CORE COURSE V

INTEGRAL EQUATIONS, CALCULUS OF VARIATIONS AND TRANSFORMS

Objectives.

1. To introduce the concept of calculus of variations and integral equations and their applications.
2. To study the different types of transforms and their properties.

UNIT I

Calculus of variations – Maxima and Minima – the simplest case – Natural boundary and transition conditions - variational notation – more general case – constraints and Lagrange's multipliers – variable end points – Sturm-Liouville problems.

UNIT – II

Fourier transform - Fourier sine and cosine transforms - Properties Convolution - Solving integral equations - Finite Fourier transform - Finite Fourier sine and cosine transforms - Fourier integral theorem - Parseval's identity.

UNIT III

Hankel Transform : Definition – Inverse formula – Some important results for Bessel function – Linearity property – Hankel Transform of the derivatives of the function – Hankel Transform of differential operators – Parseval's Theorem

UNIT IV

Linear Integral Equations - Definition, Regularity conditions – special kind of kernels – eigen values and eigen functions – convolution Integral – the inner and scalar product of two functions – Notation – reduction to a system of Algebraic equations – examples– Fredholm alternative - examples – an approximate method.

UNIT V

Method of successive approximations: Iterative scheme – examples – Volterra Integral equation – examples – some results about the resolvent kernel. Classical Fredholm Theory: the method of solution of Fredholm – Fredholm's first theorem – second theorem – third theorem.

TEXT BOOKS

[1] Ram.P.Kanwal – Linear Integral Equations Theory and Practise, Academic Press 1971.
[2] F.B. Hildebrand, Methods of Applied Mathematics II ed. PHI, ND 1972.
[3] A.R. Vasishtha, R.K. Gupta, Integral Transforms, Krishna Prakashan Media Pvt Ltd, India, 2002.

UNIT – I Chapter 2: Sections 2.1 to 2.9 of [2]
UNIT – II Chapter 7 of [3]
UNIT – III Chapter 9 of [3]; UNIT – IV -Chapters 1 and 2 of [1]
UNIT – V Chapters 3 and 4 of [1]

REFERENCES

[1] S.J. Mikhlin, Linear Integral Equations (translated from Russian), Hindustan Book Agency, 1960.
[2] I.N. Snedden, Mixed Boundary Value Problems in Potential Theory, North Holland, 1966.

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Unit - I

Def: Maxima & Minima:-

A pt of which a fun attains its maximum (or) (minimum) value is a maxima (or) (minimum) of f respectively.

Application of calculus of variation:

Application of calculus of variation's one mainly used with determing this maxima & minima of certain expression involving unknown funs.

(2m) **Necessary & sufficient condition for maxima & minima of the fun:-**

case(i) :-

The fun has only one independent variable. Consider $\boxed{y=f(x)}$.

Which is differential in an (a,b)

The necessary condition for $x_0 \in (a,b)$

to ~~Two~~ maxima (or) (minima) if $\frac{dy}{dx}=0$ at x_0.

The sufficient condition for $x_0 \in (a,b)$

to be, i) Maxima if it's $\frac{d^2y}{dx^2}<0$.

ii) Minima if it's $\frac{d^2y}{dx^2}>0$.

case (i)

The fun has two independent variable $\boxed{z = f(x,y)}$ in a region R & R. there partial derivative $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ exist and are continuous on R.

the nessary condition for a point $(x_0, y_0) \in R$ to be a maxima or (minima) of $\frac{dz}{dx} = 0$, $\frac{dz}{dy} = 0$ at (x_0, y_0)

(OR)

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy = 0 \text{ at } (x_0, y_0).$$

the sufficient condition involve certain in equalities the 2nd second order partial derivative, this concept can be similarly extented for a fun having 'n' independent variable.

i) Maxima $\frac{\partial^2 z}{\partial x^2} < 0$; $\frac{\partial^2 z}{\partial y^2} < 0$.

ii) Minima $\frac{\partial^2 z}{\partial x^2} > 0$, $\frac{\partial^2 z}{\partial y^2} > 0$.

Stationary point :-

Consider a fun of 'n' variable's say $(x_1, x_2 \ldots x_n)$ then f has a maximum (or) (minimum) value at an interior point of a region.

If $df = \frac{\partial f}{\partial x_1} \cdot dx_1 + \frac{\partial f}{\partial x_2} \cdot dx_2 + \cdots + \frac{\partial f}{\partial x_n} \cdot dx_n \longrightarrow (1)$.

Those pt's at which eqn (1) ~~is~~ satisfies is called stationary pt's and the fun if is said to be stationary.]

In otherwards, the pts at which the fun attains maximum or minimum values are called stationary pts.

Procedure for finding stationary points:-

consider a fun $f(x,y,z) \longrightarrow (1)$

subject to the constraints

$$\left.\begin{array}{l}\phi_1(x,y,z)=0\\ \phi_2(x,y,z)=0\end{array}\right\} \rightarrow (2)$$

Then the stationary values can be obtained by the following procedure

At stationary values $df=0$.

i.e.) $df = f_x\,dx + f_y\,dy + f_z\,dz = 0 \longrightarrow (3)$

where, $f_x = \frac{\partial f}{\partial x}$; $f_y = \frac{\partial f}{\partial y}$; $f_z = \frac{\partial f}{\partial z}$ $\longrightarrow$ 4a

where $\left.\phi_{1x} = \frac{\partial \phi_1}{\partial x}.\quad \phi_{2x} = \frac{\partial \phi_2}{\partial x}\right\} \rightarrow$ 4b

x^{ly} (4a) & (4b) by λ_1 & λ_2 and adding (3) we get,

$(f_x+\lambda_1\phi_{1x}+\lambda_2\phi_{2x})dx + (f_y+\lambda_1\phi_{1y}+\lambda_2\phi_{2y})dy +$

$$(f_z+\lambda_1\phi_{1z}+\lambda_2\phi_{2z})dz = 0.$$

(OR)

$f_x+\lambda_1\phi_{1x}+\lambda_2\phi_{2x} = 0 \rightarrow 5a$

$f_y+\lambda_1\phi_{1y}+\lambda_2\phi_{2y} = 0 \rightarrow 5b$

$f_z+\lambda_1\phi_{1z}+\lambda_2\phi_{2z} = 0 \rightarrow 5c.$

② ~~then~~ the eqn's 5a, 5b, 5c ~~are~~ with 2a, 2b determins x, y, z ~~and~~ with λ_1, λ_2.

Legrange multipliers:-

Ao ~~The~~ introduction of the quantities λ_1 & λ_2 in the above eqn's (5a, 5b, 5c) frequently simplyfy the soln procedure.

this quantities λ_1 & λ_2 are known as legrangean Mutipliers's,

V.V.I. Pbm:- Determine the stationary pt on the curve of intersection of the surface $z = xy+5$, $x+y+z=1$. Which is a nearest the orgion of the sphere.

Soln:- Here we must the minimize the eqn of the sphere $f = x^2+y^2+z^2 \rightarrow ①$

Using this in 2b

We get $x = -y$.

Using $z=1$ & $x=-y$ in 2a

2a $\Rightarrow$ $1 = xy+5$

$1 = (-y)\, y + 5$

$1 = -y^2 + 5$

$-y^2 = -4$

$\boxed{y = \pm 2.}$

When $y = 2$; $x = -2$

$y = -2$; $x = 2$.

$\therefore$ The stationary pts of $(2, -2, 1)$

&

$(-2, 2, 1)$.

To Be Continued.....

Unit-2.

Fourier Transforms.

Dirichlet's condition's:-

A fun's F(x) is said to satisfies dirichlet's condition's in the interval (a,b)

i) F(x) is define and single exact posibily at a finite no of pt in the interval (a,b).

ii) F(x) & F'(x) are piecewies continuous in the interval (a,b).

Fourier series:-

If f(x) periodic fun with period 2l that is F(x+2l) = F(x) and satisfies dirichlet's condition in the interval [-l, l] Then at every point of continuty.

We have,

formula:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right] \dashrightarrow ①$$

where,

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \longrightarrow ②$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx \longrightarrow ③.$$

The series eqn ① with co-eff a_n & b_n. Gn by eqn ② & ③ respectively is called the fourier series of F(x) and co-eff corresponding to F(x).

Note:-

At a point of discontinuity.

$$f(x) = \frac{1}{2}\left[f(x+0) + f(x-0)\right]$$

If the function $f(x)$ define in the interval $-l, l$ be a even fun of x.

That is $f(-x) = f(x)$

Then $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$

$$= \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = 0.$$

In this case we get fourier cosine series. Again $F(x)$ is a odd fun of x.

$$\Rightarrow F(-x) = -F(x)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} F(x)\cos\frac{n\pi x}{l}\,dx = 0$$

and

$$b_n = \frac{2}{l}\int_{0}^{l} F(x)\sin\frac{n\pi x}{l}\,dx$$

$\therefore$ In this case we get Fourier sine series.

Fourier integral formula:-

Let $f(x)$ be a fun satisfy Drichlet's condition in every finite interval $(-l \leq x \leq l)$ and defined as $\frac{1}{2}[f(x+0)+f(x-0)]$ at every pt of discontinuity for there $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges, ie $F(x)$ is obsolulety integrable that is $F(x)$ is obsolutely integrable in $(-\infty < x < \infty)$ then

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(v)\left[\int_{-\infty}^{\infty} \cos w(x-v)\,dw\right]dv \longrightarrow ①$$

(OR)

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} \cos w(x-v)F(v)\,dv.$$

The representation eqn① off $f(x)$ is known as fourier integral formula.

Another form:

We have,

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(v)\left[\int_{-\infty}^{\infty} i\sin w(x-v)\,dw\right]dv \longrightarrow ②$$

① + ② ⇒

we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\,dv\left[\int_{-\infty}^{\infty}[\cos w(x-v)+i\sin w(x-v)]\right]dw$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\,dv\left[\int_{-\infty}^{\infty} e^{iw(x-v)}\,dw\right]dv$$

Fourier transform's (or) Complex fourier transform :-

(2m) Let $f(x)$ be a fun defined on $-\infty, \infty$ and the piecewise continuous in each finite partial interval and absoultely interval $(-\infty, \infty)$. Then,

$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$ is called the fourier transform's of $F(x)$ and is denoted by $F[F(x)]$ (or) $\tilde{f}(P)$.

The fun $F(x)$ is called the inverse fourier transform's of $\tilde{f}(P)$.

$$F\{f(x)\} = \tilde{f}(P)$$

$$f(x) = F^{-1}\{\tilde{f}(P)\}.$$

(5m) Inverse thm for complex fourier tranfers:

If $\tilde{f}(P)$ is the fourier transform's of $F(x)$ & $f(x)$ satisfies the dirichelet's condition in every finite interval $[-l, l]$ and for there a $\int_{-\infty}^{\infty} |f(x)|\, dx$ is convergent then at every pt of continuity of $f(x)$.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(P)\, e^{-ipx}\, dp.$$

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proof:-

W.K.T

Fourier integral formula.

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(v)\left[\int_{-\infty}^{\infty} e^{i\omega(x-v)}d\omega\right]dv.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}d\omega\left[\int_{-\infty}^{\infty} f(v)e^{-i\omega v}dv\right]$$

Put $\omega = -p$ | $p = -\infty$ How?

$d\omega = -dp$ | $dp = \infty$.

Limit How to change?

$$\therefore F(x) = \frac{1}{2\pi}\int_{\infty}^{-\infty} e^{-ipx}(-dp)\left[\int_{-\infty}^{\infty} F(v)e^{ipv}dv\right]$$

Put $v = x$; $dv = dx$.

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}dp\left[\int_{-\infty}^{\infty} F(x)e^{ipx}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx}dp\left[\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)e^{ipx}dx}_{\tilde{f}(P)}\right]$$

$$\therefore F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx}dp\,\tilde{f}(P).$$

Note:- sum other's also defined a fourier transform in the following form's.

1. $\tilde{f}(P) = \int_{-\infty}^{\infty} e^{-ipx}f(x)dx$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{F}(P)e^{ipx}dx.$$

2. $\tilde{f}(P) = \int_{-\infty}^{\infty} e^{ipx}F(x)\,dx.$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{F}(P)e^{-ipx}.F(P)dp.$$

3. $\tilde{f}(P) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} f(x)\,dx.$

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(P) e^{ipx}\,dp.$

Fourier sine transform's :-

The infinite fourier sine transform of $f(x)$, $0 < x < \infty$ defined by $F_s\{F(x)\}$ (or) $\tilde{f}_s(P)$.

$\ni: F_s\{F(x)\} = \tilde{f}_s(P) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin px\,dx.$

The fun $f(x)$ is called the inverse fourier sine transform of $\tilde{f}_s(P)$.

$\Rightarrow F(x) = F_s^{-1}\{\tilde{f}_s(P)\}.$

Note:-

Some others also define $\tilde{f}_s(P) = \int_0^{\infty} f(x) \sin px$

Inverse formula for fourier sine transform :-

If $\tilde{f}_s(P)$ if the fourier transform of the fun $f(x)$ which satisfy the Dirichlet's conditions in every finite interval $(0, \infty)$. $\ni: \int_0^{\infty} |F(x)|\,dx$.

Then $F(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \tilde{f}_s(P) \sin px\,dp$ at every pt of continuity of $f(x)$.

$\therefore$ is an inverse

formula for finite fourier sine transform.

proof:-

W.K.T, Fourier integral formula

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(v)\left[\int_{-\infty}^{\infty} \cos w(x-v)\,dw\right]dv$$

$$= \frac{2}{2\pi}\int_{0}^{\infty} F(v)\left[\int_{-\infty}^{\infty} \cos w(x-v)\,dw\right]dv$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} dw\left[\int_{0}^{\infty} F(v)\cos w(x-v)\,dv\right]$$

Put $w=p$; $dw=dp$.

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} dp\left[\int_{-\infty}^{\infty} F(v)\cos p(x-v)\,dv\right]$$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} dp\int_{0}^{\infty} F(v)\left[\cos px\cos pv + \sin px\sin pv\right]dv$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} F(v)\,dv\left[\int_{-\infty}^{\infty}\cos px\cos pv\,dp + \int_{-\infty}^{\infty}\sin px\sin pv\,dp\right]$$

Put $v=x$

$dv=dx$.

$$F(x) = \frac{1}{\pi}\int_{0}^{\infty} F(x)\,dx\left[\int_{-\infty}^{\infty}\cos px\cos px\,dx + \int_{-\infty}^{\infty}\sin px\sin px\,dp\right]$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\cos px\,dp\int_{-\infty}^{\infty}F(x)\cos px\,dx + \frac{1}{\pi}\int_{0}^{\infty}\sin px\,dp\int_{-\infty}^{\infty}F(x)\sin px\,dx$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\cos px\,dp\int_{0}^{\infty}\underbrace{f(x)\cos px\,dx}_{\text{even}} + \frac{2}{\pi}\int_{0}^{\infty}\sin px\,dp\int_{0}^{\infty}F(x)\sin px\,dx$$

Now defind $F(x)$ in $(-\infty,\infty)$. $\ni$: $f(x)$ is an odd fun of x. $F(x)\cos px$ is an odd fun & $f(x)\sin px$ is an even fun of x.

If $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin px \, dp$.

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin px \, dx.$$

$$\therefore f(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \sin px \, dp \, \tilde{f}_s(P).$$

According to other authors to defined

$$\tilde{f}_s(P) = \int_0^\infty F(x) \sin px \, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{f}_s(P) \sin px \, dp.$$

Cosine Transform:-

The infinite fourier cosine transform of $f(x)$, x lies b/w $(0, \infty)$ is defined by $F_c\{F(x)\}$ (or) $\tilde{f}_c(P)$.

$$\ni: F_c\{F(x)\} = \tilde{F}_c(P) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty F(x) \cos px \, dx.$$

The fun $F(x)$ is called the inverse fourier cosine transform of $f(x) = F_c^{-1}\{\tilde{f}_c(P)\}$.

Note:-

$$\tilde{f}_c(P) = \int_0^\infty F(x) \cos px \, dx.$$

—— × —— × ——

Inverse formula for fourier cosine formula

statement:- $\tilde{f}_c(P)$ is the $F(x)$ which satisfies the Dirichlet's condition's in every finite interval $(0, l)$ and $\ni: \int_0^\infty |f(x)| \, dx$.

then $f(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \tilde{f}_c(p) \cos px \, dp$ at every pt of continuity of $f(x)$ this is an inverse formula for infinite fourier cosine transform.

proof:-

W.K.T

Fourier integral formula,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(v) \left[\int_{-\infty}^{\infty} \cos w(x-v) \, dw \right] dv.$$

$$= \frac{2}{2\pi} \int_0^\infty f(v) \left[\int_{-\infty}^{\infty} \cos w(x-v) \, dw \right] dv.$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} dw \left[\int_0^\infty f(v) \cos w(x-v) \, dv \right].$$

$w = p$; $dw = dp$.

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dp \left[\int_0^\infty f(v) \cos p(x-v) \, dv \right]$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} dp \left[\int_0^\infty f(v) [\cos px \cos pv + \sin px \sin pv] dv \right].$$

$$= \frac{1}{\pi} \int_0^\infty f(v) dv \left[\int_{-\infty}^{\infty} \cos px \cos pv \, dp + \int_{-\infty}^{\infty} \sin px \sin pv \, dp. \right.$$

$v = x$; $dv = dx$

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) dx \left[\int_{-\infty}^{\infty} \cos px \cos px \, dp + \int_{-\infty}^{\infty} \sin px \sin px \, dp \right].$$

$$= \frac{1}{\pi} \int_0^\infty \cos px \, dp \int_{-\infty}^{\infty} f(x) \cos px \, dx + \frac{1}{\pi} \int_0^\infty \sin px \, dp \int_{-\infty}^{\infty} f(x) \sin px \, dx$$

$$= \frac{2}{\pi} \int_0^\infty \cos px \, dp \int_0^\infty f(x) \cos px \, dx + \frac{2}{\pi} \int_0^\infty \sin px \, dp \int_0^\infty f(x) \sin px \, dx \quad (1)$$

Now defined $f(x)$ in $(-\infty, \infty)$. $\exists$: $f(x)$ is an even fun of x, $f(x)\cdot \sin px$ is an odd fun of x, $f(x)\cdot\cos px$ is an even fun of x.

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos px\, dp \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos px\, dp.$$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \tilde{f}_c(p)\cos px\, dp.$$

Linear property of fourier transforms:-

If $\tilde{f}(p), \tilde{g}(p)$ are Fourier transform of $f(x)$ & $g(x)$ respectively. Then

$$F[af(x)+bg(x)] = a\tilde{f}(p)+b\tilde{g}(p).$$

Where, a & b are constant.

proof:-

W.K.T

$$F\{f(x)\} = \tilde{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{ipx} f(x)\,dx$$

$$F[g(x)] = \tilde{g}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{ipx} g(x)\,dx.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{ipx}[af(x)+bg(x)]\,dx.$$

$$F[af(x)+bg(x)] = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{ipx} af(x)\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{ipx} bg(x)\,dx\right]$$

$$= \frac{a}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{ipx} f(x)\,dx + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{ipx} g(x)\,dx.$$

$$\therefore F[af(x)+bg(x)] = a\tilde{f}(p)+b\tilde{g}(p),$$

Hence the proved.

Change of scale property:-

① For complex Fourier Transforms:-

If $\tilde{f}(P)$ is the complex fourier transform of $f(x)$ the complex Fourier transform of $f(ax)$ is $F[F(ax)] = \frac{1}{a}\tilde{f}(P/a)$.

proof:-

W.K.T,

$$F[F(x)] = \tilde{f}(P) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)\,dx.$$

$x = ax$

$$F[F(ax)] = \tilde{f}(P) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(ax)\,dx.$$

Put $ax = t \Rightarrow x = \frac{t}{a}$

$a\,dx = dt \Rightarrow dx = \frac{dt}{a}$.

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iP(t/a)} f(t)\cdot\frac{dt}{a}$$

$$= \frac{1}{a}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iP(t/a)} f(t)\,dt\right]$$

$e^{it(P/a)}$

$$F[f(ax)] = \frac{1}{a}\tilde{f}(P/a)$$

Hence the proved.

② For Fourier sine transform:-

If $\tilde{f_s}(P)$ is the fourier sine transform of $f(x)$. Then fourier sine transform of $f(ax)$.

$$F_s[f(ax)] = \frac{1}{a}\tilde{f_s}(P/a).$$

Proof:

W.K.T

$$F_s[F(x)] = \tilde{f}_s(P) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(x) \sin px \, dx .$$

$$F_s[F(ax)] = \tilde{f}_s(P) = \frac{\sqrt{2}}{\pi} \int_0^\infty F(ax) \sin px \, dx .$$

Put $ax = t \Rightarrow x = \frac{t}{a}$

$a\,dx = dt \Rightarrow dx = \frac{dt}{a}$.

$$F[F(ax)] = \frac{\sqrt{2}}{\pi} \int_0^\infty f(t) \sin p(t/a) \frac{dt}{a}.$$

t, dt

$$= \frac{1}{a}\left[\frac{\sqrt{2}}{\pi} \int_0^\infty f(t) \sin t(P/a)\, dt\right]$$

$$F[F(ax)] = \frac{1}{a} \tilde{f}_s(P/a),,$$

(2). If $\tilde{f}_c(P)$ is the fourier cosine transform of $f(x)$. Then Fourier cosine transform of $f(x)$.

$$F_c[F(ax)] = \frac{1}{a} \tilde{f}_c(P/a).$$

Proof:

W.K.T

$$F_c[f(x)] = \tilde{f}_c(P) = \frac{\sqrt{2}}{\pi} \int_0^\infty f(x) \cos px \, dx .$$

$$F_c[f(ax)] = \tilde{f}_c(P) = \frac{\sqrt{2}}{\pi} \int_0^\infty f(ax) \cos px \, dx .$$

Put $ax = t \Rightarrow x = \frac{t}{a}$

$a\,dx = dt \Rightarrow dx = \frac{dt}{a}$

$$\therefore F_c[F(ax)] = \frac{\sqrt{2}}{\pi}\int_0^\infty F(t)\cos p(t/a)\,\frac{dt}{a}.$$

$$= \frac{1}{a}\left[\frac{\sqrt{2}}{\pi}\int_0^\infty f(t)\cos t(P/a)\,dt\right].$$

$$\therefore F_c[f(ax)] = \frac{1}{a}\tilde{f}_c(P/a)$$

④ Shifting property:-

If $\tilde{f}(P)$ is the complex fourier tranform of $f(x)$. Then complex Fourier transform $f(x-a)$ is $F\{F(x-a)\} = e^{iPa}\tilde{f}(P)$.

proof: W.K.T

$$F[f(x)] = \tilde{f}(P) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iPx} F(x)\,dx.$$

$$F[F(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iPx} f(x-a)\,dx.$$

Put $x-a=t \Rightarrow x=a+t$

$dx=dt$.

$$\therefore F[F(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iP(a+t)} f(t)\,dt$$

$$= \frac{e^{iPa}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iPt} f(t)\,dt.$$

$$= e^{iPa}\tilde{f}(P) //$$

⑤ Modulation thm:-

If $\tilde{f}(P)$ is the complex Fourier transform of $f(x)$. Then complex Fourier transform of $f(x)\cos ax$ is,

$$F\{F(x)\cos ax\} = \frac{1}{2}\{\tilde{F}(P-a)+\tilde{f}(P+a)\}.$$

proof:-

W.K.T

$$F\{f(x)\} = \tilde{f}(P) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx. \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$F\{f(x)\cos ax\} = \tilde{f}(P) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x)\cos ax\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x) \left[\frac{e^{iax}+e^{-iax}}{2}\right] dx.$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} e^{iax} f(x)\,dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} e^{-iax} f(x)\,dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(p+a)x} f(x)\,dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(p-a)x} f(x)\,dx\right].$$

$$= \frac{1}{2}\left[\tilde{f}(P+a) + \tilde{f}(P-a)\right].$$

$$\therefore F[f(x)\cos ax] = \frac{1}{2}\left[\tilde{f}(P+a) + \tilde{f}(P-a)\right]$$

Theorem:-

If $f'(P)$ & $\hat{f}(P)$ are Fourier sine & cosine Transform of $f(x)$. Then,

i) $F_s[f(x)\cos ax] = \frac{1}{2}[\tilde{f}_s(P+a) + \tilde{f}_s(P-a)]$.

ii) $F_c[f(x)\sin ax] = \frac{1}{2}[\tilde{f}_s(P+a) - \tilde{f}_s(P-a)]$.

iii) $F_s[f(x)\sin ax] = \frac{1}{2}[\tilde{f}_c(P-a) - \tilde{F}_c(P+a)]$.

proof:

i) $F_s[F(x)] = \tilde{f}_s(P) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\sin px\,dx$.

$\sqrt{\frac{2}{\pi}}$ Doute

$$F_s[F(x)\cos ax] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\cos ax \sin px.$$

$\cos A \sin B$;

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{2} f(x)[\sin(P+a)x + \sin(P-a)x]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\sin(a+P)x\,dx + \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(x)\sin(P-a)x\,dx\right]$$

$F_s[f(x)\cos ax] = \frac{1}{2}[\tilde{f}_s(p+a) + \tilde{f}_s(p-a)]$.

ii) $F_c[f(x)] = \tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos px \, dx$.

$F_c[f(x)\sin ax] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin ax \cos px \, dx$.

$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty [\sin(p+a)x - \sin(p-a)x] \, dx$

$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(p+a)x \, dx - \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(p-a)x \, dx\right]$.

$\therefore F_c\{F(x)\sin ax\} = \frac{1}{2}[\tilde{f}_s(p+a) - \tilde{f}_s(p-a)]$.

iii) $F_s[f(x)\sin ax] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin ax \sin px \, dx$

$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) [\cos(p-a)x - \cos(p+a)x \, dx]$

$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(p-a)x \, dx - \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(p+a)x \, dx\right]$.

$\therefore F_s[F(x)\sin ax] = \frac{1}{2}[\tilde{f}_c(p-a) - \tilde{f}_c(p+a)]$.

Result:- If $\phi(p)$ the Fourier sine transform of $f(x)$ for $p>0$ then

$F_s[f(x)] = -\phi(-p)$ for $p<0$.

proof:

$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin px \, dx$ (D)

$= \phi(p)$ for $p>0$.

For $p<0$, Let $p=-s$, where $s>0$.

$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(-sx) \, dx$

$= -\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$.

$= -\phi(s)$

$= -\phi(-p)$ for $p<0$.

Hence $F_s[f(x)] = \begin{cases} \phi(p) & p>0 \\ -\phi(p) & p<0. \end{cases}$

$F_s[f(x)] = \phi\,|p|\ \text{sgn}\, p.$

$\text{sgn}\, p = \begin{cases} 1 & p>0 \\ -1 & p<0. \end{cases}$

Pbm:-

Find the Fourier transform of $f(x)$ defined by $F(x) = \begin{cases} 1 & |x|<a \\ 0 & |x|>0 \end{cases}$ & Hence evaluate (a) $\int_{-\infty}^{\infty} \frac{\sin pa \cos px}{p}\,dp$ & (b) $\int_{0}^{\infty} \frac{\sin p}{p}\,dp$.

Soln:-

$$\tilde{F}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{ipx}.1.dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ipx}}{ip}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ipa}}{ip} - \frac{e^{-ipa}}{ip}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{ipa} - e^{-ipa}}{ip}\right)$$

$$= \frac{2}{p\sqrt{2\pi}} \left(\frac{e^{ipa} - e^{-ipa}}{2i}\right) = \frac{e^{ipa} - e^{-ipa}}{p\sqrt{2\pi}} \cdot \frac{2\sqrt{2}}{\sqrt{2}\cdot\sqrt{\pi}}$$

$\tilde{F}(p) = \frac{2\sin pa}{p\sqrt{2\pi}}$

$$\tilde{F}(p) = \frac{\sqrt{2}\sin pa}{p\sqrt{\pi}}, \quad p \neq 0.$$

$$\tilde{F}(p) = \frac{2a}{\sqrt{2\pi}}.$$

limit $p \to 0$.

$2\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{i}$

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

(a) W.K.T, if $\tilde{F}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ipx} dx$.

Then (inverse formula) (now take)

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{F}(p) e^{-ipx} dp \rightarrow (2) \text{ formula}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$\therefore \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2\sin pa}{p\sqrt{2\pi}} e^{-ipx} dp = \begin{cases} 1 & |x| < a \\ 0 & |x| > a. \end{cases}$$

But R.H.S $= \frac{1}{\not{2}\pi} \int_{-\infty}^{\infty} \frac{\not{2}\sin pa\,[\cos px - i\sin px]}{p} dp$. [$\because$ sub eqn (1)(2)]

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin pa \cos px}{p} dp - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin pa \sin px}{p} dp.$$

$$L.H.S = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin pa \cos px}{p} dp.$$ How?

$\therefore$ Integrand in the integral is an odd fun.

$$\therefore \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin pa \cos px}{p} dp = \begin{cases} 1 & |x| < a \\ 0 & |x| > a. \end{cases}$$

How?

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin pa \cos px}{p} dp = \begin{cases} \pi & |x| < a \\ 0 & |x| > a. \end{cases} \longrightarrow (3).$$

(b) If $x = 0$ & $a = 1$ in (3) we get

$\cos 0 = 1$

$$\int_{-\infty}^{\infty} \frac{\sin pa}{p} dp = \pi$$

$$2 \int_{0}^{\infty} \frac{\sin pa}{p} dp = \pi$$

$a = 1$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin p}{p} dp = \frac{\pi}{2}.$$

Pbm:

Find the Fourier transform of

$$F(x) = \begin{cases} 1 - x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$ & Hence Evaluate

$$\int_{0}^{\infty} \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\left(\frac{x}{2}\right) dx.$$

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Soln:-

$$\tilde{F}(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{ipx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{(1-x^2)e^{ipx}}{ip}-\frac{(-2x)e^{ipx}}{i^2p^2}+\frac{(-2)e^{ipx}}{i^3p^3}\right]_{-1}^{1}$$

$$=\frac{1}{\sqrt{2\pi}}\left[\left(0-\frac{2e^{ip}}{p^2}+\frac{2}{ip^3}e^{ip}\right)-\left(0+2\cdot\frac{e^{-ip}}{p^2}+\frac{2e^{-ip}}{ip^3}\right)\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{-2}{p^2}(e^{ip}+e^{-ip})+\frac{2}{p^3}\frac{(e^{ip}-e^{-ip})}{i}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{-4}{p^2}\left(\frac{e^{ip}+e^{-ip}}{2}\right)+\frac{4}{p^3}\left(\frac{e^{ip}-e^{-ip}}{2i}\right)\right]$$

$$=\frac{4}{\sqrt{2\pi}}\left[\frac{\sin p}{p^3}-\frac{\cos p}{p^2}\right]$$

$$\tilde{F}(p)=\frac{4}{\sqrt{2\pi}}\left[\frac{\sin p-p\cos p}{p^3}\right].$$

$$\tilde{F}(p)=\frac{-4}{\sqrt{2\pi}}\left(\frac{p\cos p-\sin p}{p^3}\right)$$

W.K.T

$$\tilde{F}(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(x)e^{ipx}\,dx$$

$$\Rightarrow F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{F}(p)e^{-ipx}\,dp.$$

$$F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{-4}{\sqrt{2\pi}}\left(\frac{p\cos p-\sin p}{p^3}\right)e^{-ipx}\,dp.$$

$$\Rightarrow \frac{-4}{2\pi}\int_{-\infty}^{\infty}\left(\frac{p\cos p-\sin p}{p^3}\right)(\cos px-i\sin px)\,dp=\begin{cases}1-x^2 & |x|\le 1\\ 0 & |x|>1\end{cases}$$

$$\Rightarrow \frac{-2}{\pi}\int_{-\infty}^{\infty}\left(\frac{p\cos p-\sin p}{p^3}\right)\cos px\,dp+\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{p\cos p-\sin p}{p^3}\right)\sin px\,dp=\begin{cases}1-x^2 & |x|\le 1\\ 0 & |x|>1\end{cases}$$

$\cos\theta=\frac{e^{ip}+e^{-ip}}{2}$ $\quad$ $\sin\theta=\frac{e^{ip}-e^{-ip}}{2i}$

$e^{i\theta}=\cos\theta+i\sin\theta$

$$\Rightarrow -\int_{-\infty}^{\infty}\left(\frac{p\cos p-\sin p}{p^3}\right)\cos px\,dx+\int_{-\infty}^{\infty}\left(\frac{p\cos p-\sin p}{p^3}\right)\sin px\,dp=\begin{cases}\pi/2(x-x^2) & |x|\le 1\\ 0 & |x|>1\end{cases}$$

Where

$$\Rightarrow -\int_{-\infty}^{\infty}\left(\frac{p\cos p-\sin p}{p^3}\right)\cos px\,dp=\begin{cases}\pi/2\,(1-x^2) & |x|\le 1\\ 0 & |x|>1.\end{cases}$$

$\therefore$ The integrand in the 2^{nd} integral on L.H.S is odd. Taking $x=\frac{1}{2}$, we have.

How?

How

$$\Rightarrow -\int_{-\infty}^{\infty}\left(\frac{p\cos p-\sin p}{p^3}\right)\cos\frac{p}{2}\,dp=\frac{\pi}{2}\left(1-\frac{1}{4}\right).$$

$$\Rightarrow 2\int_{0}^{\infty}\left(\frac{p\cos p-\sin p}{p^3}\right)\cos\frac{p}{2}\,dp=-\frac{3\pi}{8}.$$

Put $p=x$.

$$\Rightarrow \int_{0}^{\infty}\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\left(\frac{x}{2}\right)dx=-\frac{3\pi}{16}.$$

Pbm: Find the Fourier complex transform of $f(x)$ if $f(x)=\begin{cases}e^{i\omega x} & a<x<b\\ 0 & x<0, x>0.\end{cases}$

Soln:

$$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipx}f(x)\,dx.$$

$$=\frac{1}{\sqrt{2\pi}}\int_a^b e^{ipx}e^{i\omega x}\,dx.$$

$\int e^{iox}=\frac{e^{iox}}{io}$

$$=\frac{1}{\sqrt{2\pi}}\int_a^b e^{i(p+\omega)x}\,dx.$$

$=\frac{1}{\sqrt{2\pi}}\times\frac{i}{i^2}$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{i(p+\omega)x}}{i(p+\omega)}\right]_a^b$$

$=\frac{-i}{\sqrt{2\pi}}$

$|x|\le 1$

$|x|>1$

$$=\frac{-i}{\sqrt{2\pi}}\left[\frac{e^{i(p+\omega)b}-e^{i(p+\omega)a}}{p+\omega}\right]$$

$$\therefore F[F(x)] = \frac{-i}{\sqrt{2\pi}}\left[\frac{e^{i(p+\omega)b} - e^{i(p+\omega)a}}{p+\omega}\right].$$

Pbm: 2.

Find the Fourier's transform of $f(x)$ of $F(x) = \begin{cases} \sqrt{2\pi}/2\epsilon & |x| \le \epsilon \\ 0 & |x| > \epsilon. \end{cases}$

Soln:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ipx}dx \qquad = \frac{1}{\sqrt{2\pi}}\int_{-\epsilon}^{\epsilon}\frac{\sqrt{2\pi}}{2\epsilon}e^{ipx}dx$$

$$= \int_{-\epsilon}^{\epsilon}\frac{1}{2\epsilon}e^{ipx}dx. \qquad = \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}e^{ipx}dx$$

$$= \frac{1}{2ip\epsilon}\left(e^{ipx}\right)_{-\epsilon}^{\epsilon} \qquad \left[\frac{e^{ipx}}{ip}\right]$$

$$F[f(x)] = \frac{e^{ip\epsilon} - e^{-ip\epsilon}}{2ip\epsilon}. \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\therefore F[f(x)] = \frac{\sin p\epsilon}{p\epsilon}.$$

Pbm: 3:

Find the cosine transform of the fun $f(x)$ if $F(x) = \begin{cases} \cos x & 0 < x < a. \\ 0 & x > 0. \end{cases}$

Soln:

$$\tilde{f}_c(p) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F(x)\cos px\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{a}\cos x\cos px\,dx \qquad \cos A\cos B = \tfrac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\sqrt{\frac{2}{\pi}} \times \frac{1}{2} \qquad \frac{1}{\sqrt{\pi}} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{a}[\cos(1+p)x + \cos(1-p)x]dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(1+p)x}{1+p} + \frac{\sin(1-p)x}{1-p}\right]_0^{a}$$

$$\therefore \tilde{f}_c(p) = \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(1+p)a}{1+p} + \frac{\sin(1-p)a}{1-p}\right]_p$$

Pbm: 4 :-

Find the cosine transform of a fun of x which is unity for $0<x<a$ & zero for $x>a$. What is the fun. whose cosine transform is $\sqrt{\frac{2}{\pi}}\frac{\sin ap}{p}$.

Soln:-

fn $f(x) = \begin{cases} 1 & 0<x<a \\ 0 & x>a. \end{cases}$

$$\tilde{f_c}(P) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos px\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^a 1\cdot\cos px\,dx. = \sqrt{\frac{2}{\pi}}\left[\frac{\sin px}{p}\right]_0^a = \sqrt{\frac{2}{\pi}}\left[\frac{\sin ap}{p}\right]$$

Again $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \tilde{f_c}(P)\cos px\,dp.$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{\sin pa}{p}\cos px\,dx.$$

$$= \frac{2}{\pi}\left(\frac{x}{p}\right)$$

$$= \frac{1}{\pi}\int_0^\infty\left(\frac{\sin(a+x)p + \sin(a-x)p}{p}\right)dp.$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\sin(a+x)p}{p}dp + \frac{1}{\pi}\int_0^\infty \frac{\sin(a-x)p}{p}.dp.$$

How to apply the limit.

$$f(x) = \frac{1}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 1 \quad \text{if } x<a.$$

and

$$f(x) = \frac{1}{\pi}\left(\frac{\pi}{2}-\frac{\pi}{2}\right) = 0 \quad \text{if } x>a.$$

$$\left[\because \int_0^\infty \frac{\sin\alpha x}{x}dx = \frac{\pi}{2}\right]$$

Pbm: 5

Find the fourier sine transform of $f(x) = \frac{1}{x}$.

Soln:

$$\tilde{f_s}(P) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin px\,dx.$$

Eqn (1) $\tilde{f}_s(P) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \cdot \sin px \, dx$

we have $\int_0^\infty e^{-\alpha x} \sin px \, dx = \frac{p}{\alpha^2+p^2}$.

Integrating both sides with response α, b/w the limit α_1 to α_2, we have.

$$\int_0^\infty \left[\int_{\alpha_1}^{\alpha_2} e^{-\alpha x} dx \right] \sin px \, dx = \int_{\alpha_1}^{\alpha_2} \frac{P \, d\alpha}{p^2+\alpha^2}.$$

$$\int_0^\infty \frac{e^{-\alpha_1 x} - e^{-\alpha_2 x}}{x} \sin px \, dx = \tan^{-1}\frac{\alpha_2}{P} - \tan^{-1}\frac{\alpha_1}{P}.$$

Now, when $\alpha_1 = 0$ & $\alpha_2 = 0$ we have

$$\int_0^\infty \frac{\sin px}{x} dx = \frac{\pi}{2}.$$

$\therefore$ From (i) we have

$$\tilde{f}_s(P) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin px}{x} dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2}.$$

$$\tilde{f}_s(P) = \sqrt{\pi/2}.$$

Pbm: 6.

Find the Fourier sine & cosine transform of e^{-x} and using the inversion formula recover the original fun in both case.

Soln: $\int e^{-ax} \cos bx \, dx = \frac{e^{-ax}}{a^2+b^2} (-a\cos bx + b\sin bx)$

$\int e^{-ax} \sin bx \, dx = \frac{e^{-ax}}{a^2+b^2} [-a \sin bx - b\cos bx]$.

Let $f(x) = e^{-x}$.

$$\tilde{f}_s(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin px \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin px \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+p^2} (-\sin px - p\cos px) \right]_0^\infty$$

$$\sqrt{\frac{2}{\pi}} \left[-\frac{e^{-0}}{1+p^2} (-\sin p(0) - p\cos p(0)) \right]$$

$\sin 0 = 0$

$\cos 0 = 1$

$$\tilde{f}_s(p) = \frac{p}{1+p^2} \sqrt{\frac{2}{\pi}} \cdot \qquad \sqrt{\frac{2}{\pi}} \left\{ -\frac{1}{1+p^2}(0-p) \right\}$$

$$\tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos px \, dx . \qquad \sqrt{\frac{2}{\pi}} \cdot \frac{p}{1+p^2}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos px \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+p^2} (-\cos px + p\sin px) \right]_0^\infty$$

$$\sqrt{\frac{2}{\pi}} \left[-\frac{e^{-0}}{1+p^2} [-\cos p0 + p\sin 0] \right]$$

$$\tilde{f}_c(p) = \frac{1}{1+p^2} \sqrt{\frac{2}{\pi}} . \qquad = \sqrt{\frac{2}{\pi}} \left[-\frac{1}{1+p}(-1) \right]$$

Applying the inversion to the sine transform. We have,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{f}_s(p) \sin px \, dp \qquad = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{p}{1+p^2} \cdot \sqrt{\frac{2}{\pi}} \sin px \, dp$$

$$\frac{2}{\pi}$$

$$= \frac{2}{\pi} \int_0^\infty \frac{p \sin px}{1+p^2} \, dp \longrightarrow (i)$$

And applying inversion to the cosine transform, we have,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{f}_c(p) \cos px \, dp$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos px}{1+p^2} \, dp \longrightarrow (ii)$$

Now Fourier integral thm. we have

$$f(x)=\frac{1}{\pi}\int_0^\infty dp\int_{-\infty}^\infty f(v)\cos p(x-v)\,dv.$$

$$=\frac{1}{\pi}\int_0^\infty \cos px\,dp\int_{-\infty}^\infty f(v)\cos pv\,dv+\frac{1}{\pi}\int_0^\infty \sin px\,dx$$

$$+\int_{-\infty}^\infty f(v)\sin pv\,dv \text{ --- (iii)}$$

case(i)

Defining $f(x)$ in $(-\infty,0)$ $\ni$: $f(x)$ is an even fun of x from (iii) we have

$$f(x)=\frac{2}{\pi}\int_0^\infty \cos px\,dp\int_0^\infty f(v)\cos pv\,dv.$$

Taking $f(x)=e^{-x}$. we have,

$$e^{-x}=\frac{2}{\pi}\int_0^\infty \cos px\,dx\int_0^\infty e^{-v}\cos pv\,dv.$$

$$=\frac{2}{\pi}\int_0^\infty \cos px\left[\frac{e^{-p}}{1+p^2}(-\cos pv+p\sin pv)\right]_0^\infty dp.$$

$$=\frac{2}{\pi}\int_0^\infty \frac{\cos px}{1+p^2}\,dp$$

$$\therefore \int_0^\infty \frac{\cos px}{1+p^2}\,dp=\frac{\pi}{2}e^{-x}.$$

From (ii) we have,

$$f(x)=\frac{2}{\pi}\cdot\frac{\pi}{2}e^{-x}=e^{-x}.$$

case(ii)

Again defining $f(x)$ in $(-\infty,0)$ $\ni$: $f(x)$ is a odd fun of x from (iii) we have,

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin px\,dp\int_0^\infty f(v)\sin pv\,dv.$$

Taking $f(x) = e^{-x}$ & simplifying we have,

From (i) $f(x) = \frac{2}{\pi} \cdot \frac{\pi}{2} \cdot e^{-x}$

$$\boxed{\therefore f(x) = e^{-x}}.$$

Pbm: 7:- Find Fourier cosine transform of $f(x) = \frac{1}{1+x^2}$ & hence find fourier sine transform of $f(x) = \frac{x}{1+x^2}$.

Soln:-

$$\tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos px \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos px}{1+x^2} dx.$$

$\int \cos px \, dp = \cancel{x} \sin px$

Diff both sides with respect to P, we have,

$$\frac{d}{dp} \tilde{f}_c(p) = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x \sin px}{1+x^2} dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x^2+1-1}{x(1+x^2)} \sin px \, dx.$$

— How?

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin px}{x} + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin px}{x(1+x^2)} dx.$$

$\int_0^\infty \frac{\sin x}{x} = [$

$$= -\sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin px}{x(1+x^2)} dx.$$

Diff again with respect to p we have,

$$\frac{d}{dp} \tilde{f}_c(p) = +\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x \sin px}{1+x^2} dx.$$

How?

$$\frac{d^2}{dp^2} \tilde{f}_c(p) = +\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos px}{1+x^2} dx = \tilde{f}_c(p).$$

(OR)

$$(D^2-1) \tilde{f}_c(p) = 0.$$

whose general soln is

$$\tilde{f_c}(P) = Ae^{P} + Be^{-P} \longrightarrow (1)$$

Now, where $P=0$, $\tilde{f_c}(P) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{dx}{1+x^2}$.

$$= \sqrt{\frac{2}{\pi}} \left(\tan^{-1}x\right)_0^\infty$$

$$= \frac{\pi}{2} \sqrt{\frac{2}{\pi}}$$

$$= \sqrt{\frac{\pi}{2}}.$$

$P=0 \Rightarrow \frac{d}{dp} \tilde{f_c}(P) = -\sqrt{\frac{\pi}{2}}$.

$P=0$ in (i) we have

$$\sqrt{\frac{\pi}{2}} = A+B$$

$$-\sqrt{\frac{\pi}{2}} = A-B.$$

Solving $A=0$; $B = \sqrt{\pi/2}$.

side work

eqn (1) $\Rightarrow \left(\frac{d}{dp} = \tilde{f_c}(P) = -\frac{\sqrt{2}}{\pi} \int_0^\infty \frac{x \sin px}{1+x^2} dx.\right.$

$$\frac{d}{dp} \sqrt{\frac{\pi}{2}} e^{-P} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x \sin px}{1+x^2} dx.$$

$$\cancel{-}\sqrt{\frac{\pi}{2}} e^{-P} = \cancel{-}\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x \sin px}{1+x^2} dx$$

$$\sqrt{\frac{\pi}{2}} e^{-P} = \tilde{f_s}(P)$$

$$\therefore \tilde{f_s}(P) = \sqrt{\pi/2}\, e^{-P}.)$$

From (i) we have,

$$\boxed{\tilde{f_c}(P) = \sqrt{\frac{\pi}{2}}\, e^{-P}.}$$

Pbm:

Find the Fourier sine transform of $\frac{e^{-ax}}{x}$.

Soln:
$$f(x) = \frac{e^{-ax}}{x}.$$

$$\tilde{f_s}(P) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin px \, dx.$$

why

Diff both sides with respect to P, we have.

$$\frac{d}{dp} \tilde{f_s}(P) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax} \cos px \cdot x}{x} dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos px \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+p^2} (-a\cos px + p\sin px) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^0}{a^2+p^2} \left[-(-a\cos 0 + p\sin 0) \right] \right].$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{a^2+p^2} [a(1) - 0] \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+p^2} \right]$$

$$\frac{d}{dp} \tilde{f_s}(P) = \frac{a}{a^2+p^2} \sqrt{\frac{2}{\pi}}$$

$$\int \frac{a}{a^2+p^2} = \frac{1}{a} \tan^{-1}(P/a)$$

$$\int \frac{d}{dp} \tilde{f_s}(P) = \int \frac{a}{a^2+p^2} \cdot \sqrt{\frac{2}{\pi}} \, dp + C.$$

→ How?

$$\tilde{f_s}(P) = \frac{a\sqrt{2}}{\sqrt{\pi}} \int \frac{a}{a^2+p^2} + C.$$

$$\tilde{f_s}(P) = \cancel{a} \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\cancel{a}} \tan^{-1}(P/a) + C.$$

$$\tilde{f_s}(P) = \sqrt{\frac{2}{\pi}} \tan^{-1}(P/a) + C.$$

Put $P = 0$; $\tilde{f_s}(P) = 0 \Rightarrow C = 0$.

$$\tilde{f_c}(P) = \sqrt{\frac{2}{\pi}} \tan^{-1}(P/a) + 0.$$

$$\therefore \boxed{\tilde{f_s}(P) = \sqrt{\frac{2}{\pi}} \tan^{-1}(P/a)}$$

Pbm:-

Find the sine & cosine transform of $\frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}$

Soln:-

Fourier sine transform

If $f(x)=\frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}$

$$\tilde{f_s}(p)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin px\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}\sin px\,dx.$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}\cdot\frac{e^{ipx}-e^{-ipx}}{2i}\,dx.$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{1}{2i}\int_0^\infty \frac{e^{(a+ip)x}-e^{-(a+ip)x}}{e^{\pi x}-e^{-\pi x}}dx-\frac{1}{2i}\int_0^\infty\frac{e^{(a-ip)x}-e^{-(a-ip)x}}{e^{\pi x}-e^{-\pi x}}dx\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{\cancel{2}}{2i}\cdot\frac{1}{2}\tan\frac{a+ip}{2}-\frac{1}{2i}\frac{1}{2}\tan\frac{a-ip}{2}\right].$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{1}{4i}\cdot\frac{\sin\frac{a+ip}{2}}{\cos\frac{a+ip}{2}}-\frac{1}{4i}\frac{\sin\frac{a-ip}{2}}{\cos\frac{a-ip}{2}}\right].$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{\sin\frac{a+ip}{2}\cos\frac{a-ip}{2}-\sin\frac{a-ip}{2}\cos\frac{a+ip}{2}}{4i\cos\frac{a+ip}{2}\cos\frac{a-ip}{2}}\right]$$

$2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)=\sin A+\sin B$

$2\sin\left(\frac{A-B}{2}\right)\cdot\cos\left(\frac{A+B}{2}\right)=\sin A-\sin B$

$$=\sqrt{\frac{2}{\pi}}\frac{\sin a+\sin ip-(\sin a-\sin ip)}{2\cdot 2i\,[\cos ip+\cos a]}.$$

$$=\sqrt{\frac{2}{\pi}}\frac{2\cdot\sin ip}{2\cdot 2i\,(\cos ip+\cos a)}$$

$$=\frac{\sin ip}{\sqrt{2\pi}\,i\,(\cos ip+\cos a)}.$$

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$$= \frac{i \sinh p}{\sqrt{2\pi}\, i\,(\cosh p + \cos a)}$$

$\sin i\theta = i \sinh\theta$
$\cos i\theta = \cosh\theta$

$$= \frac{\sinh p}{\sqrt{2\pi}\,[\cosh p + \cos a]}$$

$$\tilde{f}_s(p) = \frac{e^{p} - e^{-p}}{\sqrt{2\pi}\,(e^{p} + e^{-p} + 2\cos a)}$$

Fourier cosine transform:-

$$\tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos px \, dx .$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}} \cos px \, dx .$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{ax} + e^{-ax}}{e^{\pi x} - e^{-\pi x}} \cdot \frac{e^{ipx} + e^{-ipx}}{2} \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{(a+ip)x} + e^{-(a+ip)x}}{e^{\pi x} - e^{-\pi x}} + \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{(a-ip)x} + e^{-(a-ip)x}}{e^{\pi x} - e^{-\pi x}} \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2} \sec \frac{a+ip}{2} + \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2} \sec \frac{a-ip}{2} .$$

$$= \frac{1}{2\sqrt{2\pi} \cos \frac{a+ip}{2}} + \frac{1}{2\sqrt{2\pi} \cos \frac{a-ip}{2}} .$$

$$= \frac{\cos \frac{a-ip}{2} + \cos a+ip}{2\sqrt{2\pi} \cos \frac{a+ip}{2} \cdot \cos \frac{a-ip}{2}} \cdot = \cos A + \cos B .$$

$$= \sqrt{\frac{2}{\pi}} \quad \frac{\cos \frac{a}{2} \cdot \cos \frac{ip}{2}}{\cos a + \cos ip} .$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\cos \frac{a}{2} \cdot \cosh \frac{p}{2}}{\cos a + \cosh p} .$$

$$\therefore \tilde{f}_c(p) = \sqrt{\frac{2}{\pi}} \cdot \frac{\cos \frac{a}{p} \, (e^{p/2} + e^{-p/2})}{2\cos a + e^{p} + e^{-p}}$$

Pbm:

Find the sine transform of $\frac{1}{e^{\pi x}-e^{-\pi x}}$ and deduce that, $F_s(\operatorname{cosech} \pi x) = \frac{1}{\sqrt{2\pi}} \tanh \frac{p}{2}$.

Soln:

If $F(x) = \frac{1}{e^{\pi x}-e^{-\pi x}}$.

$$\tilde{f_s}(p) = F_s[F(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{e^{\pi x}-e^{-\pi x}} \sin px \, dx.$$

$$= \frac{1}{i\sqrt{2\pi}} \int_0^\infty \frac{e^{ipx}-e^{-ipx}}{e^{\pi x}-e^{-\pi x}} \, dx.$$

$$= \frac{1}{i\sqrt{2\pi}} \cdot \frac{1}{2} \tan \frac{ip}{2}.$$

$$\therefore F_s\left[\frac{1}{e^{\pi x}-e^{-\pi x}}\right] = \frac{1}{2\sqrt{2\pi}} \tanh \frac{p}{2} \rightarrow (1).$$

$$= \frac{1}{2\sqrt{2\pi}} \frac{e^{p/2}-e^{-p/2}}{e^{p/2}+e^{-p/2}}$$

$$= \frac{1}{2\sqrt{2\pi}} \cdot \frac{e^p-1}{e^p+1}.$$

$\tanh \theta = \frac{1}{i} \tan i\theta$

$\frac{e^{p/2} - \frac{1}{e^{p/2}}}{e^{p/2} + \frac{1}{e^{p/2}}}$

Deduction from (1) we've

$$F_s\left\{\frac{1}{2\sinh \pi x}\right\} = \frac{1}{2\sqrt{2\pi}} \tanh \frac{p}{2}.$$

$$\therefore F_s[\operatorname{cosech} \pi x] = \frac{1}{\sqrt{2\pi}} \tanh \frac{p}{2}.$$

Pbm:

Find the fourier transform of $f(x)$.

$$f(x) = \begin{cases} 0 & 0 < x < a. \\ x & a \le x \le b. \\ 0 & x > b. \end{cases}$$

Soln:

$$\tilde{f_s}(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin px \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a 0 \cdot \sin px\, dx + \int_a^b x \sin px\, dx + \int_b^\infty 0 \cdot \sin px\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_a^b x \sin px\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(-\frac{x \cos px}{p} \right)_a^b + \left(\frac{\sin px}{p^2} \right)_a^b \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-b\cos pb + a\cos pa}{p} + \frac{\sin pb - \sin pa}{p^2} \right]$$

Pbm: Show that the FT of $f(x) = e^{-x^2/2}$ is $e^{-p^2/2}$.

Soln:-

$$F[F(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{ipx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \cdot e^{ipx} \cdot e^{-p^2/2} \cdot e^{p^2/2} \cdot dx$$

$$-\frac{x^2}{2} + ipx + \frac{p^2}{2}$$

$$-\frac{x^2 - 2ipx + p^2}{2}$$

$$= -\frac{1}{2}(x^2 - 2ipx - p^2)$$

$$-\frac{1}{2}(x - ip)^2$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-ip)^2} \cdot e^{-p^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} \sqrt{2}\, dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

Put $y^2 = \frac{1}{2}(x - ip)^2$

$y = \frac{1}{\sqrt{2}}(x - ip)$

$dy = \frac{1}{\sqrt{2}} dx$

$dx = \sqrt{2}\, dy$

$$= \frac{e^{-p^2/2}}{\cancel{\sqrt{\pi}}} \cdot \cancel{\sqrt{\pi}}$$

How?

$$F[F(x)] = e^{-p^2/2}$$

Pbm:- Find the fourier cosine transform of e^{-x^2}.

Soln:

$$F_c[F(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos px\, dx = I \longrightarrow (1)$$

$$\frac{dI}{dp} = -\sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-x^2} \sin px\, dx$$

$$= \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{-2x e^{-x^2}}_{dv} \underbrace{\sin px}_{u}\, dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\left(e^{-x^2}\sin px\right)_0^{\infty} - p\int_0^{\infty}\cos px\,dx\right]$$

$u = e^{-x^2}$, $\int dv = \int \sin px$.
$du = -e^{-x^2}\,2x$, $dv =$
$v = \cos px / p$.

$$\frac{dI}{dp} = -\frac{p}{2} I.$$

$$\frac{dI}{I} = -\frac{p}{2}dp.$$

Integrating we get

$$\log I = -\frac{p^2}{4} + \log A.$$

$$\log I - \log A = -\frac{p^2}{4}.$$

$$\log(I/A) = -p^2/4.$$

$$I = A e^{-p^2/4} \longrightarrow (ii)$$

But when $p=0$; from(i) we get

$$I = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x^2}dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{2}.$$

$$I = \frac{1}{\sqrt{2}}$$

$p=0$ from (ii) we get

$$A = \frac{1}{\sqrt{2}}.$$

$$\therefore I = F_c[F(x)] = \frac{1}{\sqrt{2}} e^{-p^2/4}.$$

Pbm:-

Find the inverse fourier transform of $\tilde{f}(p) = e^{-|p|y}$.

$-\infty \to 0 \to \infty$

Soln:-

$$|p| = \begin{cases} -p & p \le 0 \\ p & p \ge 0. \end{cases}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(p)\, e^{-ipx}dp.$$

$|p|$ - absolute value

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|p|y} \cdot e^{-ipx} dp.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{py} e^{-ipx} dp + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-py} \cdot e^{-ipx} dp.$$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{py} \cdot e^{-ipx}\right]_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[e^{-py} \cdot e^{-ipx}\right]_{0}^{\infty}.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(y-ix)p} dp + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-p(y+ix)} dp.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(y-ix)p}}{y-ix}\right]_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-p(y+ix)}}{y+ix}\right]_{0}^{\infty}.$$

limit apply for p

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(y-ix)0}}{y-ix}\right] + \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-0(y+ix)}}{(y+ix)}\right]$$

where $e^0 = 1$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{y-ix} + \frac{1}{y+ix}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{y+ix+y-ix}{y^2+x^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2y}{y^2+x^2}\right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{y}{y^2+x^2}\right].$$

$$f(x) = \sqrt{\frac{2}{\pi}} \left(\frac{y}{y^2+x^2}\right).$$

Pbm:- Find the fourier sine transform of x^{m-1}.

Soln:- $= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin px \, dx.$

$$\tilde{f}_s(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^{m-1} \sin px \, dx \longrightarrow (1).$$

W.K.T

$$\int_0^\infty e^{-xt} \sin px \, dx = \left[\frac{e^{-xt}}{t^2+p^2} (-t \sin px - p \cos px)\right]_0^\infty$$

$$= \frac{p}{t^2+p^2}$$

$$= \frac{1}{2i}\left(\frac{2ip}{t^2-p^2}\right)$$

$$= \frac{1}{2i}\left(\frac{1}{t-ip} - \frac{1}{t+ip}\right]$$

Not clear

$$= \frac{1}{2i}\left[(t-ip)^{-1} - (t+ip)^{-1}\right]$$

Diff both sides with respect to t $(m-1)$ times

$$(-1)^{m-1}\int_0^\infty x^{m-1} e^{-xt} \sin px\, dx = \frac{1}{2i}(-1)^{m-1}(m-1)!\left[(t-ip)^{-m} - (t+ip)^{-m}\right]$$

Put $t = r\cos\phi$; $p = r\sin\phi$ we get,

$$(-1)^{m-1}\int_0^\infty x^{m-1} e^{-xt} \sin px\, dx = \frac{1}{2i}(-1)^{(m-1)}(m-1)!\left[(r\cos\phi - ir\sin\phi)^{-m} - (r\cos\phi + ir\sin\phi)^{-m}\right].$$

$$= \frac{1}{2i}(-1)^{m-1}(m-1)!\, r^{-m}\left[\cos m\phi + i\sin m\phi - \cos m\phi + i\sin m\phi\right]$$

$$= \frac{1}{2i}(-1)^{m-1}(m-1)!\, r^{-m}\, 2i\sin m\phi.$$

$$= (-1)^{m-1}(m-1)!\,\frac{1}{r^m}\sin m\phi.$$

$$= (-1)^{m-1}(m-1)!\,\frac{1}{(t^2+p^2)^{m/2}}\left[\sin\{m\tan^{-1}(p/t)\}\right].$$

$$\left[\because t^2+p^2 = r^2 \;\&\; \frac{p}{t} = \tan\phi\right].$$

$$\therefore \int_0^\infty x^{m-1} e^{-xt} \sin px\, dx = \frac{(m-1)!}{(t^2-p^2)^{m/2}} \sin\left[m\tan^{-1} p/t\right]$$

Now taking $t = 0$.

$$\int_0^\infty x^{m-1} \sin px\, dx = \frac{\sqrt{m}}{p^m}\sin\left(\frac{m\pi}{2}\right).$$

Hence from (1) we get,

$$\tilde{f}_s(p) = \frac{\Gamma m}{p^m}\sqrt{\frac{2}{\pi}}\sin\left(\frac{m\pi}{2}\right).$$

Pbm: Find $f(x)$ if $\tilde{f}_c(p) = p^n . e^{-ap}$.

Soln:

Using Fourier cosine inverse formula.

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty p^n . e^{-ap}\cos px\,dp \longrightarrow (\text{II}).$$

$$\int_0^\infty e^{-ap}\cos px\,dp = \frac{a}{a^2+x^2}.$$

Diff with respect to a n times.

$$(-1)^n\int_0^\infty p^n . e^{-ap}\cos px\,dp = \frac{d^n}{da^n}\left(\frac{a}{a^2+x^2}\right)$$

$$= \frac{1}{2}\frac{d^n}{da^n}\left(\frac{2a}{a^2+x^2}\right)$$

$$= \frac{1}{2}\frac{d^n}{da^n}\left[\frac{1}{a-ix}+\frac{1}{a+ix}\right]$$

$$= \frac{1}{2}(-1)^n n!\left[\frac{1}{(a+ix)^{n+1}}+\frac{1}{(a+ix)^{n+1}}\right].$$

Put $x = r\cos\theta$; $a = r\sin\theta$.

$$\int_0^\infty p^n e^{-ap}\cos px\,dp = \frac{n!}{2}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} + \frac{1}{(r\cos\theta + i\sin\theta)^{n+1}}\right].$$

$$= \frac{n!}{2r^{n+1}}\left[2\cos(n+1)\theta\right].$$

$$\int_0^\infty p^n e^{-ap}\cos px\,dp = \frac{n!}{r^{n+1}}\left[\cos(n+1)\theta\right].$$

$$f(x) = \sqrt{\frac{2}{\pi}}.\int_0^\infty p^n e^{-ap}\cos px\,dp.$$

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$$= \sqrt{\frac{2}{\pi}} \cdot \frac{n!}{r^{n+1}} \cos(n+1)\theta .$$

(OR)

$$f(x) \sqrt{\frac{2}{\pi}} \cdot \frac{n! \cos\left[(n+1)\tan^{-1}\frac{x}{a}\right]}{(a^2+x^2)^{\frac{n+1}{2}}} .$$

Pbm:-

i) Find $f(x)$ if $\tilde{f_s}(p) = p^n e^{-ap}$.

Ans:- $f(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{n! \sin(n+1)\theta}{(a^2+x^2)^{n+1/2}}$

ii) Find $f(x)$ if its cosine trans is

$$\tilde{f_s}(p) = \begin{cases} \frac{1}{\sqrt{2\pi}}\left(a - \frac{p}{2}\right) & \text{if } p < 2a \\ 0 & \text{if } p \geq 2a . \end{cases}$$

Ans:- $\frac{1 - \cos 2ax}{2\pi x^2} = \pi^{-1} x^2 \sin^2 ax .$

Pbm:

Use the sine inverse formula to obtain $f(x)$ if $\tilde{f_s}(p) = \frac{p}{1+p^2}$.

Soln:- $= \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{f}(p) \sin px \, dx$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{p}{1+p^2} \sin px \, dp .$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{p^2+1-1}{p(1+p^2)} \sin px \, dp .$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin px}{p} dp - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin px}{p(1+p^2)} dp .$$

$$f(x) = \sqrt{\frac{\pi}{2}} - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin px}{p(1+p^2)} dp \longrightarrow (1) .$$

$$\left[\because \int_0^\infty \frac{\sin px}{p} dp = \frac{\pi}{2} \right]$$

Diff with respect to x in (i) we get,

$$\frac{df}{dx} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos px}{1+p^2}\, dp \longrightarrow \text{(ii)}$$

$$\frac{d^2f}{dx^2} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{p \sin px}{1+p^2}\, dp.$$

$$= f.$$

$$\Rightarrow \frac{d^2f}{dx^2} - f = 0.$$

Whose soln is $f = Ae^{x} + Be^{-x}$.

$$\frac{df}{dx} = Ae^{x} - Be^{-x} \longrightarrow \text{(iii)}$$

Now, when $x=0$ in (i) & (iii) we get

$$f = \sqrt{\frac{\pi}{2}}. \text{ and}$$

$$\frac{df}{dx} = -\sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{dp}{1+p^2}.$$

$$= -\sqrt{\frac{\pi}{2}} \quad \left[\because \int_0^\infty \frac{dx}{1+p^2} = \frac{\pi}{2}.\right].$$

Put $x=0$ in (ii) we get,

$$\sqrt{\frac{\pi}{2}} = A+B.$$

$$-\sqrt{\frac{\pi}{2}} = A-B.$$

Solving we get, $A=0$; $B = \sqrt{\frac{\pi}{2}}$.

$\therefore$ Hence $f(x) = \sqrt{\frac{\pi}{2}}\, e^{-x}$

Pbm:- Find $F(x)$ if $\tilde{f}_s(p) = \frac{e^{-ap}}{p}$. Hence

Deduce $F_s^{-1}\left[\frac{1}{p}\right]$.

Soln:-

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ap}}{p} \sin px\, dp \longrightarrow (1)$$

Diff w.r to x we get,

$$\frac{df}{dx} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ap} \cos px \, dp.$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2+x^2}.$$

$\int$ing both sides the above eqn we get,

$$f = \sqrt{\frac{2}{\pi}} \int \frac{a}{a^2+x^2} dx + A.$$

$$\Rightarrow f = \sqrt{\frac{2}{\pi}} \tan^{-1}(x/a) + A \longrightarrow (2),$$

But, when $x=0$ in (1) we get,

$$f=0.$$

(2) $\Rightarrow$ $A=0$.

Hence $f(x) = F_s^{-1}\left[\frac{e^{-ap}}{p}\right]$

$$= \sqrt{\frac{2}{\pi}} \tan^{-1}\frac{x}{a} \longrightarrow (3)$$

Now putting $a=0$ in (3) we get,

$$f = F_s^{-1}\left[\frac{e^{-0p}}{p}\right] = \sqrt{\frac{2}{\pi}} \tan^{-1}\frac{x}{0}.$$

$$\therefore F_s^{-1}\left[\frac{1}{p}\right] = \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right) = \sqrt{\frac{\pi}{2}}.$$

Pbm:- Find the finite F.S & C.T of $f(x)=1$.

Soln:-

$$\tilde{f}_s(p) = \int_0^\pi f(x) \sin px \, dx = \int_0^\pi 1 \cdot \sin px \, dx.$$

$$= \left(-\frac{\cos px}{p}\right)_0^\pi = \frac{1-(-1)^p}{p}.$$

How?

and

$$\tilde{f_c}(P) = \int_0^{\pi} f(x) \cos px \, dx$$

$\cos 0 = 1$
$\cos \pi = -1$
$\sin \pi = 0$
$\sin 0 = 0$

$$= \int_0^{\pi} 1 \cdot \cos px \, dx$$

$$= \left(\frac{\sin px}{p}\right)_0^{\pi} = 0 \quad \text{if } P = 1, 2, \ldots$$

and if $P = 0$ then $\tilde{f_c}(P) = \int_0^{\pi} 1 \cdot dx = \pi$.

Pbm:
Find the finite F.S & C.T of $f(x) = x$.

Soln:
$$\tilde{f_s}(P) = \int_0^{\pi} f(x) \sin px \, dx = \int_0^{\pi} \underset{u}{x} \cdot \underset{v}{\sin px} \, dx.$$

$$= \left[-\frac{x \cos px}{p}\right]_0^{\pi} + \frac{1}{p} \int_0^{\pi} \cos px \, dx.$$

$u = x$ | $\int dv = \int \sin px$
$du = dx$ | $v = -\frac{\cos px}{p}$

$u = x$ | $dv = \sin px$
$du = dx$ | $V = -\frac{\cos px}{p}$.

$\int u \, dv = uv - \int v \, du$.

$$= \frac{\pi (-1)^{p+1}}{p} + \left(\frac{\sin px}{p^2}\right)_0^{\pi}.$$

How?

$$= \frac{\pi (-1)^{p+1}}{p}. \quad P = 0. \quad \boxed{\tilde{f_s}(P) = 0}$$

and
$$\tilde{f_c}(P) = \int_0^{\pi} f(x) \cos px \, dx$$

$$= \int_0^{\pi} x \cos px \, dx$$

$-\frac{1}{p}\left[-\frac{\cos px}{p}\right]_0^{\pi}$

$$= \left[\frac{x \sin px}{p}\right]_0^{\pi} - \frac{1}{p} \int_0^{\pi} \sin px \, dx.$$

$$= \left(\frac{\cos px}{p^2}\right]_0^{\pi} = \frac{(-1)^p - 1}{p^2} \quad \text{if } P = 1, 2, \ldots$$

If $P = 0$ then $\tilde{f_c}(P) = \int_0^{\pi} x \cdot 1 \cdot dx = \frac{\pi^2}{2}$.

$\left[\frac{x^2}{2}\right]$

Pbm:- Find the finite F.S & C.T of the fun

$f(x) = 2x \; ; 0 < x < 4.$

Soln:-

$$\tilde{f}_s(P) = \int_0^l f(x) \sin \frac{P\pi x}{l} dx$$

$$= \int_0^4 2x \cdot \sin \frac{P\pi x}{4} dx \; ; \text{ as } l = 4.$$

$$= \left[\frac{-2x \cos \frac{P\pi x}{4}}{\frac{P\pi}{4}} \right]_0^4 + 2 \int_0^4 \frac{\cos\left(\frac{P\pi x}{4}\right)}{\frac{P\pi}{4}} dx.$$

$$= -\frac{32}{P\pi} \cos P\pi + \frac{8}{P\pi} \left(\frac{\sin \frac{P\pi x}{4}}{P\pi/4} \right]_0^4$$

$$= -\frac{32}{P\pi} \cos P\pi$$

$$\tilde{f}_c(P) = \int_0^l f(x) \cos \frac{P\pi x}{l} dx$$

$$= \int_0^4 2x \cos \frac{P\pi x}{4} dx \quad \text{as } l = 4.$$

$$= \left(\frac{2x \sin \frac{P\pi x}{4}}{P\pi/4} \right)_0^4 - 2 \int_0^4 \frac{\sin(P\pi x/4)}{P\pi/4} dx$$

$$= -\frac{8}{P\pi} \left(\frac{-\cos \frac{P\pi x}{4}}{P\pi/4} \right)_0^4$$

$$= \frac{32}{P^2\pi^2} (\cos P\pi - 1) \text{ if } P > 0.$$

and if $P = 0$ then $\tilde{f}_c(P) = \int_0^4 2x \cdot 1 \cdot dx$

$= 16.$

2021/25

Pbm: Find the finite C.T of $(1-\frac{x}{\pi})^2$

Soln:

$$\tilde{f_c}(P) = \int_0^{\pi} (1-\frac{x}{\pi})^2 \cos px \, dx$$

$$= \left[(1-\frac{x}{\pi})^2 \cdot \frac{\sin px}{p}\right]_0^{\pi} + \frac{2}{p\pi}\int_0^{\pi} (1-\frac{x}{\pi}) \sin px \, dx$$

$$= \frac{2}{\pi p}\left[-(1-\frac{x}{\pi})\frac{\cos px}{p}\right]_0^{\pi} - \frac{2}{p\pi} \cdot \frac{1}{p\pi}\int_0^{\pi} \cos px \, dx$$

$$= \frac{2}{\pi p^2} - \frac{2}{p^2\pi^2}\left(\frac{\sin px}{p}\right)_0^{\pi}$$

$$= \frac{2}{\pi p^2} \text{ if } P>0,$$

And if $P=0$ then

$$\tilde{f_c}(P) = \int_0^{\pi} (1-\frac{x}{\pi})^2 dx$$

$$= \left[-\frac{\pi}{3}(1-\frac{x}{\pi})^3\right]_0^{\pi}$$

$$\boxed{\therefore \tilde{f_c}(P) = \frac{\pi}{3}}$$

$u = 1-\frac{x}{\pi}$, $du = -\frac{dx}{\pi}$ | $\int dv = \sin px$, $v = -\frac{\cos px}{p}$

$uv - \int v\,du$

$= \frac{2}{p\pi}\left[(1-\frac{x}{\pi})(-\frac{\cos px}{p}) - \int \frac{-\cos px}{p}(-\frac{1}{\pi})\right]$

$= \frac{2}{\pi p}\left[\left[(1-\frac{x}{\pi})(-\frac{\cos px}{p})\right]_0^{\pi} - \frac{1}{\pi}\int \cos px \, dx\right]$

$= \frac{2}{\pi p}\left[(1-\frac{\pi}{\pi})(-\frac{\cos p\pi}{p}) + (1-0)\frac{\cos p0}{p}\right] - 0$

$= \frac{2}{\pi p}\left[0 + \frac{1}{p}\right]$

$= \frac{2}{\pi p^2}$

Pbm: Show that the finite S.T of $\frac{x}{\pi}$ is $(-1)^{P+1}\frac{1}{P}$.

$u = \frac{x}{\pi}$, $du = \frac{dx}{\pi}$ | $\int dv = \int \sin px \, dx$, $v = -\frac{\cos px}{p}$

Soln:

$$\tilde{f_s}(P) = F_s\left[\frac{x}{\pi}\right] = \int_0^{\pi} \frac{x}{\pi} \sin px \, dx$$

$$= \left(-\frac{x}{\pi p}\cos px\right)_0^{\pi} + \frac{1}{p\pi}\int_0^{\pi} 1 \cdot \cos px \, dx$$

$= -\frac{\pi}{\pi p}\cos p\pi + 0$

$$= -\frac{1}{p}\cos p\pi + \frac{1}{\pi p^2}(\sin px)_0^{\pi}$$

$\sin \pi = 0$, $\sin 0 = 0$

$= -\frac{1}{p}\cos p\pi$,

$$\tilde{f_s}(P) = (-1)^{P+1} \cdot \frac{1}{P}.$$ How?

Pbm:-

Find the finite C.T of $f(x)$ if

$$f(x) = \begin{cases} 1 & 0 < x < \frac{\pi}{2} \\ -1 & \pi/2 < x < \pi \end{cases}$$

Soln:-

$$\tilde{f_c}(P) = \int_0^{\pi} f(x) \cos px \, dx$$

$$= \int_0^{\pi/2} 1 \cdot \cos px \, dx + \int_{\pi/2}^{\pi} -1 \cdot \cos px \, dx$$

$$= \left(\frac{1}{P} \sin px\right)_0^{\pi/2} - \left(\frac{1}{P} \sin px\right)_{\pi/2}^{\pi}$$

$$= \frac{1}{P} \sin \frac{P\pi}{2} + \frac{1}{P} \sin \frac{P\pi}{2}$$

$$= \frac{2}{P} \sin \frac{P\pi}{2} \quad \boxed{P > 0.}$$

But if $P = 0$ then,

$$\tilde{f_c}(P) = \int_0^{\pi} f(x) \cdot 1 \, dx = \int_0^{\pi/2} 1 \cdot dx + \int_{\pi/2}^{\pi} -1 \cdot dx$$

$$= (x)_0^{\pi/2} - (x)_{\pi/2}^{\pi}$$

$$\therefore \tilde{f_c}(P) = 0$$

Pbm:-

Find the finite S.T of $f(x)$ if

i) $f(x) = \cos kx$ ii) $f(x) = x^3$; iii) $f(x) = e^{cx}$.

Soln:-

$$\tilde{f_s}(P) = \int_0^{\pi} f(x) \sin px \, dx.$$

i) $$\tilde{f_s}(P) = \int_0^{\pi} \cos kx \sin px \, dx$$

$$= \frac{1}{2} \int_0^{\pi} [\sin (P+k)x + \sin (P-k)x] \, dx$$

$$= \frac{1}{2} \left[-\frac{\cos (k+P)x}{k+P} - \frac{\cos (P-k)x}{P-k} \right]_0^{\pi}.$$

$$= \frac{1}{2}\left[-\frac{\cos(K+P)\pi}{K+P} - \frac{\cos(P-K)\pi}{P-K} + \frac{1}{K+P} + \frac{1}{P-K}\right]$$

$$= \frac{1}{2}\left[\frac{-(P-K)\cos(K+P)\pi + \cos(P-K)\pi\cdot(K+P)}{P^2-K^2} + \frac{P-K+K+P}{P^2-K^2}\right]$$

$$= \frac{1}{2(P^2-K^2)}\left[-P\{\cos(K+P)\pi + \cos(P-K)\pi\} + K\{\cos(K+P)\pi - \cos(P-K)\pi\} + 2P\right].$$

$\frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$\cos A \cos B$

$$= \frac{1}{P^2-K^2}\left[-P\cdot\cos K\pi\cos P\pi - K\sin K\pi\sin P\pi + P\right]$$

$$= \frac{P}{P^2-K^2}\left[1-\cos K\pi\cdot\cos P\pi\right]$$

$$\tilde{f_s}(P) = \frac{P}{P^2-K^2}\left[1-(-1)^P\cos K\pi\right]$$

(ii) $\tilde{f_s}(P) = \int_0^\pi \underset{u}{x^3}\,\underset{dv}{\sin px\,dx}$

$u = x^3,\ du = 3x^2dx$ | $dv = \sin px$, $v = -\frac{\cos px}{p}$

$$= \left(-x^3\frac{1}{P}\cos px\right)_0^\pi + \frac{3}{P}\int_0^\pi \underset{u}{x^2}\,\underset{dv}{\cos px\,dx}.$$

$$= -\frac{\pi^3}{P}\cos P\pi + 0 + \frac{3}{P}\left[\left(\frac{x^2}{P}\sin px\right)_0^\pi - \frac{2}{P}\int_0^\pi x\sin px\,dx\right].$$

$\to 0 + \frac{3}{P}\times -\frac{2}{P}\int_0^\pi x\sin px\,dx.$

$$= -\frac{\pi^3}{P}\cos p\pi - \frac{6}{P^2}\left[-\frac{x}{P}\cos px + \frac{\sin px}{P^2}\right]_0^\pi$$

$= -\frac{\pi^3}{P}\cos p\pi + \frac{6}{P^3}[\pi\cos p\pi]_0^{\pi}$

$$= \pi\left(\frac{6}{P^3} - \frac{\pi^2}{P}\right)\cos p\pi$$

$= \pi\left[-\frac{\pi^2}{P}\cos p\pi + \frac{6}{P^3}[\cos p\pi - 1]\right]$

$$= \pi(-1)^P\left(\frac{6}{P^3} - \frac{\pi^2}{P}\right).$$

(iii) $\tilde{f_s}(P) = \int_0^\pi e^{cx}\sin px\,dx$

$$= \left[\frac{e^{cx}}{c^2+P^2}(c\sin px - p\cos px)\right]_0^\pi$$

$$= \frac{P}{c^2+P^2}\left[1-\cos p\pi\, e^{c\pi}\right]$$

$$= \frac{P}{c^2+P^2}\left[1-(-1)^P e^{c\pi}\right].$$

Pbm: Find the finite C.T of $f(x)$ if

i) $f(x) = \frac{\pi}{3} - x + \frac{x^2}{2\pi}$; ii) $f(x) = \sin nx$.

Soln:

i) $\tilde{f_c}(P) = \int_0^{\pi} f(x) \cos px \, dx$.

$$\tilde{f_c}(P) = \int_0^{\pi} \left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right) \cos px \, dx.$$

$$= \left[\left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right) \frac{1}{p} \sin px\right]_0^{\pi} - \frac{1}{p}\int_0^{\pi} \left(-1 + \frac{x}{\pi}\right) \sin px \, dx$$

$$= -\frac{1}{p}\left[-\left(-1 + \frac{x}{\pi}\right)\frac{1}{p}\cos px\right]_0^{\pi} + \frac{1}{p^2}\int_0^{\pi} \frac{1}{\pi} \cos px \, dx$$

$$= \frac{1}{p^2} - \frac{1}{p^3 \pi} (\sin px)_0^{\pi}$$

$$= \frac{1}{p^2} \text{ if } p > 0.$$

$P = 0 \Rightarrow$

$$\tilde{f_c}(P) = \int_0^{\pi} \left(\frac{\pi}{3} - x + \frac{x^2}{2\pi}\right) dx = 0.$$

ii) $\tilde{f_c}(P) = \int_0^{\pi} \sin nx \cos px \, dx$.

$$= \frac{1}{2}\int_0^{\pi} [\sin(n+p)x + \sin(n-p)x] \, dx$$

$$= \frac{1}{2}\left[-\frac{\cos(n+p)x}{n+p} - \frac{\cos(n-p)x}{n-p}\right]_0^{\pi}$$

$$\tilde{f_c}(P) = \frac{1}{2}\left[-\frac{\cos(n+p)\pi}{n+p} - \frac{\cos(n-p)\pi}{n-p} + \frac{1}{n+p} + \frac{1}{n-p}\right]$$

If $(n-p)$ is even then $(n+p)$ is also even.

$$\therefore \tilde{f_c}(P) = \frac{1}{2}\left[-\frac{1}{n+p} - \frac{1}{n-p} + \frac{1}{n+p} + \frac{1}{n-p}\right]$$

$= 0$

And if $n-p$ is odd then $n+p$ is also odd.

$$\therefore \tilde{f_c}(p) = \frac{1}{2}\left[\frac{2}{n+p} + \frac{2}{n-p}\right] = \frac{2n}{n^2-p^2}$$

$\therefore \tilde{f_c}(p) = 0$ (or) $\frac{2n}{n^2-p^2}$ according as $n-p$ is even (or) odd.

Pbm: Find the finite C.T of $f(x)$ if

$$f(x) = -\frac{\cos k(\pi - x)}{k\sin k\pi}.$$

Soln:

$$\tilde{f_c}(p) = -\int_0^{\pi} \frac{\cos[k(\pi-x)]}{k\sin k\pi}\cos px\,dx.$$

$$= -\frac{1}{2k\sin k\pi}\int_0^{\pi}\left[\cos\{k(\pi-x)+px\} + \cos\{k(\pi-x)-px\}\right]dx$$

$$= -\frac{1}{2k\sin k\pi}\left[\frac{\sin(k\pi - kx + px)}{p-k} - \frac{\sin(k\pi - kx - px)}{p+k}\right]_0^{\pi}$$

$$= -\frac{1}{2k\sin k\pi}\left[\frac{\sin p\pi}{p-k} - \frac{\sin(-p\pi)}{p+k} - \frac{\sin k\pi}{p-k} + \frac{\sin k\pi}{p+k}\right]$$

$$= -\frac{1}{2k\sin k\pi}\left[\frac{\sin k\pi}{p-k} - \frac{\sin k\pi}{p+k}\right] = \frac{\sin k\pi}{2k\sin k\pi}\left[\frac{1}{p-k} - \frac{1}{p+k}\right]$$

$$= \frac{1}{2k}\left(\frac{1}{p-k} - \frac{1}{p+k}\right)$$

$\sin(-\theta) = -\sin\theta$

$$\tilde{f_c}(p) = \frac{1}{p^2-k^2} \text{ if } k \neq 0, 1, 2, \dots$$

Pbm: Find $f(x)$ if $\tilde{f_c}(p) = \frac{\cos(2p\pi/3)}{(2p+1)^2}$ if $0<x<1$.

Soln:

$$f(x) = \frac{1}{l}\tilde{f_c}(0) + \frac{2}{l}\sum_{p=1}^{\infty}\tilde{f_c}(p)\cos\frac{p\pi x}{l}.$$

$$= \frac{1}{1}\cdot 1 + \frac{2}{1}\sum_{p=1}^{\infty}\tilde{f_c}(p)\cos\frac{p\pi x}{1}$$

$$f(x) = 1 + 2\sum_{P=1}^{\infty} \frac{\cos(2P\pi/3)}{(2P+1)^2} \cos P\pi x.$$

Pbm:-

Find $f(x)$ if $\tilde{f_c}(P) = \dfrac{6\left(\sin\frac{P\pi}{2} - \cos P\pi\right)}{(2P+1)\pi}$

For $P = 1, 2 \ldots$ and $\frac{2}{\pi}$ for $P=0$. Where $0<x<4$.

Soln:-

$$f(x) = \frac{1}{l}\tilde{f_c}(0) + \frac{2}{l}\sum_{P=1}^{\infty} \tilde{f_c}(P) \cos\frac{P\pi x}{l}$$

$$= \frac{1}{4}\cdot\frac{2}{\pi} + \frac{2}{4}\sum_{P=1}^{\infty} 6\cdot\frac{\left(\sin\frac{P\pi}{2} - \cos P\pi\right)}{2P+1}\cos\left(\frac{P\pi x}{4}\right)$$

$$f(x) = \frac{1}{2\pi} + \frac{3}{\pi}\sum_{P=1}^{\infty} \frac{(\sin P\pi - \cos P\pi)}{2P+1}\cos\left(\frac{P\pi x}{4}\right).$$

Pbm:-

Find $f(x)$ if the finite S.T is gn

by $\tilde{f_s}(P) = \dfrac{1-\cos P\pi}{P^2\pi^2}$. Where $0<x<\pi$.

Soln:-

$$f(x) = \frac{2}{\pi}\sum_{P=1}^{\infty} \tilde{f_s}(P) \sin px$$

$$= \frac{2}{\pi}\sum_{P=1}^{\infty}\left(\frac{1-\cos P\pi}{P^2\pi^2}\right)\sin px$$

$$f(x) = \frac{2}{\pi^3}\sum_{P=1}^{\infty}\left(\frac{1-\cos P\pi}{P^2}\right)\sin px.$$

Pbm:-

Find $f(x)$ if it's finite S.T is gn

by $\tilde{f_s}(P) = \dfrac{2\pi(-1)^{P-1}}{P^3}$; $P = 1, 2 \ldots$ Where $0<x<\pi$.

Soln:-

$$f(x) = \frac{2}{\pi}\sum_{P=1}^{\infty} \tilde{f_s}(P) \sin px.$$

$$= \frac{2}{\pi} \sum_{P=1}^{\infty} \frac{2\pi(-1)^{P-1}}{p^3} \sin px.$$

$$f(x) = 4 \sum_{P=1}^{\infty} \frac{(-1)^{P-1}}{p^3} \sin px.$$

Pbm: When $f(x) = \sin mx$. Where 'm' is a +ve integer show that $\tilde{f}_s(P) = 0$ if $P \neq m$ and show that $\tilde{f}_s(P) = \frac{\pi}{2}$ if $P = m$.

Soln:

$$\tilde{f}_s(P) = \int_0^{\pi} f(x) \sin px \, dx$$

$$= \int_0^{\pi} \sin mx \sin px \, dx$$

$$= \frac{1}{2} \int_0^{\pi} [\cos(m-P)x - \cos(m+P)x] \, dx.$$

$$= \frac{1}{2} \left[\frac{\sin(m-P)x}{m-P} - \frac{\sin(m+P)x}{m+P} \right]_0^{\pi}$$

$$= 0 \text{ if } m \neq P.$$

If $m = P$, then

$$\tilde{f}_s(p) = \int_0^{\pi} \sin^2 px \, dx.$$

$$= \frac{1}{2} \int_0^{\pi} (1 - \cos 2px) \, dx$$

$$= \frac{1}{2} \left[x - \frac{\sin 2px}{2p} \right]_0^{\pi}$$

$$= \frac{1}{2} \left[\pi - \frac{\sin 2p\pi}{2p} + \frac{\sin 0}{2p} \right]$$

$$= \frac{1}{2} [\pi - 0]$$

$$\tilde{f}_s(P) = \pi/2$$

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Pbm: Find finite F.S.T of $f(x)$ of

$$f(x) = \frac{\pi \sin kx}{2k\sin^2 k\pi} - \frac{x\cos k(\pi - x)}{2k\sin k\pi}.$$

Soln:-

$$\tilde{f_s}(P) = \int_0^\pi f(x)\sin px\,dx$$

$$= \int_0^\pi \left[\frac{\pi \sin kx}{2k\sin^2 k\pi} - \frac{x\cos k(\pi - x)}{2k\sin k\pi}\right]\sin px\,dx$$

$$= \frac{\pi}{2k\sin^2 k\pi}\int_0^\pi \sin kx \sin px\,dx - \frac{1}{2k\sin k\pi}\int_0^\pi x\cos k(\pi - x)\sin px\,dx$$

$$\tilde{f_s}(P) = \frac{\pi}{4k\sin^2 k\pi}\int_0^\pi [\cos(P-k)x - \cos(P+k)x]\,dx$$

$$- \frac{1}{4k\sin k\pi}\int_0^\pi x[\sin(k\pi - kx + Px) + \sin(Px - k\pi + kx)].$$

$$= \frac{\pi}{4k\sin^2 k\pi}\left[\frac{\sin(P-k)x}{P-k} - \frac{\sin(P+k)x}{P+k}\right]_0^\pi$$

$$- \frac{1}{4k\sin k\pi}\left[x\left\{\frac{-\cos(k\pi - kx + Px)}{P-k} - \frac{\cos(Px - k\pi + kx)}{P+k}\right\}\right]_0^\pi$$

$$+ \frac{1}{4k\sin k\pi}\int_0^\pi 1\cdot\left\{-\frac{\cos(k\pi - kx + Px)}{P-x} - \frac{\cos(Px - k\pi + kx)}{P+k}\right\}dx$$

$$= \frac{\pi}{4k\sin^2 k\pi}\left[\frac{\sin(P-k)\pi}{P-k} - \frac{\sin(P+k)\pi}{P+k}\right] +$$

$$\frac{\pi}{4k\sin k\pi}\left[\cos P\pi\left(\frac{1}{P-k} + \frac{1}{P+k}\right)\right] -$$

$$\frac{1}{4k\sin k\pi}\left[\frac{\sin(k\pi - kx + Px)}{(P-k)^2} + \frac{\sin(Px - k\pi + kx)}{(P+k)^2}\right]_0^\pi$$

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$$= \frac{\pi}{4k\sin^2 k\pi}\left[\sin p\pi \cos k\pi\left(\frac{1}{p-k}-\frac{1}{p+k}\right)-\cos p\pi \sin k\pi\left(\frac{1}{p-k}+\frac{1}{p+k}\right)\right]$$

$$+\frac{\pi}{4k\sin k\pi}\cdot\frac{2p}{p^2-k^2}\cos p\pi+\frac{1}{4k\sin k\pi}\left[\sin k\pi\left\{\frac{1}{(p-k)^2}-\frac{1}{(p+k)^2}\right\}\right].$$

$$= \frac{1}{4k}\,\frac{(p+k)^2-(p-k)^2}{(p^2-k^2)^2}$$

$$\therefore \tilde{f}_s(p) = \frac{p}{(p^2-k^2)^2} \qquad [\because (k) \neq 0;\ k=1,2\ldots]$$

Pbm:- Find the finite F.S.T of $f(x)$ if

$$f(x) = \begin{cases} -x & x < c \\ \pi - x & x > c \end{cases} \quad \text{Where } 0 \le x \le \pi.$$

Soln:-

$$\tilde{f}_s(p) = \int_0^\pi f(x)\sin px\,dx$$

$$= \int_0^c -x\sin px\,dx + \int_c^\pi (\pi - x)\sin px\,dx.$$

$$= \left(\frac{x\cos px}{p}\right)_0^c - \frac{1}{p}\int_0^c \cos px\,dx - \left[\frac{(\pi-x)}{p}\cos px\right]_c^\pi - \frac{1}{p}\int_c^\pi \cos px\,dx$$

$$= \frac{c\cos pc}{p} - \frac{1}{p^2}(\sin px)_0^c + \left(\frac{\pi - c}{p}\right)\cos pc - \frac{1}{p^2}(\sin px)_c^\pi$$

$$= \frac{c\cos pc}{p} - \cancel{\frac{\sin pc}{p^2}} + \frac{(\pi - c)}{p}\cos pc - \frac{1}{p^2}\sin p\pi + \cancel{\frac{1}{p^2}\sin pc}$$

$$= \cancel{\frac{c\cos pc}{p}} + \frac{\pi}{p}\cos pc - \cancel{\frac{c\cos pc}{p}} - \frac{1}{p^2}\sin p\pi.$$

$$\therefore \tilde{f}_s(p) = \frac{\pi}{p}\cos pc.$$

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Finite Fourier tfms:-

U.Q (2m) Finite Fourier sine tfms:-

Let $f(x)$ denote a fun that is sectionally ~~conts~~ over some finite interval $(0, l)$ of the variable x. The finite fourier sine tfm of $f(x)$ on this interval is defined as

$$\tilde{f_s}(p) = \int_0^l f(x) \sin \frac{p\pi x}{l} \, dx.$$

Where, p is an integer.

By the proper choice of the origin and the unit of length, if the end pts of the interval become $x=0$ & $x=\pi$.

Then, $\tilde{f_s}(p) = \int_0^\pi f(x) \sin px \, dx$.

The transformation sets up a correspondence b/w funs $f(x)$ on the interval $0 < x < \pi$ & sequence of no's.

$\tilde{f_s}(p)$, $(p = 1, 2, \ldots)$

The fun $f(x)$ is called the inverse finite fourier sine tfms of $\tilde{f_s}(p)$,

i.e., $f(x) = f_s^{-1}\{\tilde{f_s}(p)\}$.

Inversion formula for finite sine tfm:-

If $\tilde{f_s}(p)$ is the finite fourier sine tfm of $f(x)$ over the interval

$(0, l)$ then the inversion formula for sine tfm is gn by

$$f(x) = \frac{2}{l} \sum_{p=1}^{\infty} \tilde{f}_s(p) \sin \frac{p\pi x}{l}.$$

(or)

$$f(x) = \frac{2}{\pi} \sum_{p=1}^{\infty} \tilde{f}_s(p) \sin px.$$

If $(0, \pi)$ is the interval consider for $\tilde{f}_s(p)$.

Proof: Defining $f(x)$ in the interval $(-l, 0)$.

$\ni$: $f(x)$ is an odd fun of x in $(-l, l)$ by fourier series, we've $f(x) = \sum_{p=1}^{\infty} b_p \sin \frac{p\pi x}{l}$.

where,

$$b_p = \frac{2}{l} \int_0^l f(x) \sin \frac{p\pi x}{l} dx$$

$$= \frac{2}{l} \tilde{f}_s(p).$$

where, $\tilde{f}_s(p)$ is the finite Fourier sin tfm of $f(x)$.

Hence $f(x) = \frac{2}{l} \sum_{p=1}^{\infty} \tilde{f}_s(p) \sin \frac{p\pi x}{l}.$

(or)

$$f(x) = \frac{2}{\pi} \sum_{p=1}^{\infty} \tilde{f}_s(p) \sin px.$$

Where $f(x)$ is an odd fun of x in the interval $(-\pi, \pi)$. $\ni$:

$$\tilde{f}_s(p) = \int_0^{\pi} f(x) \sin px. //$$

Q U.Q (am)

Finite Fourier cosine tfms:-

Let $f(x)$ denote a fun that is sectionally cts over some finite interval $(0,l)$, the variable x. The finite cosine tfm of $f(x)$ on this interval is defined as

$$\tilde{f}_c(P) = \int_0^l f(x) \cos \frac{P\pi x}{l} dx .$$

where, P is an integer.

On the interval $(0,\pi)$, $\tilde{f}_c(P)$ is defined as, $\tilde{f}_c(P) = \int_0^\infty f(x) \cos px \, dx$.

The fun $f(x)$ is called the inverse finite fourier cosine tfm of $\tilde{f}_c(P)$.

i.e) $f(x) = F_c^{-1}\{ f_c^{-1}(P) \}$.

Inversion formula for cosine tfm:-

If $\tilde{f}_c(P)$ is the finite fourier cosine tfm of $f(x)$ over the interval $(0,l)$ then the inversion formula for cosine tfm is gn by,

$$f(x) = \frac{1}{l} \tilde{f}_c(0) + \frac{2}{l} \sum_{P=1}^{\infty} \tilde{f}_c(P) \cos \frac{P\pi x}{l} .$$

where, $\tilde{f}_c(0) = \int_0^l f(x) dx$.

It π is taken as the upper limit for the finite cosine tfm then the inversion is gn by,

$$f(x) = \frac{1}{\pi}\tilde{f_c}(0) + \frac{2}{\pi}\sum_{P=1}^{\infty}\tilde{f_c}(P)\cos px.$$

where,

$$\tilde{f_c}(0) = \int_0^{\pi} f(x)\,dx.$$

proof:

Defining $f(x)$ in the interval $(-l,0)$.

$\therefore f(x)$ is an even fun of x in $(-l,l)$, by the fourier, we have.

$$f(x) = \frac{a_0}{2} + \sum_{P=1}^{\infty} a_p \cos\frac{p\pi x}{l}.$$

where

$$a_p = \frac{2}{l}\int_0^l f(x)\cos\frac{p\pi x}{l}\,dx = \frac{2}{l}\tilde{f_c}(P).$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\tilde{f_c}(0).$$

$$\therefore f(x) = \frac{1}{l}\tilde{f_c}(0) + \frac{2}{l}\sum_{P=1}^{\infty}\tilde{f_c}(P)\cos\frac{p\pi x}{l}.$$

If $f(x)$ is an even fun of x in the interval $(-\pi,\pi)$, then.

$$f(x) = \frac{1}{\pi}\tilde{f_c}(0) + \frac{2}{\pi}\sum_{P=1}^{\infty}\tilde{f_c}(P)\cos px.$$

where,

$$\tilde{f_c}(P) = \int_0^{\pi} f(x)\cos px\,dx.$$

and

$$\tilde{f_c}(0) = \int_0^{\pi} f(x)\,dx.$$

Multiple Fourier tfms:-

Let $f(x,y)$ be a fun of two variables x & y regarding $f(x,y)$, temporarily, as a fun of x, its fourier tfm is,

$$\tilde{f}(p,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,y)\,e^{ipx}\,dx.$$

Now regarding $\tilde{f}(p,y)$ as a fun of y.

its Fourier tfm is

$$\tilde{F}(p,q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(p,y) e^{iqy} dy.$$

(or)

$$\tilde{F}(p,q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{i(px+qy)} dx\, dy.$$

which is fourier tfm of $f(x,y)$.

Inversion formula for multiple fourier tfm: question:-

Using Inversion formula for fourier tfms. we've

$$f(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(p,y) e^{-ipx} dq.$$

and

$$\tilde{f}(p,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{F}(p,q) e^{-pqy} dq.$$

Hence

$$f(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{F}(p,q) e^{-i(px+qy)} dq\, dq.$$

which is the ~~inversion~~ formula for the fourier tfm of ~~f(x,y)~~ $\tilde{F}(p,q)$.

Convolution:- Let $f(x)$ and $g(x)$ then fun, F

$$H(x) = F * G = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) \cdot G(x-u) du.$$

is called the convolution (or) Falting of two integrable funs F & G over the integral $(-\infty, \infty)$.

U.Q (8m)

The convolution (or) Falting thm for fourier thm

If $F\{g(x)\}$ & $F\{f(x)\}$ are the fourier tfm of the funs $f(x)$ & $g(x)$ respectively then the fourier tfm of the convolution of $f(x)$ & $g(x)$ is the product of the fourier tfms.

ie, $F[f(x) * g(x)] = F[f(x)] \cdot F[g(x)]$.

proof:- We have,

$$F[f(x)*g(x)] = F\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, g(x-u)\,du\right] e^{ipx}\,dx\right.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(x-u)\, e^{ipx}\,dx\right]dx.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[e^{ipu}\int_{-\infty}^{\infty} g(y)\, e^{ipy}\,dy\right]du.$$

where, $x-u=y$.

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\left[e^{ipu}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\, e^{ipx}\,dx\right]du.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\left[e^{ipa} F(g(x))\right]du.$$

$$F[f(x)*g(x)] = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{ipu}\,du\right] F[g(x)]$$

$$= \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{ipx}\,dx\right] F[g(x)].$$

$$\therefore F[f(x)*g(x)] = F[f(x)] \cdot F[g(x)].$$

Hence thm proved.

U.Q (Am)

Parseval's identity for transforms (or) Plancherel's thm (or) Rayleigh's Thm.

If $\tilde{f}(p)$ is the fourier tfm of $f(x)$

then $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{f}(p)|^2 dp$.

proof:-

Let $f^*(x)$ be the complex conjugate of the fun, $f(x)$. If $f^*(p)$ is the fourier tfm of $f^*(x)$. then we've

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) f^*(x) dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) f^*(x) . e^{xp'} dx.$$

$p' = 0$.

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} f(x) e^{xp'} dx . \int_{-\infty}^{\infty} f^*(x) e^{xp'} dx \right]$$

$$= \tilde{f}(p') * \tilde{f}^*(-p')!$$

$\therefore$ F-T of the product is the convolution of the F.T.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(p) . \tilde{f}^*(p-p') dp. \quad p' = 0.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(p) \tilde{f}^*(p) dp.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\tilde{f}(p)|^2 dp.$$

Hence

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{f}(p)|^2 dp.$$

Note:-

The thm is also referred as Plancherel's thm (or) Rayleigh's thm. Some authors also define it as.

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\bar{f}(p)|^2 dp.$$

(A) (2m) Relation b/w Fourier & laplace tfms:-

Let us consider the fun,

$$f(t) = \begin{cases} e^{-xt} g(t) ; & t>0 \\ 0 ; & t<0 \end{cases} \rightarrow (1).$$

NOT the F.T of $f(t)$ is gn by,

$$F[f(t)] = \int_{-\infty}^{\infty} e^{ipt} f(t) dt .$$

[Taking non - symmetrical form of F.T] ~$\frac{1}{2\pi}$ omitted

$$= \int_{-\infty}^{0} 0 . e^{ipt} dt + \int_{0}^{\infty} e^{-xt} g(t) e^{ipt} dt$$

$$= \int_{0}^{\infty} e^{(ip-x)t} g(t) dt .$$

Put $x - ip = s$.

$$= \int_{0}^{\infty} e^{-st} g(t) dt .$$

$$= L[g(t)]$$

$[\because L[f(t)] = \int_{0}^{\infty} e^{-st} f(t) dt]$

Hence the F.T of the fun $f(t)$ define by (1) is the laplace tfm of the fun $g(t)$.

—x—

Completed for unit 2.

UNIT-III.

Hankel Transform:-

The Hankel Transform of a fun $f(x), -\infty < x < \infty$ is defined as $H_n\{f(x)\} =$

$$\boxed{\tilde{f}(P) = \int_0^\infty f(x)\, x\, J_n P(x)\, dx.}$$

Where, $J_n P(x)$ is the Bessel's fun of the 1st kind of order n and it's denoted by,

$H[f(x)]$ (or) $H_n[f(x);P]$ (or) $H_n[f(x)]$ (or) $\tilde{f}(P)$.

Note:-

$xJ_n P(x)$ is called the kernal of the transformation.

Some important result's of the Bessel's fun.

i) Bessel's fun of 1st kind,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}.$$

ii) Recurrence formula for $J_n(x)$,

1) $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

2) $xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$.

3) $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$.

4) $2J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$

5) $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$.

6) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$.

iii) Infinite integrals involving Bessel's fun

1) $\int_0^\infty e^{-ax} J_0 P(x)\, dx = (a^2+p^2)^{-1/2}$

2) $\int_0^\infty e^{-ax} J_1 P(x) dx = \frac{1}{P} - \frac{a}{P(a^2+p^2)^{1/2}}$.

3) $\int_0^\infty x e^{-ax} J_0 P(x) dx = a(a^2+p^2)^{-3/2}$

4) $\int_0^\infty x e^{-ax} J_1 P(x)\, dx = p\,(a^2+p^2)^{-3/2}$

5) $\int_0^\infty \frac{1}{x} e^{-ax} J_1 P(x) dx = \frac{(a^2+p^2)^{1/2} - a}{p}$.

Linear property:-

If $f(x)$ & $g(x)$ are two fun a, b are two constants. Then

$H_n[af(x)+bg(x)] = a[H_n f(x)] + b[H_n g(x)]$.

proof: W.K.T, $H_n[F(x)] = \int_0^\infty f(x) \cdot x J_n P(x) dx$.

$H_n[af(x)+bg(x)] = \int_0^\infty [af(x)+bg(x)] \cdot x J_n P(x) dx$

$= \int_0^\infty \{a[f(x) \cdot x J_n P(x) dx] + b[g(x) \cdot x J_n P(x) dx]\}$

$= \int_0^\infty a[f(x)\, x J_n P(x)] dx + \int_0^\infty b[g(x) \cdot x J_n P(x)] dx$.

$= a\int_0^\infty f(x) \cdot x J_n P(x) dx + b\int_0^\infty g(x) \cdot x J_n P(x) dx$.

$= a\, H_n[f(x)] + b\, H_n[g(x)]$,

Hence the thm,,

$H[f(x)] = \int f(x) x \cdot J(px) dx$

Thm:-

Find the Hankel Transform of $J_0 P(x)$ i) e^{-x}; ii) e^{-x}/x; iii) $\frac{e^{-ax}}{x}$ taking $x J_0 e\, p(x)$ as the Kernal of the Tfmation:-

Proof:-

i) Let $f(x) = e^{-x}$

W.K.T

$$H_n[f(x)] = \int_0^\infty f(x) \cdot x\, J_n P(x)\, dx.$$

$$H_n[f(x)] = \int_0^\infty x \cdot e^{-x} J_0 P(x)\, dx.$$

W.K.T

$$\int_0^\infty x e^{-ax} J_0 P(x)\, dx = a(a^2+p^2)^{-3/2}.$$

put $a = 1$.

$$\int_0^\infty x e^{-x} J_0 P(x)\, dx = 1(1^2+p^2)^{-3/2}$$

$$\therefore \int_0^\infty x e^{-x} J_0 P(x)\, dx = (1+p^2)^{-3/2}$$

$$\therefore H_n[F(x)] = (1+p^2)^{-3/2}$$

ii) Let $f(x) = \frac{e^{-x}}{x}$.

W.K.T

$$H_n[f(x)] = \int_0^\infty f(x) \cdot x J_n P(x)\, dx.$$

$$H_n[f(x)] = \int_0^\infty \frac{e^{-x}}{\cancel{x}} \cdot \cancel{x}\, J_0 P(x)\, dx$$

W.K.T

$$\int_0^\infty e^{-ax} J_0 P(x)\, dx = (a^2+p^2)^{-1/2}$$

Put $a = 1$

$$\int_0^\infty e^{-x} J_0 P(x)\, dx = (1^2+p^2)^{-1/2}$$

$\therefore \int_0^\infty e^{-x} J_0 P(x)\, dx = (1^2+p^2)^{-1/2}.$

$\therefore H_n[f(x)] = (1^2+p^2)^{-1/2}.$

iii) Let $f(x) = \dfrac{e^{-ax}}{x}.$

W.K.T

$$H_n[F(x)] = \int_0^\infty f(x) \cdot x J_n P(x) dx.$$

$$H_n[F(x)] = \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cdot \cancel{x} J_0 P(x) dx$$

$$= \int_0^\infty e^{-ax} J_0 P(x) dx.$$

$$\therefore H_n[F(x)] = (a^2+p^2)^{-1/2}$$

Problem:- Find the Hankel transform of e^{-ax} taking $xJ_0P(x)$ as the Kernal of the trf.

proof: Gn $f(x) = e^{-ax}$

W.K.T

$$H_n[f(x)] = \int_0^\infty f(x) \cdot x J_n P(x) dx.$$

$$H_n[f(x)] = \int_0^\infty e^{-ax} \cdot x J_0 P(x) dx$$

$$= a(a^2+p^2)^{-3/2}$$

$$\therefore H_n[F(x)] = a(a^2+p^2)^{-3/2}$$

Pbm: Find the Henkel tfm of,

$$f(x) = \begin{cases} 1 & 0<x<a, \; n=0 \\ 0 & x>a, \; n=0. \end{cases}$$

Soln:- W.K.T

$$H_n[F(x)] = \int_0^\infty x f(x) J_n P(x) dx.$$

How choose the formula

$$= \int_0^a x \cdot 1 \, J_0 \, P(x) \, dx$$

$$= \int_0^a x J_0 P(x) \, dx \longrightarrow (1).$$

Recurrence (vi) formula w.k.T

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x).$$

Put $n=1$

$$\frac{d}{dx}[x J_1(x)] = x^1 J_{1-1}(x)$$

$$= x J_0(x).$$

Replace x by Px.

$$x = Px$$

$$dx = p\,dx.$$

$$\frac{1}{P} \cdot \frac{d}{dx}[Px J_1(Px)] = Px J_0(Px)$$

$$\frac{d}{dx}[x J_1(Px)] = Px J_0(Px)$$

$$x J_0 Px = \frac{1}{P} \cdot \frac{d}{dx}[x J_1(Px)] \longrightarrow (2).$$

Sub eqn (2) in (1).

We get,

$$H_n[f(x)] = \int_0^a \frac{1}{P} \cdot \frac{d}{dx}[x J_1 P(x)] \, dx.$$

$$= \frac{1}{P}[x J_1 P(x)]_0^a$$

$$= \frac{1}{P}\{[a J_1 P(a)] - [0 J_1 P(0)]\}$$

$$= \frac{1}{P}[a J_1[Pa]] - 0.$$

$$= \frac{1}{P}[a J_1(Pa)]$$

Pbm:- Find the Hankel tfm $x^{-2}e^{-x}$ taking $xJ_1P(x)$ as the kernal of tfm.

Soln:- Gn $f(x) = x^{-2}e^{-x}$

W.K.T

$$H_n[f(x)] = \int_0^\infty f(x) \cdot xJ_n P(x)dx.$$

$$H_n[f(x)] = \int_0^\infty x^{-2}e^{-x} \cdot x J_1 P(x)dx.$$

$$= \int_0^\infty x \cdot x^{-2} \cdot e^{-x} J_1 P(x)dx.$$

$$= \int_0^\infty x^{-1}e^{-x} J_1 P(x)dx.$$

$$H_n[f(x)] = \int_0^\infty \frac{1}{x} e^{-x} J_1 P(x) dx.$$

W.K.T

$$\int_0^\infty \frac{1}{x} e^{-ax} J_1 P(x) dx = \frac{(a^2+p^2)^{1/2} - a}{p}.$$

Put $a=1$

$$\int_0^\infty \frac{1}{x} e^{-x} J_1 P(x) dx = \frac{(1^2+p^2)^{1/2} - 1}{p}$$

Pbm: Find the Hankel tfm e^{-ax} taking $xJ_1P(x)$ as the kernal of tfm.

Soln: Gn $f(x) = e^{-ax}$.

W.K.T

$$H_n[f(x)] = \int_0^\infty f(x) \cdot x \cdot J_n P(x)dx.$$

$$H_n[f(x)] = \int_0^\infty e^{-ax} x J_1 P(x)dx.$$

$$= \int_0^\infty x e^{-ax} J_1 P(x)dx.$$

$$= P(a^2+P^2)^{-3/2}$$

Pbm: Find the kernal transform of e^{-5x} taking $xJ_1P(x)$ as the kernal of tfm.

Soln:- Let $f(x) = e^{-5x}$

W.K.T

$$H_n\{f(x)\} = \int_0^\infty f(x)\cdot x\cdot J_n P(x)dx.$$

$$= \int_0^\infty e^{-5x}\cdot x J_0 P(x)dx.$$

$$= \int_0^\infty x e^{-5x} J_0 P(x)dx.$$

W.K.T

$$\int_0^\infty x e^{-ax} J_0 P(x) dx = a(a^2+p^2)^{-3/2}.$$

Put $a=5$,

$$\int_0^\infty x e^{-5x} J_0 P(x) dx = 5(5^2+p^2)^{-3/2}$$

$$= 5(25+p^2)^{-3/2}.$$

$$\therefore H_n[f(x)] = 5(25+p^2)^{-3/2}.$$

Pbm:- Find the kernal tfm $\frac{e^{-ax}}{x}$ taking $xJ_1P(x)$ as the kernal of tfm.

Soln:- Gn $f(x) = \frac{e^{-ax}}{x}$

W.K.T

$$H_n\{f(x)\} = \int_0^\infty f(x)\cdot x J_n P(x)dx$$

$$= \int_0^\infty \frac{e^{-ax}}{\cancel{x}}\cdot \cancel{x} J P(x)dx$$

$$= \int_0^\infty e^{-ax} J_1 P(x)dx.$$

W.K.T

$$\int_0^\infty e^{-ax} J_1 p(x)\,dx = \frac{1}{p} - \frac{a}{p(a^2+p^2)^{1/2}}$$

$$\therefore H_n\{f(x)\} = \frac{1}{p} - \frac{a}{p(a^2+p^2)^{1/2}}$$

Pbm:- Find the Kernal tfm $\frac{e^{-ax}}{x^2}$ taking $x J_1(px)$ ~~$x J_1 p(x)$~~ as the Kernal of tfm.

Soln:- Gn $f(x) = \frac{e^{-ax}}{x^2}$.

W.K.T

$$H_n[f(x)] = \int_0^\infty f(x) \cdot x J_n p(x)\,dx.$$

$$= \int_0^\infty \frac{e^{-ax}}{x^{\cancel{2}}} \cdot \cancel{x} J_1 p(x)\,dx$$

W.K.T

$$\Rightarrow \int_0^\infty \frac{1}{x} e^{-ax} J_1 p(x)\,dx = \frac{(a^2+p^2)^{1/2} - a}{p}$$

$$\therefore H_n\{f(x)\} = \frac{(a^2+p^2)^{1/2} - a}{p}$$

Pbm:- Find the Hankel tfm of the fn

$$f(x) = \begin{cases} a^2 - x^2, & 0 < x < a,\ n = 0 \\ 0, & x > a,\ n = 0. \end{cases}$$

Soln:- Let $H_n[f(x)] = \int_0^\infty x \cdot f(x) \cdot J_n(px)\,dx.$

$$= \int_0^a x(a^2 - x^2) J_0 p(x)\,dx.$$

$$= \int_0^a (a^2 x - x^3) J_0 p(x)]\,dx$$

$$= a^2 \int_0^\infty x J_0 p(x)\,dx - \int_0^a x^3 J_0(px)\,dx$$

$$= I_1 - I_2.$$

$I_1 = a^2 \int x J_0 P(x) dx.$

W.K.T,

$$\int_0^a x J_0 P(x) dx = \frac{a J_1 P(a)}{P} \text{ [By previous thm]}$$

$$\therefore I_1 = \frac{a^2}{P}[a J_1 P(a)].$$

$$= \frac{a^3 J_1 P(a)}{P}.$$

$$I_2 = \int_0^a x^3 J_0 P(x) dx.$$

$$= \int_0^a x^2 [x J_0 P(x) dx]$$

$$= \int_0^a x^2 \cdot \frac{1}{P} \frac{d}{dx}[x J_1 P(x)] dx \quad \text{[By eqn(2)]}.$$

$$= \frac{1}{P} \int_0^a x^2 \frac{d}{dx}[x J_1 P(x)] dx.$$

Let $u = x^2$; $dv = \int \frac{d}{dx}(x J_1 P(x)] dx$

$du = 2x dx$; $v = x J_1 P(x)$.

$$\int u dv = uv - \int v du.$$

$$\therefore I_2 = \frac{1}{P}\left\{ [x^2 \cdot x J_1 P(x)]_0^a - \int_0^a x J_1 P(x) \cdot 2x dx. \right.$$

$$= \frac{1}{P}\left\{ [a^3 J_1 P(a)] - 2\int_0^a x^2 J_1 P(x) dx \right\} \rightarrow (A)$$

$$= \frac{1}{P}[a^3 J_1 P(a) + I_3].$$

Recurrence formula $\underline{\text{vi}}$.

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x).$$

Put $n = 2$.

$$\Rightarrow \frac{d}{dx}[x^2 J_2(x)] = x^2 J_1(x).$$

Replace x by Px.

$$\frac{1}{P}\cdot\frac{d}{dx}\left[P^2x^2J_2\,P(x)\right] = P^2x^2J \cdot P(x).$$

$$P\cdot\frac{d}{dx}\left[x^2J_2(P(x))\right] = P^2x^2J_1(Px).$$

$$\Rightarrow x^2J_1(P(x)) = \frac{1}{P}\cdot\frac{d}{dx}\left\{x^2J_2(Px)\right] \rightarrow (B).$$

Consider $I_3 = -2\int_0^a x^2J_1P(x)\,dx.$

Sub eqn (B) in I_3.

$$I_3 = -2\int_0^a \frac{1}{P}\cdot\frac{d}{dx}\left[x^2J_2P(x)\,dx\right].$$

$$= -\frac{2}{P}\int_0^a \frac{d}{dx}\left[x^2J_2(Px)\right]dx.$$

$$= -\frac{2}{P}\left[x^2J_2P(x)\right]_0^a$$

$$= -\frac{2}{P}\left[a^2J_2P(a)\right].$$

Eqn (A) Becomes,

$$I_2 = \frac{1}{P}\left[a^3J_1P(a) - \frac{2}{P}\left\{a^2J_2P(a)\right]\right.$$

$$= \frac{a^3J_1P(a)}{P} - \frac{2}{P^2}a^2J_2P(a).$$

From I_1 & I_2 we get,

$$H_n[f(x)] = \frac{a^3J_1P(a)}{P} - \frac{a^3J_1P(a)}{P} + \frac{2}{P^2}\left[a^2J_2P(a)\right].$$

$$H_n\{f(x)\} = \frac{2}{P^2}\left[a^2J_2P(a)\right] \rightarrow (C).$$

$\therefore$ Using Recurrence Relation IV.

$$2nJ_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

Put $n=1$

$$2(1)J_1(x) = x[J_{1-1}(x) + J_{1+1}(x)].$$

$$2J_1(x) = x[J_0(x) + J_2(x)].$$

Replace x by pa.

$$2J_1(pa) = pa[J_0(pa) + J_2(pa)].$$

$$J_2(pa) = \frac{2J_1(pa)}{pa} - \frac{paJ_0 p(a)}{pa}.$$

$$= \frac{2}{pa} J_1(pa) - J_0(pa)$$

Applying in eqn ©.

$$H_n[f(x)] = \frac{2}{p^2}\left[a^2\left(\frac{2}{p^2}J_1(pa) - J_0(pa)\right)\right].$$

$$= \frac{4a^2 J_1(pa)}{p^3(a)} - \frac{2a^2 J_0 p(a)}{p}$$

$$\therefore H_n[f(x)] = \frac{4aJ_1(pa)}{p^3} - \frac{2a^2 J_0 p(a)}{p}$$

$\therefore$ Hence the proof.

Pbm:- Find the Hankel transform $\frac{\sin ax}{x}$ taking $xJ_0 p(x)$ as the kernal.

Soln:-

Let $H_n[f(x)] = \int_0^\infty x f(x) \cdot J_n p(x) dx$.

$$= \int_0^\infty x \cdot \frac{\sin ax}{x} J_0(px) dx$$

$$= \int_0^\infty \sin ax J_0(px) dx.$$

Not clear

$[\because e^{-iax} = \cos ax - i\sin ax]$.

$\sin ax = -$ Imaginary part of $[e^{-iax}]$.

$= -$ Imaginary part of $\int_0^\infty e^{-iax} J_0(px)\,dx$.

Bessel's fun formula in ①.

$$\int_0^\infty e^{-ax} J_0(px)\,dx = (a^2+p^2)^{-1/2}$$

Replace a by ia: $[(ia)^2+p^2]^{-1/2}$.

$= -$ Imaginary part of $[(-a^2+p^2)^{-1/2}]$

$$Hn\left[\frac{\sin ax}{x}\right] = \begin{cases} 0 & \text{if } p>a \\ (a^2-p^2)^{-1/2} & \text{if } 0<p<a \end{cases}.$$

[$-$ Imaginary part of].

$$(-a^2+p^2)^{-1/2} = (-1)^{-1/2}(a^2-p^2)^{-1/2}$$

$$= -\frac{1}{(-1)^{1/2}}(a^2-p^2)^{-1/2}$$

How?

$$= -\frac{1}{i^{2\cdot 1/2}}(a^2-p^2)^{-1/2}$$

$$= -\frac{1}{i}(a^2-p^2)^{-1/2}.$$

i ⊗ ly.

$$(a^2+p^2)^{-1/2} = -\frac{i}{i^2}[-a^2-p^2]^{-1/2}$$

$$Hn[f(x)]. = \cancel{i}[a^2-p^2]^{-1/2}$$

Pbm:-

Find the Hankel tfm $f(x) = \begin{cases} x^n, & 0<x<a \\ 0, & x>a. \end{cases}$

$n>-1$ taking $x J_n(px)$ as kernal.

Soln:-

Let $Hn[f(x)] = \int_0^\infty x f(x) \cdot J_n(px)\,dx$

$$= \int_0^a x \cdot x^n J_n(px)\,dx.$$

$$H_n[x^n] = \int_0^a x^{n+1} J_n(px)\,dx \longrightarrow (1)$$

Using Recurrence (vi) relation

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x).$$

Put $n = n+1$.

$$\frac{d}{dx}[x^{n+1} \cdot J_{n+1}(x)] = x^{n+1} J_n(x).$$

Replace x by px.

$$\frac{1}{p} \cdot \frac{d}{dx}[p^{n+1} \cdot x^{n+1} \cdot J_{n+1}(px)] = p^{n+1} \cdot x^{n+1} \cdot J_n(px).$$

$$x^{n+1} J_n p(x) = \frac{1}{p} \cdot \frac{d}{dx}[x^{n+1} \cdot J_{n+1} p(x)] \longrightarrow (2)$$

Apply eqn (2) in (1).

$$= \frac{1}{p} \int_0^a \frac{d}{dx}[x^{n+1} \cdot J_{n+1}(px)]\,dx.$$

$$= \frac{1}{p}[x^{n+1} \cdot J_{n+1}(px)]_0^a$$

$$= \frac{1}{p}[a^{n+1} \cdot J_{n+1}(pa)] - 0.$$

$$\therefore \frac{1}{p}[a^{n+1} \cdot J_{n+1}(pa)]$$

Inverse formula for Hankel transform.

If $\bar{f}(p)$ is the Hankel transform of the fun $f(x)$.

i.e.) $\bar{f}(p) = H_n[f(x)]$

$$= \int_0^\infty x f(x) \cdot J_n(px)\,dx.$$

Then $f(x) = \int_0^\infty p \cdot \bar{f}(p) \cdot J_n(px)\,dp$ is called the inverse formula for

Hankel transform of $\tilde{f}(p)$ and we write it as.

$$f(x) = H_n^{-1}\{\tilde{f}(p)\}.$$

Pbm: Find $H^{-1}\left[\frac{e^{-ap}}{p}\right]$ when $n=1$.

Soln:-
$$f(x) = \int_0^\infty p\,\tilde{f}(p)\,J_n(px)\,dp.$$

$$= \int_0^\infty \cancel{p}\cdot\frac{e^{-ap}}{\cancel{p}}\,J_1(px)\,dp.$$

$$= \int_0^\infty e^{-ap}J_1(px)\,dp.$$

$$\left[\because \int_0^\infty e^{-ax}J_1(px)\,dx = \frac{1}{p} - \frac{a}{p(a^2+p^2)^{1/2}}\right].$$

Replace p by x. [inverse all replace must be doing].

$$f(x) = \frac{1}{p} - \frac{a}{p(a^2+p^2)^{1/2}}$$

$$\therefore f(x) = \frac{1}{x} - \frac{a}{x(a^2+x^2)^{1/2}}$$

Pbm: Find $H^{-1}[p^{-2}e^{-ap}]$ taking $n=1$.

Soln:-
$$f(x) = \int_0^\infty p\,\tilde{f}(p)\,J_n(px)\,dp.$$

$$= \int_0^\infty p\cdot p^{-2}e^{-ap}J_1(px)\,dp$$

$$= \int_0^\infty p^{-1}e^{-ap}J_1(px)\,dp$$

$$= \int_0^\infty \frac{e^{-ap}}{p}J_1(px)\,dp$$

$$= \int_0^\infty \frac{1}{p}\,e^{-ap}J_1(px)\,dp.$$

$$\left[\because \int_0^\infty \frac{1}{x} e^{-ax} J_1 p(x)\, dx = \frac{(a^2+p^2)^{1/2}-a}{p}\right].$$

Replace p by x.

$$\int_0^\infty \frac{1}{x} e^{-ax} J_1 p(x)\, dp = \frac{(a^2+x^2)^{1/2}-a}{x}$$

$$\therefore f(x) = \frac{(a^2+x^2)^{1/2}-a}{x}$$

Pbm:- Find $H^{-1}\left[\frac{e^{-ap}}{p}\right]$ when $n=0$.

Soln:-

$$f(x) = \int_0^\infty p\tilde{f}(p)\, J_n p(x)\, dp.$$

$$= \int_0^\infty \cancel{p}\cdot \frac{e^{-ap}}{\cancel{p}} J_0 (p(x))\, dp.$$

$$= \int_0^\infty e^{-ap} J_0 (px)\, dp.$$

$$\left[\because \int_0^\infty e^{-ax} J_0 p(x)\, dx = (a^2+p^2)^{-1/2}\right].$$

Replace p by x.

$$\int_0^\infty e^{-ax} J_0 p(x)\, dp = (a^2+x^2)^{-1/2}$$

$$f(x) = (a^2+x^2)^{-1/2}$$

Pbm: Find the Hankel $\frac{\cos ax}{x}$ taking $x J_0 p(x)$ as the kernal.

Soln:-

Let $H_n = [f(x)] = \int_0^\infty x f(x) J_n p(x)\, dx$

$$= \int_0^\infty \cancel{x}\cdot \frac{\cos ax}{\cancel{x}} J_0 p(x)\, dx$$

$$= \int_0^\infty \cos ax\, J_0 p(x)\, dx.$$

not complete

~~Pb~~ Thm:-

Find the Hankel tfm of the derivative of a fun.

proof:- The Hankel transform of order n of the fun $f(x)$ is gn by,

$$\tilde{f}_n(P) = \int_0^\infty x\, f(x) \cdot J_n(Px)\, dx \longrightarrow (1).$$

If $f_n'(P)$ is the Hankel Transform of $\frac{df}{dx}$,

then $$\tilde{f}_n'(P) = \int_0^\infty x \cdot \frac{df}{dx} J_n(Px)\, dx \rightarrow (2).$$

Now $\int$ing the eqn (2), the integral on R.H.S by part,

Taking $u = x J_n(Px)$

$du = [x \cdot J_n'(Px) \cdot P + J_n(Px)]$.

$\int dv = \int \frac{df}{dx} \cdot dx$. $\rightarrow$ how?

$\therefore v = f(x)$.

$\therefore$ Eqn (2) becomes. $\int u\,dv = uv - \int v\,du$

$$\tilde{f}_n'(P) = [x J_n(Px) f(x)]_0^\infty - \int_0^\infty f(x)[Px J_n'(Px) + J_n(Px)]\, dx.$$

$$= -\int_0^\infty f(x)\, [Px J_n'(Px) + J_n(Px)]\, dx \rightarrow (3').$$

[Assuming that $xf(x) \to 0$, when $x \to 0$ (or) when $x \to \infty$].

~~Then~~ By the recurrence relation (3).

formula $x J_n'(x) = -nJ_n(x) + x J_{n-1}(x)$.

Replace x by Px.

$$Px J_n'(Px) = -nJ_n(Px) + Px J_{n-1}(Px) \longrightarrow (4)$$

Sub (4) in (3).

$$\tilde{f_n}'(P) = -\int_0^\infty f(x)\left[-nJ_n(Px) + PxJ_{n-1}(Px) + J_n(Px)\right]dx$$

$$= -\int_0^\infty \left[-nf(x)J_n P(x)\,dx + Px\,f(x)J_{n-1}P(x)\,dx + f(x)J_n(Px)\,dx\right]$$

$$= -\int_0^\infty (1-n)f(x)J_n(Px)\,dx - \int_0^\infty Px\,f(x)J_{n-1}(Px)\,dx.$$

$$= (n-1)\int_0^\infty f(x)\cdot J_n(Px)\,dx - \int_0^\infty Px\cdot f(x)\,J_{n-1}P(x)\,dx \rightarrow (5)$$

Then the recurrence relation (4).

$$2nJ_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right].$$

Replace x by Px.

$$2nJ_n(Px) = Px\left[J_{n-1}(Px) + J_{n+1}(Px)\right]$$

$$\Rightarrow J_n(Px) = \frac{Px}{2n}\left[J_{n-1}(Px) + J_{n+1}(Px)\right] \rightarrow (6),$$

Sub eqn (6) in (5).

$$\tilde{f_n}'(P) = (n-1)\int_0^\infty f(x)\left[\frac{Px}{2n}\left(J_{n-1}(Px) + J_{n+1}(Px)\right)\right]dx$$

$$- \int_0^\infty Px\,f(x)J_{n-1}(Px)\,dx.$$

$$\tilde{f_n}'(P) = \frac{(n-1)}{2n}\int_0^\infty Px\,f(x)J_{n-1}(Px)\,dx +$$

$$+ \frac{(n-1)}{2n}\int_0^\infty Px\,f(x)J_{n+1}(Px)\,dx - \int_0^\infty Px\,f(x)J_{n-1}(Px)\,dx \rightarrow (7)$$

$$= \left[\frac{(n-1)}{2n} - 1\right]\int_0^\infty Px\,f(x)J_{n-1}(Px)\,dx + \frac{(n-1)}{2n}\int_0^\infty Px\,f(x)J_{n+1}(Px)\,dx$$

$$\frac{n-1-2n}{2n} = \frac{-1-n}{2n} = -\frac{(n+1)}{2n}$$

$$= -P\left[\frac{n+1}{2n}\int_0^\infty x\,f(x)J_{n-1}(Px)\,dx - \frac{(n-1)}{2n}\int_0^\infty x\,f(x)J_{n+1}(Px)\,dx\right]$$

$$\tilde{f}_n'(P) = -P\left[\frac{n+1}{2n}\tilde{f}_{n-1}(P) - \frac{(n-1)}{2n}\tilde{f}_{n+1}(P)\right] \to (*)$$

$$\left[\because \tilde{f}_n(P) = \int_0^\infty x f(x)\, p(x)\, dx\right.$$

$$\int_0^\infty x f(x) J_{n-1}(Px)\, dx = \tilde{f}_{n-1}(P).$$

$$\tilde{f}_n''(P) = -P\left[\frac{n+1}{2n}\tilde{f}_{n-1}'(P) - \frac{(n-1)}{2n}\tilde{f}_{n+1}'(P)\right] \to (8)$$

Sub $n = n-1$ in eqn $(*)$.

$$\tilde{f}_{n-1}'(P) = -P\left[\frac{n}{2(n-1)}\tilde{f}_{n-2}(P) - \frac{(n-2)}{2(n-1)}\tilde{f}_n(P)\right] \to (9)$$

III^{ly} also replace n by $n+1$ in $(*)$

$n = n+1$

$$\tilde{f}_{(n+1)}'(P) = -P\left[\frac{n+2}{2(n+1)}\tilde{f}_n(P) - \frac{n}{2(n+1)}\tilde{f}_{n+2}(P)\right] \to (10)$$

Sub eqn (9) & (10) in eqn (8).

$$\tilde{f}_n''(P) = -P^2\left\{\frac{n+1}{2n}\left[\frac{n}{2(n-1)}\tilde{f}_{n-2}(P) - \frac{(n-2)}{2(n-1)}\tilde{f}_n(P)\right] - \frac{(n-1)}{2n}\left[\frac{n+2}{2(n+1)}\tilde{f}_n(P) - \frac{n}{2(n+1)}\tilde{f}_{n+2}(P)\right]\right\}$$

$$= P^2\left[\frac{(n+1)n}{2n \times 2(n-1)}\tilde{f}_{n-2}(P) - \frac{(n+1)(n-2)}{2n \times 2(n-1)}\tilde{f}_n(P)\right] -$$

$$= \frac{P^2}{4}\left\{\frac{(n+1)}{(n-1)}\tilde{f}_{n-2}(P) - \frac{(n+1)(n-2)}{n(n-1)}\tilde{f}_n(P) - \frac{(n-1)(n+2)}{n(n+1)}\tilde{f}_n(P) + \frac{(n-1)}{(n+1)}\tilde{f}_{n+2}(P)\right\} \to (11)$$

$(7) \to$

$$= \frac{P^2}{4}\left\{\frac{(n+1)}{(n-1)}\tilde{f}_{n-2}(P) + \frac{(n-1)}{(n+1)}\tilde{f}_{n+2}(P) - \left[\frac{(n+1)(n-2)}{n(n-1)}\tilde{f}_n(P) + \frac{(n-1)(n+2)}{n(n+1)}\tilde{f}_n(P)\right]\right\} \to (11)$$

take this value, again sub.

$$\Rightarrow \tilde{f}_n(P)\left[\frac{(n+1)(n-2)}{n(n-1)} + \frac{(n-1)(n+2)}{n(n+1)}\right].$$

$$\Rightarrow \frac{n^2-2n+n-2}{n(n-1)} + \frac{n^2+2n-n-2}{n(n+1)}$$

$$\Rightarrow \frac{(n+1)(n^2-2-n)}{n(n+1)(n-1)} + \frac{(n-1)(n^2+n-2)}{n(n+1)(n-1)}$$

$$\Rightarrow \frac{n^3-2n-n^2+n^2-2-n+n^3+n^2-2n-n^2-n+2}{n(n+1)(n-1)}$$

$$= \frac{2n^3-6n}{n(n+1)(n-1)}$$

$$\Rightarrow \frac{2n(n^2-3)}{n(n-1)(n+1)}$$

$$\Rightarrow \frac{2(n^2-3)}{n^2-1} \longrightarrow (a).$$

Sub eqn (a) in (11).

$$f_n''(p) = \frac{p^2}{4}\left[\frac{n+1}{n-1}\tilde{f}_{n-2}(p) - \frac{2(n^2-3)}{n^2-1}\tilde{f}_n(p) + \frac{(n-1)}{(n+1)}\tilde{f}_{n+2}(p)\right]$$

Proceeding similary we can find the Hankel tfm of the derivative any order, Deduction's.

Put $n = 1, 2, \ldots 3 \ldots n$ (9).

$$\Rightarrow f_n'(p) = -p\left[\frac{n+1}{2n} f_{n-1}'(p) - \frac{(n-1)}{2n} f_{n+1}(p)\right].$$

$n=1$.

$$\Rightarrow f_1'(p) = -p[\tilde{f}_0(p) - 0]$$

$$f_1'(p) = -p\tilde{f}_0(p)$$

$$f_1'(p) = -pf_0(p).$$

$n=2$

$$f_2'(p) = -p\left[\tfrac{3}{4}\tilde{f}_1(p) - \tfrac{1}{4}f_3(p)\right]$$

$n=3$,

$$\therefore f_3'(p) = -p\left[\tfrac{2}{3}f_2(p) - \tfrac{1}{3}f_4(p)\right]$$

Perseval's theorem:-

(R) (Em) r.v.I.

Statement:-

If $\tilde{f}(P)$ & $\tilde{g}(P)$ are the Hankel tfm of the fun $f(x)$ & $g(x)$ respectively. then

$$\int_0^\infty x\cdot f(x)\, g(x)\,dx = \int_0^\infty p\,\tilde{f}(P)\cdot\tilde{g}(P)\,dp \longrightarrow (1).$$

proof:

We have $\tilde{f}(P) = \int_0^\infty x f(x)\, J_n(Px)\,dx \longrightarrow (2)$

And

$$\tilde{g}(P) = \int_0^\infty x g(x)\, J_n(Px)\,dx \longrightarrow (3)$$

Consider the R.H.S on the eqn (1).

$$\int_0^\infty p\tilde{f}(P)\,\tilde{g}(P)\,dp = \int_0^\infty p\tilde{f}(P)\,dp \int_0^\infty x\, g(x)\, J_n(Px)\,dx$$ [in eqn (3)].

[By changing the order of integration]

We've

$$\int_0^\infty p\tilde{f}(P)\cdot\tilde{g}(P)\,dp = \int_0^\infty x g(x)\,dx \int_0^\infty p\tilde{f}(P)\, J_n(Px)\,dx$$

$$= \int_0^\infty x\, g(x) f(x)\,dx$$ [By Inverse formula]

$$\therefore \int_0^\infty p\tilde{f}(P)\,\tilde{g}(P)\,dp = \int_0^\infty x\cdot f(x)\, g(x)\,dx.$$

Hence the thm.

Pbm: Hankel transform $\frac{d^2f}{dx^2} + \frac{1}{x}\cdot\frac{df}{dx}\cdot\frac{n^2}{x^2}f$.

Soln: Let $H\left\{\frac{d^2f}{dx^2}\right\} = \int_0^\infty x\cdot\frac{d^2f}{dx^2}\, J_n\, P(x)\,dx$.

$u = x J_n\, P(x)$ | $dv = \int \frac{d^2f}{dx^2}\,dx$

$du = x\cdot J_n'(Px)\, P + J_n(Px)\,(1)$. | $v = \frac{df}{dx}$.

$\therefore H\left[\frac{d^2f}{dx^2}\right] = \left[x \cdot J_n(px) \cdot \frac{df}{dx}\right]_0^\infty - \int_0^\infty \frac{df}{dx}\left[px J_n'(px) + J_n(px)\right]$ → (*)

Assuming $x \cdot f'(x) \to 0$; when $x \to 0$; $x \to \infty$.

$$H\left[\frac{d^2f}{dx^2}\right] = -\int_0^\infty \frac{df}{dx}\left[px J_n'(px) + J_n(px)\right]dx \to (1)$$

$$\therefore \int_0^\infty \frac{d^2f}{dx^2} \cdot x J_n(px)\,dx = -\int_0^\infty \frac{df}{dx}\, px J_n'(px)\,dx - \int_0^\infty \frac{df}{dx} J_n(px)\,dx.$$

$$\therefore \int_0^\infty \left(\frac{d^2f}{dx^2} + \frac{1}{x}\cdot\frac{df}{dx}\right) x J_n(px)\,dx$$

$$= -\int_0^\infty \frac{df}{dx}\, px J_n'(px)\,dx \to (2).$$

Integrating the ϕ on the R.H.S parts taking $x J_n''(px)$ as the 1st fun.

$u = x \cdot J_n'(px)$ $\qquad \int dv = \int \frac{df}{dx} \cdot dx$

$du = \frac{d}{dx}\left[x \cdot J_n'(px)\right]$ $\qquad v = f(x).$

Sub eqn in (2),

$$= -p\left\{\left[x \cdot J_n'(px) f(x)\right]_0^\infty - \int_0^\infty f(x) \frac{d}{dx}\left(x J_n' p(x)\right) dx\right\}.$$

Assuming that $x f(x) \to 0$, when $x \to 0$ (or) when $x \to \infty$.

$$\int_0^\infty \left(\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right) x J_n P(x)\,dx$$

$$= P\int f(x) \cdot \frac{d}{dx}\left[x \cdot J_n'(px)\,dx\right] \to (3).$$

Since $J_n(x)$ satisfies Bessel's fun.

$$\frac{d}{dx}\left(x\cdot\frac{dy}{dx}\right)+\left(1-\frac{n^2}{x^2}\right)xy=0,$$ $y=J_n(x)$

$$\therefore \frac{d}{dx}\left[x\cdot\frac{d\,J_n(x)}{dx}\right]+\left(1-\frac{n^2}{x^2}\right)xJ_n(x)=0$$

$$\frac{d}{dx}\left[x\cdot J_n'(x)\right]+\left(1-\frac{n^2}{x^2}\right)x\,J_n(x)=0.$$

Replace x by px.

$$\frac{d}{dx}\left[px\cdot J_n'(px)\right]+\left(1-\frac{n^2}{p^2x^2}\right)pxJ_n(px)=0$$

$$\frac{1}{p}\cdot\frac{d}{dx}\left[px\cdot J_n'(px)\right]+\left(1-\frac{n^2}{p^2x^2}\right)pxJ_n(px)=0$$

$$\frac{d}{dx}\left[xJ_n'(px)\right]+\frac{1}{p^2}\left(p^2-\frac{n^2}{x^2}\right)\cancel{p}xJ_n(px)=0$$

$$\frac{d}{dx}\left[xJ_n'(px)\right]+\left(\cancel{p}^2-\frac{n^2}{x^2}\right)\frac{x}{p}J_n(px)=0 \longrightarrow (4).$$

Since

$$\left(1-\frac{n^2}{p^2x^2}\right)px=\left(\frac{p^2x^2-n^2}{p^2x^2}\right)p\cancel{x}.$$

$$=\frac{p^2x^2-n^2}{px}.$$

(a) $$\frac{d}{dx}\left[xJ_n'(px)\right]=-\left(p^2-\frac{n^2}{x^2}\right)-\frac{x}{p}J_n(px)\longrightarrow (5).$$

Sub eqn (5) in (3).

$$\int_0^\infty\left(\frac{d^2f}{dx^2}+\frac{1}{x}\cdot\frac{df}{dx}\right)xJ_n(px)dx=-\int_0^\infty\left(p^2-\frac{n^2}{x^2}\right)f(x).\,xJ_n(px)dx.$$

Then Rearrange the term

$$\int_0^\infty\left(\frac{d^2f}{dx^2}+\frac{1}{x}\cdot\frac{df}{dx}-\frac{n^2}{x^2}f\right)xJ_nP(x)d x=-p^2\int_0^\infty xf(x)J_n\,P(x)\,dx$$

$$\therefore H\left(\frac{d^2f}{dx^2}+\frac{1}{x}\cdot\frac{df}{dx}-\frac{n^2}{x^2}f\right)=-p^2\cancel{H}\left[\cancel{f}(x)\right].$$

$$=-p^2f_n(P)\longrightarrow (*).$$

Deductions

$n=0 \Rightarrow H\int_0^\infty \left[\frac{d^2f}{dx^2} + \frac{1}{x}\cdot\frac{df}{dx}\right] = -p^2 \bar{f}_0(p).$

Where $f_0(p)$ is the H.T of fun of zero order.

$n=1 \Rightarrow H\int_0^\infty \left[\frac{d^2f}{dx^2} + \frac{1}{x}\cdot\frac{df}{dx} - \frac{1}{x^2}f\right] = -p^2 \bar{f}_1(p).$

$n=2 \Rightarrow H\int_0^\infty \left[\frac{d^2f}{dx^2} + \frac{1}{x}\cdot\frac{df}{dx} - \frac{4}{x^2}f\right] = -p^2 f_2(p).$

Pbm:-

Find the Hankel tfm of $\frac{df}{dx}$ when $f = \frac{e^{-ax}}{x}$ and $n=1$.

Soln:-

$H\left[\frac{df}{dx}\right] = \int_0^\infty x\cdot\frac{df}{dx} J_1(px)\,dx.$

$\tilde{J}_n'(p) = \int_0^\infty x\,\frac{df}{dx} J_n(px)\,dx$

$n=1,$

$\tilde{J}_n'(p) = -p\tilde{f}_0(p).$

$= -p\tilde{f}_0(p).$

$= -p\int_0^\infty x\cdot\frac{e^{-ax}}{x}\cdot J_0(px)\,dx$

$= -p\left[(a^2+p^2)^{-1/2}\right].$

Pbm:-

Find the Hankel tfm of $\frac{d^2f}{dt^2}$ where f is a fun of x & t.

Soln:-

$H\left[\frac{d^2f}{dt^2}\right] = \int_0^\infty x\cdot\frac{d^2f}{dt^2} J_n(px)\,dx.$

$= \frac{d^2}{dt^2}\int_0^\infty x\cdot f(x,t)\, J_n(px)\,dx$

$= \frac{d^2}{dt^2}\tilde{f}(p,t).$

Pbm:-

Evaluate $\int_0^{\infty} r\left(\frac{d^2f}{dr^2}+\frac{1}{r}\cdot\frac{df}{dr}\right]J_0(px)\,dx$.

where $f(r)=\frac{e^{-ax}}{r}$.

recall

proof:

W.K.T

why? How?

$$\int_0^{\infty}\left(\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}-\frac{n^2}{x^2}f\right)xJ_n(px)\,dx=-p\tilde{f}_n(p).$$

Put $n=0$ and Replace x by r.

$$\int_0^{\infty}\left(\frac{d^2f}{dr^2}+\frac{1}{r}\cdot\frac{df}{dr}-\frac{0^2}{r^2}f\right)rJ_0(pr)\,dr=-p\tilde{f}_0(p).$$

$$=-p\int_0^{\infty} r\cdot f(r)J_0(pr)\,dr.$$

$$=-p\int_0^{\infty} r\cdot\frac{e^{-ar}}{r}J_0(pr)\,dr.$$

$$=-p\int_0^{\infty} e^{-ar}J_0(pr)\,dr.$$

$$=-p\left[a^2+p^2\right]^{-1/2}$$

Unit - IV.

Linear integral equation's:-

Integral equation:-

An integral eqn is an eqn in which on unknown fun appears and one are more integral sign's.

For Ex:-

For $a \le s \le b$; $a \le t \le b$.

$$g(s) = \int_a^b K(s,t)\, g(t)\, dt \longrightarrow (1).$$

$$g(s) = f(s) + \int_a^b K(s,t) \cdot g(t)\, dt.$$

$$g(s) = \int_a^b K(s,t)\, [g(t)]^2\, dt \longrightarrow (2).$$

Where the fun $g(s)$ is the unknown fun, ~~why~~ while all the other function are known integral eqn's.

These funs may be complex valued fun's of the real variables 's' & 't'.

Linear integral Equation:-

An integral eqn is called linear. If only linear operation's are perform in it upon the Unknown fun's.

Ex :-

$$g(s) = \int_a^b k(s,t)\, g(t)\, dt.$$

$$g(s) = f(s) + \int_a^b k(s,t) \cdot g(t)\, dt.$$

which are the linear eqn.

$$g(s) = \int_a^b k(s,t)\, [g(t)]^2\, dt \longrightarrow (2)$$

Which is non linear integral eqn.

Note:- The most general type of linear integral eqn, is of the form,

$$h(s)\, g(s) = f(s) + \lambda \int_a^b k(s,t)\, g(t)\, dt \longrightarrow (*)$$

where ~~the~~ the upper limit may be either variable ~~are~~ or fixed (constant).

The funs h, f & K are known funs ~~why~~ While g is to be determined and λ is the non-zero real ~~of~~ or complex perameter.

The fun $K(s,t)$ is called the kernal.

Consider the integral eqn,

$$h(s)\, g(s) = f(s) + \lambda \int_a^b k(s,t)\, g(t)\, dt.$$

The upper limit of the integral eqn is be fixed ~~are~~ or constant. So the eqn of the types are called ~~Freedom~~ Fredholm integral eqn. i) If $h(s) = 0$.

Then $f(s) = -\lambda \int_a^b K(s,t)\, g(t)\, dt = 0$.

This is called Fredholm integral eqn of 1st kind.

ii) $h(s) = 1$.

Then $g(s) = f(s) + \lambda \int_0^a K(s,t)\, g(t)\, dt$.

This is called Fredholm Integral eqn of 2nd kind.

iii) Special case of 2nd kind, in this case $f(s) = 0$.

$$g(s) = \lambda \int_a^b K(s,t)\, g(t)\, dt.$$

This is called the homogenous fredholm integral eqn of 2nd kind:

Voltera Eqn:-

Consider the eqn

$h(s)\, g(s) = f(s) + \lambda \int_a^b K(s,t)\, g(t)\, dt$ if the upper limit is not fixed integration is the variable then the eqn's are called Voltera integral eqn's.

i) If $h(s) = 0$.

then $f(s) + \lambda \int_a^b K(s,t)\, g(t)\, dt$.

This is called voltra integral eqn of 1st kind.

ii) $h(s)=1$.

Then $g(s) = f(s) + \lambda \int_a^b k(s,t)\, g(t)\, dt$.

this is called Voltera integral eqn 2nd kind.

iii) special case of 2nd kind, in this case $f(s)=0$.

$\therefore g(s) = \lambda \int_a^b k(s,t)\, g(t)\, dt$.

this is called the homogeneous Voltera integral eqn of 2nd kind.

When one more both lts of integral become infinite or when the kernel becomes infinite at one or more pt with in the range of integration, the integral eqn is called singular integral eqn.

Ex:-

i) $g(s) = f(s) + \lambda \int_{-\infty}^{\infty} e^{-|s-t|} g(t)\, dt$.

ii) $g(s) = \lambda \int_0^s \frac{1}{(s-t)^2} \cdot g(t)\, dt$ are singular integral eqn.

L2 functions:-

The integral fun $g(t)$.

(ie) $\int_a^b |g(t)|^2 dt < \infty$ which is called the square integrable fun (or) L2 functions.

Special kind of Kernal's:-

Seperable (or) Degenerate Kernal:-

A Kernal $K(s,t)$ is called the seperable or degenerate if it can be expressed as the sum of a finite no terms.

Each of which is a product of a fun of "s" only and a fun's of 't' only

i.e) $K(s,t) = \sum_{i=1}^{n} a_i(s) b_i(t)$.

Here the fun $a_i(s)$ can be assume linearly independent.

Otherwise the no of terms in this relation can be reduced symmetric (or) Hermitian kernal.

A complex valued fun $K(s,t)$ is called a symmentric (or) Hermitian kernal.

If $K(s,t) = K(t,s)$

where $*$ denotes the complex conjugate

Note:-

For a real Kernal is co-off with the definition $K(s,t) = K(t,s)$.

Characteristic fun:-

Consider the homogenous 2nd kind of Fredholm integral eqn,

$$g(x) = \lambda \int_a^b K(s,t)\, g(t)\, dt.$$

$$\frac{1}{\lambda} g(s) = \int_a^b K(s,t)\, g(t)\, dt.$$

(i.e.) $\int_a^b K(s,t)\, g(t)\, dt = \mu \cdot g(s)$. where $\mu = \frac{1}{\lambda}$.

So we've the classical eigen value or characteristic value pbm and μ is the eigen and $g(s)$ is the corresponding eigen fun.

Convolution Integral:-

Consider the integral eqn in which the kernal $K(s,t)$ is the function of the difference $(s-t)$ only.

(i.e.) $K(s,t) = K(s-t)$.

Where K is the certain fun of one variable.

Consider the integral eqn each of which is a product of a fun.

$g(s) = f(s) + \lambda \int_a^s K(s-t)\, g(t)\, dt$ and the corresponding fredholm integral eqn called the integral eqn of convolution

The fun defined by the integral $\int_a^s k(s-t)\, g(t)\, dt = \int_a^s k(s-t) \cdot g(s-t)\, dt$ is called the convolution or falting of the fun k & g.

The scalar or Inner product of 2 fun's:-

The inner or scalar product of (ϕ, ψ) of two complex la fun's ϕ & ψ of a real variable s, $a \leq s \leq b$ is defined $(\phi, \psi) = \int_a^b \phi(t) \cdot \psi^*(t)\, dt$.

Orthogonal:-

Two fun's are called orthogonal and ~~there~~ their inner product is zero.

(ie) ϕ & ψ are two orthogonal funs.

If $(\phi, \psi) = 0$.

Norm:-

The norm of a fun $\phi(t)$ is gn the relation

$$\|\phi(t)\| = \left[\int_a^b \phi(t)\, \phi^*(t)\, dt \right]^{1/2}$$

$$= \left[\int_a^b |\phi(t)|^2 dt \right]^{1/2}.$$

Normaliser:-

A fun ϕ is called a normalised if $\|\phi\| = 1$.

Reduction to a system of algebric eqn:-

Consider the fredholm integral eqn of 2nd kind $g(s) = f(s) + \lambda \int_a^b K(s,t)\, g(t)\, dt$ → (1).

Where $K(s,t)$ is the seperable (or) degenerate kernal then $K(s,t)$ is written as, $K(s,t) = \sum_{i=1}^{n} a_i(s), b_i(t)$.

Where the fun $a_1(s), a_2(s) \dots a_n(s)$ and the funs $b_1(t), b_2(t) \dots b_n(t)$ all are linearly independent.

∴ The eqn (1) becomes.

$$g(s) = f(s) + \lambda \int_a^b \sum_{i=1}^{n} a_i(s) b_i(t) g(t) dt.$$

$$g(s) = f(s) + \lambda \cdot \sum_{i=1}^{n} a_i(s) \int_a^b b_i(t) \cdot g(t) dt \rightarrow (2).$$

Then, $g(s) = f(s) + \lambda \sum_{i=1}^{n} a_i(s) \underbrace{\int_a^b b_i(t) \cdot g(t) dt}_{c_i}$

Where, $c_i = \int_a^b b_i(t) \cdot g(t) dt$.

ie) $g(s) = f(s) + \lambda \sum_{i=1}^{n} c_i a_i(s) \rightarrow (3)$.

Replace s by t.

$$g(t) = f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t) \rightarrow (4)$$

Sub eqn (4) in (2).

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} a_i(s) \int_a^b b_i(t) \left[f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t) \right] dt$$

$$= f(s) + \lambda \sum_{i=1}^{n} a_i(s) \left[\int_a^b b_i(t) f(t) dt + \int_a^b b_i(t) \sum_{k=1}^{n} c_k a_k(t) dt \right]$$

→ (5).

$$f(s)+\lambda\sum_{i=1}^{n} c_i a_i(s) = f(s)+\lambda\sum_{i=1}^{n} a_i(s)\left[\int_a^b b_i(t)\cdot f(t)\,dt+\int_a^b b_i(t)\sum_{k=1}^{n} c_k a_k(t)\,dt\right]$$

$$\lambda\sum_{i=1}^{n} c_i a_i(s) = \lambda\sum_{i=1}^{n} a_i(s)\left[\int_a^b b_i(t)\cdot f(t)+\int_a^b b_i(t)\,\lambda\sum_{k=1}^{n} c_k a_k(t)\right]dt.$$

$$\sum_{i=1}^{n} a_i(s)c_i-\sum_{i=1}^{n} a_i(s)\left[\int_a^b b_i(t)f(t)+\int_a^b b_i(t))\lambda\sum_{k=1}^{n} c_k a_k(t)\right]dt=0.$$

$$\sum_{i=1}^{n} a_i(s)\left[c_i-\int_a^b b_i(t)f(t)\,dt-\lambda\int_a^b b_i(t)\sum_{k=1}^{n} c_k a_k(t)\,dt\right]=0$$

Since ~~given~~ $a_1(s), a_2(s) \ldots a_n(s)$ are linear independent

$$\therefore c_i=\int_a^b b_i(t)f(t)\,dt+\lambda\int_a^b b_i(t)\sum_{k=1}^{n} c_k a_k(t)\,dt$$

$$c_i = f_i+\lambda\sum_{k=1}^{n} c_k a_{ik}=0.$$

Where, $f_i=\int_a^b b_i(t)f(t)\,dt.$

$$a_{ik}=\int_a^b b_i(t)\,a_k(t).$$

$$\Rightarrow c_i-\lambda\sum_{k=1}^{n} c_k a_{ik}=f_i \rightarrow (6).$$

$i=1$ to n.

Put $i=1$ in eqn (6).

$$\Rightarrow c_1-\lambda\sum_{k=1}^{n} c_k a_{1k}=f_1$$

$$c_1-\lambda\left[c_1a_{11}+c_2a_{12}+\ldots+c_na_{1n}\right]=f_1$$

$$\Rightarrow (1-\lambda a_{11})c_1-\lambda(c_2a_{12}+c_3a_{13}+\ldots+c_na_{1n})=f_1$$

Put $i=2$ in (6).

$$\Rightarrow c_2-\lambda\sum_{k=1}^{n} c_k a_{2k}=f_2.$$

$\Rightarrow c_2 - \lambda c_1 a_{21} - \lambda c_2 a_{22} - \lambda c_3 a_{23} - \cdots \lambda c_n a_{2n} = f_2.$

$\Rightarrow -\lambda c_1 a_{21} + (1-\lambda_{22}) c_2 - \lambda c_3 a_{23} \cdots \lambda, c_n a_{2n} = f_2.$

Put $i=n$ in (6) we get proceeding like way.

$-\lambda a_{n1} c_1 - \lambda a_{n2} c_2 \cdots + (1-\lambda a_{nn}) c_n = f_n.$

This eqn is called algebric system of eqns

$\therefore (1-\lambda a_{11}) c_1 - \lambda c_2 a_{12} \cdots - \lambda c_n a_{1n} = f_1$

$-\lambda a_{21} c_1 + (1-\lambda a_{22}) c_2 \cdots - \lambda c_n a_{2n} = f_2.$

$\vdots$

$-\lambda a_{n1} c_1 - \lambda a_{n2} c_2 \cdots + (1-\lambda a_{nn}) c_n = f_n.$

Which is known as system of n algebric eqn for the unknown's $c_1, c_2 \cdots c_n$.

The determined $D(\lambda)$ of this system gn by,

$$D(\lambda) = \begin{bmatrix} (1-\lambda a_{11}) & -\lambda a_{12} \cdots & -\lambda a_{1n} \\ -a_{21} & (1-\lambda a_{22}) \cdots & \lambda a_{2n} \\ \vdots & & \\ \lambda a_{n1} & -\lambda a_{n2} \cdots & (1-\lambda a_{nn}) \end{bmatrix} \rightarrow (7)$$

Which is the polynomial in λ of degree atmost n moreover it is not identically zero.

Since when $\lambda = 0$.

It reduces to unity for all values of λ for which $D(\lambda)=1$.

The algebric system eqn (7) and there

by integral eqn ① has unique solution

on the other hand for all values of

λ for which $D(\lambda)$ becomes equal to zero

The algebric system eqn ④ and

with if the integral eqn ①.

Either is in solvable (or) has an

infinite no/- of solution.

Setting $\lambda = \frac{1}{\mu}$ in eqn (6).

We've the Eigen value pbm of matrix

theory.

The Eigen values of gn by the

polynomial $D(\lambda)=0$.

They are also the Eigen values

of our integral eqn.

Hence the thm.

Pbm:

(A) (10m) Solve the fredholm integral eqn

of the 2^{nd} kind $g(s) = s + \lambda \int_0^1 (st^2 + s^2 t) g(t)\,dt$.

Soln:-

Gn $g(s) = s + \lambda \int_0^1 (st^2 + s^2 t) g(t)\,dt \longrightarrow ①$

~~Hence~~ Then the kernal.

$K(s,t) = \sum_{i=1}^{n} a_i(s), b_i(t)$.

~~The~~ ~~kernal~~ Since the kernal $k(s,t)$ is seperable

(1) $\therefore k(s,t) = st^2 + s^2 t$.

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$\therefore st^2+s^2t = a_1(s)\, b_1(t) + a_2(s) b_2(t)$.

$\therefore a_1(s) = s \quad ; \quad b_1(t) = t^2$

$a_2(s) = s^2 \quad ; \quad b_2(t) = t$.

W.k.T

$c_i = \int b_i(t)\, g(t)\, dt \quad , \quad i = 1 \text{ to } n.$

$i=1,2$

$c_1 = \int_0^1 b_1(t)\, g(t)\, dt = \int_0^1 t^2 g(t)\, dt \longrightarrow (2)$

$c_2 = \int_0^1 b_2(t) g(t)\, dt = \int_0^1 t\, g(t)\, dt \longrightarrow (3).$

$\therefore$ (1) Becomes,

$g(s) = s + \lambda \left[s \int_0^1 t^2 g(t)\, dt + s^2 \int_0^1 t\, g(t)\, dt \right]$

$g(s) = s + \lambda \left[s c_1 + s^2 c_2 \right] \longrightarrow (4)$ [From (2) & (3)].

Replace s by t.

$g(t) = t + \lambda \left[t c_1 + t^2 c_2 \right] \longrightarrow (5)$

Eqn (5) in (1).

$$g(s) = s + \lambda \int_0^1 (st^2 + s^2 t) \left[t + \lambda (t c_1 + t^2 c_2) \right] dt.$$

$$= s + \lambda \int_0^1 st^3 + s^2t^2 + \lambda (st^3 c_1 + st^4 c_2 + s^2 t^2 c_1 + s^2 t^3 c_2) dt$$

$$= s + \lambda \left[s \left(\frac{t^4}{4} \right)_0^1 + s^2 \left(\frac{t^3}{3} \right)_0^1 + \lambda s c_1 \left(\frac{t^4}{4} \right)_0^1 + \lambda s \left(\frac{t^5}{5} \right)_0^1 c_2 + \lambda c_1 s^2 \left(\frac{t^3}{3} \right)_0^1 + \lambda s^2 c_2 \left(\frac{t^4}{4} \right)_0^1 \right].$$

$$= s + \lambda \left[s\left(\frac{1}{4}\right) + s^2\left(\frac{1}{3}\right) + \lambda s c_1 \left(\frac{1}{4}\right) + \lambda s \left(\frac{1}{5}\right) c_2 + \lambda c_1 s^2 (1/3) + \lambda s^2 c_2 (1/4) \right].$$

$$g(s) = s + \lambda \left[s\left(\frac{1}{4} + \lambda \frac{c_1}{4} + \frac{\lambda c_2}{5} \right) + s^2 \left(\frac{1}{3} + \frac{\lambda c_1}{3} + \frac{\lambda c_2}{4} \right) \right] \longrightarrow (6).$$

Equating ④ & ⑥.

$$s+\lambda[sc_1+s^2c_2] = s+\lambda\left[s\left(\frac{1}{4}+\lambda\frac{c_1}{4}+\lambda\frac{c_2}{5}\right)+s^2\left(\frac{1}{3}+\lambda\frac{c_1}{3}+\lambda\frac{c_2}{4}\right)\right]$$

Equating the co-eff of s & s^2.

$$c_1 = \frac{1}{4}+\frac{\lambda c_1}{4}+\frac{\lambda c_2}{5}$$

$$c_2 = \frac{1}{3}+\frac{\lambda c_1}{3}+\frac{\lambda c_2}{4}.$$

$$c_1 = \frac{5+5\lambda c_1+4\lambda c_2}{20}.$$

$$20c_1-5\lambda c_1-4\lambda c_2-5=0.$$

$$\boxed{(20-5\lambda)c_1-4\lambda c_2-5=0} \rightarrow ⑦.$$

$$c_2 = \frac{4+4\lambda c_1+3\lambda c_2}{12}$$

$$12c_2-3\lambda c_2-4\lambda c_1-4=0.$$

$$\boxed{(12-3\lambda)c_2-4\lambda c_1-4=0} \rightarrow ⑧$$

$$⑦\times 4\lambda \Rightarrow 4\lambda(20-5\lambda)c_1-16\lambda^2c_2-20\lambda=0.$$

$$⑧\times(20-5\lambda) \Rightarrow -4\lambda(20-5\lambda)c_1+(12-3\lambda)(20-5\lambda)c_2-4(20-5\lambda)=0$$

$$[-16\lambda^2+(12-3\lambda)(20-5\lambda)]c_2-20\lambda-4(20-5\lambda)=0$$

$$[-16\lambda^2+240-60\lambda-60\lambda+15\lambda^2]c_2-\cancel{20\lambda}-80+\cancel{20\lambda}=0$$

$$[-\lambda^2-120\lambda+240]c_2-80=0$$

$$c_2 = \frac{80}{240-120\lambda-\lambda^2}.$$

$$⑦\times(12-3\lambda) \Rightarrow (12-3\lambda)(20-5\lambda)c_1-(12-3\lambda)4\lambda c_2-5(12-3\lambda)=0$$

$$⑧\times 4\lambda \Rightarrow 4\lambda(12-3\lambda)c_2-16\lambda^2c_1-16\lambda=0$$

$$[(12-3\lambda)(20-5\lambda)-16\lambda^2]c_1-5(12-3\lambda)-16\lambda=0.$$

$$(240-60\lambda-60\lambda+15\lambda^2-16\lambda^2)c_1-60+15\lambda-16\lambda=0$$

$$(240-120\lambda-\lambda^2)\,c_1-60-\lambda=0.$$

$$c_1=\frac{60+\lambda}{240-120\lambda-\lambda^2}.$$

Sub c_1, c_2 in eqn ④.

$$g(s)=s+\lambda\left[s\left(\frac{60+\lambda}{240-120\lambda-\lambda^2}\right)+s^2\left(\frac{80}{240-120\lambda-\lambda^2}\right)\right].$$

$$=s+\frac{\lambda s60+\lambda^2 s}{240-120\lambda-\lambda^2}+\frac{s^2\lambda 80}{240-120\lambda-\lambda^2}.$$

$$=\frac{(240-120\lambda-\lambda^2)s+\lambda s60+\lambda^2 s+s^2\lambda 80}{240-120\lambda-\lambda^2}.$$

$$g(s)=\frac{80s^2\lambda+s(240-60\lambda)}{240-120\lambda-\lambda^2}\;//.$$

⑧ (5m) **Pbm:-** Find the integral eqn $g(s)=f(s)+\lambda\int_0^1(s+t)g(t)\,dt$ at find the eigen values and resolvent Kernal.

Soln:- Consider the fredholm 2nd kind of integral eqn.

$$g(s)=f(s)+\lambda\int k(s,t)g(t)\,dt \longrightarrow ⑧$$

Here the kernal $k(s,t)=\sum_{i=1}^{n}a_i(s)\,b_i(t)$

$$k(s,t)=a_1(s)b_1(t)+a_2(s)b_2(t).$$

$$s+t=a_1(s)b_1(t)+a_2(s)b_2(t).$$

$a_1(s)=s$; $b_1(t)=1$

$a_2(s)=1$; $b_2(t)=t$.

$g(s)=f(s)+\lambda\int[a_1(s)b_1(t)+a_2(s)b_2(t)]$

$=f(s)+\lambda[\int a_1(s)b_1(t)\,g(t)\,dt$

$\int s\, g(t)\,dt$

W.K.T

$f_i = \int b_i(t) f(t)\, dt \longrightarrow (1) \quad i = 1 \text{ to } u.$

$a_{ik} = \int b_i(t) a_k(t)\, dt \longrightarrow (2)$

where $i = 1, 2 \ldots u$, $k = 1, 2, \ldots u$.

From (2).

$a_{11} = \int_0^1 b_1(t) a_1(t) dt.$

$= \int_0^1 1 \cdot t\, dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}.$

$a_{11} = \frac{1}{2}.$

$c_1 = \int b_1(t) g(t)\, dt$

$a_{12} = \int_0^1 b_1(t)\, a_2(t) dt$

$= \int_0^1 1 \cdot 1 dt = (t)_0^1 = 1.$

$a_{12} = 1.$

$a_{21} = \int_0^1 b_2(t)\, a_1(t) dt = \int_0^1 t \cdot t \cdot dt = \int_0^1 t^2 dt$

$= \left(\frac{t^3}{3}\right)_0^1 = \frac{1}{3}.$

$a_{21} = \frac{1}{3}$

$a_{22} = \int_0^1 b_2(t) a_2(t) dt = \int_0^1 t \cdot 1 \cdot dt = \left(t^2/2\right)_0^1$

$a_{22} = 1/2$

From (1) $\Rightarrow$

$f_1 = \int_0^1 b_1(t) f(t)\, dt = \int_0^1 1 \cdot f(t) dt.$

1114 $f_2 = \int_0^1 b_2(t) f(t) dt = \int_0^1 t \cdot f(t)\, dt \longrightarrow (A).$

Consider the system of eqn,

Now, $1-\lambda a_{11} c_1 - \lambda a_{12} c_2 = f_1 \longrightarrow (3)$: Sys of Algebraic equations.

$-\lambda_{21} c_1 + 1 - \lambda a_{22} c_2 = f_2 \longrightarrow (4)$.

18/06/2014

To find Eigen values

$D(\lambda) = |1-\lambda a_{21} + 1 - \lambda a_{22}|$

$$= \begin{vmatrix} 1-\lambda/2 & -\lambda \\ -\lambda/3 & 1-\lambda/2 \end{vmatrix}$$

$D(\lambda) = (1-\lambda/2)^2 - \lambda^2/3.$

$\therefore$ Eigen value $D(\lambda)=0$.

$\Rightarrow (1-\lambda/2)^2 - \lambda^3/3 = 0.$

$\Rightarrow 1+\lambda^2/4 - \lambda - \lambda^2/3 = 0.$

$\Rightarrow \dfrac{12+3\lambda^2 - 12\lambda - 4\lambda^3}{12} = 0.$

$\Rightarrow -\lambda^2 - 12\lambda + 12 = 0.$

$\Rightarrow \lambda^2 + 12\lambda - 12 = 0.$

$\lambda = \dfrac{-12 \pm \sqrt{144 - 4 \times 1 \times (-12)}}{2}.$

$= \dfrac{-12 \pm \sqrt{144+48}}{2}$

$= \dfrac{-12 \pm \sqrt{192}}{2}.$

$\lambda = \dfrac{12 \pm 8\sqrt{3}}{2}.$

$\lambda = -6 \pm 4\sqrt{3}$ //.

$\begin{vmatrix} 1-\lambda/2 & -\lambda \\ -\lambda/3 & 1-\lambda/2 \end{vmatrix}$

$g(x) = f(x) + \lambda \int (\bar{a}_1(x) b_1(t) + a_2(x) b_2(t)) g(t) dt$

$= \int s\, g(t) dt + \int (t)\, g(t) dt$

$\Rightarrow$

$f_1 = \int b_1(t) f(t) dt$

$f_1 = \int 1\, f(t) dt$

$\therefore$ The Eigen values are

$$\lambda_1 = -6+4\sqrt{3} \; ; \; \lambda_2 = -6-4\sqrt{3}.$$

For ~~this~~ These two values of λ then the homogeneous eqn has a non-trivial soln.

While the integral eqn (*) is in general non-solvable then λ differs from this values the soln of the above algebric system. we can find.

solve the eqn from 5 & 6.

$$(1-\lambda/2)c_1 - \lambda c_2 = f_1 \longrightarrow 5$$

$$-\frac{\lambda}{3}c_1 + (1-\lambda/2)c_2 = f_2 \longrightarrow 6.$$

$$5 \times \lambda/3 \Rightarrow \frac{\lambda}{3}(1-\frac{\lambda}{2})c_1 - \lambda^2/3\, c_2 = \lambda/3 f_1$$

$$6 \times (1-\frac{\lambda}{2}) \Rightarrow -\frac{\lambda}{3}(1-\frac{\lambda}{2})c_1 + (1-\frac{\lambda}{2})^2 c_2 = (1-\frac{\lambda}{2})f_2$$

$$\left[(1-\frac{\lambda}{2})^2 - \frac{\lambda^2}{3}\right]c_2 = \frac{\lambda}{3}f_1 + (1-\frac{\lambda}{2})f_2.$$

$$\left(1+\frac{\lambda^2}{4} - \lambda - \frac{\lambda^2}{3}\right)c_2 = \frac{\lambda}{3}f_1 + \left(\frac{2-\lambda}{2}\right)f_2.$$

$$\left(\frac{12+3\lambda^2-12\lambda-4\lambda^2}{12}\right)c_2 = \frac{\lambda}{3}f_1 + \left(\frac{2-\lambda}{2}\right)f_2.$$

$$\left(\frac{-\lambda^2-12\lambda+12}{12}\right)c_2 = \frac{2\lambda f_1 + (6-3\lambda)f_2}{6}.$$

$$(-\lambda^2-12\lambda+12)c_2 = 2\left[2\lambda f_1 + (6-3\lambda)f_2\right].$$

$$c_2 = \frac{4\lambda f_1 + (12-6\lambda)f_2}{[12-12\lambda-\lambda^2]} \longrightarrow 7.$$

$$(1)\times\left(1-\frac{\lambda}{2}\right) \Rightarrow \left(1-\frac{\lambda}{2}\right)^2 c_1 + \lambda\left(1-\frac{\lambda}{2}\right)c_2 = \left(1-\frac{\lambda}{2}\right) f_1$$

$$(2)\times\lambda \Rightarrow -\frac{\lambda^2}{3} c_1 + \lambda\left(1-\frac{\lambda}{2}\right)c_2 = \lambda f_2$$

$$\left[\left(1-\frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{3}\right]c_1 = \left(1-\frac{\lambda}{2}\right)f_1 + \lambda f_2 .$$

$$\left(1+\frac{\lambda^2}{4}-\lambda-\frac{\lambda^2}{3}\right)c_1 = \left(\frac{2-\lambda}{2}\right)f_1 + \lambda f_2 .$$

$$\left(\frac{12+3\lambda^2-12\lambda-4\lambda^2}{12}\right)c_1 = \frac{(2-\lambda)f_1 + 2\lambda f_2}{2}.$$

$$(-\lambda^2-12\lambda+12)\,c_1 = 6(12-\lambda)f_1 + 12\lambda f_2 .$$

$$(-\lambda^2-12\lambda+12)\,c_1 = (12-6\lambda)f_1 + 12\lambda f_2 .$$

$$c_1 = \frac{(12-6\lambda)f_1 + 12\lambda f_2}{[12-12\lambda-\lambda^2]} \longrightarrow (3).$$

W.K.T

$$c_i = \int b_i(t)\,g(t)\,dt .$$

$$c_1 = \int_0^1 b_1(t)\,g(t)\,dt = \int_0^1 1\cdot g(t)\,dt .$$

$$c_2 = \int_0^1 b_2(t)\,g(t)\,dt = \int_0^1 t\cdot g(t)\,dt .$$

Gn that.

$$g(s) = f(s) + \lambda\int_0^1 (s+t)\,g(t)\,dt .$$

$$= f(s) + \lambda\left[\int_0^1 s\,g(t)\,dt + \int_0^1 t\,g(t)\,dt\right]$$

$$= f(s) + \lambda\,[s c_1 + c_2].$$

$$\therefore\ g(s) = f(s) + \lambda\left\{\left[\frac{s(12-6\lambda)f_1 + 12\lambda f_2}{12-12\lambda-\lambda^2}\right] + \left[\frac{4\lambda f_1 + (12-6\lambda)f_2}{12-12\lambda-\lambda^2}\right]\right\}$$

$$g(s) = f(s) + \left[\frac{\lambda s(12-6\lambda)f_1 + 12\lambda f_2}{12-12\lambda-\lambda^2}\right] + \lambda\left[\frac{4\lambda f_1 + (12-6\lambda)f_2}{12-12\lambda-\lambda^2}\right].$$

To find Resolvent kernal.

$$g(s) = f(s) + \frac{\lambda}{12-12\lambda-\lambda^2}\left[(12s-6\lambda s+4\lambda)f_1 + (12\lambda s+12-6\lambda)f_2\right]$$

$$= f(s) + \frac{\lambda}{12-12\lambda-\lambda^2}\left\{\left[(12s-6\lambda s+4\lambda)\int_0^1 f(t)\,dt\right]\right.$$

$$\left. + \left[(12\lambda s+12-6\lambda)\int_0^1 t f(t)\,dt\right]\right\} \text{ By} \rightarrow (A)$$

$$g(s) = f(s) + \frac{\lambda}{12-12\lambda-\lambda^2}\left[\int_0^1 12s-6\lambda s+4\lambda + t(12\lambda s+12-6\lambda) f(t)\,dt\right]$$

$$= f(s) + \frac{\lambda}{12-12\lambda-\lambda^2}\left[\int_0^1 12(s+t) - 6\lambda(s+t) + 4\lambda + 12\lambda st\right] f(t)\,dt.$$

$$g(s) = f(s) + \frac{\lambda}{12-12\lambda-\lambda^2}\left[\int_0^1 \Gamma(s,t,\lambda) f(t)\,dt\right]$$

Which is called Resolvant Kernal.

ie, $g(s) = f(s) + \lambda \int_0^1 \Gamma(s,t,\lambda)\, f(t)\,dt.$

SM N.V-I

Pbm: Find the resolve and kernal for the integral eqn $g(s) = f(s) + \lambda \int_{-1}^{1} (st + s^2t^2) g(t)\,dt.$

Soln: Fredholm II kind of integral eqn,

$$g(s) = f(s) + \lambda \int_a^b K(s,t)\, g(t)\,dt.$$

Where $K(s,t) = st + s^2t^2.$

$K(s,t) = a_1(s) \cdot b_1(t) + a_2(s)\, b_2(t).$

$\therefore a_1(s) = s \; ; \; b_1(t) = t$

$a_2(s) = s^2 \; ; \; b_2(t) = t^2.$

w.k.t.

$$f_i = \int b_i(t) f(t)\,dt$$

$$a_i(k) = \int b_i(t)\, a_k(t)\,dt$$

lyn that

$$c_i = \int b_i(t)\, g(t)\,dt.$$

To find a_{ik}.

$$a_{11} = \int b_1(t)\, a_1(t)\,dt = \int_{-1}^{1} t \cdot t\,dt = \left(\frac{t^3}{3}\right)_{-1}^{1}$$

$$a_{11} = \left(\frac{1}{3}+\frac{1}{3}\right) = \frac{2}{3}.$$

$$\boxed{a_{11} = \frac{2}{3}}$$

$$a_{12} = \int_{-1}^{1} b_1(t)\, a_2(t)\,dt = \int_{-1}^{1} t \cdot t^2\,dt = \int_{-1}^{1} t^3\,dt = \left(\frac{t^4}{4}\right)_{-1}^{1}$$

$$= \left(\frac{1}{4} - \frac{1}{4}\right) = 0.$$

$$\boxed{a_{12} = 0.}$$

$$a_{21} = \int_{-1}^{1} b_2(t) \cdot a_1(t)\,dt = \int_{-1}^{1} t^2 \cdot t\,dt = \int_{-1}^{1} t^3\,dt = \left(\frac{t^4}{4}\right)_{-1}^{1}$$

$$= \frac{1}{4} - \frac{1}{4} = 0$$

$$\boxed{a_{21} = 0}$$

$$a_{22} = \int_{-1}^{1} b_2(t) \cdot a_2(t)\,dt = \int_{-1}^{1} t^2 \cdot t^2\,dt = \int_{-1}^{1} t^4\,dt = \left(\frac{t^5}{5}\right)_{-1}^{1}$$

$$= \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

$$\boxed{a_{22} = \frac{2}{5}}$$

To find c_1 & c_2.

$$c_1 = \int b_1(t)\, g(t)\,dt.$$

$$= \int_{-1}^{1} t \cdot g(t)\,dt \rightarrow (1).$$

$C_2 = \int b_2(t)\, g(t)\, dt$

$C_2 = \int t^2 g(t)\, dt \longrightarrow (2)$

Given that

$g(s) = f(s) + \lambda \int_{-1}^{1} (st - s^2t^2)\, g(t)\, dt$.

$g(s) = f(s) + \lambda \left[s \int_{-1}^{1} t\, g(t)\, dt + s^2 \int_{-1}^{1} t^2 g(t)\, dt \right]$.

$g(s) = f(s) + \lambda [sc_1 + s^2 c_2] \longrightarrow (*)$ [By (1) & (2)].

Consider the system of eqn.

$(1-\lambda a_{11}) c_1 - \lambda a_{12} c_2 = f_1 \longrightarrow (3)$

$-\lambda a_{21} c_1 + (1-\lambda a_{22}) c_2 = f_2 \longrightarrow (4)$.

Sub the values of a_{ik}.

$(3) \Rightarrow \left(1 - \frac{2}{3}\lambda\right) c_1 = f_1$

$$\boxed{c_1 = \frac{f_1}{\left(1 - \frac{2}{3}\lambda\right)}}$$

$(4) \Rightarrow -\lambda(0) + [1 - \lambda(2/5)]\, c_2 = f_2$

$$\boxed{c_2 = \frac{f_2}{\left(1 - 2/5\,\lambda\right)}}$$

Sub in eqn $(*)$.

$g(s) = f(s) + \lambda \left[s\left(\frac{f_1}{1 - 2/3\,\lambda}\right) + s^2 \left(\frac{f_2}{1 - 2/5\,\lambda}\right) \right]$

To find the resolvant kernal

$f_i = \int b_i(t) \cdot f(t)\, dt$.

Such that

$f_1 = \int b_1(t) f(t)\, dt$.

$$= \int_{-1} t \cdot f(t)\, dt \to (5)$$

$$f_2 = \int b_2(t) \cdot f(t)\, dt = \int_{-1}^{1} t^2 f(t)\, dt \to (6)$$

Sub these in eqn (*).

$$g(s) = f(s) + \lambda \left[\frac{s}{1-\frac{2}{3}\lambda} \int_{-1}^{1} t\, f(t)\, dt + \frac{s^2}{(1-\frac{2}{5}\lambda)} \int_{-1}^{1} t^2 f(t)\, dt \right]$$

$$= f(s) + \lambda \int_{-1}^{1} \left[\frac{st}{(1-\frac{2}{3}\lambda)} + \frac{s^2 t^2}{(1-\frac{2}{5}\lambda)} \right] f(t)\, dt.$$

$$g(s) = f(s) + \lambda \int_{-1}^{1} \Gamma(s,t,\lambda)\, f(t)\, dt.$$

Where $\Gamma(s,t,\lambda) = \dfrac{st}{(1-\frac{2}{3}\lambda)} + \dfrac{s^2 t^2}{(1-\frac{2}{5}\lambda)}$

$\therefore$ Which is the resolvant kernal.

Pbm: Find the Eigen value and eigen fun of the homogenous integral eqn

$$g(s) = \lambda \int_{1}^{2} \left[st + \frac{1}{st} \right) g(t)\, dt.$$

Soln:- Consider the Fredholm II kind integral eqn. $g(s) = \lambda \int_a^b k(s,t)\, g(t)\, dt.$

Where $k(s,t)$ is seperable.

Gn that $k(s,t) = st + \frac{1}{st}$.

$$K(s,t) = a_1(s)\, b_1(t) + a_2(s)\, b_2(t).$$

$\therefore a_1(s) = s \quad ; \quad b_1(t) = t$

$a_2(s) = \frac{1}{s} \quad ; \quad b_2(t) = 1/t.$

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W.K.T

$f_i = \int b_i(t) \cdot f(t)\, dt.$

$a_{ik} = \int b_i(t)\, a_k(t)\, dt.$

$c_i = \int b_i(t)\, g(t)\, dt.$

To find a_{ik} :-

$a_{11} = \int b_1(t)\, a_1(t)\, dt = \int_1^2 t \cdot t\, dt = \left(\frac{t^3}{3}\right)_1^2$

$= \frac{8}{3} - \frac{1}{3}$

$\boxed{a_{11} = \frac{7}{3}.}$

$a_{22} = \int b_2(t)\, a_2(t)\, dt = \int_1^2 \frac{1}{t} \cdot \frac{1}{t} \cdot dt = \int_1^2 \frac{1}{t^2} dt.$

$= \int_1^2 t^{-2} dt = \left[\frac{t^{-1}}{-1}\right]_1^2 = \left(\frac{2^{-1}}{-1} - \frac{1^{-1}}{1}\right)$

$= -\frac{1}{2} + \frac{1}{1} = \left(-\frac{1}{2} + 1\right) = \frac{1}{2}.$

$\boxed{a_{22} = \frac{1}{2}}$

$a_{21} = \int b_2(t)\, a_1(t)\, dt = \int_1^2 \left(\frac{1}{t} \cdot t\right) dt = \int_1^2 dt = (t)_1^2$

$a_{21} = (2-1) = 1.$

$a_{12} = \int b_1(t)\, a_2(t)\, dt = \int_1^2 t \cdot \frac{1}{t}\, dt = (t)_1^2 = 2-1 = 1$

$\boxed{a_{12} = 1}$

Consider the system of Eqn :-

$(1 - \lambda a_{11})\, c_1 - \lambda_{12} c_2 = f_1$

$-\lambda a_{21} c_1 + (1 - \lambda_{22})\, c_2 = f_2$

$\left(1 - \lambda \cdot \frac{7}{3}\right) c_1 - \lambda c_2 = f_1$

$-\lambda c_1 + \left(1 - \frac{\lambda}{2}\right) c_2 = f_2,$

Consider, $D(\lambda) = \begin{vmatrix} 1-\frac{7}{3}\lambda & -\lambda \\ -\lambda & 1-\lambda \end{vmatrix}$

$D(\lambda) = 0.$

$\Rightarrow (1-\frac{7}{3}\lambda)(1-\frac{\lambda}{2}) - \lambda^2 = 0$

$\Rightarrow -\lambda^2 + 1 - \frac{\lambda}{2} - \frac{7\lambda}{3} + \frac{7\lambda^2}{6} = 0$

$\Rightarrow \frac{6 - 3\lambda - 14\lambda - 7\lambda^2 - 6\lambda^2}{6} = 0.$

$\Rightarrow \lambda^2 - 17\lambda + 6 = 0$

$\lambda = \frac{17 \pm \sqrt{289-24}}{2}.$

$= \frac{17 \pm \sqrt{265}}{2} = \frac{17 \pm 16.2788}{2} = \frac{17 + 16.27}{2}$

$\boxed{\lambda_1 = 16.6394}$

$\lambda_2 = \frac{17 - 16.6394}{2} = 0.3606$

$\boxed{\lambda_2 = 0.3606}.$

$\therefore$ The Eigen values are $\lambda_1 = 16.6394$
$\lambda_2 = 0.3606.$

To Find the eigen function:-

Let $f_1 = \int_1^2 t \cdot f(t)\,dt = 0$

$f_2 = \int_1^2 \frac{1}{t} \cdot f(t)\,dt = 0.$

Then $(1-\frac{7}{3}\lambda)c_1 - \lambda c_2 = 0 \longrightarrow (1)$

$-\lambda c_1 + (1-\frac{\lambda}{2})c_2 = 0 \longrightarrow (2).$

Put $\lambda = 16.6394$ in eqn (1).

$[1 - \frac{7}{3}(16.6394)]c_1 - 16.6394 c_2 = 0.$

$\boxed{c_1 = -0.4399 c_2}.$

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The general eqn.

$$g(s) = \lambda \int_1^2 \left(st + \frac{1}{st}\right) g(t)\, dt.$$

$$c_i = \int b_i(t)\, g(t)\, dt$$

$$c_1 = \int_1^2 t\, g(t)\, dt \; ; \quad c_2 = \int_1^2 \frac{1}{t} g(t)\, dt.$$

$$\therefore g(s) = \lambda \left[s \int_1^2 t\, g(t)\, dt + \frac{1}{s} \int_1^2 \frac{1}{t} \cdot g(t)\, dt \right]$$

$$g(s) = \lambda \left[s c_1 + \frac{1}{s} c_2 \right] \rightarrow (*)$$

$$= 16.6394 \left[s(-0.4399\, c_2) + \frac{1}{s} c_2 \right]$$

$$g(s) = \left[-7.31973 + \frac{16.6394}{s} \right] c_2.$$

Which is the Eigen fun corresponding to the eigen value 16.6394.

Put $\lambda = 0.3606$ in (2).

$$-0.3606 c_1 + \left(1 - \frac{0.3606}{2}\right) c_2 = 0.$$

$$0.8197\, c_2 = 0.3606\, c_1$$

$$c_2 = \frac{0.3606}{0.8197} c_1$$

$$\boxed{c_2 = 0.4399 c_1}$$

Sub in eqn (*)

$$g(s) = 0.3606 \left[s c_1 + \frac{1}{s} 0.4399\, c_1 \right]$$

$$g(s) = \left[0.3606 s + \frac{0.1586}{s} \right] c_1$$

Which is the eigen fun correspondi to the eigen value 0.3606.

Invert integral equation:-

$$g(s) = f(s) + \lambda \int_0^{2\pi} (\underset{a_1}{\sin s}\,\underset{b_1}{\cos t})\, g(t)\,dt \longrightarrow (1).$$

Soln:-

Let $K(s,t) = \sin s \cos t$

$a_1(s) = \sin s$

$b_1(t) = \cos t$.

$c_i = \int b_i(t) g(t)\,dt$.

$$c_1 = \int_0^{2\pi} \cos t\, g(t)\,dt.$$

$(1) \Rightarrow g(s) = f(s) + \lambda c_1 \sin s \longrightarrow (2)$

$g(t) = f(t) + \lambda c_1 \sin t$.

$$\int_0^{2\pi} \cos t\, g(t)\,dt = \int_0^{2\pi} \cos t\, f(t)\,dt + \int_0^{2\pi} \lambda c_1 \sin t \cos t\,dt$$

$$c_1 = \int_0^{2\pi} \cos t f(t)\,dt + \frac{\lambda c_1}{2}\int_0^{2\pi} 2\sin t \cos t\,dt.$$

$$= \int_0^{2\pi} \cos t f(t)\,dt + \frac{\lambda c_1}{2}\int_0^{2\pi} \sin 2t\,dt.$$

$$= \int_0^{2\pi} \cos t f(t)\,dt + \frac{\lambda c_1}{2}\left(-\frac{\cos 2t}{2}\right)_0^{2\pi}$$

$$= \int_0^{2\pi} \cos t f(t)\,dt + \frac{\lambda c_1}{2}\left[-\left(\tfrac{1}{2} - \tfrac{1}{2}\right)\right]$$

$$c_1 = \int_0^{2\pi} \cos t f(t)\,dt.$$

$$\therefore g(t) = f(t) + \lambda \int_0^{2\pi} \cos t \cdot \sin t\, f(t)\,dt.$$

Pbm! (sum) Solve the homogeneous fredholm integral eqn, $g(s) = \lambda \int_0^{1} e^{s} e^{t} g(t)\,dt$.

Soln:- Gn that,

$$g(s) = \lambda \int_0^{1} e^{s} e^{t} g(t)\,dt \longrightarrow (1).$$

Define $c = \int_0^1 e^t g(t)\, dt$

$$g(s) = \lambda c e^s \longrightarrow ②.$$

Sub ② in ①.

$$\lambda c e^s = \lambda \int_0^1 e^s e^t \lambda c e^t dt.$$

$$\lambda c e^s = \lambda^2 \int_0^1 e^s e^{2t} dt.$$

$$= \lambda^2 c \int_0^1 e^s e^{2t} dt.$$

$$\lambda c e^s = \lambda^2 c e^s \int_0^1 e^{2t} dt.$$

$$= \lambda^2 c e^s \left(\frac{e^{2t}}{2}\right)_0^1$$

$$= \lambda^2 c e^s \left(\frac{e^2}{2} - \frac{1}{2}\right)$$

$$\cancel{\lambda} \cancel{ce^s} = \lambda^{\cancel{2}} \cancel{ce^s} \left(\frac{e^2-1}{2}\right)$$

If $c = 0$ (or) $\lambda = 0$, Then $g = 0$.

Assume that neither $c = 0$ (or) $\lambda = 0$.

$$\cancel{\lambda c} = \lambda^2 c \left[ce^2-1)/2\right)$$

$\cancel{\lambda c}$

$$1 = \lambda \left(\frac{e^2-1}{2}\right)$$

$$\frac{1}{\left(\frac{e^2-1}{2}\right)} = \lambda$$

$$\boxed{\lambda = \frac{2}{e^2-1}}$$

Only for this value $\lambda = \frac{2}{e^2-1}$ $\exists$ a non trivial soln of eqn ①.

Eqn ② $\Rightarrow$: $g(s) = \frac{2}{e^2-1} c e^s$.

$$= \frac{2c}{e^2-1} e^s = \frac{2}{e^2-1}$$

~~Thus to the Eigen $\lambda = \frac{2}{e^2-1}$.~~

~~Thus corresponding Eigen fun e^s.~~

Theorem :- Fredholm Alternative

Fredholm thm :-

The inhomogenoues fredholm integral eqn. $g(s) = f(s) + \lambda \int K(s,t) g(t) dt \rightarrow (1)$.

with a seperable Kernal has one and only gn by,

$$g(s) = f(s) + \lambda \int \Gamma(s,t,\lambda) f(t)\, dt \longrightarrow (2).$$

The resolvant kernal $\Gamma(s,t,\lambda)$ consider with a Quitient $\frac{D(s,t,\lambda)}{D(\lambda)}$ of two polynonials.

$$\Gamma(s,t,\lambda) = \frac{D(s,t,\lambda)}{D(\lambda)}$$

proof:

If $D(\lambda) = 0$ then the inhomogeneous Fredholm eqn (1) has no soln in general, because an algebric ~~system with~~ system vanishing determine can be solved only for some particular values of f_i.

To discuss this case we write the algebric system.

$c_i - \lambda \sum_{k=1}^{n} a_{ik} c_k = f_i$ (where $i = 1, 2 \ldots n$)

has $(I - \lambda A) c = f$. where I is the unit matrix of order n & A is the matrix $\forall a_{ij}$.

Now, when $D(\lambda) = 0$ we observe that for each non trivial soln of the homogeneous. Algebric system,

$(I - \lambda A) = 0 \rightarrow$ ③ is correspond a non trivial soln of a homo integral eqn.

$\therefore g(s) = \lambda \int k(s,t)\, g(t)\, dt \rightarrow$ ④

Further more if λ consider with a certain eigen value λ_0 for which the determined.

$D(\lambda_0) = |1 - \lambda_0 A|$ has the rank p,

$1 \leq p \leq n$. Then there are $r = n - p$ linearly independent soln of the algebric system.

Where, r is called the index of the eigen value λ_0.

The same holds for the homo integral eqn ④.

Now, 'p' denote these 'r' linearly independent soln as,

$g_{01}(s), g_{02}(s), \ldots g_{0r}(s)$.

And ~~A.T~~ Assume that they are Normalized.

Then to each eigen value λ_0 of index $r = n-p$.

These corresponds a soln $g_0(s)$ of the homo eqn ④ are from.

$$g_0(s) = \sum_{K=1}^{r} \alpha_K g_{0K}(s).$$

where, α_K are arbitary constant

Let 'm' be the multiplicity of the eigan value λ_0.

i.e.) $\lambda_0 = 0$, $D(\lambda) = 0$ has 'm' equal roots λ_0.

Thus the rank p of $D(\lambda_0)$ is greater than are equal to $n-m$. thus $r = n-p$

$$\leq n-(n-m)$$
$$= m.$$

And the equality holds only. when $a_{ij} = a_{ji}$. Thus we have to prove the thm of fredholm then $\lambda = \lambda_0$ is a root of multiplicity $m \geq 0$ of the eqn $D(\lambda) = 0$.

Then the homo' integral eqn ④ has ~~are~~ linearly independent soln r is the eigen value $\ni : 1 \leq r \leq m$.