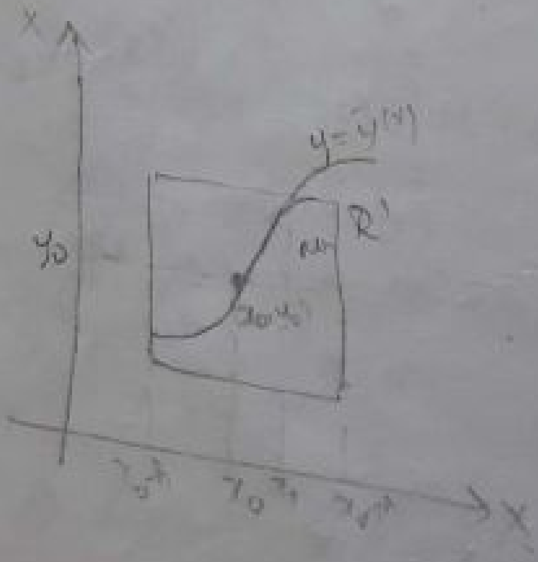
$$\text{I.F. } e^{\int P dx} = e^{-\int dx} = e^{-x}$$

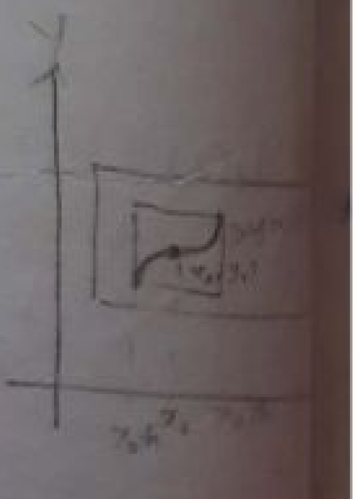
$$ye^{-x} = \int e^{-x} x \, dx$$

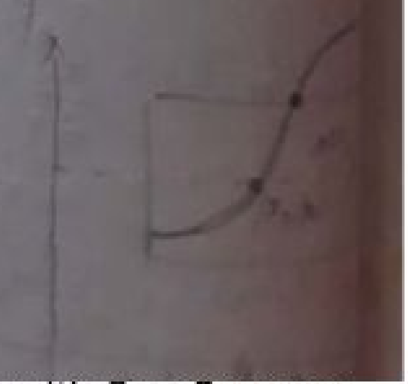
$$ye^{-x} = -e^{-x}[x+1] + c$$







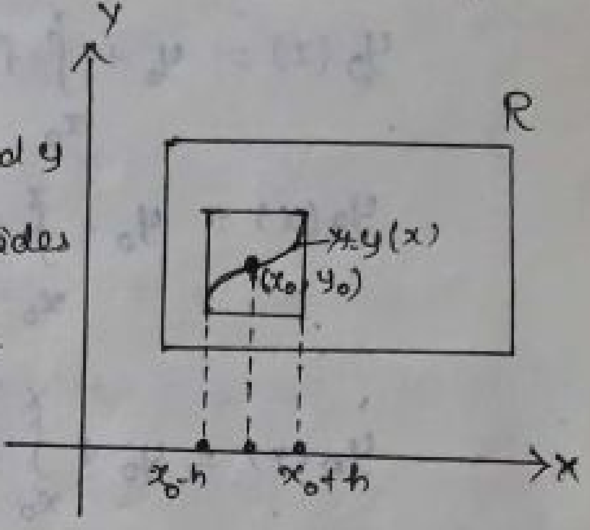




10 marks. V.V.

Picards Theorem [Local Existence & Uniqueness Theorem].

~~If~~ Let $f(x,y)$ and $\partial f/\partial y$ be continuous functions of x and y on a closed rectangle R with sides parallel to the axis. If (x_0, y_0) is any interior point of R, then there exists a number $h>0$ with the property that the initial value problem



$$y' = f(x,y)\ ,\ y(x_0) = y_0$$

has one and only one solution $y = y(x)$ on the interval $|x - x_0| \le h$.

Proof :
$$y' = f(x,y)\ ,\ y(x_0) = y_0 \quad — ①$$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt \quad ← ②$$

(Methods of successive approximation)

① has a unique soln. on the interval $|x - x_0| \le h$

iff ② has a unique continuous soln in the same interval.

Consider the sequence of fn $y_n(x)$ defined by

$y_0(x_0) = y_0$

i.e.,
$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt$$

$$\left.\begin{aligned} y_1(x) &= y_0 + \int_{x_0}^{x} f(t, y_0(t))\,dt \\ y_2(x) &= y_0 + \int_{x_0}^{x} f(t, y_1(t))\,dt \\ y_3(x) &= y_0 + \int_{x_0}^{x} f(t, y_2(t))\,dt \\ &\vdots \\ y_n(x) &= y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt \end{aligned}\right\} \quad ③$$

$\vdots$

$u_n = v_n - v_{n-1}$

$u_{n-1} = v_{n-1} - v_{n-2}$

$u_{n-2} = v_{n-2} - v_{n-3}$

$\vdots$

$u_2 = v_2 - v_1$

$u_1 = v_1 - v_0$

$\boxed{u_1 + u_2 + \dots + u_n = v_n - v_0}$

Now $y_n(x)$ is the n^{th} partial sum of the series of fns.

$\Delta y_n(x) = y_n(x) - y_{n-1}(x)$

Type 1: $U_n = V_n - V_{n-1}$ (To find the sum of

$$y_0(x) + \sum_{n=1}^{\infty} [y_n(x) - y_{n-1}(x)] = y_0(x) + [y_1(x) - y_0(x)] + [y_2(x) - y_1(x)] + \dots + [y_n(x) - y_{n-1}(x)] + \quad ④$$

So, the convergence of the sequence ③ is equivalent to the convergence of the series ④.

Inorder to complete the proof, we (produce) take a number $h > 0$ defining the interval $|x - x_0| \leq h$ and then we S.T the following statements are true

i) the series ④ converges to a fn. $y(x)$

ii) $y(x)$ is a continuous soln of ②

iii) $y(x)$ is the only continuous soln of ②.

Since $f(x,y)$ & $\frac{\partial f}{\partial y}$ are continuous functions on the rectangle R. But R is closed (includes its boundary) and bounded, so each of these fns is bounded on R.

This means that $\exists$ constants M and k such that

$$|f(x,y)| \le M \quad — ⑤$$

and $$\left|\frac{\partial}{\partial y} f(x,y)\right| \le k \quad — ⑥ \quad \forall \text{ points } (x,y) \text{ in } R.$$

If (x, y_1) and (x, y_2) are distinct points in R with the same x co-ordinate, then by mean value theorem

Lagrange's mean value theorem: $f(b) - f(a) = (b-a) f'(c)$

$$|f(x,y_1) - f(x,y_2)| = \left|\frac{\partial}{\partial y} f(x,y^*)\right| |y_1 - y_2| \quad — ⑦$$

where y^* is some number b/w y_1 & y_2.

From ⑥ & ⑦

$$|f(x,y_1) - f(x,y_2)| \le k|y_1 - y_2| \quad — ⑧$$

We now choose h (to be any +ve number) $\ni kh < 1$ — ⑨ $h > 0$

and the rectangle R' defined by the inequalities $|x - x_0| \le h$ and $|y - y_0| \le Mh$ is contained in R.

Now we've defined another rectangle R'.

To prove (i) : Inorder to prove the series ④ is convergent, it is enough to prove the series

$$|y_0(x)| + |y_1(x) - y_0(x)| + |y_2(x) - y_1(x)| + \dots + |y_n(x) - y_{n-1}(x)| + \dots$$

is convergent. ⑩

Let us first prove $y = y_n(x)$ has a graph that lies in R' and hence in R.

→ method of successive approximation

$y_1(x) = y_0$

From ③ $|y_1(x) - y_0| \le \int_{x_0}^{x} |f(t, y_0)| \, dt$

$|f(x, y_0)| \le M$ (by 5)

$\le M(x - x_0)$ $\left[\because [x, y_0(x)] \text{ is a pt in } R', \; |f(x, y_0)| \le M\right]$

$\le Mh$

$\therefore$ $y = y_1(x)$ lies in R'

$|||^{ly}$ $|y_2(x) - y_0| \le \int_{x_0}^{x} |f(t, y_1)| \, dt$

$\le M(x - x_0)$ $\left\{\because [x, y_1(x)] \text{ is a pt in } R', \; |f(x, y_1)| \le M\right.$

$\le Mh$

$\therefore$ $y = y_2(x)$ lies in R' and so on.

$\therefore$ $y_1(x)$ is continuous. and a continuous function on a closed interval has a maximum.

We define a constant 'a' by

$$a = \max |y_1(x) - y_0|.$$

$\therefore$ $|y_1(x) - y_0| \le a$.

Next, the points $[t, y_1(t)]$ and $[t, y_0(t)]$ lie in R',

so from ⑧

$$|f(t, y_1(t)) - f(t, y_0(t))| \leq k\,|y_1(t) - y_0(t)|$$

$$\leq ka$$

$y_2(x) - y_0 - y_1(x) + y_0$

from ③ $$|y_2(x) - y_1(x)| \leq \left|\int_{x_0}^{x} (f(t, y_1(t)) - f(t, y_0(t)))\,dt\right|$$

$|f(t, y_1(t)) - f(t, y_0(t))| \int_{x_0}^{x} dt$

$$\leq ka(x - x_0)$$

$$\leq kah$$

$|||^{ly}$ $$|f(t, y_2(t)) - f(t, y_1(t))| \leq k\,|y_2(t) - y_1(t)|$$

$\leq k\,(kah)$

$$\leq k^2 ah$$

So, $$|y_3(x) - y_2(x)| = \left|\int_{x_0}^{x} (f(t, y_2(t) - f(t, y_1(t)))\,dt\right|$$

$$\leq \int_{x_0}^{x} |f(t, y_2(t) - f(t, y_1(t)))|\,dt$$

$$\leq k^2 ah\,(x - x_0)$$

$$\leq k^2 ah\,(h)$$

$$\leq a(kh)^2$$

By continuing this way, we get

$$|y_n(x) - y_{n-1}(x)| \leq a(kh)^{n-1}$$

∴ Each of term of the series ⑩ is ≤ the corresponding term of the series of constants.

$$|y_0| + a + a(kh) + a(kh)^2 + \dots + a(kh)^{n-1} + \dots$$

which is convergent. [∵ $kh < 1$.

∴ Series ⑩ is convergent & hence series ④ is convergent to a sum which we denote by $y(x)$ and $y_n(x) \to y(x)$.

Since the graph of $y_1(x), y_2(x) \dots y_n(x), \dots$ lies in R' and hence in R the graph of $y(x)$ also lies in R'.

To prove (ii), We must show that

$$\boxed{y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]\,dt = 0} \quad - ⑪$$

But W.k.T

$$y_n(x) - y_0 - \int_{x_0}^{x} f[t, y_{n-1}(t)]\,dt = 0 \qquad ⑫$$

⑪ - ⑫ gives,

$$y(x) - y_n(x) - \int_{x_0}^{x} f(t, y(t))\,dt + \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt = 0$$

⑪ $\Rightarrow$ $0 = y(x) - y_0 - \int_{x_0}^{x} f(t, y(t))\,dt$

$$\therefore y(x) - y_0 - \int_{x_0}^{x} f(t, y(t))\,dt = y(x) - y_n(x) + \int_{x_0}^{x} [f(t, y_{n-1}(t)) - f(t, y(t))]\,dt$$

~~i.e., $\left|y(x) - y_0 - \int_{x_0}^{x} [f(t, y(t))]\, dt\right| \le |y(x) - y_n(x)| + \int_{x_0}^{x} |f(t, y_{n-1}(t)) - f(t, y(t))| dt$~~

i.e., $\left|y(x) - y_0 - \int_{x_0}^{x} [f(t, y(t))]\, dt\right| \le |y(x) - y_n(x)| + \int_{x_0}^{x} |f(t, y_{n-1}(t)) - f(t, y(t))| dt$

$$\le |y(x) - y_n(x)| + \int_{x_0}^{x} k\, |y_{n-1}(t) - y(t)| dt \quad [\text{from } ⑧]$$

$$\le |y(x) - y_n(x)| + k \max |y_{n-1}(x) - y(x)| \int_{x_0}^{x} dt$$

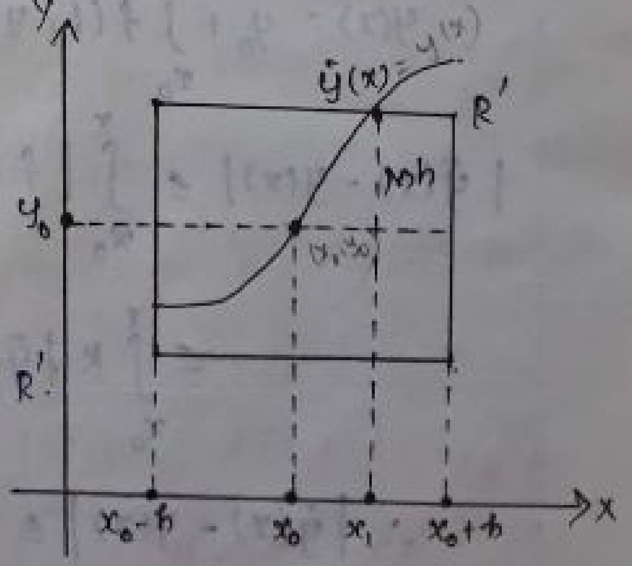
$$\le |y(x) - y_n(x)| + kh \max |y_{n-1}(x) - y(x)|$$

$\to 0$ when n is sufficiently large

$\therefore$ ① is true

To prove (iii): Let $\bar{y}(x)$ is also a continuous soln of ⑤ on the interval $|x - x_0| \le h$ and we show that

$$\bar{y}(x) = y(x) \ \forall \ x \text{ in the interval}$$



First let us prove that graph of $\bar{y}(x)$ lies in R' & hence in R

If the graph $\bar{y}(x)$ leaves R'.

Then by the properties of this function [continuity and $\bar{y}(x_0) = y_0$]

$\Rightarrow \exists$ an $x_1 \ni |x_1 - x_0| < h$, $|\bar{y}(x_1) - y_0| = Mh$

and $|\bar{y}(x) - y_0| < Mh$ if $|x - x_0| < |x_1 - x_0|$

$$\Rightarrow \frac{|\bar{y}(x_1)-y_0|}{|x_1-x_0|} = \frac{Mh}{|x_1-x_0|} > \frac{Mh}{h} > M \quad \text{—} ⑬$$

$$\left[\because x_1 - x_0 < h \quad \frac{1}{x_1-x_0} > \frac{1}{h}\right]$$

By mean value theorem,

$|\bar{y}(x_1)-y_0| = |x_1-x_0|\,|\bar{y}'(x^*)|$ where x^* is a number b/w x_0 and x_1

such that. $\dfrac{|\bar{y}(x_1)-y_0|}{|x_1-x_0|} = |\bar{y}'(x^*)|$ $[\because y' = f(x,$

$$= |f(x^*, \bar{y}(x^*))| \le M \quad ⑭$$

From ⑬ & ⑭ we meet a contradiction.

Hence $\bar{y}(x)$ lies in R'.

To complete the proof of (iii),

∵ $\bar{y}(x)$ and $y(x)$ are solns of ⑧

$$\bar{y}(x) = y_0 + \int_{x_0}^{x} f(t, \bar{y}(t))\,dt \text{ and}$$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$

$$|\bar{y}(x) - y(x)| \le \int_{x_0}^{x} |f(t, \bar{y}(t)) - f(t, y(t))|\,dt$$

$$\le \int_{x_0}^{x} K\,|\bar{y}(t) - y(t)|\,dt \quad [\text{from } ⑧]$$

$$\therefore |\bar{y}(x) - y(x)| \le Kh \max|\bar{y}(x) - y(x)|$$

So,

$$\max|\bar{y}(x) - y(x)| \le Kh \max|\bar{y}(x) - y(x)|$$

$\Rightarrow \max|\bar{y}(x) - y(x)| = 0$, for otherwise $1 \le Kh$ [i.e, $Kh>1$

$\Rightarrow$ contradicts ⑨

$\therefore \bar{y}(x) = y(x)$ for every x in the interval $|x - x_0| \le h$

Hence the proof.

Note:

In Picard's theorem, we are used the result

$$\left| \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} \right| \le k$$ which is called Lipschitz and k is called Lipschitz constant. We can also prove picard's theorem without using Lipschitz condition. But in this case the differential eqn need not have the unique soln.

mark.

Problem

Q7 the pbm $y' = 3y^{2/3}$, $y(0)=0$, on $|x| \le 1$, $|y| \le 1$ has more than one soln. why?

1) Show that the initial value problem $y' = 3y^{2/3}$, $y(0)=0$ does not satisfy Lipschitz condition on the rectangle $|x| \le 1$, $|y| \le 1$.

mark

y(0)=0 have a unique soln? why?

$y(0) = 0 \Rightarrow x_0 = 0, y_0 = 0$

soln:
$$\left| \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} \right| = \left| \frac{f(0, y) - f(0, 0)}{y - 0} \right| = \frac{3y^{2/3} - 0}{y}$$

$$= \frac{3}{y^{1/3}} \to \infty \text{ when } y \to 0$$

$$\therefore \left| \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} \right| \not\le k.$$

Note:

Lipschitz condn. is not satisfied we get sol.

i) $y = x^3$ ii) $y = 0$.

Problems.

5 mark

1) Show that $f(x,y) = y^{1/2}$ does not satisfies Lipschitz condition on the rectangle $|x| \le 1$ & $0 \le y \le 1$.

Soln:

$$\left|\frac{f(x,y_1) - f(x,y_2)}{y_1 - y_2}\right| = \left|\frac{f(0,y) - f(0,0)}{y - 0}\right| = \frac{y^{1/2} - 0}{y}$$

$$= \frac{1}{y^{1/2}} \Rightarrow \infty \text{ when } y \to 0$$

5 mark

$$\therefore \left|\frac{f(x,y_1) - f(x,y_2)}{y_1 - y_2}\right| \not\le k.$$

2 mark

S.T the pbm $y' = y^{1/2}$, $y(0) = 0$ on $|x| \le 1$ $|y| \le 1$ has more than one soln. Explain

2) Test whether $f(x,y) = y^{1/2}$ satisfies Lipschitz condn on the rectangle $|x| \le 1$ & $c \le y \le d$ where $0 < c < d$

Soln:

$$\left|\frac{f(x,y) - f(x,y_1)}{y - y_1}\right| = \left|\frac{f(x,y) - f(x,0)}{y - 0}\right| = \frac{y^{1/2} - 0}{y} = \frac{1}{y^{1/2}}$$

$$\le \frac{1}{c^{1/2}} \text{ [This is Lipschitz constant]}$$

max value of $y = d$
min value of $y = c$
$\frac{1}{\sqrt{d}} < \frac{1}{\sqrt{c}}$

$\therefore f(x,y) = y^{1/2}$ satisfies the Lipschitz condition.

3) Show that $f(x,y) = xy^2$ satisfies the Lipschitz condn on any rectangle $a \leq x \leq b$ & $c \leq y \leq d$.

Soln:
$$\left|\frac{f(x,y_1) - f(x,y_2)}{y_1 - y_2}\right| = \left|\frac{x(y_1^2 - y_2^2)}{y_1 - y_2}\right| = |x(y_1 + y_2)|$$

$$\leq |2bd|$$

This is Lipschitz constant.

$\therefore f(x,y) = xy^2$ satisfied the Lipschitz condn.

4) Show that $f(x,y) = xy^2$ does not satisfy Lipschitz condn on any strip $a \leq x \leq b$, $-\infty < y < \infty$

Soln:
$$\left|\frac{f(x,y) - f(x,0)}{y-0}\right| = \frac{xy^2 - 0}{y} = xy < \infty \text{ when } x \to \infty \text{ \& } y \to \infty$$

$\therefore f(x,y) = xy^2$ does not satisfies the Lipschitz condn.

5) Show that $f(x,y) = xy$ satisfies Lipschitz condn on any rectangle $a \leq x \leq b$, $c \leq y \leq d$.

Soln:
$$\left|\frac{f(x,y_1) - f(x,y_2)}{y_1 - y_2}\right| = \left|\frac{xy_1 - xy_2}{y_1 - y_2}\right| = \left|\frac{x(y_1 - y_2)}{y_1 - y_2}\right| = |x|$$

$= |b| \to$ This is Lipschitz constant

$\therefore f(x,y) = xy$ satisfies the Lipschitz condn.

6) Show that $f(x,y) = xy$ satisfies the Lipschitz condn on any strip $a \le x \le b$ & $-\infty < y < \infty$.

Soln: $\left|\frac{f(x,y)-f(x,0)}{y-0}\right| = \left|\frac{xy-0}{y}\right| = \left|\frac{xy}{y}\right| = |x| = |b|$

$\therefore f(x,y) = xy$ satisfies the Lipschitz condn.

Theorem-7:

Let $f(x,y)$ be a continuous function that satisfies Lipschitz condn. $|f(x,y_1) - f(x,y_2)| \le k|y_1 - y_2|$ on a strip defined by $a \le x \le b$, $-\infty < y < \infty$. If (x_0, y_0) is any point of the strip then the initial value problem $y' = f(x,y)$, $y(x_0) = y_0$ has one & only one soln $y = y(x)$ on the interval $a \le x \le b$.

proof: From Picard's theorem, write upto the convergence of seq ③ is equivalent to the convergence of the series ④.

We define M_0, M_1 & M by $M_0 = |y_0|$, $M_1 = \max|y_1(x)|$,

$M = M_0 + M_1$

$\therefore$ We see that $|y_0(x)| \le M$ and $|y_1(x) - y_0(x)| \le M$

Let x_0 be a point $\ni x_0 \le x \le b$

$$|y_2(x) - y_1(x)| = \left|\int_{x_0}^{x} \{f[t, y_1(t)] - f[t, y_0(t)]\} dt\right|$$

$$\leq \int_{x_0}^{x} |f(t, y_1(t)) - f(t, y_0(t))| dt$$

$$\leq K \int_{x_0}^{x} |y_1(t) - y_0(t)| dt$$

$$\leq KM(x - x_0)$$

$$|y_3(x) - y_2(x)| = \left|\int_{x_0}^{x} \{f(t, y_2(t)) - f(t, y_1(t))\} dt\right|$$

$$\leq \int_{x_0}^{x} |f(t, y_2(t)) - f(t, y_1(t))| dt$$

$$\leq K \int_{x_0}^{x} |y_2(t) - y_1(t)| dt \leq K \int_{x_0}^{x} KM(t - x_0) dt$$

$$\leq K^2 M \left[\frac{(t - x_0)^2}{2}\right]_{x_0}^{x}$$

$$\leq K^2 M \frac{(x - x_0)^2}{2!}$$

$|||^{ly}$ $$|y_n(x) - y_{n-1}(x)| \leq \frac{K^{n-1} M (x - x_0)^{n-1}}{(n-1)!}$$

$|||^{ly}$ for $a \leq x \leq x_0$,

$$|y_n(x) - y_{n-1}(x)| \leq \frac{K^{n-1} M (x_0 - x)^{n-1}}{(n-1)!}$$

∴ for all x in this interval $a \le x \le b$.

$$|y_n(x) - y_{n-1}(x)| \le \frac{K^{n-1} M |x-x_0|^{n-1}}{(n-1)!} \le \frac{K^{n-1} M (b-a)^{n-1}}{(n-1)!}$$

Now consider the series,

$$|y_0(x)| + |y_1(x) - y_0(x)| + |y_2(x) - y_1(x)| + \cdots + |y_n(x) - y_{n-1}(x)| + \cdots \quad ⑤$$

Each term of ⑤ is less than or equal to the correspond term of $M + M + KM(b-a) + K^2 \frac{M(b-a)^2}{2!} + \cdots + K^{n-1} \frac{M(b-a)^{n-1}}{(n-1)!} + \cdots$

which is uniformly convergent on the interval $a \le x \le b$ to limit function $y(x)$

To prove uniqueness:

Let $\bar{y}(x)$ be also a soln of the given d.e.

Then $\bar{y}(x)$ is continuous & satisfies $\bar{y}(x) = y_0 + \int_{x_0}^{x} f(t, \bar{y}(t))dt$

Let $A = \max|\bar{y}(x) - y_0|$ Then for $x_0 \le x \le b$ we've

$$|\bar{y}(x) - y_1(x)| \le \int_{x_0}^{x} |f(t, \bar{y}(t)) - f(t, y_0(t))| dt$$

$$\le k \int_{x_0}^{x} |\bar{y}(t) - y_0| dt \le kA(x - x_0)$$

$$|\bar{y}(x) - y_2(x)| \le \int_{x_0}^{x} |f(t, \bar{y}(t)) - f(t, y_1(t))| dt$$

$$\le k \int_{x_0}^{x} |\bar{y}(t) - y_1(t)| dt \le k^2 A \int_{x_0}^{x} (t - x_0) dt$$

$$\leq k^2 A \frac{(x-x_0)^2}{2!}$$

III'ly $$|\bar{y}(x) - y_n(x)| \leq k^n A \frac{(x-x_0)^n}{n!}$$

III'ly for $a \leq x \leq x_0$ we've

$$|\bar{y}(x) - y_n(x)| \leq k^n A \frac{(x_0-x)^n}{n!}$$

$\forall$ x in $a \leq x \leq b$ we've

$$|\bar{y}(x) - y_n(x)| \leq k^n A \frac{(x-x_0)^n}{n!} \leq k^n A \frac{(b-a)^n}{n!} \to 0 \text{ as } n \to \infty$$

$\therefore \bar{y}(x) = y(x)$ $\forall$ x in $a \leq x \leq b$.

In the picards theorem, if you drop the Lipschitz condn. & assume only $f(x,y)$ is continuous on R then it is still possible to prove the initial value problem has a soln. This result is known as peanos theorem.

Unit -III is over

Unit - IV

OSCILLATION THEORY AND BOUNDARY VALUE PROBLEMS.

Qualitative properties of solutions.

Consider the second order linear equation

$y'' + p(x) y' + Q(x) y = 0$. — ①

• It is rarely possible to solve this equation in terms of familiar elementary functions.

Now consider the equation $y'' + y = 0$ — ②

Solve eqn ② by elementary method, we have

$y'' + y = 0$

Auxiliary eqn is: $m^2 + 1 = 0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm i$

[∵ the general soln is $y(x) = c_1 y_1(x) + c_2 y_2$

∴ solution is $y = c_1 \sin x + c_2 \cos x$

Here $y_1(x) = \sin x$ and $y_2(x) = \cos x$ are two linearly Independent solutions of ② and the initial conditions are

$y_1(0) = 0$ and $y_2(0) = 1$

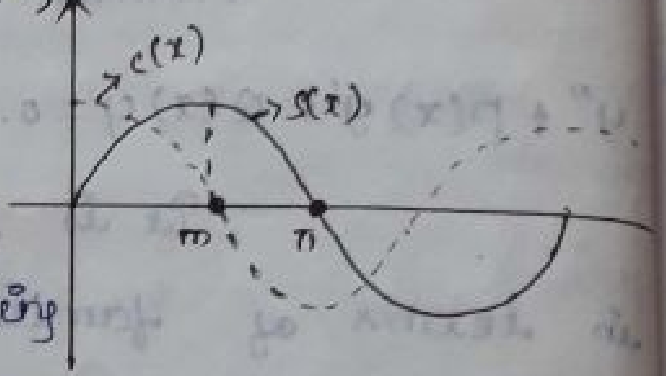
$y_1'(0) = 1$ $y_2'(0) = 0$ and the

general solution is $y = c_1 \sin x + c_2 \cos x$.

let $y = s(x)$ be defined as solution of ② a

the initial conditions are $s(0)=0$ and $s'(0)=1$

Now we draw the graph of $s(x)$ by letting



x increase from 0, the initial conditions gives us to start the curve at the origin and let it rise with slope beginning at 1 and we have from eqn ②

$$y'' + s(x) = 0 \quad [\because y = s(x)]$$

$$y'' = -s(x)$$

$$s''(x) = -s(x) \quad [\because y'' = s''(x)]$$

So when the curve is above the x axis, $s''(x)$ i a negative number that (increase in magnitude) decreases as the curve rise Since $s''(x)$ is the rate of change of the slope $s'(x)$, this slope decreases at an increasing rate as the cu lifts and it must reach zero at some point $x=m$.

As x continues to increase, the curve fal towards the x axis, $s'(x)$ decreases at a decreasing rate and the curve crosses the x axis at a poi we can define to be π.

Since $s'(x)$ depends only on $s(x)$, from the graph between $x=0$ & $x=\pi$ is symmetric about the line $x=m$, so $m=\frac{\pi}{2}$ and $s'(\pi)=-1$.

A similar argument shows that the next portion of the curve is an inverted replica of the first arch and so on indefinitely.

Now we introduce $y=c(x)$ as the solution of ② and the initial conditions $c(0)=1$ and $c'(0)=0$

From the conditions we [illegible] have the graph of $c(x)$ starts at the point $(0,1)$ and moves to the right with slope beginning at 0.

From eqn ②, we have

$$y''+y=0$$

$$y''+c(x)=0 \quad [\because y=c(x)]$$

$$\Rightarrow y''=-c(x)$$

$$\Rightarrow c''(x)=-c(x), [\because y''=c''(x)]$$

The same reasoning as before shows that the curve bends down and crosses the x-axis. The height of the first arch of $s(x)=1$ and the first zero of $c(x)$ is $\pi/2$.

To prove that $s'(x) = c(x)$ & $c'(x) = -s(x)$ ③

From eqn ② $y''+y=0$

$$\Rightarrow y'''+y'=0$$

$$(y')''+y'=0 \quad \text{—— } ⊛$$

$\Rightarrow$ the derivative of any solution of ② is again solution of ⊛.

Thus $s'(x)$ and $c(x)$ are both solutions of ②

To show that they have the same values and the same derivatives at $x=0$, we've

$$s'(0)=1,\quad c(0)=1 \ \& \ s''(0)=-s(0)=0,\quad c'(0)=0.$$

$$\Rightarrow s'(0) = 1 = c(0)$$

$$\Rightarrow \underline{s'(x) = c(x)} \text{ and}$$

$$c'(0) = 0 = s''(0) = -s(0)$$

$$\underline{\underline{c'(x) = -s(x)}}$$

Now using ③ we have to prove

$$s(x)^2 + c(x)^2 = 1 \quad \text{—— } ④$$

Since the derivative of L.H.S of ④ is

$$2s(x)s'(x)+2c(x)c'(x) = 0$$
$$2s(x)c(x)-2c(x)s(x) = 0$$

We see that $s(x)^2+c(x)^2$ equals a constant and th constant must be 1.

Becoz,

$$s(0)^2 + c(0)^2 = 1$$

⇒ ④ is completed.

To show $s(x)$ and $c(x)$ are linearly independent, for their Wronskian is,

$$W[s(x), c(x)] = s(x)\,c'(x) - c(x)\,s'(x)$$

$$= -s(x)^2 - c(x)^2 = -1.$$

We have the following results,

i) $s(x+a) = s(x)\,c(a) + c(x)\,s(a)$

ii) $c(x+a) = c(x)\cdot c(a) - s(x)\,s(a)$

iii) $s(2x) = 2\,s(x)\cdot c(x)$

iv) $c(2x) = c(x^2) - s(x)^2$

v) $s(x+2\pi) = s(x)$

vi) $c(x+2\pi) = c(x)$

From the above results that the positive zeros of $s(x)$ and $c(x)$ are respectively $\pi, 2\pi, 3\pi$ .. and $\pi/2$, $\pi/2+\pi$, $\pi/2 + 2\pi$ . . .

The fact that they oscillate in such a manner that their zeros are distinct and occur alternatively.

S.mau

2mar 15.

Theorem-1: Sturm separation theorem:

SII If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of

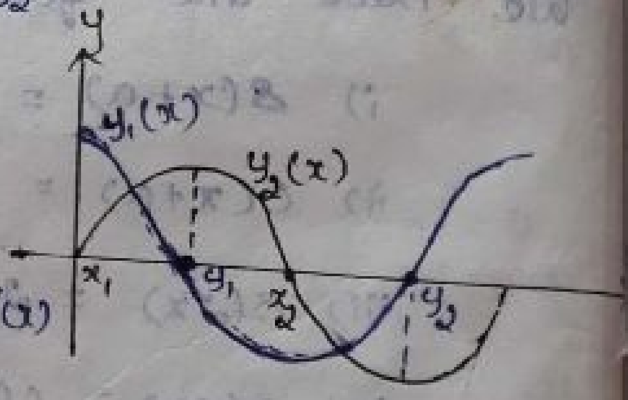
$$y'' + p(x)y' + Q(x)y = 0,$$

then the zeros of these functions are distinct and occu alternately in the sense that $y_1(x)$ vanishes exactly one between any two successive zeros of $y_2(x)$ and convers

proof:

Given that $y_1(x)$ and $y_2(x)$ are linearly independent, then their Wronskian is not zero.

(ie) $W(y_1, y_2) = y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x)$ does not vanish.



Since, it is continuous and therefore must have constant sign.

Let y_1 & y_2 cannot have common zero.

Suppose y_1 & y_2 have common zero, then their Wronskian will vanish at that point, which is impossible. Therefore y_1 & y_2 cannot have a common zero.

Now we assume that x_1 & x_2 are successive zeros of y_2 and show that y_1 vanishes between these points.

The Wronskian clearly reduces to $y_1(x) y_2'(x)$ at x_1 and x_2, so both factors $y_1(x)$ and $y_2'(x)$ are $\neq 0$ at each of these points.

Furthermore, $y_2'(x_1)$ and $y_2'(x_2)$ must have opposite signs, ~~and therefore~~ because if y_2 is increasing at x_1 it must be decreasing at x_2 and y_2 is increasing at x_2 it must be decreasing at x_1.

Since the Wronskian has constant sign, $y_1(x_1)$ and $y_1(x_2)$ must also have opposite signs and therefore by continuity $y_1(x)$ must vanish at some point between x_1 & x_2.

Note that y_1 cannot vanish more than once between x_1 & x_2; for if it does, then the same argument shows that y_2 must vanish between these zeros of y_1, which is contradiction to the original assumption that x_1 & x_2 are successive zeros of y_2.

Hence the proof.

what is the normal form of $y''+p(x)y'$

Now consider the equation

$$y''+p(x)y'+Q(x)y=0 \quad \text{—} ①$$

can be written as

$$\underline{u''+q(x)u=0} \quad \text{—} ②$$

by a simple change of the dependent variable

Eqn ① is known as the standard form and

Ⓠ 2 marks

Eqn ② is known as the normal form of a homogeneous second order linear eqn.

To write ① in normal form, we put $y(x)=u(x)$

$y'=uv'+u'v$ [1st derivative]

$y''=uv''+2u'v'+u''v$ [2nd derivative]

Substitute these in ①

$$(uv''+2u'v'+u''v)+p(uv'+u'v)+Q(uv)=0$$

$$vu''+(2v'+pv)u'+(v''+pv'+Qv)u=0 \quad \text{—} ③$$

Equating the coeff of u' to zero

$$2v'+pv=0$$

$$\Rightarrow v'+\frac{p}{2}v=0$$

$\frac{dy}{dx}+py=Q$

$\boxed{y'+py=Q}$

$ye^{\int pdx}=\int e^{\int pdx}Q\,dx$

$P = \frac{p}{2}$, $Q = 0$

(8) 2marks What is the normal form of $y'' + x^2 y' + xy = 0$.

$$v e^{\int P/2 \, dx} = 0$$

$y(x) = u(x) v(x)$

$y'(x) = uv' + u'v$

$y''(x) = uv'' + 2u'v' + u''v$

$$v e^{\frac{1}{2}\int p \, dx} = 0$$

$uv'' + 2u'v' + u''v + x^2(uv' + u'v) + x(uv) = 0$

$$\boxed{v = e^{-\frac{1}{2}\int p \, dx}} \longrightarrow ④$$

$u''v + (x^2 v + 2v')u' + (x^2 v' + xv)u$

pu

reduces ③ to the normal form ② with

$$q(x) = Q(x) - \frac{1}{4} p(x)^2 - \frac{1}{2} p'(x)$$

Since $v(x)$ in ④ never vanishes, the above transformation of ① into ② has no effect whatever on the zeros of solutions.

with thm 3

If $q(x)$ in ② is a negative function, then the solutions of this equation do not oscillate at all.

2marks When does a non trivial soln of $u'' + q(x)u = 0$ have atmost one zero? Justify.

Theorem-2: If $q(x) < 0$ and $u(x)$ is a nontrivial soln of $u'' + q(x)u = 0$, then $u(x)$ has atmost one zero. ①

repeated

proof: Let x_0 be a zero of $u(x)$, so that $u(x_0) = 0$.

And we've $u(x)$ has a nontrivial soln.

(ie) $u(x)$ is not identically zero.

Then by thm A (unit-1) $\Rightarrow u'(x_0) \neq 0$.

Now assume that, $u'(x_0) > 0$, so that $u(x)$ is positive over some interval to the right of x_0. [by ①]

$u''(x)+q(x)u=0.$

Since $q(x) < 0$, $u''(x) = -q(x)u(x)$ is a positive function on the same interval $\left[\because q(x) < 0 \Rightarrow -q(x) = -(-q(x))u(x)\right.$

$\Rightarrow$ That the slope $u'(x)$ is an increasing fun so $u(x)$ cannot have a zero to the right of x_0.

In the sameway we cannot have a zero the left of x_0.

A similar argument holds when $u'(x_0) <$ so $u(x)$ has either no zeros at all or only one.

And the proof is completed.

Note:

Let $u(x)$ be a non-trivial soln with $q(x$ then $u''(x) = -q(x) \cdot u(x)$ is negative.

$\Rightarrow$ that the curve is concave and the slop $u'(x)$ is decreasing.

If this slope ever becomes negative, then the curve crosses the x-axis somewhere to the right and we get a zero.

with the previous thm

Theorem-3: Let $u(x)$ be any nontrivial solution of $u'' + q(x)u = 0$, where $q(x) > 0$ $\forall$ $x > 0$. If

$$\int_0^{\infty} q(x)\,dx = \infty,$$

then $u(x)$ has infinitely many zeros on the positive x axis.

Proof: Suppose assume that contrary,

(ie) $u(x)$ vanishes atmost a finite number of times for $0 \le x < \infty$

Let $x_0 > 1$ be any point then $\exists$ $u(x)$ $\ni$ $u(x) \neq 0$ $\forall$ $x \geq x_0$.

Without loss of generality, assume that $u(x) > 0$ $\forall x \geq x_0$

Since $u(x)$ can be replaced by its $-$ve if necessary

To complete the proof, it is sufficient to show that $u'(x)$ is $-$ve somewhere to the right of x_0

Let $v(x) = -\dfrac{u'(x)}{u(x)} = -u'(x)[u(x)]^{-1}$ for $x \geq x_0$, ①

then $v'(x) =$ ~~q(x)~~ $-u''(x)[u(x)]^{-1} + u'(x)[u(x)]^{-2}u'(x)$

$$= -\frac{u''(x)}{u(x)} + \frac{[u'(x)]^2}{[u(x)]^2}$$

$u'' + q(x)u = 0$

$q(x) = \dfrac{-u''}{u}$

$$v'(x) = q(x) + v(x)^2$$

$v'(x) = q(x) + v(x)^2$

$\int$ing both sides from x_0 to x, where $x > x_0$, we

$$\int_{x_0}^{x} v'(x) = \int_{x_0}^{x} q(x)\,dx + \int_{x_0}^{x} [v(x)]^2\,dx$$

$$[v(x)]_{x_0}^{x} = \int_{x_0}^{x} q(x)\,dx + \int_{x_0}^{x} (v(x)]^2\,dx$$

$$v(x) - v(x_0) = \int_{x_0}^{x} q(x)\,dx + \int_{x_0}^{x} v(x)^2\,dx \quad ②$$

But we've

$$\int_{1}^{\infty} q(x)\,dx = \infty \quad ③$$

from ② & ③ we conclude that,

$v(x)$ is +ve if x is large

from ① $\Rightarrow$ $v(x) = -\frac{u'(x)}{u(x)}$ $u(x)$ & $u'(x)$ have opposite signs if x sufficiently large. But we've $u(x) > 0 \Rightarrow u'(x) = 0$

$\Rightarrow$ that the slope ever becomes -ve. Then the curve cross the x-axis somewhere to the right of x_0 and we get a zero for $u(x)$.

which is contradiction to our assumptio

Hence the proof.

THE STURM COMPARISON THEOREM

Theorem-4 : Let $y(x)$ be a non trivial soln of $y'' + q(x)y = 0$ ①, where $q(x)$ is +ve on a closed interval $[a, b]$. Then $y(x)$ has atmost a finite number of zeros in this interval.

Proof: Suppose assume the contrary,

(ie) $y(x)$ has an infinite number of zeros in $[a, b]$

Then ∃ a point x_0 in $[a, b]$ and a sequence of zeros $x_n \neq x_0$ ∋ $x_n \to x_0$.

Since $y(x)$ is continuous and differentiable at x_0, we've

$$\lim_{x_n \to x_0} y(x_n) = y(x_0) = 0$$

$$y(x_0) = \lim_{x_n \to x_0} y(x_n) = 0$$

and
$$y'(x_0) = \lim_{x_n \to x_0} \frac{y(x_n) - y(x_0)}{x_n - x_0} = 0$$

(ie) $y'(x_0) = 0$

u(x) for non trivial soln, $u'(x_0) \neq 0$

Then by thm A (unit-1), $y(x)$ is the trivial soln of ①.

Which is contradiction to our assumption

Hence the theorem.

Q. 5 marks 10 marks

Expected

Theorem-5 : Sturm comparison theorem.

(Stt) : Let $y(x)$ and $z(x)$ be nontrivial solns

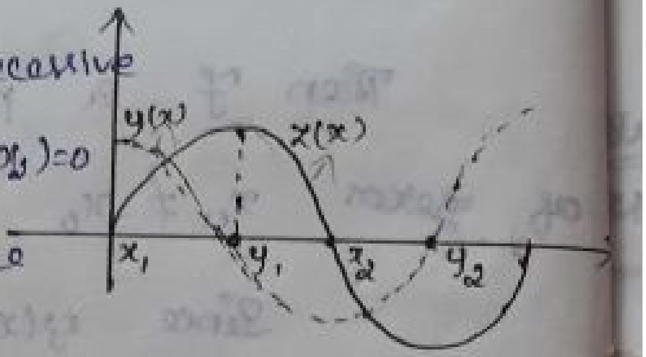
5 marks

$$y'' + q(x)y = 0 \quad \&$$

$$z'' + r(x)z = 0$$

(e) $q(x)$ & $q(x) > 0$

where $q(x)$ and $r(x)$ are positive functions ∋ $q(x) > r(x)$. Then $y(x)$ vanishes atleast once betw any two successive zeros of $z(x)$.



proof : Let x_1 and x_2 be successive zeros of $z(x)$, so that $z(x_1) = z(x_2) = 0$ and $z(x)$ does not vanish on the open interval (x_1, x_2).

We prove the theorem by contradiction metho

Let us assume that $y(x)$ does not vanish on $(x_1,$

Suppose that both $y(x)$ & $z(x)$ are +ve on (x_1, x_2) for either function can be replaced by its -ve if necessary.

The Wronskian $W(y,z) = y(x)z'(x) - z(x)y'(x)$ is a fn of x by writing it $W(x)$, then

$$\frac{dW(x)}{dx} = y'(x)\cancel{z'(x)} + z''(x)y(x) - z(x)$$

$$- z(x)y''(x)$$

$$= z''(x)y(x) - z(x)y''(x)$$

(iv) $\frac{dW(x)}{dx} = yz'' - zy''$ $z' + r(x)z = 0$ $\boxed{z'' = -rz}$ & $y''+q(x)y$ $y' = -qy$.

$= y(-rz) - (-qy)z$

$= y(-rz) + (qy)z$

$q > r$.

$= zy(q-r) > 0$ on (x_1, x_2)

∫ing on both sides of this inequality from x_1 to x_2

$W(x.$ $W(x_2) - W(x_1) > 0$ $\int_{x_1}^{x_2} \frac{dW(x)}{dx} > 0$

$\Rightarrow W(x_2) > W(x_1)$ —— ①

But we have,

$W(y,z) = yz' - zy'$ at x_1 & x_2, both factors $y(x)$ & $z'(x)$ are not zero at x_1 & x_2.

Furthermore, $z'(x_1)$ & $z'(x_2)$ must have opposite sign because if z is increasing at x, it must be decreasing at x_2

$\Rightarrow$ that $W(x_1) \geq 0$ & $W(x_2) \nleq 0$

which is a contradiction.

So our assumption is wrong.

$y(x)$ vanishes at x_1 & x_2 exactly.

Theorem-6 :

Let $y_p(x)$ be a nontrivial soln of Be
eqn on the positive x axis. If $0 \le p < 1/2$, then ev
integ interval of length π contains atleast one ze
of $y_p(x)$; if $p = 1/2$, then the distance between
successive zeros of $y_p(x)$ is exactly π; and if $p > 1/$
then every interval of length π contains atmost o
zero of $y_p(x)$.

EIGEN VALUES, EIGEN FUNCTIONS AND THE VIBRA STR

Let $y(x)$ be a non-trivial soln of the
equation $y'' + \lambda y = 0$ —①

where x is a parameter and satisfies the
boundary conditions $y(0) = 0$ and $y(\pi) = 0$ —②.

Our aim is to solve λ :

If λ is positive then we get only the
trivial soln by the theorem

i) If $q(x) < 0$ and $u(x)$ is non-trivial soln o
$u''(x) + q(x) u = 0$, then $u(x)$ has atmost one ze

ii) If $\lambda = 0$, then we get the soln is $c_1 x + c_2$ and we get a trivial soln.

iii) If λ is +ve ($\lambda > 0$), then the general soln of ① is

$$y(x) = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x$$

Given that $y(0) = 0$

$$\Rightarrow \quad y(x) = c_1 \sin\sqrt{\lambda}\,x \quad \text{—③}$$

eqn ① has a soln it must be of ③.

Note:

i) The values of λ are called eigen values and the corresponding solns $\sin x, \sin 2x \ldots$ are called eigen functions.

ii) The eigen values form an increasing sequence of positive numbers that approaches ∞.

iii) The n^{th} eigen function $\sin nx$ vanishes at the endpoints of the interval $[0, \pi]$ and has exactly $(n-1)$ zeros inside this interval.

Problems :-

1) Find the eigen function and eigen values of $y''+$

$y(0)=0$, $y(\pi/2)=0$. (H.W.) 2 mark $y(0)=0$, $y(2\pi)=0$

Soln:

Given $y''+\lambda y=0$ — ① with boundary conditio

$y(0)=0$ & $y(\pi/2)=0$ — ②

case (i) : Let $\lambda=0$

$$y''+\lambda y=0$$

$$m^2=-\lambda$$

$$m^2=-0 \quad [\because \lambda=0]$$

$$m=\pm 0$$

The soln is $y(x)=Ax+B$ — ③

using eqn ② we've $y(0)=0+B$

$$0=B$$

$$y(\pi/2)=A\pi/2+B$$

$$0=\pi/2 \cdot A+B$$

But we've $B=0$

$$\pi/2 \cdot A=0$$

$$\Rightarrow A=0$$

The eqn ③ reduces to $y(x)=0$.

Since $y(x)\neq 0$, so there is no eigen fn corresponding to $\lambda=0$.

case (ii) : Let $\lambda = -\mu^2$, when $\mu \neq 0$ & λ is -ve.

$$y'' + \lambda y = 0$$

$$\Rightarrow m^2 = -\lambda$$

$$m^2 = \mu^2$$

$$m = \pm\mu$$

The soln is $y(x) = Ae^{mx} + Be^{-mx}$ ← ④

Using boundary conditions, $y(0) = 0$.

$$y(0) = A + B$$

$$A + B = 0 \leftarrow ⑤$$

and the condition $y(\pi/2) = 0$

$$y(\pi/2) = Ae^{\mu\pi/2} + Be^{-\mu\pi/2}$$

$$Ae^{\mu\pi/2} + Be^{-\mu\pi/2} = 0 \leftarrow ⑥$$

Solving ⑤ & ⑥

$$⑤ \times e^{\mu\pi/2} \Rightarrow Ae^{\mu\pi/2} + Be^{\mu\pi/2} = 0$$

$$⑥ \times 1 \Rightarrow \underline{\overset{-}{A}e^{\mu\pi/2} \overset{-}{+} Be^{-\mu\pi/2} = 0}$$

$$B\left(e^{\mu\pi/2} - e^{-\mu\pi/2}\right) = 0$$

$$\Rightarrow B = 0$$

from ⑤ ⇒ $A = 0$

∴ ④ reduces to $y(x) = 0$.

Since $y(x) \neq 0$. So there is no eigen fn corresponding to $\underline{\underline{\lambda = -\mu^2}}$

Case (iii) :

Let $\lambda = \mu^2$, where $\mu \neq 0$ ∀ x is +ve.

$y'' + \lambda y = 0$

$\Rightarrow m^2 = -\lambda \Rightarrow m^2 = -\mu^2$

$m = \pm i\mu$

The soln is $y(x) = A\cos\mu x + B\sin\mu x$ —⑦ using boundary condition $y(0)=0$.

$\Rightarrow y(0) = A \quad [\because y(0) = A\cos 0 + B\sin 0].$

$\Rightarrow A = 0$ —⑧

& $y(\pi/2) = 0$

$\Rightarrow y(\pi/2) = B\sin\mu\pi/2 \quad [\because y(\pi/2) = A\cos\mu\pi/2 + B\sin$

$0 = B\sin\mu\pi/2 = 0$ ← ⑨

But $\mu \neq 0 \Rightarrow B\sin\mu\pi/2 = 0$

If $B=0$, then $A=0$.

Then eqn ⑦ reduces to $y(x) = 0$ which is not a eigen function. so $B \neq 0$ for the existence of eigen function.

Since $B \neq 0$, then ⑨ reduces to

$\sin\mu\pi/2 = 0$.

So that $\mu = 2n$, $n = 1, 2, 3 \ldots$

We've $A=0$, & $\mu=2n$ then eqn $\overset{①}{⑦}$ reduces to

$y(x) = B \cdot \sin 2nx$, $n=1,2,\ldots$

$$\lambda = \mu^2 = 4n^2, \quad n=1,2,\ldots$$

So the required eigen function is $y_n(x)$ with corresponding eigen values λ_n are

$$y_n(x) = B_n \sin 2nx$$

$$\lambda_n = 4n^2, \quad n=1,2,\ldots$$

2 marks

Find the eigen function and eigen values of $y''+\lambda y=0$ and $y(0)=0$ & $y(L)=0$ when $L>0$.

Proof: Given $y''+\lambda y=0$ — ① with the boundary conditions $y(0)=0$ and $y(L)=0$ — ②

Case (i): Let $\lambda=0$

$$y''+\lambda y=0$$

$$m^2+\lambda=0$$

$$m^2=-\lambda$$

$$m^2=0$$

The soln is $y(x) = Ax+B$ — ③

using the boundary conditions

$$y(0)=0$$

③ $\Rightarrow$ $y(0) = 0+B \Rightarrow \boxed{B=0}$

$y(L)=0$

③ $\Rightarrow$ $y(L)=AL+B$

$0=AL+0$ $[\because B=0]$

$AL=0$

$\Rightarrow \boxed{A=0}$ $(\because L>0)$

Eqn ③ reduces to $y(x)=0$.

Since $y(x)\neq 0$. There is no eigen function correspo

to $\lambda=0$.

Case (ii): $\lambda=-\mu^2, \mu\neq 0, \lambda$ is -ve

$y''+\lambda y=0$

$m^2+\lambda=0$

$m^2=-\lambda$

$m^2=\mu^2$

$m=\pm\mu$

The solution is $y(x)=Ae^{\mu x}+Be^{-\mu x}$ — ④

Using the boundary conds,

$y(0)=0$

④ $\Rightarrow$ $y(0)=A+B$

$A+B=0$ — ⑤

$y(L)=0$

④ $\Rightarrow$ $y(L)=Ae^{\mu L}+Be^{-\mu L}$

$Ae^{\mu L}+Be^{-\mu L}=0$ — ⑥

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Solving ⑤ & ⑥

⑤ $\times e^{\mu t} \Rightarrow \cancel{Ae^{\mu t}} + Be^{\mu t} = 0$

⑥ $\times 1 \Rightarrow \underline{\cancel{Ae^{\mu t}} \overset{-}{+} Be^{-\mu t} = 0}$

$$B(e^{\mu t} - e^{-\mu t}) = 0$$

But we have $\mu \neq 0$

$$e^{\mu t} - e^{-\mu t} \neq 0$$

$$\therefore \boxed{B = 0}$$

From ⑤ we've $\boxed{A = 0}$

$A = 0$ & $B = 0$, then eqn ④ reduces to $y(x) = 0$.

since $y(x) \neq 0$ there is no eigen fn corresponding to λ is negative.

Case (iii): Let $\lambda = \mu^2, \mu \neq 0$ and λ is +ve.

$$y'' + \lambda y = 0$$

$$\Rightarrow m^2 + \lambda = 0$$

$$m^2 = -\lambda$$

$$m = \pm \mu i$$

The soln is

$$y(x) = A\cos \mu x + B \sin \mu x$$

using boundary conds $y(0) = 0$

$$y(0) = A \quad [\because y(0) = A\cos 0 + B \sin 0].$$

$\Rightarrow \boxed{A=0}$

& $y(L)=0$

$$y(L) = A\cos \mu L + B \sin \mu L$$

$$0 = 0 + B\sin \mu L \quad [\because A=0]$$

$$\therefore B\sin\mu L = 0 \quad — ⊛$$

If $B=0$, then $A=0$ then eqn ① reduces to $y(x)$ which is not a eigen function. So $B \neq 0$ for the existence of eigen fns. since $B \neq 0$, then ⊛ reduces

$$\sin \mu L = 0$$

so that $\mu = \frac{n\pi}{L}$, $n = 1, 2, 3 \ldots$

$$\therefore \lambda = \mu^2 = \frac{n^2\pi^2}{L^2}$$

$$\text{ie } \mu^2 = \frac{n^2\pi^2}{L^2}$$

So the required eigen function is $y_n(x)$ with corresponding eigen values λ_n are -

$$y_n(x) = B_n \sin \lambda_n x$$

$$y_n(x) = B_n \sin \frac{n\pi}{L}x$$

2 mark.

③ $y''+\lambda y=0$ & $y(0)=0$ & $y(1)=0$ - find eigen values & fn.

Soln Given $y''+\lambda y=0$ — ① with the boundary condn

$y(0)=0$ & $y(1)=0$ — ②

Case (i) : let $\lambda=0$

$y''+\lambda y=0$

$m^2+\lambda=0 \Rightarrow m^2=-\lambda$

$m^2=0$

The soln is $y(x)=Ax+B$ — ③

Using the boundary condns,

$y(0)=0$

③ $\Rightarrow y(0)=A(0)+B$

$\Rightarrow \boxed{B=0}$

$y(1)=0$

③ $\Rightarrow y(1)=A(1)+B$

$0=A+B$

$\boxed{A=0}$

Eqn ③ reduces to $y(x)=0$

Since $y(x)\neq 0$, there is no eigen fn corresponding to $\lambda=0$.

Case (ii) : $\lambda=-\mu^2$, $\mu\neq 0$, λ is -ve

$y''+\lambda y=0$

$m^2 = -\lambda$

$m^2 = \mu^2$

$m = \pm\mu$

∴ The soln is $y = Ae^{\mu x} + Be^{-\mu x}$

using $y(0) = 0$

we've $A + B = 0$ — ⑤ $\left[\because y(0) = Ae^0 + Be^0\right]$

using $y(1) = 0$

$Ae^{\mu} + Be^{-\mu} = 0$ — ⑥ $\left[\because y(1) = Ae^{\mu} + Be^{-\mu}\right]$

consider ⑤ & ⑥

⑤ × e^{μ} ⇒ $A\cancel{e^{\mu}} + Be^{\mu} = 0$

⑥ × 1 ⇒ $\underline{\overset{-}{A}e^{\mu} + Be^{-\mu} = 0}$

$B(e^{\mu} - e^{-\mu}) = 0.$

Since $\mu \neq 0 \Rightarrow e^{\mu} - e^{-\mu} \neq 0$

∴ $B = 0$

Hence $y(x) = 0$. There is no eigen values corresponding to λ is -ve.

Case (iii): let $\lambda = \mu^2$, $\mu \neq 0$ & λ is +ve

$y'' + \lambda y = 0$

$m^2 = -\lambda$

$m^2 = -\mu^2$

$m = \pm\mu i$

Thus the soln is $y(x) = A\cos \mu x + B\sin \mu x$ — ⑦

using $y(0)=0 \Rightarrow$ $\boxed{A=0}$ $\left[\because y(0) = A\cos 0 + B\sin 0\right]$ — ⑧

using $y(1)=0 \Rightarrow$ $0 = B\sin\mu$ — ⑨ $\left[\because y(1) = 0 + B\sin\mu \quad \because A=0\right]$

But $\mu \neq 0 \Rightarrow B\sin\mu = 0$

If $B=0$, then $A=0$.

Then eqn ③ reduces to $y(x)=0$ which is not an eigen fn. So $B\neq 0$ for the existence of eigen fn.

Since $B\neq 0$, eqn ⑨ reduces to $\sin\mu(1) = 0$

so that $\mu = n\pi$, $n=1,2,\ldots$

we've $A=0$, $\mu = n\pi$ the eqn ⑦ reduces to

$y(x) = B\sin n\pi x$, $n=1,2,\ldots$

$\lambda = \mu^2 = n^2\pi^2$, $n=1,2,\ldots$

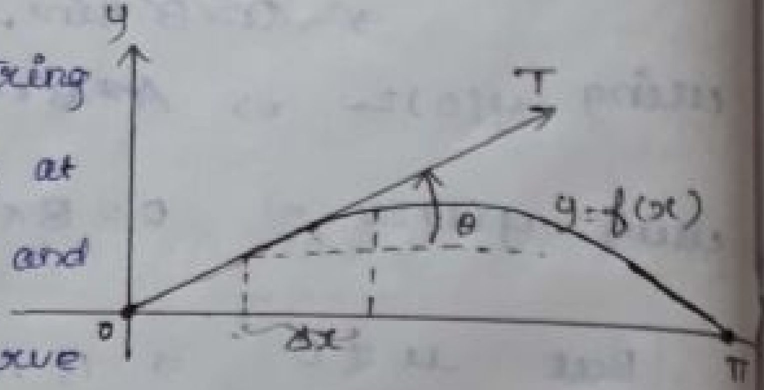
So the required eigen fn is $y_n(x)$ with corresponding eigen values

Thus $y_n(x) = B_n \sin n\pi x$

$\lambda_n = n^2\pi^2$, $n=1,2\ldots$

5marks &
10 marks

One dimensional wave equations



Let a flexible string is pulled on the x-axis at the two points be $x=0$ and $x=\pi$. Then we got a curve $y=f(x)$ in the xy plane. In order to obtain the equation of motion, we make several assumptions.

i) At each point of the string has constant x co-ordinate, so that its y coordinate dep only on x and the time t.

ii) The time derivatives $\frac{\partial y}{\partial t}$ and $\frac{\partial^2 y}{\partial t^2}$ represent th string's velocity and acceleration.

iii) We consider the motion of small piece wh in its equilibrium position has length Δx

If the linear mass density of the string is $m=m(x)$, so that the mass of the piece is $m\Delta x$, by Newton's II law of motion, the transverse fo F acting on it is given by

$$F = m\Delta x \frac{\partial^2 y}{\partial t^2} \quad \text{①}$$

F=ma

Since the string is flexible, the tension $T = T(x)$ at any point is directed along the tangent and has $T \sin\theta$ as its y component.

We next assume that the motion of the string is due solely (alone) to the tension in it.

Let F is the difference between the values of $T \sin\theta$ at the ends of our piece, (ie) $\Delta(T\sin\theta)$

$$① \Rightarrow \quad \Delta(T\sin\theta) = m\,\Delta x \frac{\partial^2 y}{\partial t^2} \longleftarrow ②$$

If the vibrations are relatively small, so that θ is small and sin θ is approximately equal to

~~sinθ~~
tanθ ~~tanθ~~ $= \partial y/\partial x$, then

$$② \Rightarrow \quad \Delta\left(T\frac{\partial y}{\partial x}\right) = m\,\Delta x \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\Delta\left(T\,\partial y/\partial x\right)}{\Delta x} = m\frac{\partial^2 y}{\partial t^2} \longleftarrow ③$$

and when Δx is allowed to approach 0, we obtain

$$\frac{\partial}{\partial x}\left(T\cdot\frac{\partial y}{\partial x}\right) = m\frac{\partial^2 y}{\partial t^2} \longleftarrow ④$$

where m and T are constants.

$\therefore$ ④ $\Rightarrow$ $\boxed{a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}}$ ⟵ ⑤ $\frac{T}{m} \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$. $a^2 = T$

with $a = \sqrt{T/m}$. $a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$

⑤ is called the one-dimensional wave equa

Find the Fourier series :

We've to show that the soln $y(x, t)$ that satisfies the boundary conditions

$y(0, t) = 0$ ⑥ and $y(\pi, t) = 0$ ⑦

and the initial conditions,

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ — ⑧ and $y(x, 0) = f(x)$ ⑨

from ⑥ & ⑦ we've the ends of the are perman fixed string is motionless at the points $x = 0$ & $x =$

from ⑧ & ⑨ we've the string is motionless when it is released and that $y = f(x)$ is its shape at that moment.

Let the soln of ⑤ by the method of "separation of variables" in the form

$$y(x,t) = u(x)\,v(t) \longleftarrow ⑩$$

$$\frac{\partial^2 y}{\partial x^2} = u''(x)\,v(t) \text{ --- } ⑪$$

$$\frac{\partial^2 y}{\partial t^2} = u(x)\,v''(t) \longleftarrow ⑫$$

using ⑪ & ⑫ in ⑤, we've

$$⑤ \Rightarrow \quad a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \quad a^2 \left[u''(x)\,v(t)\right] = u(x)\,v''(t)$$

$$\Rightarrow \quad \frac{u''(x)}{u(x)} = \frac{v''(t)}{a^2 v(t)} \longleftarrow ⑬$$

Now equating each term to the constant $-\lambda$, we get

$$\frac{u''(x)}{u(x)} = -\lambda$$

$$u''(x) + \lambda u(x) = 0 \text{ --- } ⑭$$

Illy $\quad v''(t) + \lambda a^2 v(t) = 0 \longleftarrow ⑮ \quad \left[\because \frac{v''(t)}{a^2 v(t)} = -\lambda\right].$

Solve ⑭ by using ⑥ & ⑦. $\left[\begin{array}{l} y(0,t)=0 \\ y(\pi,t)=0 \end{array} \right.$

⑥ & ⑦ $\Rightarrow$ $u(0) = u(\pi) = 0$

But W.K.T, ⑭ has a nontrivial soln iff $\lambda = n^2$ for some +ve integer n and the corresp eigen functions (solns) are

$$u_n(x) = \sin nx.$$

111'y, for these λ's (the eigen values) the general sol of ⑮ is

$$v(t) = c_1 \sin nat + c_2 \cos nat$$

$$v'(t) = c_1 an \cos nat + c_2 an(-\sin nat)$$

(ie) $v'(t) = c_1 an \cos nat - c_2 an \sin nat.$ ⑯

using ⑧ $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

$$\Rightarrow v'(0) = 0$$

$$\Rightarrow 0 = c_1 an(1) - t_2 a_n(0)$$

$$\Rightarrow 0 = c_1 an$$

where a & n are constant values.

$$\boxed{\therefore c_1 = 0}$$

Then the soln is $v_n(t) = \cos nat.$

The corresponding products of the form ⑩ are

$y_n(x,t) = \sin nx \cos nat$, $n = 1, 2, \ldots$ ⑰

⑰ satisfies 6, 7, 8 & ⑰ is true for any finite sum of constant multiplies of y_n's.

$$y_n(x,t) = a_1 \sin x \cos at + a_2 \sin 2x \cos 2at + \cdots + a_n \sin nx \cos nat$$

$$y_n(x,t) = \sum_{n=1}^{\infty} a_n \sin nx \cos nat .$$

$$= a_1 \sin x \cos at + a_2 \sin 2x \cos 2at + \cdots$$ ⑱

which is also a soln that satisfies 6, 7, 8, 9.

from ⑨ $\Rightarrow y(x,0) = f(x)$

$$f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots$$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx$$ ⑲

Eqn ⑲ is called the "Fourier sine series of f(x)".

Find eigen functions.

Let the eigen functions $u_m(x)$ and $u_n(x)$

(ie) sin mx and sin nx satisfy the eqns

$$u_m'' = -m^2 u_m \longleftarrow ①$$

$$u_n'' = -n^2 u_n \longrightarrow ②$$

① × u_n ⇒ $u_n u_m'' = -m^2 u_n u_m$

② × u_m ⇒ $\underline{\overset{-}{u_m} u_n'' = \overset{(+)}{-} n^2 u_n u_m}$

$$u_m'' u_n - u_m u_n'' = u_n u_m (n^2 - m^2)$$

$$(u_n u_m' - u_m u_n')' = u_n u_m (n^2 - m^2)$$

$\int$ing on both sides from 0 to π

$$\int_0^\pi (n^2 - m^2) u_n \cdot u_m \, dx = \left[u_n u_m' - u_m u_n' \right]_0^\pi$$

$$(n^2 - m^2) \int_0^\pi \sin nx \sin mx \, dx = [m \cos mx \cdot \sin x - n \cos nx \sin m$$

$$n^2 - m^2 \int_0^\pi \sin nx \sin mx \, dx = 0 \quad \text{if } m \neq n.$$

W.K.T $\quad f(x) = \sum_{n=1}^{\infty} a_n \sin nx$

Multiplying both sides by sin nx,

$$f(x) \sin nx = \sum_{n=1}^{\infty} a_n \sin^2 nx$$

$\int$ing on both sides from 0 to π

$$\int_0^{\pi} f(x) \sin nx \, dx = a_n \int_0^{\pi} \sin^2 nx \, dx$$

$$= a_n \int_0^{\pi} \left(\frac{1 - \cos 2nx}{2} \right) dx$$

$$= \frac{a_n}{2} \left[x - \sin 2nx \right]_0^{\pi}$$

$$\therefore \int_0^{\pi} f(x) \sin nx \, dx = \frac{a_n}{2} \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \text{ ——— } ③$$

Thes a_n's are called the Fourier coefficients of $f(x)$

and ③ is called Euler's formula.

— x — .

Note:

i) Eqn ⑲ is called the Fourier sine series of $f(x)$ or the eigenfunction expansion of $f(x)$ in terms of eigenfns $\sin nx$.

ii) Eqn ⑱ is called Bernoulli's soln of the wave eqn.

UNIT - V

Definition;

Non-linear eqn ;

Consider, the second order non-trivial eqn of the form $\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right)$.

Definition;

Phase ;

The value of x & dx/dt which at each incident characterize the state of system are called in phase.

Phase plane ;

The plane of the variables x & dx/dt is called the phase plane.

Autonomous system ;

$$\frac{dx}{dt} = F(x, y)$$

$$\frac{dy}{dt} = G(x, y)$$, where F & G are continuous.

The system in which independent variables t does not appear in the funt. F & G on the write is said to be autonomous.

Path of the system (or) directed curve ;-

If $f(x)$ & $y(x)$ are not both constant funt. Then $x = x(t)$ & $y = y(t)$ defines a curve in the phase plane is called a path of the system.

Critical points:

A point (x_0, y_0) is said to be critical point if both F & G vanishes.

That is $F[x_0, y_0] = 0$ & $G[x_0, y_0] = 0$.

Isolated critical point:

A critical point (x_0, y_0) is said to be isolated if there exist a circle center on $[x_0, y_0]$. That contains no other critical points.

Problem:

1. To find the critical points of $dx/dt = y^2 - 5x + 6$ & $\frac{dy}{dt} = x - y$

Soln:

Given the autonomous system are

$$\frac{dx}{dt} = y^2 - 5x + 6 \ \& \ \frac{dy}{dt} = x - y$$

$$F(x,y) = y^2 - 5x + 6, \quad G(x,y) = x - y.$$

To find the critical points.

$$F(x,y) = 0 \ \& \ G(x,y) = 0$$

$$x - y = 0 \Rightarrow x = y.$$

$$F(x,y) = 0$$

$$y^2 - 5x + 6 = 0$$

$$(y-2)(y-3) = 0$$

$$y = 2 \ \& \ y = 3.$$

If $y = 2$ then $x = 2$

If $y = 3$ then $x = 3$

The critical points are $(2,2), (3,3)$

2. Find the critical point $\frac{d^2x}{dt^2} + \frac{dx}{dt} - (x^3 + x^2 - 2x) = 0$

Soln:

Let $y = \frac{dx}{dt} \Rightarrow \frac{d^2x}{dt^2} = \frac{dy}{dt}$

Given, $\frac{d^2x}{dt^2} + \frac{dx}{dt} - (x^3+x^2-2x) = 0.$

$$\frac{d^2x}{dt^2} + y - (x^3+x^2-2x) = 0$$

$$\frac{dy}{dt} + y - (x^3+x^2-2x) = 0$$

$$\frac{dy}{dt} = x^3+x^2-2x-y$$

∴ The autonomous system are $\frac{dx}{dt} = y.$

$$\frac{dy}{dt} = x^3+x^2-2x-y.$$

$F(x,y) = y.$

$G(x,y) = x^3+x^2-2x-y.$

To find the critical points.

$F(x,y)=0, \; G(x,y)=0$

$\Rightarrow F(x,y) = y \Rightarrow y=0$

$\Rightarrow x^3+x^2-2x=0$

$\Rightarrow x(x^2+x-2)=0$ $\quad x=0, \; x^2+x-2=0$ $\quad -2$ $\quad -1 \,|\, 2$ $\quad 1$

$\quad (x-1)(x+2)=0$

$\quad x = 1, -2$

$x=0, \; x=\pm1, \; x=-2.$

∴ The critical points are $(0,0)(1,0)(2,0)$.

3. Find the critical points of $\frac{d^2x}{dt^2} + g/a \sin x = 0$

Soln:

Let $y = \frac{dx}{dt}, \; \frac{d^2x}{dt^2} = \frac{dy}{dt}$

$$\frac{dy}{dt} + g/a \sin x = 0.$$

$$\frac{dy}{dt} = -g/a \sin x.$$

The autonomous system are

$$\frac{dx}{dt} = y.$$

$$\frac{dy}{dt} = -g/a \sin x.$$

$F(x,y) = y.$

$G(x,y) = -g/a \sin x.$

$F(x,y) = 0$ & $G(x,y) = 0.$

$F(x,y) = 0 \Rightarrow y = 0.$

$-g/a \sin x = 0$

$\sin x = 0$

$\sin x = \sin n\pi$

$x = n\pi, \; n = 1, 2, 3 \ldots$

put $n = 1 \Rightarrow x = \pi$

$n = 2 \Rightarrow x = 2\pi$

The critical points are $(\pi, 0)(2\pi, 0)(3\pi, 0) \ldots$

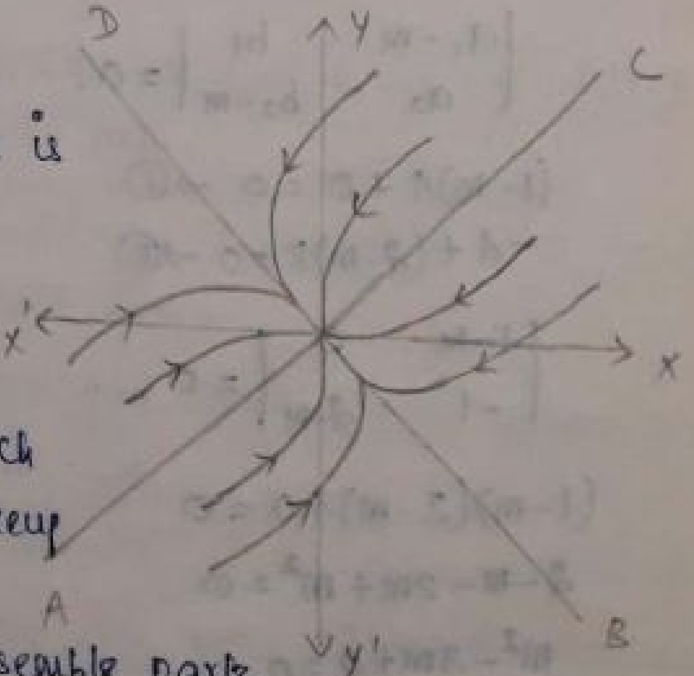
TYPES OF CRITICAL POINTS STABILITY

NODES:

A critical point like that in the figure is called node.

For the Node There are four of line paths.

Ao, Bo, Co, Do which together with origin makeup the lines AB & CD.



All other paths resemble parts and as each of these paths approaches o its slope approaches that of the line AB.

Examples:

1. consider the system $\frac{dx}{dt} = x, \frac{dy}{dt} = -x+2y$.

Solu:

Given the autonomous system are,

$$\left.\begin{aligned}\frac{dx}{dt} &= x\\ \frac{dy}{dt} &= -x+2y\end{aligned}\right\} \rightarrow ⓐ$$

$F(x,y) = x.$

$G(x,y) = -x+2y.$

$F(x,y) = 0$ & $G(x,y) = -x+2y$

$F(x,y) = 0.$

$\Rightarrow x = 0$ & $\Rightarrow -x+2y = 0.$

$2y = 0$

$y = 0$

$\therefore$ The critical points are $(0,0)$

$(a_1 - m)A + b_1 B = 0$

$a_2 A + (b_2 - m)B = 0$

$a_1 = 1,\ b_1 = 0,\ a_2 = -1,\ b_2 = 2.$

$$\begin{vmatrix} a_1 - m & b_1 \\ a_2 & b_2 - m \end{vmatrix} = 0. \Rightarrow \begin{vmatrix} 1-m & 0 \\ -1 & 2-m \end{vmatrix} = 0$$

1-

$(1-m)A + 0 = 0 \rightarrow ①$

$-A + (2-m)B = 0 \rightarrow ②$

$$\begin{vmatrix} 1-m & 0 \\ -1 & 2-m \end{vmatrix} = 0$$

$(1-m)(2-m) + 0 = 0$

$2 - m - 2m + m^2 = 0$

$m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1,\ m = 2.$

∴ The roots are real & distinct take $m=1$.

$(1-m)A = 0$

$-A+(2-m)B = 0$

$(1-1)A = 0$

$-A+(2-1)B = 0$

$-A+B = 0$

Let $A=1$, $B=1$.

$x(t) = \overset{Ae^t}{e^t}$, $y(t) = \overset{Be^t}{e^t}$

$m=2$, $-A=0$

$\Rightarrow A=0$, $B=2$

$\therefore x(t) = 0e^{2t}$; $y(t) = 2e^{2t}$

$\therefore$ Solns are $x(t) = c_1 e^t$

$$\left.\begin{array}{l} \\ y(t) = c_1 e^t + 2c_2 e^{2t} \end{array}\right\} \rightarrow ②$$

Case (i):

when $c_1 = 0$ we have $x=0$ & $y = 2c_2 e^{2t}$. In this case the path is the +ve y-axis. when $c_2 > 0$. The path is the -ve y-axis. Then $c_2 > 0$ and each path approaches and enter the origin $t \to -\infty$.

Case (ii):

when $c_2 = 0$, we have $x = c_1 e^t$ & $y = c_1 e^t$. ie) $x=y$. This path is the half line path.

If $x>0$ and $c_1 > 0$.

also this path is the half line.

If $x<0$ and $c_1 < 0$ and again 0 pase approaches and enter the origin $t \to -\infty$.

Case (iii):

when c_1 & $c_2 \neq 0$. The path lie on the parabola's

ie) $x(t) = c_1 e^t$

ie) $y(t) = c_1 e^t + 2c_2 e^{2t}$

$y = x + \frac{2C_2}{C_1} x^2$.

The path of the parabola's are then $x>0$ & $C_1>0$, $x<0$ & $C_1<0$.

∴ By the above discussion the critical point (0,0) is a Node. Also, we can find the type of a critical point by solving the differential eqn.

⊛ $\Rightarrow \frac{dy}{dx} = \frac{-x+2y}{x} \Rightarrow \frac{dy}{dx} = -1 + 2y/x$

$\Rightarrow \frac{dy}{dx} - \frac{2y}{x} = -1$

This is of the form, $\frac{dy}{dx} + Py = Q$

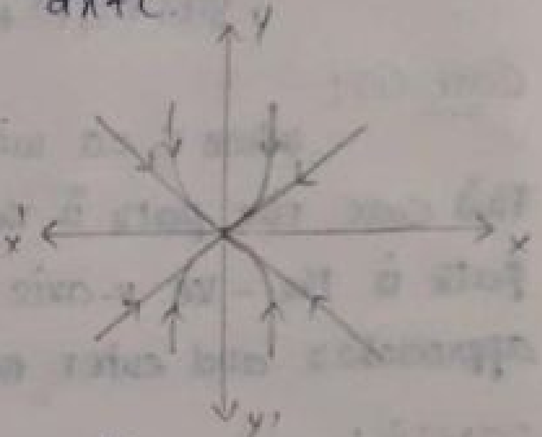
∴ The soln is $ye^{-\int P(x)dx} = \int Q e^{\int P dx} dx + c$

$ye^{\int 2/x\, dx} = \int -e^{\int 2/x\, dx} dx + c$

$yx^2 = -\int x^2 dx + c$

$yx^2 = -\frac{x^3}{3} + c$

$y = -\frac{x}{3} + \frac{c}{x^2}$



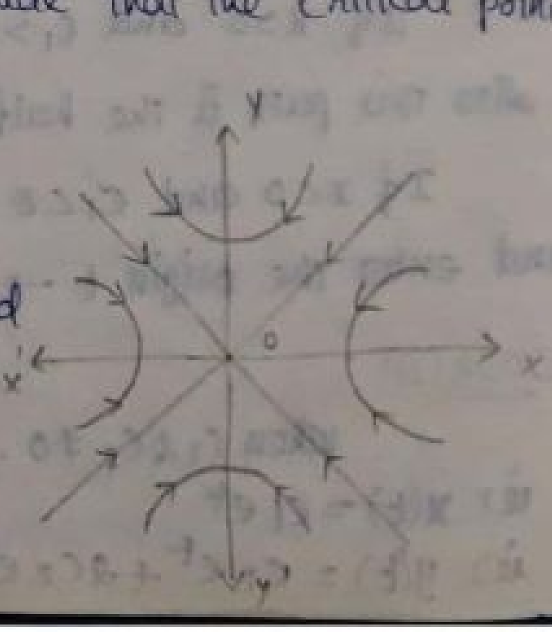
This procedure gives no information above the manner in which the paths are trased out.

From the soln. we conclude that the critical point (0,0) is a node.

Saddle point:

A critical point like that in the figure is called a saddle point.

It is approaches and entered by two half line



path A_0, B_0 as $t \to \infty$ and these two paths lie on a line AB.

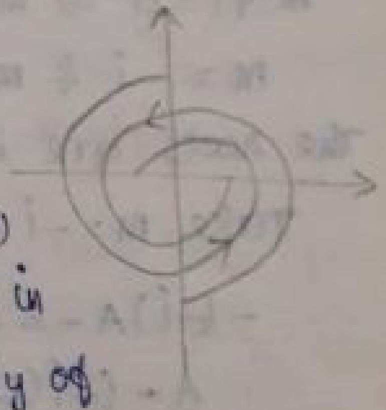
It is also approach and entered by two half line paths C_0 & D_0 as $t \to \infty$ and these two paths lies no another line CD, b/w the four half line paths these are four region each contains a family of paths resembling hyperbola.

These paths to not approach 0 as $t \to \infty$ (or) $t \to -\infty$.

Center (or) Vertex : ⇒

A center is a critical point that is surrounded by a family of closed paths. It is not approached by any path as $t \to \infty$ (or) $t \to -\infty$.

Spiral (or) Focus :



A critical point like that in the figure is called spiral (or) Focus. such a point is approach in a spiral like manner by a family of paths that wind arround it and infinite no. of lines as $t \to \infty$ (or) $t \to -\infty$.

Problem :

1. Consider the system $\frac{dx}{dt} = -y$, $\frac{dy}{dt} = x$. find the critical point and find the type of critical point.

Soln :

Given the autonomous system are,

$$\left.\begin{aligned}\frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x\end{aligned}\right\} \rightarrow ①$$

$F(x,y) = -y, \; G(x,y) = x.$

$F(x,y) = 0 \Rightarrow -y = 0 \Rightarrow y = 0 \Rightarrow x = 0.$

The critical points are (0,0)

$(a_1 - m)A + b_1 B = 0$

$a_1 A + (b_1 - m)B = 0$

$a_1 = 0, \; b_1 = -1, \; a_2 = 1, \; b_2 = 0$

$-mA - B = 0 \rightarrow ②$

$A - mB = 0 \rightarrow ③$

$$\begin{vmatrix} -m & -1 \\ 1 & -m \end{vmatrix} = 0$$

$m^2 + 1 = 0 \Rightarrow m = \pm i$

$m = -i \;\&\; m = i$

The roots are distinct and complex root.

Take $m = -i$ & $A = 1$.

$-(-i)A - B = 0$

$A - (-i(B)) = 0$

$\Rightarrow i - B = 0$

$B = i$

$1 + iB = 0$

let $A = 1, \; B = -1.$

$x(t) = -c_1 \sin t + c_2 \cos t \rightarrow ④$

$y(t) = c_1 \cos t + c_2 \sin t \rightarrow ⑤$

$t=0,\ x(0)=0.$

$0=C_2$

$\therefore x(t) = -C_1 \sin t$

$y(0)=C_1 \Rightarrow C_1=-1.$

$\therefore y(t) = -\cos t = \sin t(t-\pi/2) = \sin[-(\pi/2 - t)]$

$x(t) = \sin t = \cos(t-\pi/2)$ $\sin\theta = \cos(\theta-90^\circ)$

$x(t)^2 + y(t)^2 = \sin^2(t-\pi/2) + \cos^2(t-\pi/2)$

$= 1.$

$\frac{dy}{dx} = -x/y.$ $\frac{dx}{dt} = -y,\ \frac{dy}{dt} = x.$

$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{x}{-y} \Rightarrow \frac{dy}{dt}\cdot\frac{dt}{dx} = -\frac{x}{y} \Rightarrow dy/dx = -x/y$

$dy.y = -x.dx$

$\int y.dy = -\int x.dx$

$y^2/2 = -x^2/2 + c^2/2.$

$x^2+y^2=c^2$ is the circle with centre (0,0) and radius is c.

$\therefore$ The critical point (0,0) is the centre.

2. If 'a' is an orbitrary constant then the system $dy/dt = x+ay$, $\frac{dx}{dt} = ax-y$. find the critical point and it 'y' type.

Soln:

Given the autonomous system are

$$\left.\begin{array}{l}\frac{dy}{dt} = x+ay \\ dx/dt = ax-y\end{array}\right\} \longrightarrow (*)$$

$F(x,y) = ax-y,\ G(x,y) = x+ay$

$F(x,y)=0 \Rightarrow ax-y=0 \Rightarrow -y=0 \Rightarrow y=0.$

$G(x,y)=0 \Rightarrow x+ay=0 \Rightarrow \cancel{-y/-1/0} \Rightarrow x=0$

∴ The critical points are (0,0)

Let $x = r\cos\theta,\ y = r\sin\theta.$ and

$\theta = \tan^{-1}(y/x)$ be the polar co-ordinates.

$$\frac{dy}{dx} = \frac{x+ay}{ax-y} \rightarrow ①$$

$$x^2+y^2 = r^2 \rightarrow ②$$

Diff!- w.r.to 'x' in eqn ②,

$$2x + 2y\,dy/dx = 2r.\,dr/dx \rightarrow ③$$

$$\theta = \tan^{-1}(y/x)$$

$$\tan\theta = y/x\ .$$

Diff!- w.r.to 'x'

$$\sec^2\theta . \frac{d\theta}{dx} = \frac{x.\,dy/dx - y.1}{x^2}$$

$$\sec^2\theta = 1+\tan^2\theta = \frac{1+y^2}{x^2} = \frac{x^2+y^2}{x^2}$$

$$\sec^2\theta = \frac{r^2}{x^2}$$

$$\frac{r^2}{x^2} . \frac{d\theta}{dx} = \frac{x.\,dy/dx - y}{x^2}$$

$$r^2 . \frac{d\theta}{dx} = x.\,dy/dx - y \rightarrow ④$$

$$\frac{③}{④} \Rightarrow \frac{r.\,dr/dx}{r^2.\,d\theta/dx} = \frac{x+y.\,dy/dx}{x.\,dy/dx - y}$$

$$= \frac{x+y.\left(\frac{x+ay}{ax-y}\right)}{x\left(\frac{x+ay}{ax-y}\right)-y}$$

$$\frac{③}{④} \Rightarrow \frac{x+y\,dy/dx}{x.\,dy/dx-y}$$

$$\Rightarrow \frac{x+y\left(\frac{x+ay}{ax-y}\right)}{x\left(\frac{x+ay}{ax-y}\right)-y} \Rightarrow \frac{\frac{ax^2-\cancel{xy}+\cancel{xy}+ay^2}{ax-y}}{\frac{x^2+\cancel{axy}-\cancel{axy}+y^2}{ax-y}}$$

$$\Rightarrow \frac{ax^2+ay^2}{x^2+y^2}$$

$$\Rightarrow \frac{a(x^2+y^2)}{1(x^2+y^2)}$$

$$\Rightarrow a$$

$$= \frac{\frac{ax^2 - xy + xy + ay^2}{ax-y}}{\frac{x^2 + axy - axy + y^2}{ax-y}}$$

$$= \frac{ax^2 + ay^2}{x^2+y^2} = \frac{a(x^2+y^2)}{(x^2+y^2)} = \frac{a(r^2)}{r^2}$$

$$\frac{dr/dx}{r . d\theta/dx} = a.$$

$$\frac{1}{r}\frac{dr}{d\theta} = a$$

$$\frac{dr}{r} = a\,d\theta$$

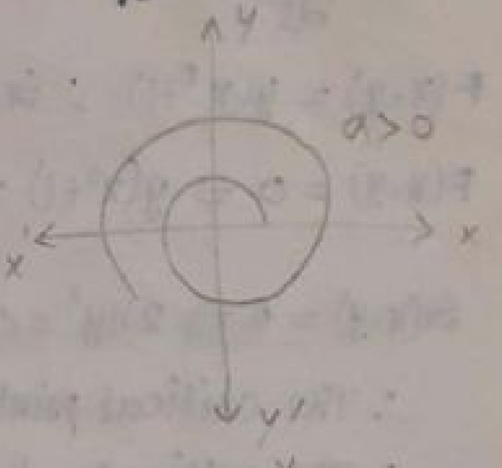
$$\int \frac{dr}{r} = \int a.d\theta$$

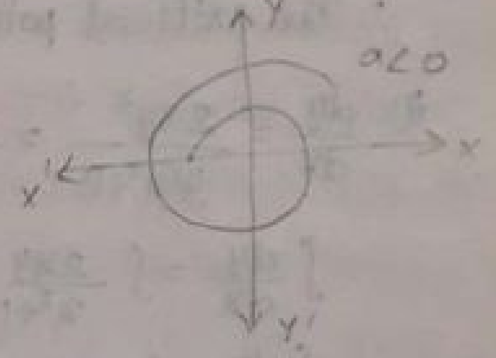
$$\log r = a\theta + \log c.$$

$$\log(r/c) = a\theta$$

$$r/c = e^{a\theta}$$

$$r = ce^{a\theta}$$





∴ The critical points are the spiral. If $a>0$, if $a<0$, if $a=0$; Then $r=c$.

ie) $\Rightarrow x^2+y^2+c^2$

∴ If $a=0$ the critical point is the centre.

If $a<0$, $a>0$ the critical point is the spiral.

3. For each of the following non-linear system.

i) Find the critical points. $\Rightarrow \frac{dx}{dt} = y(x^2+1), \frac{dy}{dt} = 2xy^2$

ii) Find the D.E of the data $dy/dx \Rightarrow \frac{dx}{dt} = -x, \frac{dy}{dt} = 2x^2y^2$

iii) solve the eqn of find the paths.

iv) Sketch a few of the paths and show the deviation of increasing t.

Soln:

i) Given the autonomous system are,

$$\left.\begin{aligned}\frac{dx}{dt} &= y(x^2+1)\\ \frac{dy}{dt} &= 2xy^2\end{aligned}\right\} \rightarrow ①$$

$F(x,y) = y(x^2+1)$; $G(x,y) = 2xy^2$

$F(x,y) = 0 \Rightarrow y(x^2+1) = 0 \Rightarrow y = 0$ & $x^2+1 = 0$

$\Rightarrow y = 0$ & $x = \pm i$

$G(x,y) = 0 \Rightarrow 2xy^2 = 0$

$\therefore$ The critical points are $(i,0)(-i,0)$

$\therefore$ The critical points lie on the x-axis.

ii) $\frac{dy}{dx} = \frac{2xy^2}{y(x^2+1)} = \frac{2xy}{x^2+1} \Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2+1}$

$\int \frac{dy}{dx} = \int \frac{2xy}{x^2+1}$

$\int \frac{dy}{y} = \int \frac{2x}{x^2+1}\, dx$

$\log y = \log(x^2+1) + \log c$

$y = c(x^2+1)$ is that soln of diffl. eqn.

$y = cx^2+c$

$y - c = cx^2$

$\therefore$ The path is a parabola.

iii) $\frac{dx}{dt} = -x$, $\frac{dy}{dt} = 2x^2y^2$.

$F(x,y) = -x$; $G(x,y) = 2x^2y^2$

$F(x,y) = 0 \Rightarrow -x = 0 \Rightarrow x = 0$

$G(x,y) = 0 \Rightarrow 2x^2y^2 = 0$.

$\therefore$ The critical paths lie on y- axis.

$$\frac{dy}{dx} = -2xy^2 \Rightarrow \frac{dy}{y^2} = -2x\,dx$$

$$\int \frac{dy}{y^2} = \int -2x\,dx$$

$$\frac{y^{-2+1}}{-2+1} = -2\left(x^2/2\right) + C$$

$$\frac{y^{-1}}{-1} = -x^2 + C$$

$$1/y = x^2 + C$$

$$y = \frac{1}{x^2 + C}$$

$$\Rightarrow \int y^{-2} dx = -\int 2x\,dx$$

$$\frac{y^{-2+1}}{-2+1} = -2x^2/2 + c$$

$$\frac{y^{-1}}{-1} = -x^2 + c$$

$$1/y = x^2 + c$$

From the soln we conclude that the node.

Stable :

Consider an isolated critical point of the System $dx/dt = F(x,y)$ & $dy/dt = G(x,y)$ and assume that this point its located at the origin $(0,0)$ of the Phase plane, this critical point is said to be Stable if for each +ve nol- R. There exist a +ve nol- $r \leq R$. Such that every path while is inside the circle $x^2+y^2 = r^2$ for some $t = t_0$ remains inside the circle $x^2+y^2 = R^2$ $\forall\, t > 0$.

Unstable :

If above critical point is not stable then it is called unstable.

Asymptotically stable :

A critical point is said to be asymptotically Stable, if it stable and ∃ a circle $x^2+y^2 = r_0^2$ such that every path which is inside this circle for some $t = t_0$ approaches the origin as $t \to \infty$.

Ex:1 : Centre is stable but not asymptotically stable.

Ex:2 : Saddle point, spiral are unstable.

Ex:3 : Node is stable and asymptotically stable.

Problem:

1. Each of the following linear system has the origin as an isolated critical point.

   i) Find the general soln.

   ii) Find the diffl- eqn of the path.

   iii) Solve the eqn found in (ii) and sketch a few of the paths showing the direction of increasing t.

   iv) Discuss the stability of the critical point

$$\frac{dx}{dt} = x, \frac{dy}{dt} = -y.$$

Soln: Consider, $\frac{dx}{dt} = x, \ \frac{dy}{dt} = -y.$

$a_1 = 1, \ b_1 = 0, \ a_2 = 0, \ b_2 = -1$

$(a_1 - m)A + b_1 B = 0$

$a_2 A + (b_2 - m)B = 0.$

$(1-m)A = 0 \rightarrow ①$

$(-1-m)B = 0 \rightarrow ②$

$$\begin{vmatrix} 1-m & 0 \\ 0 & -1-m \end{vmatrix} = 0$$

$(1-m)(-1-m) - 0 = 0$

$-1 - \cancel{m} + \cancel{m} + m^2 = 0$

$m^2 - 1 = 0$

$m^2 = 1$

$\Rightarrow m = \pm 1.$

Take $A = 1$, Choose $m = 1$.

$(-1-1)B = 0$

$-2B = 0$

$B = 0$

$x(t) = e^{t}$ & $y(t) = 0$

$m = -1$, $B = 1$ Choose,

$(1+1)A = 0 \Rightarrow 2A = 0 \Rightarrow A = 0.$

$x(t) = 0$ & $y(t) = c_2 e^{-t} = e^{-t}$ Ae^{mt}

$\therefore$ General soln is $x(t) = c_1 e^{t}$

$y(t) = c_2 e^{-t}$

$$\frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{+y} = \frac{-dx}{x}$$

$$+\int \frac{dy}{y} = \int \frac{-dx}{x}$$

$$\log y = -\log x + \log c$$

$$y = c/x$$

$\therefore$ The critical points is unstable.

2. $\frac{dx}{dt} = -x$, $\frac{dy}{dt} = -2y$.

<u>Solu:</u>

$a_1 = -1$, $b_1 = 0$ & $a_2 = 0$, $b_2 = -2$.

$(a_1 - m)A + b_1 B = 0$

$a_2 A + (b_2 - m)B = 0$

$(-1-m)A + 0 = 0 \rightarrow ①$

$0 + (-2-m)B = 0 \rightarrow ②$

$$\begin{vmatrix} -1-m & 0 \\ 0 & -2-m \end{vmatrix} = 0$$

$(-1-m)(-2-m) - 0 = 0$

$2 + m + 2m + m^2 = 0$

$m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1, -2.$

Put $m = -1$, take $A = 1$.

$(-2-1)B = 0$

$-3B = 0$

$B = 0$

$x(t) = e^{-t}$. & $y(t) = 0$

Put $m = -2$, take $B = 1$

$(-1-2)A = 0$

$-3A = 0$

$A = 0$

$x(t) = 0$ & $y(t) = e^{-2t}$

The general soln is $x(t) = c_1 e^{-t}$

$y(t) = c_2 e^{-2t}$

Diffl- eqn is, $\frac{dy}{dx} = \frac{2y}{x}$

$\frac{dy}{y} = \frac{2dx}{x}$

$\int dy/y = 2\int dx/x$

$\log y = 2 \log x + \log c$

$y = x^2 + c$

$x^2 = y/c$.

∴ The critical path is asymptotically stable.