CORE COURSE IV

GRAPH THEORY

Objectives

1. To give a rigorous study of the basic concepts of Graph Theory.
2. To study the applications of Graph Theory in other disciplines.

***Note*: Theorems, Propositions and results which are starred are to be omitted.**

Unit I Basic Results

Basic Concepts - Subgraphs - Degrees of Vertices - Paths and Connectedness-Operations on Graphs - Directed Graphs: Basic Concepts - Tournaments.

Unit II Connectivity

Vertex Cuts and Edge Cuts - Connectivity and Edge - Connectivity, Trees:Definitions, Characterization and Simple Properties - Counting the Number of Spanning Trees - Cayley's Formula.

Unit III Independent Sets and Matchings

Vertex Independent Sets and Vertex Coverings - Edge Independent Sets -Matchings and Factors - Eulerian Graphs - Hamiltonian Graphs.

Unit IV Graph Colourings

Vertex Colouring - Critical Graphs - Triangle - Free Graphs - Edge Colourings of Graphs - Chromatic Polynomials.

Unit V Planarity

Planar and Nonplanar Graphs - Euler Formula and its Consequences - K5 and K3,3 are Nonplanar Graphs - Dual of a Plane Graph - The Four-Colour Theorem and the Heawood Five-Colour Theorem-Kuratowski's Theorem.

Textbook

1. R. Balakrishnan, K. Ranganathan, A Textbook of Graph Theory, Springer International Edition, New Delhi, 2008.

UNIT I	Chapter I & II: 1.1 to 1.4, 1.7, 2.1, 2.2
UNIT II	Chapter III & IV: 3.1, 3.2, 4.1, 4.3 to 4.4
UNIT III	Chapter V & VI: 5.1 to 5.4, 6.1, 6.2
UNIT IV	Chapter VII: 7.1 to 7.4, 7.7
UNIT V	Chapter VIII: 8.1 to 8.6

References

1. J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Mac MilanPress Ltd., 1976.
2. Gary Chartrand, Linda Lesniak, Ping Zhang, Graphs and Digraph,CRC press,2010.
3. F.Harary, Graph Theory, Addison - Wesley, Reading, Mass., 1969.

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GRAPH THEORY

UNIT-1

BASIC CONCEPTS

Defn 1:- A graph is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a nonempty set whose elements are called vertices (or nodes or points) of G and $E(G)$ is a set of unordered pairs of elements of $V(G)$ whose elements are called edges (or lines) of G.

Note:- I_G is an incidence relation associate with each element of $E(G)$.

For the edge e of G, $I_G(e) = \{u, v\} = uv$

Ex:

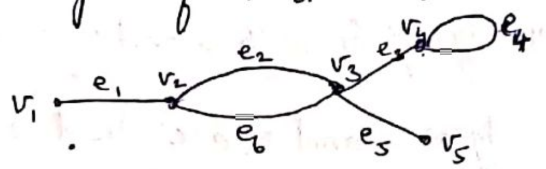


Fig: 1

$V(G) = \{v_1, v_2, v_3, v_4, v_5\}$

$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

I_G :- $I_G(e_1) = \{v_1, v_2\}$, $I_G(e_2) = \{v_2, v_3\}$

$I_G(e_3) = \{v_3, v_4\}$, $I_G(e_4) = \{v_4, v_4\}$, $I_G(e_5) = \{v_3, v_5\}$

$I_G(e_6) = \{v_2, v_3\}$

Defn 2:- If $I_G(e) = \{u, v\}$ then the vertices u & v are called the end vertices or ends of the edge e.

Note:- e is incident with each one of its ends.

The vertices u & v are incident with e.

Defn 3:- A set of two or more edges of a graph G is called a set of multiple or parallel edges if they have the same pair of distinct ends.

An edge for which the two ends are the same is called a loop.

Defn 4:- A vertex u is a neighbor of v in G, if uv is an edge of G, and u ≠ v.

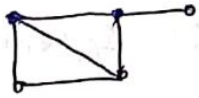
The set of all neighbors of v is the open neighborhood of v or the neighbor set of v, denoted by N(v).

The set N[v] = N(v) ∪ {v} is the closed neighborhood of v in G.

Defn 5:- Vertices u and v are adjacent to each other in G iff there is an edge of G with u and v as its ends.

Two distinct edges e_1 and e_2 are said to be adjacent iff they have a common end vertex.

Defn 6:- A graph is simple if it has no loops and no multiple edges. Ex:



Ex: In Fig. 1:- Edge $e_2 = v_2v_3 = e_6$ ~~form~~ multiple edges

$e_4 \to$ Loop at v_4

$N(v_3) = \{v_2, v_4, v_5\}$, $N(v_4) = \{v_3\}$

$N[v_3] = \{v_2, v_3, v_4, v_5\}$, $N[v_4] = \{v_3, v_4\}$

$v_2, v_3 \to$ Adjacent vertices

$e_2, e_5 \to$ Adjacent edges

Defn 7:- A graph is finite if both V(G) and E(G) are finite. A graph that is not finite is called infinite graph.

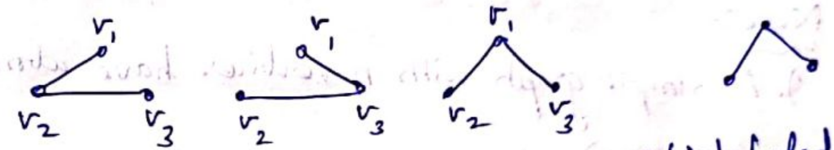
The number of vertices and number of edges of the graph G are denoted by $n(G)$ and $m(G)$ respectively.

$n(G)$ is called order of G and
$m(G)$ " " size of G.

Note: We denote $V(G)$, $E(G)$, $n(G)$ and $m(G)$ simply as V, E, n and m respectively.

Defn 8 :- A graph is said to be labeled if its n vertices are distinguished from one another by labels such as $v_1, v_2, \ldots, v_n$.

Ex :

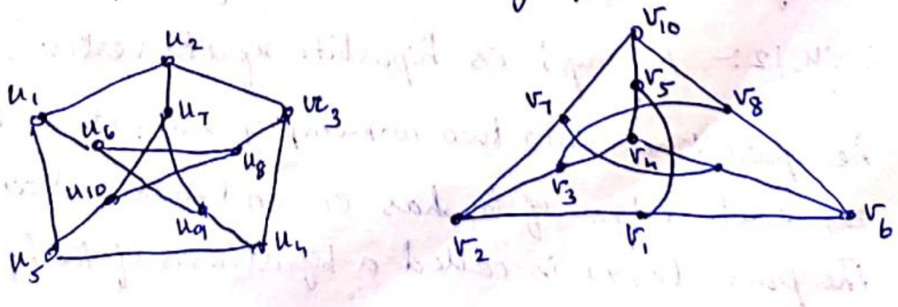


Labeled Graphs Unlabeled Graph.

Isomorphism of Graphs :- Let G and H be two graphs. A graph isomorphism from G to H is a pair (ϕ, θ), where $\phi : V(G) \to V(H)$ and $\theta : E(G) \to E(H)$ are bijections with the property that $I_G(e) = \{u, v\}$ iff $I_H(\theta(e)) = \{\phi(u), \phi(v)\}$. Isomorphism is denoted as $G \simeq H$.

Note :- If (ϕ, θ) is a graph isomorphism then (ϕ^{-1}, θ^{-1}) is also a graph isomorphism.

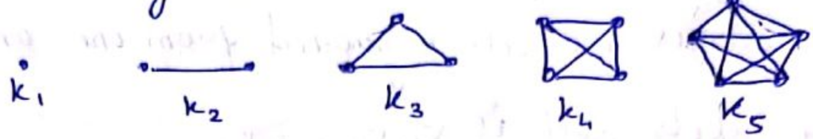
Ex :



Simple Graphs & Isomorphisms :- If G & H are simple graphs, any bijection $\phi : V(G) \to V(H)$ ∋: u & v are adjacent in G iff $\phi(u)$ & $\phi(v)$ are adjacent in H induces a bijection $\theta : E(G) \to E(H)$ satisfying the condition that $\psi_G(e) = \{u, v\}$ iff $\psi_H(\theta(e)) = \{\phi(u), \phi(v)\}$. Hence, ϕ is an isomorphism of G & H.

Defn. 9 :- A simple graph G is said to be complete if every pair of distinct vertices of G are adjacent in G. It is denoted by K_n.

Ex:



Note :-

1). A simple graph with n vertices have atmost $nC_2 = \frac{n(n-1)}{2}$ edges.

2) A simple graph G with n vertices, $0 \le m(G) \le \frac{n(n-1)}{2}$

Defn 10:- A graph with no edges is called a totally disconnected graph.

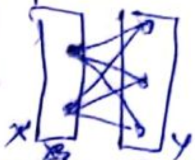
Ex :-

Defn 11 :- A graph is trivial if its vertex set is a singleton and it contains no edges.

Defn. 12 :- A graph is bipartite if its vertex set can be partitioned into two non-empty subsets X and Y ∋: each edge of G has one end in X and other in Y. The pair (X, Y) is called a bipartition of the bipartite graph.

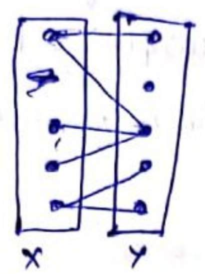
A simple bipartite graph $G(X, Y)$ is complete if each vertex of X is adjacent to all vertices of Y.

If $G(X,Y)$ is complete with $|X| = p$ and $|Y| = q$, then $G(X,Y)$ is denoted by $K_{p,q}$

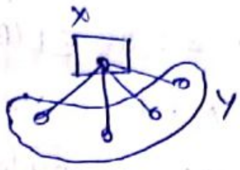


A complete bipartite graph of the form $K_{1,q}$ is called a star.

Ex:



Bipartite Graph

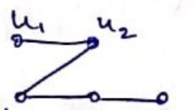


$K_{1,4}$

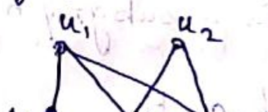
Defn.12:- Let G be a simple graph. Then the complement G^c of G is defined by taking $V(G^c) = V(G)$ and making two vertices u & v adjacent in G^c iff they are nonadjacent in G.

Note: G^c is a simple graph & $(G^c)^c = G$.

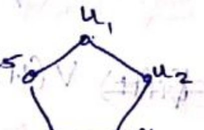
Ex:-



G_1



G_1^c



G_2

G_2^c

Note: If $|V(G)| = n$ then, $|E(G)| + |E(G^c)| = |E(K_n)| = \frac{n(n-1)}{2}$

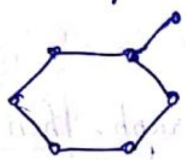
Defn 13:- A simple graph G is called self-complementary if $G \simeq G^c$.

Ex:

Home Work :-

1) If G and H are simple graphs and if $\phi : V(G) \to V(H)$ is a bijection $\ni$: $uv \in E(G) \Rightarrow \phi(u)\phi(v) \in E(H)$.
S.T ϕ need not be an isomorphism from G to H with by means of an example.

2) Find the complement of the simple graph:



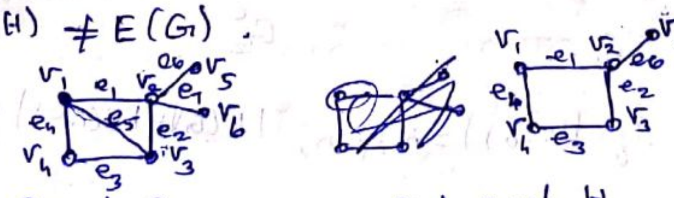
SUBGRAPHS :-

Defn 1: A graph H is called a subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$.

If H is a subgraph of G, then G is said to be a supergraph of H.

A subgraph H of a graph G is a proper subgraph of G if either ~~$V(H) \neq G(H)$~~ $V(H) \neq V(G)$ or $E(H) \neq E(G)$.

Ex:



Graph G　　　　subgraph H

Defn 2:- A subgraph H of G is said to be induced subgraph of G if each edge of G having its ends in V(H) is also an edge of H.

Defn:-3 A subgraph H of G is a spanning subgraph of G if V(H) = V(G). The induced subgraph of G with vertex set $S \subseteq V(G)$ is called the subgraph of G induced by S and is denoted by G[S].

Defn:-4 Let E' be a subset of E and let S be the subset of V consisting of all the end vertices in G of edges in E'. Then the graph $(S, E', I_G|_{E'})$ is the subgraph of G induced by the edge set E' of G. It is denoted by G[E'].

Defn:-5 A clique of G is a complete subgraph of G. A clique of G is a maximal clique of G if it is not properly contained in another clique of G.

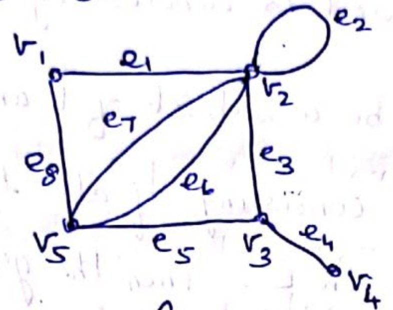
Defn:-6 Deletion of vertices and edges in a graph:

Let G be a graph, S a proper subset of V and E' a subset of E. The subgraph G[V\S] is said to be obtained from G by the deletion of S. This subgraph is denoted by G − S.

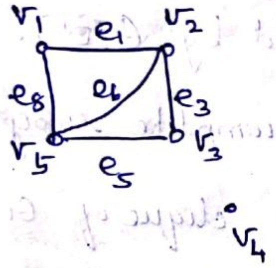
If S = {v}, G − S is simply denoted by G − v. The spanning subgraph of G with the edge set E\E' is the subgraph obtained from G by deleting the edge subset E'. This subgraph is denoted by G – E'. Whenever E' = {e}, G – E' is simply denoted by G – e.

Note: Deletion of an edge from G does not affect the vertices of G.

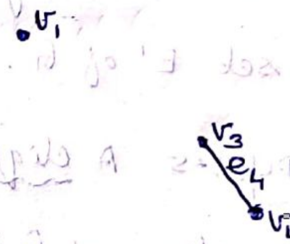
Deletion of vertices & edges from G



G



$G - \{e_2, e_4, e_7\}$



$G - \{v_2 - v_5\}$

DEGREES OF VERTICES :-

Defn:1 Let G be a graph and $v \in V$. The no. of edges incident at v in G is called the degree (or valency) of the vertex v in G and is denoted by $d_G(v)$ (or) $d(v)$.

Note:- 1) A loop at v is counted twice for the degree of v.

2) The minimum and maximum degrees of the vertices of a graph G is denoted by $\delta(G)$ (or) δ and $\Delta(G)$ (or) Δ respectively.

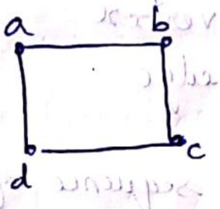
Defn : 2 A graph G is called k-regular if every vertex of G has degree k.

A graph is said to be regular if it is k-regular for some non-negative integer k.

A 3-regular graph is called a cubic graph.

Defn : 3 A spanning 1-regular subgraph of G is said called a 1-factor (or) perfect matching of G.

Ex :-



G

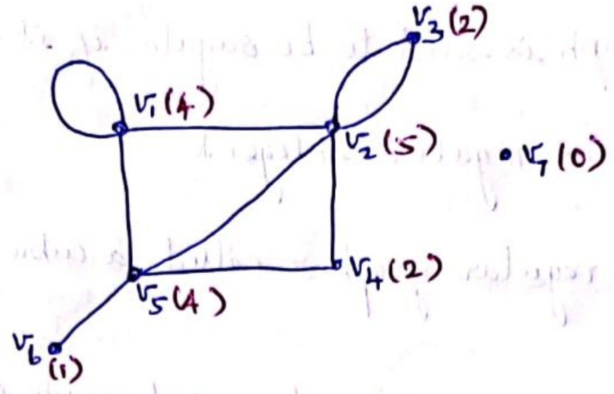
In the graph G, each pair {ab, cd} and {ad, bc} is a 1-factor of G.

Defn : 4 A vertex of degree 0 is an isolated vertex of G.

A vertex of degree 1 is called a pendant vertex of G and the unique edge of G incident to such a vertex of G is a pendant edge of G.

A sequence formed by the degrees of the vertices of G, when the vertices are taken in the same order, is called a degree sequence of G.

Ex: Degrees of vertices of graph G



G

In G, v_7 → Isolated vertex

v_6 → Pendant vertex

v_5v_6 → Pendant edge

(0,1,2,2,4,4,5) → Degree sequence of G.

Chapter 1
Basic Results

1.1 Introduction

Graphs serve as mathematical models to analyze many concrete real-world problems successfully. Certain problems in physics, chemistry, communication science, computer technology, genetics, psychology, sociology, and linguistics can be formulated as problems in graph theory. Also, many branches of mathematics, such as group theory, matrix theory, probability, and topology, have close connections with graph theory.

Some puzzles and several problems of a practical nature have been instrumental in the development of various topics in graph theory. The famous Königsberg bridge problem has been the inspiration for the development of Eulerian graph theory. The challenging Hamiltonian graph theory has been developed from the "Around the World" game of Sir William Hamilton. The theory of acyclic graphs was developed for solving problems of electrical networks, and the study of "trees" was developed for enumerating isomers of organic compounds. The well-known four-color problem formed the very basis for the development of planarity in graph theory and combinatorial topology. Problems of linear programming and operations research (such as maritime traffic problems) can be tackled by the theory of flows in networks. Kirkman's schoolgirl problem and scheduling problems are examples of problems that can be solved by graph colorings. The study of simplicial complexes can be associated with the study of graph theory. Many more such problems can be added to this list.

1.2 Basic Concepts

Consider a road network of a town consisting of streets and street intersections. Figure 1.1a represents the road network of a city. Figure 1.1b denotes the corresponding graph of this network, where the street intersections are represented by

R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*,
Universitext, DOI 10.1007/978-1-4614-4529-6_1,

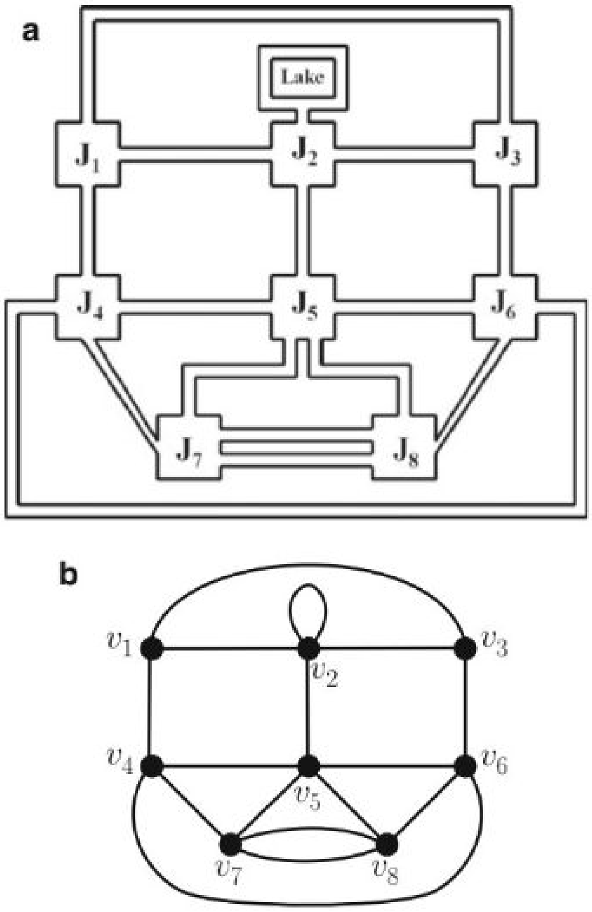


Fig. 1.1 (**a**) A road network and (**b**) the graph corresponding to the road network in (a)

points, and the street joining a pair of intersections is represented by an arc (not necessarily a straight line). The road network in Fig. 1.1 is a typical example of a graph in which intersections and streets are, respectively, the "vertices" and "edges" of the graph. (Note that in the road network in Fig. 1.1a, there are two streets joining the intersections J_7 and J_8, and there is a loop street starting and ending at J_2.)

We now present a formal definition of a graph.

Definition 1.2.1. A *graph* is an ordered triple $G = (V(G), E(G), I_G)$, where $V(G)$ is a nonempty set, $E(G)$ is a set disjoint from $V(G)$, and I_G is an "incidence" relation that associates with each element of $E(G)$ an unordered pair of elements (same or distinct) of $V(G)$. Elements of $V(G)$ are called the *vertices* (or *nodes* or *points*) of G, and elements of $E(G)$ are called the *edges* (or *lines*) of G. $V(G)$ and $E(G)$ are the *vertex set* and *edge set* of G, respectively. If, for the edge e of G, $I_G(e) = \{u, v\}$, we write $I_G(e) = uv$.

Example 1.2.2. If $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, and I_G is given by $I_G(e_1) = \{v_1, v_5\}$, $I_G(e_2) = \{v_2, v_3\}$, $I_G(e_3) = \{v_2, v_4\}$, $I_G(e_4) = \{v_2, v_5\}$, $I_G(e_5) = \{v_2, v_5\}$, $I_G(e_6) = \{v_3, v_3\}$, then $(V(G), E(G), I_G)$ is a graph.

Diagrammatic Representation of a Graph 1.2.3. Each graph can be represented by a diagram in the plane. In this diagram, each vertex of the graph is represented

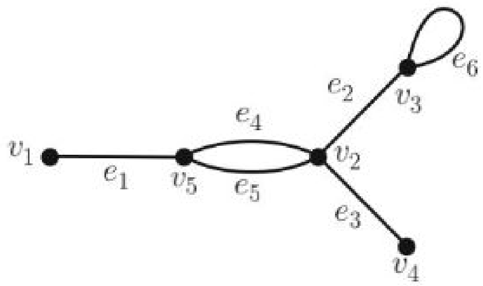


Fig. 1.2 Graph $(V(G), E(G), I_G)$ described in Example 1.2.2

by a point, with distinct vertices being represented by distinct points. Each edge is represented by a simple "Jordan" arc joining two (not necessarily distinct) vertices. The diagrammatic representation of a graph aids in visualizing many concepts related to graphs and the systems of which they are models. In a diagrammatic representation of a graph, it is possible that two edges intersect at a point that is not necessarily a vertex of the graph.

Definition 1.2.4. If $I_G(e) = \{u, v\}$, then the vertices u and v are called the *end vertices* or *ends* of the edge e. Each edge is said to join its ends; in this case, we say that e is *incident* with each one of its ends. Also, the vertices u and v are then *incident* with e. A set of two or more edges of a graph G is called a set of *multiple* or *parallel edges* if they have the same pair of distinct ends. If e is an edge with end vertices u and v, we write $e = uv$. An edge for which the two ends are the same is called a *loop* at the common vertex. A vertex u is a *neighbor* of v in G, if uv is an edge of G, and $u \neq v$. The set of all neighbors of v is the *open neighborhood* of v or the *neighbor set* of v, and is denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v in G. When G needs to be made explicit, these open and closed neighborhoods are denoted by $N_G(v)$ and $N_G[v]$, respectively. Vertices u and v are *adjacent* to each other in G if and only if there is an edge of G with u and v as its ends. Two distinct edges e and f are said to be *adjacent* if and only if they have a common end vertex. A graph is *simple* if it has no loops and no multiple edges. Thus, for a simple graph G, the incidence function I_G is one-to-one. Hence, an edge of a simple graph is identified with the pair of its ends. A simple graph therefore may be considered as an ordered pair $(V(G), E(G))$, where $V(G)$ is a nonempty set and $E(G)$ is a set of unordered pairs of elements of $V(G)$ (each edge of the graph being identified with the pair of its ends).

Example 1.2.5. In the graph of Fig. 1.2, edge $e_3 = v_2v_4$, edges e_4 and e_5 form multiple edges, e_6 is a loop at v_3, $N(v_2) = \{v_3, v_4, v_5\}$, $N(v_3) = \{v_2\}$, $N[v_2] = \{v_2, v_3, v_4, v_5\}$, and $N[v_2] = N(v_2) \cup \{v_2\}$. Further, v_2 and v_5 are adjacent vertices and e_3 and e_4 are adjacent edges.

Definition 1.2.6. A graph is called *finite* if both $V(G)$ and $E(G)$ are finite. A graph that is not finite is called an *infinite* graph. Unless otherwise stated, all graphs considered in this text are finite. Throughout this book, we denote by $n(G)$ and $m(G)$ the number of vertices and edges of the graph G, respectively. The number $n(G)$ is called the *order* of G and $m(G)$ is the *size* of G. When explicit reference to

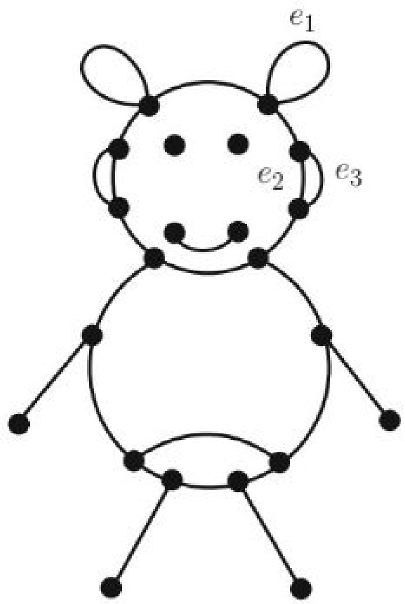


Fig. 1.3 A graph diagram; e_1 is a loop and $\{e_2, e_3\}$ is a set of multiple edges

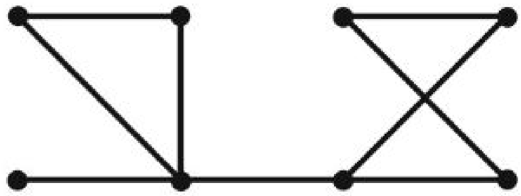


Fig. 1.4 A simple graph

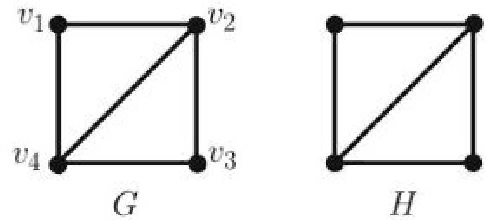


Fig. 1.5 A labeled graph G and an unlabeled graph H

the graph G is not needed, $V(G)$, $E(G)$, $n(G)$, and $m(G)$ will be denoted simply by V, E, n, and m, respectively.

Figure 1.3 is a graph with loops and multiple edges, while Fig. 1.4 represents a simple graph.

Remark 1.2.7. The representation of graphs on other surfaces such as a sphere, a torus, or a Möbius band could also be considered. Often a diagram of a graph is identified with the graph itself.

Definition 1.2.8. A graph is said to be *labeled* if its n vertices are distinguished from one another by labels such as $v_1, v_2, \ldots, v_n$ (see Fig. 1.5).

Note that there are three different labeled simple graphs on three vertices each having two edges, whereas there is only one unlabeled simple graph of the same order and size (see Fig. 1.6).

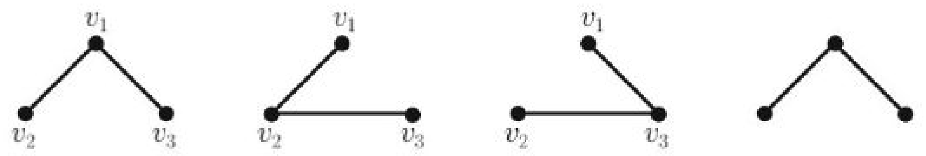


Fig. 1.6 Labeled and unlabeled simple graphs on three vertices

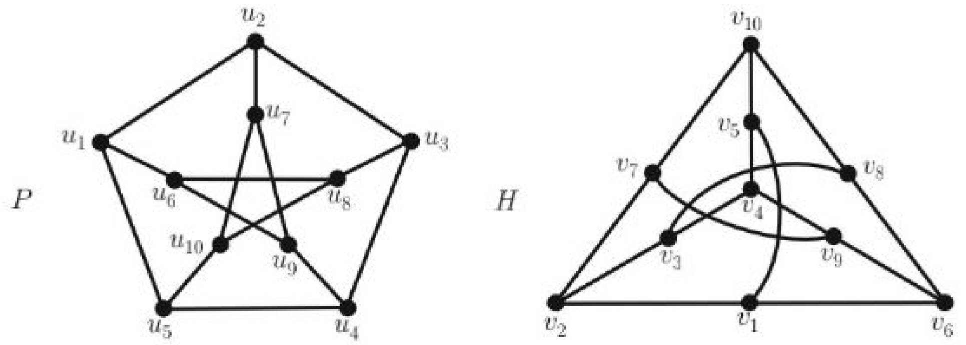


Fig. 1.7 Isomorphic graphs

Isomorphism of Graphs 1.2.9. A graph isomorphism, which we now define, is a concept similar to isomorphism in algebraic structures. Let $G = (V(G), E(G), I_G)$ and $H = (V(H), E(H), I_H)$ be two graphs. A *graph isomorphism* from G to H is a pair (ϕ, θ), where $\phi : V(G) \to V(H)$ and $\theta : E(G) \to E(H)$ are bijections with the property that $I_G(e) = \{u, v\}$ if and only if $I_H(\theta(e)) = \{\phi(u), \phi(v)\}$. If (ϕ, θ) is a graph isomorphism, the pair of inverse mappings (ϕ^{-1}, θ^{-1}) is also a graph isomorphism. Note that the bijection ϕ satisfies the condition that *u and v are end vertices of an edge e of G if and only if $\phi(u)$ and $\phi(v)$ are end vertices of the edge $\theta(e)$ in H*. It is clear that isomorphism is an equivalence relation on the set of all graphs. Isomorphism between graphs is denoted by the symbol $\simeq$ (as in algebraic structures).

Simple Graphs and Isomorphisms 1.2.10. If graphs G and H are simple, any bijection $\phi : V(G) \to V(H)$ such that u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H induces a bijection $\theta : E(G) \to E(H)$ satisfying the condition that $I_G(e) = \{u, v\}$ if and only if $I_H(\theta(e)) = \{\phi(u), \phi(v)\}$. Hence, ϕ itself is referred to as an isomorphism in the case of simple graphs G and H. Thus, if G and H are simple graphs, an isomorphism from G to H is a bijection $\phi : V(G) \to V(H)$ such that u and v are adjacent in G if and only if $\phi(v)$ and $\phi(v)$ are adjacent in H. Figure 1.7 exhibits two isomorphic graphs P and H, where P is the well-known Petersen graph. We observe that P is a simple graph.

Exercise 2.1. Let G and H be simple graphs and let $\phi : V(G) \to V(H)$ be a bijection such that $uv \in E(G)$ implies that $\phi(u)\phi(v) \in E(H)$. Show by means of an example that ϕ need not be an isomorphism from G to H.

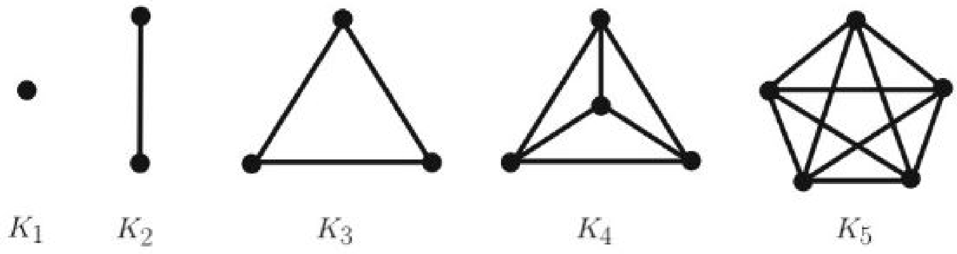


Fig. 1.8 Some complete graphs

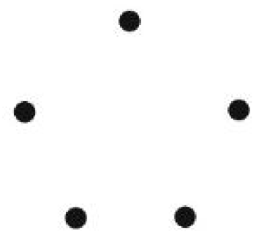


Fig. 1.9 A totally disconnected graph on five vertices

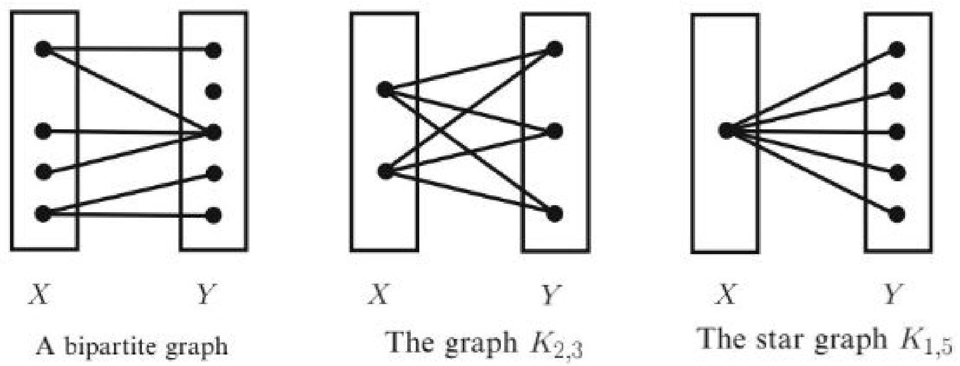


Fig. 1.10 Bipartite graphs

Definition 1.2.11. A simple graph G is said to be *complete* if every pair of distinct vertices of G are adjacent in G. Any two complete graphs each on a set of n vertices are isomorphic; each such graph is denoted by K_n (Fig. 1.8).

A simple graph with n vertices can have at most $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. The complete graph K_n has the maximum number of edges among all simple graphs with n vertices. At the other extreme, a graph may possess no edge at all. Such a graph is called a *totally disconnected graph* (see Fig. 1.9). Thus, *for a simple graph G with n vertices, we have* $0 \leq m(G) \leq \frac{n(n-1)}{2}$.

Definition 1.2.12. A graph is *trivial* if its vertex set is a singleton and it contains no edges. A graph is *bipartite* if its vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called a *bipartition* of the bipartite graph. The bipartite graph G with bipartition (X, Y) is denoted by $G(X, Y)$. A simple bipartite graph $G(X, Y)$ is *complete* if each vertex of X is adjacent to all the vertices of Y. If $G(X, Y)$ is complete with $|X| = p$ and $|Y| = q$, then $G(X, Y)$ is denoted by $K_{p,q}$. A complete bipartite graph of the form $K_{1,q}$ is called a *star* (see Fig. 1.10).

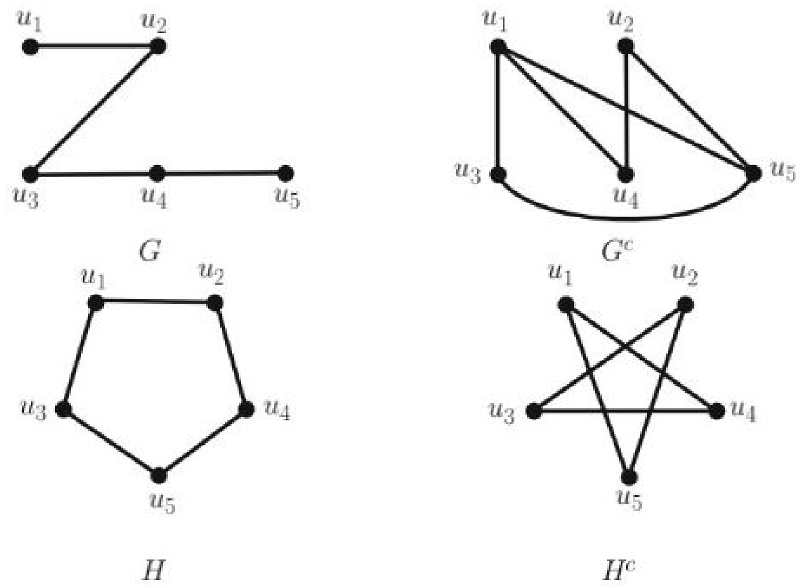
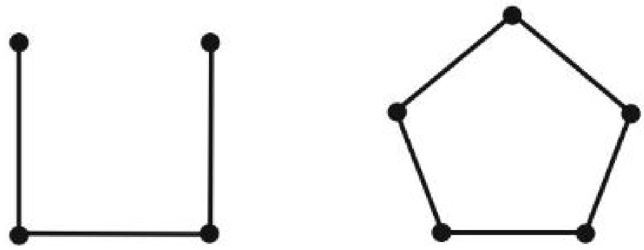


Fig. 1.11 Two simple graphs and their complements

Fig. 1.12 Self-complementary graphs



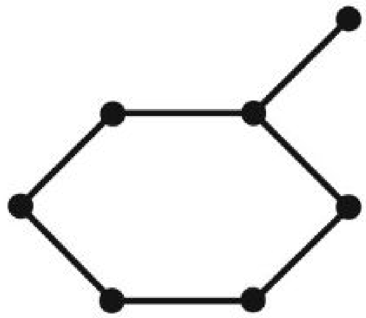
Definition 1.2.13. Let G be a simple graph. Then the *complement* G^c of G is defined by taking $V(G^c) = V(G)$ and making two vertices u and v adjacent in G^c if and only if they are nonadjacent in G (see Fig. 1.11). It is clear that G^c is also a simple graph and that $(G^c)^c = G$.

If $|V(G)| = n$, then clearly, $|E(G)| + |E(G^c)| = |E(K_n)| = \frac{n(n-1)}{2}$.

Definition 1.2.14. A simple graph G is called *self-complementary* if $G \simeq G^c$.

For example, the graphs shown in Fig. 1.12 are self-complementary.

Exercise 2.2. Find the complement of the following simple graph:



1.3 Subgraphs

Definition 1.3.1. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and I_H is the restriction of I_G to $E(H)$. If H is a subgraph of G, then G is said to be a *supergraph* of H. A subgraph H of a graph G is a *proper subgraph* of G if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$. (Hence, when G is given, for any subgraph H of G, the incidence function is already determined so that H can be specified by its vertex and edge sets.) A subgraph H of G is said to be an *induced subgraph* of G if each edge of G having its ends in $V(H)$ is also an edge of H. A subgraph H of G is a *spanning subgraph* of G if $V(H) = V(G)$. The induced subgraph of G with vertex set $S \subseteq V(G)$ is called the *subgraph of* G *induced by* S and is denoted by $G[S]$. Let E' be a subset of E and let S denote the subset of V consisting of all the end vertices in G of edges in E'. Then the graph $(S, E', I_G|_{E'})$ is the *subgraph of* G *induced by the edge set* E' *of* G. It is denoted by $G[E']$ (see Fig. 1.13). Let u and v be vertices of a graph G. By $G + uv$, we mean the graph obtained by adding a new edge uv to G.

Definition 1.3.2. A *clique* of G is a complete subgraph of G. A clique of G is a *maximal clique* of G if it is not properly contained in another clique of G (see Fig. 1.13).

Definition 1.3.3. *Deletion of vertices and edges in a graph:* Let G be a graph, S a proper subset of the vertex set V, and E' a subset of E. The subgraph $G[V \backslash S]$ is said to be obtained from G by the *deletion* of S. This subgraph is denoted by $G - S$. If $S = \{v\}$, $G - S$ is simply denoted by $G - v$. The spanning subgraph of G with the edge set $E \backslash E'$ is the subgraph obtained from G by deleting the edge subset E'. This subgraph is denoted by $G - E'$. Whenever $E' = \{e\}$, $G - E'$ is

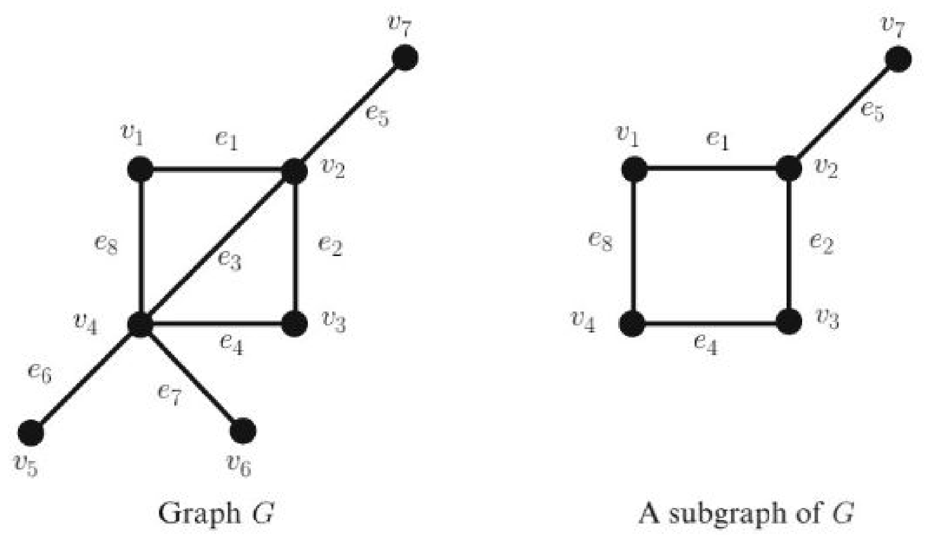


Fig. 1.13 Various subgraphs and cliques of G

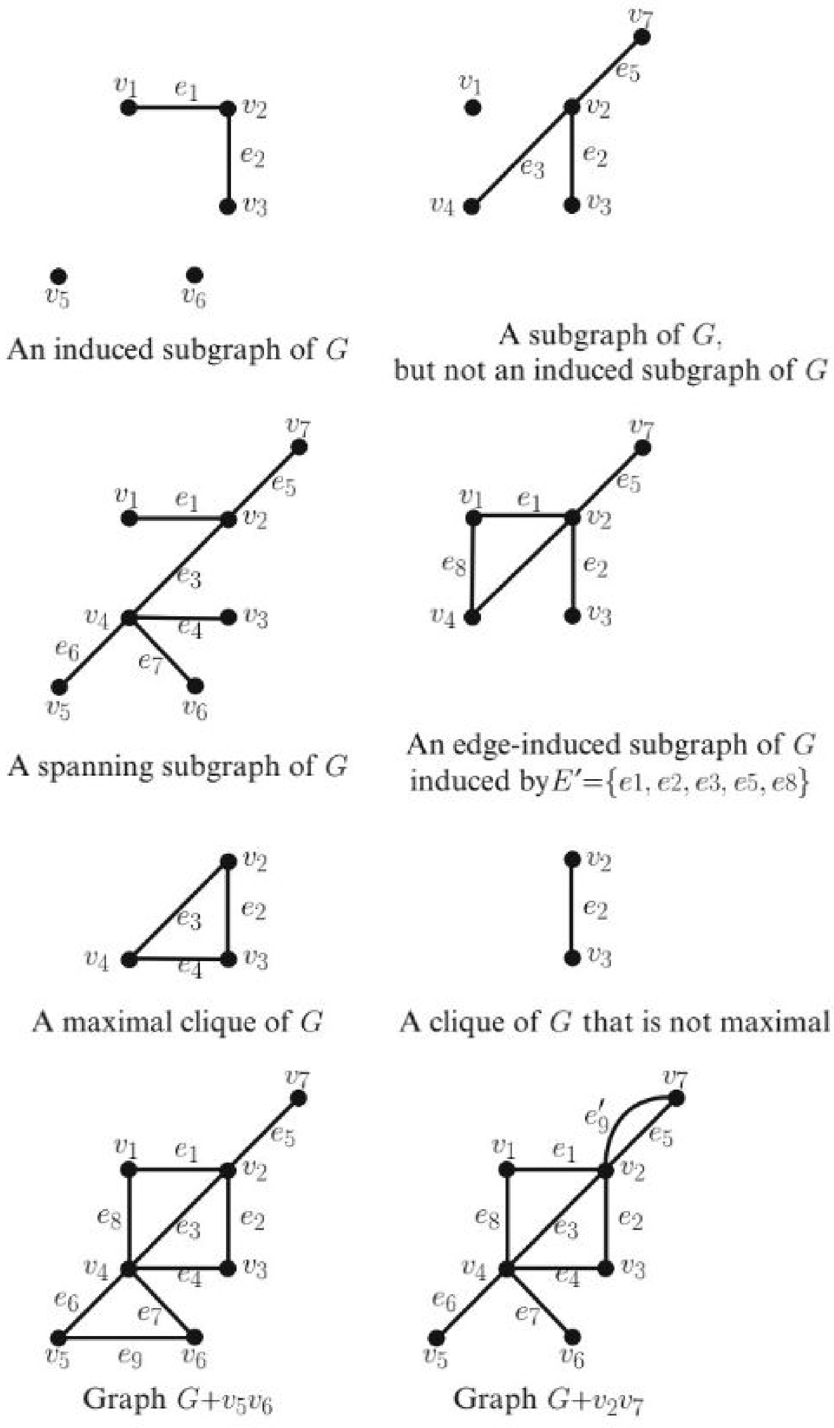


Fig. 1.13 (continued)

simply denoted by $G - e$. Note that when a vertex is deleted from G, all the edges incident to it are also deleted from G, whereas the deletion of an edge from G does not affect the vertices of G (see Fig. 1.14).

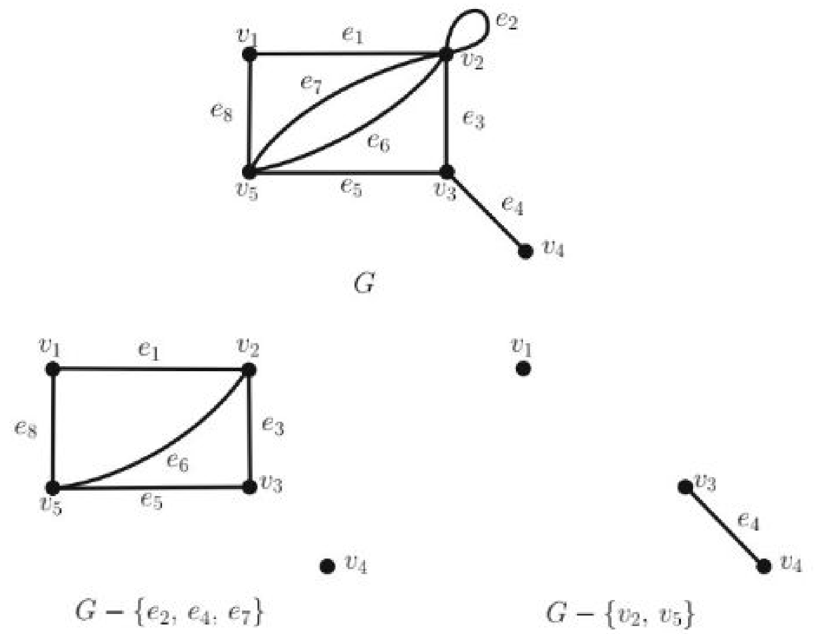


Fig. 1.14 Deletion of vertices and edges from G

1.4 Degrees of Vertices

Definition 1.4.1. Let G be a graph and $v \in V$. The number of edges incident at v in G is called the *degree* (or *valency*) of the vertex v in G and is denoted by $d_G(v)$, or simply $d(v)$ when G requires no explicit reference. A loop at v is to be counted twice in computing the degree of v. The minimum (respectively, maximum) of the degrees of the vertices of a graph G is denoted by $\delta(G)$ or δ (respectively, $\Delta(G)$ or Δ). A graph G is called k*-regular* if every vertex of G has degree k. A graph is said to be *regular* if it is k-regular for some nonnegative integer k. In particular, a 3-regular graph is called a *cubic graph.*

Definition 1.4.2. A spanning 1-regular subgraph of G is called a 1*-factor* or a *perfect matching* of G. For example, in the graph G of Fig. 1.15, each of the pairs $\{ab, cd\}$ and $\{ad, bc\}$ is a 1-factor of G.

Definition 1.4.3. A vertex of degree 0 is an *isolated vertex* of G. A vertex of degree 1 is called a *pendant vertex* of G, and the unique edge of G incident to such a vertex of G is a *pendant edge* of G. A sequence formed by the degrees of the vertices of G, when the vertices are taken in the same order, is called a *degree sequence* of G. It is customary to give this sequence in the nonincreasing or nondecreasing order, in which case the sequence is unique.

In the graph G of Fig. 1.16, the numbers within the parentheses indicate the degrees of the corresponding vertices. In G, v_7 is an isolated vertex, v_6 is a pendant vertex, and v_5v_6 is a pendant edge. The degree sequence of G is $(0, 1, 2, 2, 4, 4, 5)$.

Fig. 1.15 Graph with 1-factors

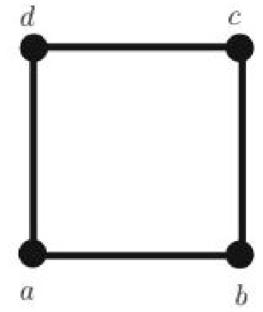
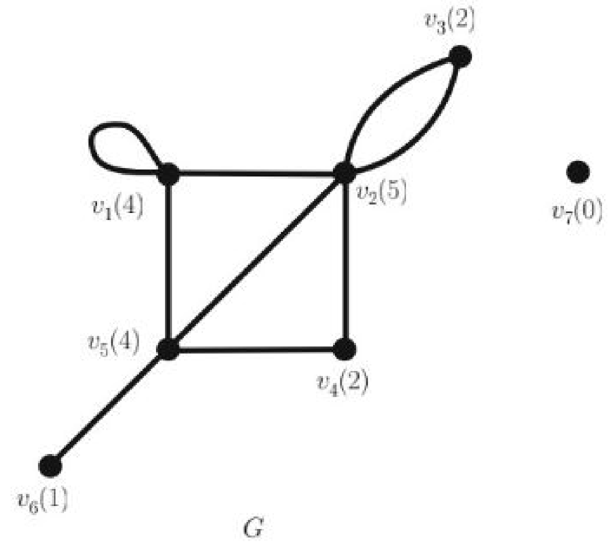


Fig. 1.16 Degrees of vertices of graph G



The very first theorem of graph theory was due to Leonhard Euler (1707–1783). This theorem connects the degrees of the vertices and the number of edges of a graph.

Theorem 1.4.4 (Euler). *The sum of the degrees of the vertices of a graph is equal to twice the number of its edges.*

Proof. If $e = uv$ is an edge of G, e is counted once while counting the degrees of each of u and v (even when $u = v$). Hence, each edge contributes 2 to the sum of the degrees of the vertices. Thus, the m edges of G contribute $2m$ to the degree sum. □

Remark 1.4.5. If $d = (d_1, d_2, \ldots, d_n)$ is the degree sequence of G, then the above theorem gives the equation $\sum_{i=1}^{n} d_i = 2m$, where n and m are the order and size of G, respectively.

Corollary 1.4.6. *In any graph G, the number of vertices of odd degree is even.*

Proof. Let V_1 and V_2 be the subsets of vertices of G with odd and even degrees, respectively. By Theorem 1.4.4,

$$2m(G) = \sum_{v \in V} d_G(v) = \sum_{v \in V_1} d_G(v) + \sum_{v \in V_2} d_G(v).$$

As $2m(G)$ and $\sum_{v \in V_2} d_G(v)$ are even, $\sum_{v \in V_1} d_G(v)$ is even. Since for each $v \in V_1$, $d_G(v)$ is odd, $|V_1|$ must be even. □

Exercise 4.1. Show that if G and H are isomorphic graphs, then each pair of corresponding vertices of G and H has the same degree.

Exercise 4.2. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of a graph and r be any positive integer. Show that $\sum_{i=1}^{n} d_i^r$ is even.

Definition 1.4.7. *Graphical sequences*: A sequence of nonnegative integers $d = (d_1, d_2, \ldots, d_n)$ is called *graphical* if there exists a simple graph whose degree sequence is d. Clearly, a necessary condition for $d = (d_1, d_2, \ldots, d_n)$ to be graphical is that $\sum_{i=1}^{n} d_i$ is even and $d_i \geq 0$, $1 \leq i \leq n$. These conditions, however, are not sufficient, as Example 1.4.8 shows.

Example 1.4.8. The sequence $d = (7, 6, 3, 3, 2, 1, 1, 1)$ is not graphical even though each term of d is a nonnegative integer and the sum of the terms is even. Indeed, if d were graphical, there must exist a simple graph G with eight vertices whose degree sequence is d. Let v_0 and v_1 be the vertices of G whose degrees are 7 and 6, respectively. Since G is simple, v_0 is adjacent to all the remaining vertices of G, and v_1, besides v_0, should be adjacent to another five vertices. This means that in $V - \{v_0, v_1\}$ there must be at least five vertices each of degree at least 2; but this is not the case. □

Exercise 4.3. If $d = (d_1, d_2, \ldots, d_n)$ is any sequence of nonnegative integers with $\sum_{i=1}^{n} d_i$ even, show that there exists a graph (not necessarily simple) with d as its degree sequence.

We present a simple application whose proof just depends on the degree sequence of a graph.

Application 1.4.9. *In any group of n persons ($n \geq 2$), there are at least two with the same number of friends.*

Proof. Denote the n persons by $v_1, v_2, \ldots, v_n$. Let G be the simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ in which v_i and v_j are adjacent if and only if the corresponding persons are friends. Then the number of friends of v_i is just the degree of v_i in G. Hence, to solve the problem, we must prove that there are two vertices in G with the same degree. If this were not the case, the degrees of the vertices of G must be $0, 1, 2, \ldots, (n-1)$ in some order. However, a vertex of degree $(n-1)$ must be adjacent to all the other vertices of G, and consequently there cannot be a vertex of degree 0 in G. This contradiction shows that the degrees of the vertices of G cannot all be distinct, and hence at least two of them should have the same degree. □

Exercise 4.4. Let G be a graph with n vertices and m edges. Assume that each vertex of G is of degree either k or $k+1$. Show that the number of vertices of degree k in G is $(k+1)n - 2m$.

1.5 Paths and Connectedness

Definition 1.5.1. A *walk* in a graph G is an alternating sequence W : $v_0e_1v_1e_2v_2 \ldots e_pv_p$ of vertices and edges beginning and ending with vertices in which v_{i-1} and v_i are the ends of e_i; v_0 is the *origin* and v_p is the *terminus* of W. The walk W is said to join v_0 and v_p; it is also referred to as a v_0-v_p walk. If the graph is simple, a walk is determined by the sequence of its vertices. The walk is *closed* if $v_0 = v_p$ and is *open* otherwise. A walk is called a *trail* if all the edges appearing in the walk are distinct. It is called a *path* if all the vertices are distinct. Thus, a path in G is automatically a trail in G. When writing a path, we usually omit the edges. A *cycle* is a closed trail in which the vertices are all distinct. The *length* of a walk is the number of edges in it. A walk of length 0 consists of just a single vertex.

Example 1.5.2. In the graph of Fig. 1.17, $v_5e_7v_1e_1v_2e_4v_4e_5v_1e_7v_5e_9v_6$ is a walk but not a trail (as edge e_7 is repeated) $v_1e_1v_2e_2v_3e_3v_2e_1v_1$ is a closed walk; $v_1e_1v_2e_4v_4e_5v_1e_7v_5$ is a trail; $v_6e_8v_1e_1v_2e_2v_3$ is a path and $v_1e_1v_2e_4v_4e_6v_5e_7v_1$ is a cycle. Also, $v_6v_1v_2v_3$ is a path, and $v_1v_2v_4v_5v_6v_1$ is a cycle in this graph. Very often a cycle is enclosed by ordinary parentheses.

Definition 1.5.3. A cycle of length k is denoted by C_k. Further, P_k denotes a path on k vertices. In particular, C_3 is often referred to as a *triangle*, C_4 as a *square*, and C_5 as a *pentagon*. If $P = v_0e_1v_1e_2v_2 \ldots e_kv_k$ is a path, then $P^{-1} = v_ke_kv_{k-1}e_{k-1}v_{k-2} \ldots v_1e_1v_0$ is also a path and P^{-1} is called the *inverse* of the path P. The subsequence $v_ie_{i+1}v_{i+1} \ldots e_jv_j$ of P is called the v_i-v_j *section* of P.

Definition 1.5.4. Let G be a graph. Two vertices u and v of G are said to be *connected* if there is a u-v path in G. The relation "connected" is an equivalence relation on $V(G)$. Let $V_1, V_2, \ldots, V_\omega$ be the equivalence classes. The subgraphs $G[V_1]$, $G[V_2]$, $\ldots$, $G[V_\omega]$ are called the *components* of G. If $\omega = 1$, the graph G is *connected*; otherwise, the graph G is *disconnected* with $\omega \geq 2$ components (see Fig. 1.18).

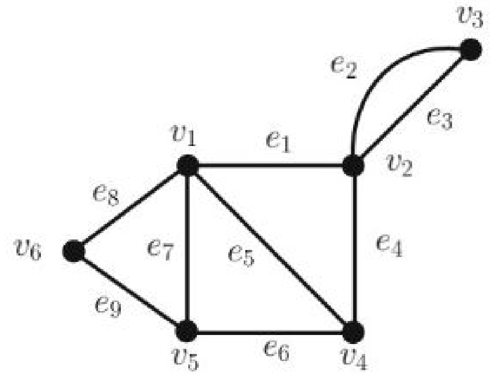


Fig. 1.17 Graph illustrating walks, trails, paths, and cycles

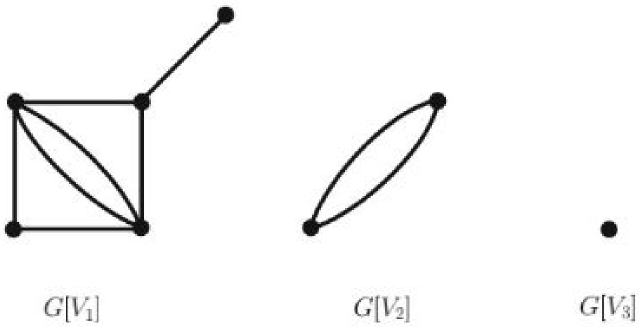


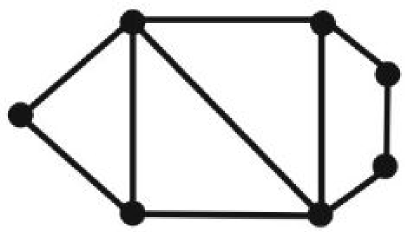
Fig. 1.18 A graph G with three components

Definition 1.5.5. The components of G are clearly the maximal connected subgraphs of G. We denote the number of components of G by $\omega(G)$. Let u and v be two vertices of G. If u and v are in the same component of G, we define $d(u, v)$ to be the length of a shortest u-v path in G; otherwise, we define $d(u, v)$ to be ∞. If G is a connected graph, then d is a distance function or metric on $V(G)$; that is, $d(u, v)$ satisfies the following conditions:

(i) $d(u, v) \geq 0$, and $d(u, v) = 0$ if and only if $u = v$.
(ii) $d(u, v) = d(v, u)$.
(iii) $d(u, v) \leq d(u, w) + d(w, v)$, for every w in $V(G)$.

Exercise 5.1. Prove that the function d defined above is indeed a metric on $V(G)$.

Exercise 5.2. In the following graph, find a closed trail of length 7 that is not a cycle:



We now give some results relating to connectedness of graphs.

Proposition 1.5.6. *If G is simple and $\delta \geq \frac{n-1}{2}$, then G is connected.*

Proof. Assume the contrary. Then G has at least two components, say G_1, G_2. Let v be any vertex of G_1. As $\delta \geq \frac{n-1}{2}$, $d(v) \geq \frac{n-1}{2}$. All the vertices adjacent to v in G must belong to G_1. Hence, G_1 contains at least $d(v) + 1 \geq \frac{n-1}{2} + 1 = \frac{n+1}{2}$ vertices. Similarly, G_2 contains at least $\frac{n+1}{2}$ vertices. Therefore G has at least $\frac{n+1}{2} + \frac{n+1}{2} = n + 1$ vertices, which is a contradiction. □

Exercise 5.3. Give an example of a nonsimple disconnected graph with $\delta \geq \frac{n-1}{2}$.

Exercise 5.4. Show by means of an example that the condition $\delta \geq \frac{n-2}{2}$ for a simple graph G need not imply that G is connected.

Exercise 5.5. In a group of six people, prove that there must be three people who are mutually acquainted or three people who are mutually nonacquainted.

Our next result shows that of the two graphs G and G^c, at least one of them must be connected.

Theorem 1.5.7. *If a simple graph G is not connected, then G^c is connected.*

Proof. Let u and v be any two vertices of G^c (and therefore of G). If u and v belong to different components of G, then obviously u and v are nonadjacent in G and so they are adjacent in G^c. Thus u and v are connected in G^c. In case u and v belong to the same component of G, take a vertex w of G not belonging to this component of G. Then uw and vw are not edges of G and hence they are edges of G^c. Then uwv is a u-v path in G^c. Thus G^c is connected. □

Exercise 5.6. Show that if G is a self-complementary graph of order n, then $n \equiv 0$ or 1 (mod 4).

Exercise 5.7. Show that if a self-complementary graph contains a pendant vertex, then it must have at least another pendant vertex.

The next theorem gives an upper bound on the number of edges in a simple graph.

Theorem 1.5.8. *The number of edges of a simple graph of order n having ω components cannot exceed $\frac{(n-\omega)(n-\omega+1)}{2}$.*

Proof. Let $G_1, G_2, \ldots, G_\omega$ be the components of a simple graph G and let n_i be the number of vertices of G_i, $1 \le i \le \omega$. Then $m(G_i) \le \frac{n_i(n_i-1)}{2}$, and hence $m(G) \le \sum_{i=1}^{\omega} \frac{n_i(n_i-1)}{2}$. Since $n_i \ge 1$ for each i, $1 \le i \le \omega$, $n_i = n - (n_1 + \ldots + n_{i-1} + n_{i+1} + \ldots + n_\omega) \le n - \omega + 1$. Hence, $\sum_{i=1}^{\omega} \frac{n_i(n_i-1)}{2} \le \sum_{i=1}^{\omega} \frac{(n-\omega+1)(n_i-1)}{2} = \frac{(n-\omega+1)}{2} \sum_{i=1}^{\omega} (n_i - 1) = \frac{(n-\omega+1)}{2} \left[\left(\sum_{i=1}^{\omega} n_i \right) - \omega \right] = \frac{(n-\omega+1)(n-\omega)}{2}$. □

Definition 1.5.9. A graph G is called *locally connected* if, for every vertex v of G, the subgraph $N_G(v)$ induced by the neighbor set of v in G is connected.

A cycle is *odd* or *even* depending on whether its length is odd or even. We now characterize bipartite graphs.

Theorem 1.5.10. *A graph is bipartite if and only if it contains no odd cycles.*

Proof. Suppose that G is a bipartite graph with the bipartition (X, Y). Let $C = v_1e_1v_2e_2v_3e_3 \ldots v_ke_kv_1$ be a cycle in G. Without loss of generality, we can suppose that $v_1 \in X$. As v_2 is adjacent to v_1, $v_2 \in Y$. Similarly, v_3 belongs to X, v_4 to Y, and so on. Thus, $v_i \in X$ or Y according as i is odd or even, $1 \le i \le k$. Since v_kv_1 is an edge of G and $v_1 \in X$, $v_k \in Y$. Accordingly, k is even and C is an even cycle.

Conversely, let us suppose that G contains no odd cycles. We first assume that G is connected. Let u be a vertex of G. Define $X = \{v \in V \mid d(u,v) \text{ is even}\}$ and $Y = \{v \in V \mid d(u,v) \text{ is odd}\}$. We will prove that (X, Y) is a bipartition of G. To prove this we have only to show that no two vertices of X as well as no two

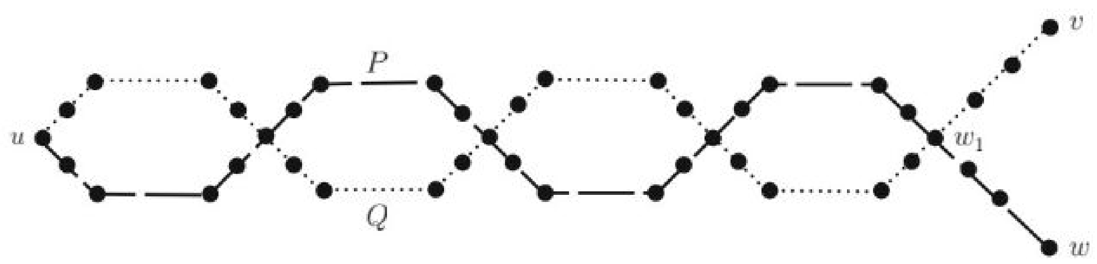


Fig. 1.19 Graph for proof of Theorem 1.5.10

vertices of Y are adjacent in G. Let v, w be two vertices of X. Then $p = d(u, v)$ and $q = d(u, w)$ are even. Further, as $d(u, u) = 0$, $u \in X$. Let P be a u-v shortest path of length p and Q, a u-w shortest path of length q. (See Fig. 1.19.) Let w_1 be a vertex common to P and Q such that the w_1-v section of P and the w_1-w section of Q contain no vertices common to P and Q. Then the u-w_1 sections of both P and Q have the same length.

Hence, the lengths of the w_1-v section of P and the w_1-w section of Q are both even or both odd. Now if $e = vw$ is an edge of G, then the w_1-v section of P followed by the edge vw and the w-w_1 section of the w-u path Q^{-1} is an odd cycle in G, contradicting the hypothesis. This contradiction proves that no two vertices of X are adjacent in G. Similarly, no two vertices of Y are adjacent in G. This proves the result when G is connected.

If G is not connected, let $G_1, G_2, \ldots, G_\omega$ be the components of G. By hypothesis, no component of G contains an odd cycle. Hence, by the previous paragraph, each component G_i, $1 \leq i \leq \omega$, is bipartite. Let (X_i, Y_i) be the bipartition of G_i. Then (X, Y), where $X = \bigcup_{i=1}^{\omega} X_i$ and $Y = \bigcup_{i=1}^{\omega} Y_i$, is a bipartition of G, and G is a bipartite graph. □

Exercise 5.8. Prove that a simple nontrivial graph G is connected if and only if for any partition of V into two nonempty subsets V_1 and V_2, there is an edge joining a vertex of V_1 to a vertex of V_2.

Example 1.5.11. Prove that in a connected graph G with at least three vertices, any two longest paths have a vertex in common.

Proof. Suppose $P = u_1u_2\ldots u_k$ and $Q = v_1v_2\ldots v_k$ are two longest paths in G having no vertex in common. As G is connected, there exists a u_1-v_1 path P' in G. Certainly there exist vertices u_r and v_s of P', $1 \leq r \leq k$, $1 \leq s \leq k$ such that the u_r-v_s section P'' of P' has no internal vertex in common with P or Q.

Now, of the two sections u_1-u_r and u_r-u_k of P, one must have length at least $\frac{k}{2}$. Similarly, of the two sections v_1-v_s and v_s-v_k of Q, one must have length at least $\frac{k}{2}$. Let these sections be P_1 and Q_1, respectively. Then $P_1 \cup P'' \cup Q_1$ is a path of length at least $\frac{k}{2} + 1 + \frac{k}{2}$, contradicting that k is the length of a longest path in G (see Fig. 1.20). □

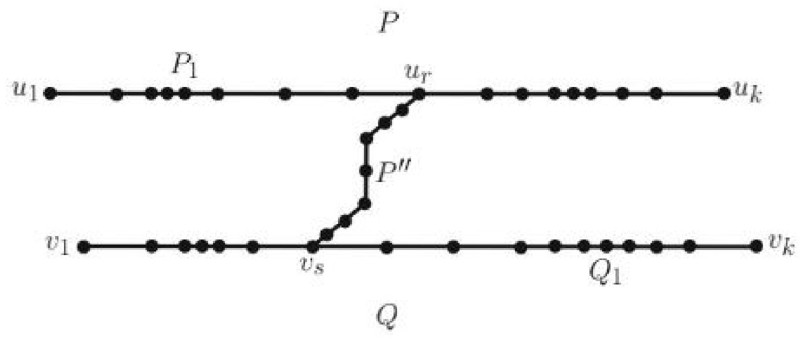


Fig. 1.20 Graph for the solution to Example 1.5.11

Exercise 5.9. Prove that in a simple graph G, the union of two distinct paths joining two distinct vertices contains a cycle.

Exercise 5.10. Show by means of an example that the union of two distinct walks joining two distinct vertices of a simple graph G need not contain a cycle.

Exercise 5.11. If a simple connected graph G is not complete, prove that there exist three vertices u, v, w of G such that uv and vw are edges of G, but uw is not an edge of G.

Exercise 5.12. (see reference: [174]) Show that a simple connected graph G is complete if and only if for some vertex v of G, $N[v] = N[u]$ for every $u \in N[v]$.

Exercise 5.13. A simple graph G is called *highly irregular* if, for each $v \in V(G)$, the degrees of the neighbors of v are all distinct. (For example, P_4 is a graph with this property.) Prove that there exist no connected highly irregular graphs of orders 3 and 5.

Exercise 5.14. The generalized Petersen graph $P(n,k)$ is defined by taking

$$V(P(n,k)) = \{a_i, b_i \; : \; 0 \le i \le n-1\}$$

and

$$E(P(n,k)) = \{a_i a_{i+1},\ a_i b_i,\ b_i b_{i+k},\ 0 \le i \le n-1\},$$

where the subscripts are integers modulo n, $n \ge 5$ and $1 \le k \le \lfloor \frac{n-1}{2} \rfloor$. Prove that if n is even and k is odd, then $P(n,k)$ is bipartite.

Example 1.5.12. If G is simple and $\delta \ge k$, then G contains a path of length at least k.

Proof. Let $P = v_0 v_1 \ldots v_r$ be a longest path in G. Then the vertices adjacent to v_r can only be from among $v_0, v_1, \ldots, v_{r-1}$. Hence, the length of $P = r \ge d_G(v_r) \ge \delta \ge k$. □

paragraph (i), the star at u' in G' is transformed by ϕ_1^{-1} either to the star at u in G or to the star at v in G. But as the star at v in G is mapped to the star at v' in G' by ϕ_1, ϕ_1^{-1} should map the star at u' in G' to the star at u in G only. As ϕ_1^{-1} is 1-1, this means that $d_G(u) \geq 2$, a contradiction. Therefore, $d_{G'}(u') = 1$, and so $S(u)$ in G is mapped onto $S(u')$ in G'.

We now define $\phi : V(G) \to V(G')$ by setting $\phi(u) = u'$ if $\phi_1(S(u)) = S(u')$. Since $S(u) = S(v)$ only when $u = v$ $(G \neq K_2, G' \neq K_2)$, ϕ is 1-1. ϕ is also onto since, for v' in G', $\phi_1^{-1}(S(v')) = S(v)$ for some $v \in V(G)$, and by the definition of ϕ, $\phi(v) = v'$. Finally, if uv is an edge of G, then $\phi_1(uv)$ belongs to both $S(u')$ and $S(v')$, where $\phi_1(S(u)) = S(u')$ and $\phi_1(S(v)) = S(v')$. This means that $u'v'$ is an edge of G'. But $u' = \phi(u)$ and $v' = \phi(v)$. Consequently, $\phi(u)\phi(v)$ is an edge of G'. If u and v are nonadjacent in G, $\phi(u)\phi(v)$ must be nonadjacent in G'. Otherwise, $\phi(u)\phi(v)$ belongs to both $S(\phi(u))$ and $S(\phi(v))$, and hence $\phi_1^{-1}(\phi(u)\phi(v)) = uv \in E(G)$, a contradiction. Thus, G and G' are isomorphic under ϕ. □

Definition 1.7.5. A graph H is called a *forbidden subgraph* for a property P of graphs if it satisfies the following condition: If a graph G has property P, then G cannot contain an induced subgraph isomorphic to H.

Beineke [17] obtained a forbidden-subgraph criterion for a graph to be a line graph. In fact, he showed that a graph G is a line graph if and only if the nine graphs of Fig. 1.25 are forbidden subgraphs for G. However, for the sake of later reference, we prove only the following result.

Theorem 1.7.6. *If G is a line graph, then $K_{1,3}$ is a forbidden subgraph of G.*

Proof. Suppose that G is the line graph of graph H and that G contains a $K_{1,3}$ as an induced subgraph. If v is the vertex of degree 3 in $K_{1,3}$ and v_1, v_2, and v_3 are the neighbors of v in this $K_{1,3}$, then the edge e corresponding to v in H is adjacent to the three edges e_1, e_2, and e_3 corresponding to the vertices v_1, v_2, and v_3. Hence, one of the end vertices of e must be the end vertex of at least two of e_1, e_2, and e_3 in H, and hence v together with two of v_1, v_2, and v_3 form a triangle in G. This means that the $K_{1,3}$ subgraph of G considered above is not an induced subgraph of G, a contradiction. □

1.8 Operations on Graphs

In mathematics, one always tries to get new structures from given ones. This also applies to the realm of graphs where one can generate many new graphs from a given set of graphs. In this section we consider some of the methods of generating new graphs from a given pair of graphs.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs.

Definition 1.8.1. *Union of two graphs:* The graph $G = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, is called the union of G_1 and G_2 and is denoted by $G_1 \cup G_2$.

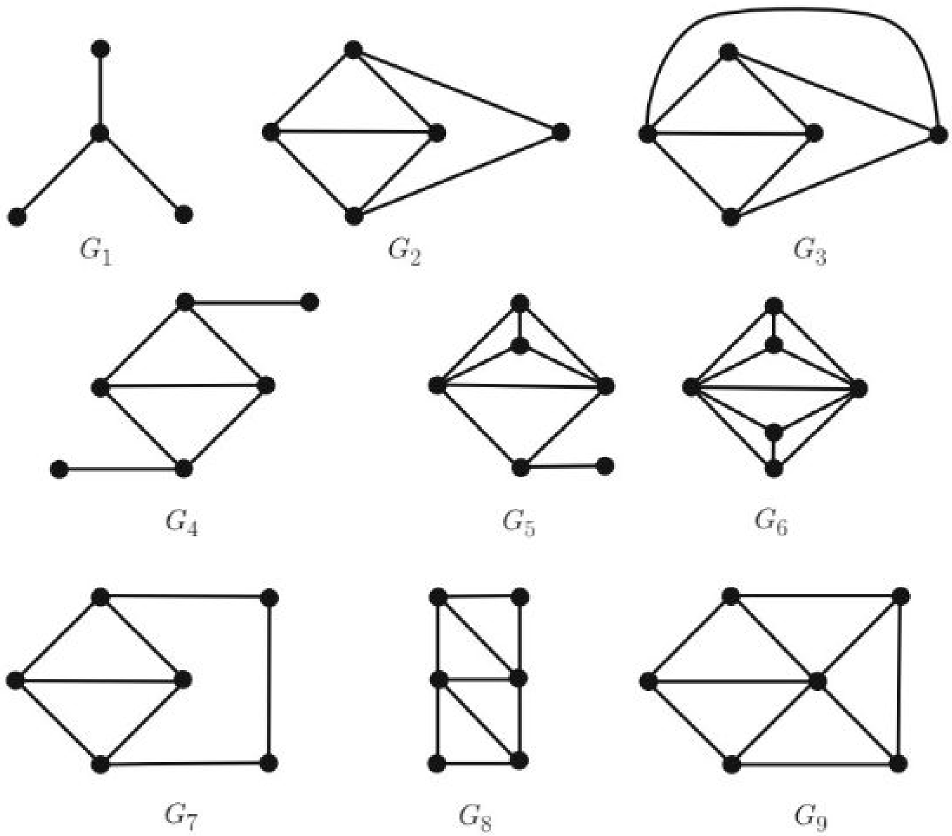


Fig. 1.25 Nine graphs of Bieneke [17]

When G_1 and G_2 are vertex disjoint, $G_1 \cup G_2$ is denoted by $G_1 + G_2$ and is called the *sum* of the graphs G_1 and G_2.

The finite union of graphs is defined by means of associativity; in particular, if $G_1, G_2, \ldots, G_r$ are pairwise vertex-disjoint graphs, each of which is isomorphic to G, then $G_1 + G_2 + \ldots + G_r$ is denoted by rG.

Definition 1.8.2. *Intersection of two graphs*: If $V_1 \cap V_2 \neq \emptyset$, the graph $G = (V, E)$, where $V = V_1 \cap V_2$ and $E = E_1 \cap E_2$ is the *intersection* of G_1 and G_2 and is written as $G_1 \cap G_2$.

Definition 1.8.3. *Join of two graphs*: Let G_1 and G_2 be two *vertex-disjoint* graphs. Then the *join* $G_1 \vee G_2$ of G_1 and G_2 is the supergraph of $G_1 + G_2$ in which each vertex of G_1 is also adjacent to every vertex of G_2.

Figure 1.26 illustrates the graph $G_1 \vee G_2$. If $G_1 = K_1$ and $G_2 = C_n$, then $G_1 \vee G_2$ is called the *wheel* W_n. W_5 is shown in Fig. 1.27.

It is worthwhile to note that $K_{m,n} = K_m^c \vee K_n^c$ and $K_n = K_1 \vee K_{n-1}$.

It follows from the above definitions that

(i) $n(G_1 \cup G_2) = n(G_1) + n(G_2) - n(G_1 \cap G_2)$, $m(G_1 \cup G_2) = m(G_1) + m(G_2) - m(G_1 \cap G_2)$.
(ii) $n(G_1 + G_2) = n(G_1) + n(G_2)$, $m(G_1 + G_2) = m(G_1) + m(G_2)$ and
(iii) $n(G_1 \vee G_2) = n(G_1) + n(G_2)$, $m(G_1 \vee G_2) = m(G_1) + m(G_2) + n(G_1)n(G_2)$.

Fig. 1.26 $G_1 \vee G_2$

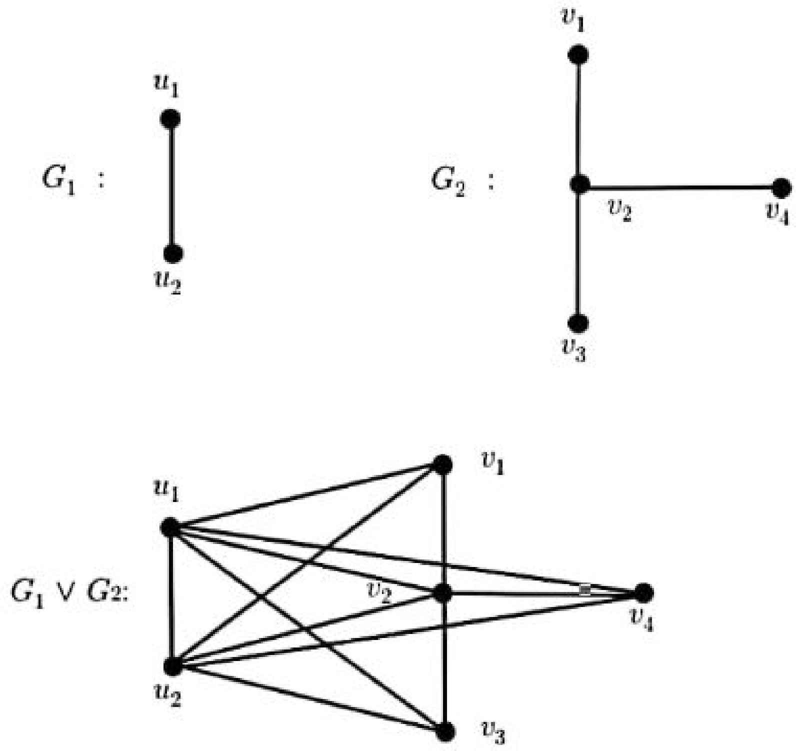
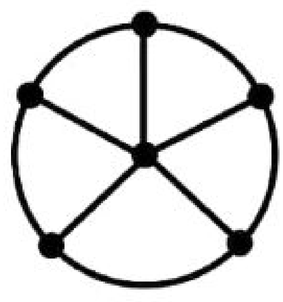


Fig. 1.27 Wheel W_5



Chapter 2
Directed Graphs

2.1 Introduction

Directed graphs arise in a natural way in many applications of graph theory. The street map of a city, an abstract representation of computer programs, and network flows can be represented only by directed graphs rather than by graphs. Directed graphs are also used in the study of sequential machines and system analysis in control theory.

2.2 Basic Concepts

Definition 2.2.1. A *directed graph* D is an ordered triple $(V(D), A(D), I_D)$, where $V(D)$ is a nonempty set called the set of *vertices* of D; $A(D)$ is a set disjoint from $V(D)$, called the set of *arcs* of D; and I_D is an *incidence map* that associates with each arc of D an ordered pair of vertices of D. If a is an arc of D, and $I_D(a) = (u, v)$, u is called the *tail* of a, and v is the *head* of a. The arc a is said to join v with u. u and v are called the *ends* of a. A directed graph is also called a *digraph*.

With each digraph D, we can associate a graph G (written $G(D)$ when reference to D is needed) on the same vertex set as follows: Corresponding to each arc of D, there is an edge of G with the same ends. This graph G is called the *underlying graph* of the digraph D. Thus, every digraph D defines a unique (up to isomorphism) graph G. Conversely, given any graph G, we can obtain a digraph from G by specifying for each edge of G an order of its ends. Such a specification is called an *orientation* of G.

Just as with graphs, digraphs have a diagrammatic representation. A digraph is represented by a diagram of its underlying graph together with arrows on its edges, the arrow pointing toward the head of the corresponding arc. A digraph and its underlying graph are shown in Fig. 2.1.

R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*,
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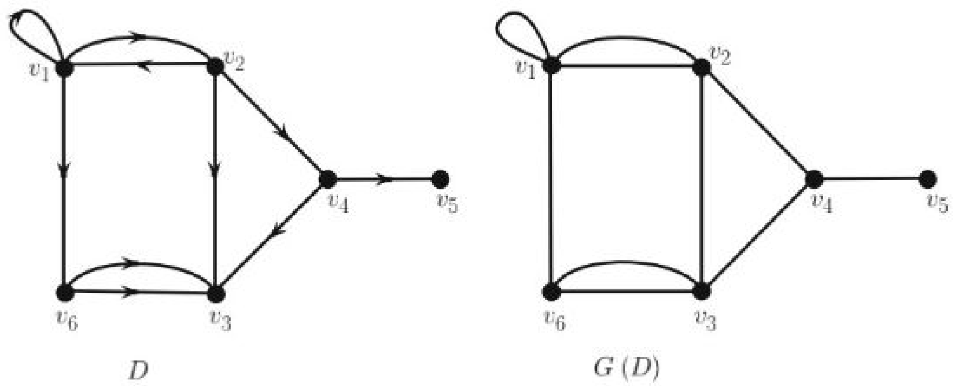


Fig. 2.1 Digraph D and its underlying graph $G(D)$

Many of the concepts and terminology for graphs are also valid for digraphs. However, there are many concepts of digraphs involving the notion of orientation that apply only to digraphs.

Definition 2.2.2. If $a = (u, v)$ is an arc of D, a is said to be *incident out of* u and *incident into* v. v is called an *outneighbor* of u, and u is called an *inneighbor* of v. $N_D^+(u)$ denotes the set of outneighbors of u in D. Similarly, $N_D^-(u)$ denotes the set of inneighbors of u in D. When no explicit reference to D is needed, we denote these sets by $N^+(u)$ and $N^-(u)$, respectively. An arc a is *incident with* u if it is either incident into or incident out of u. An arc having the same ends is called a *loop* of D. The number of arcs incident out of a vertex v is the *outdegree* of v and is denoted by $d_D^+(v)$ or $d^+(v)$. The number of arcs incident into v is its *indegree* and is denoted by $d_D^-(v)$ or $d^-(v)$.

For the digraph D of Fig. 2.2, we have $d^+(v_1) = 3$, $d^+(v_2) = 3$, $d^+(v_3) = 0$, $d^+(v_4) = 2$, $d^+(v_5) = 0$, $d^+(v_6) = 2$, $d^-(v_1) = 2$, $d^-(v_2) = 1$, $d^-(v_3) = 4$, $d^-(v_4) = 1$, $d^-(v_5) = 1$, and $d^-(v_6) = 1$. (The loop at v_1 contributes 1 each to $d^+(v_1)$ and $d^-(v_1)$.)

The *degree* $d_D(v)$ of a vertex v of a digraph D is the degree of v in $G(D)$. Thus, $d(v) = d^+(v) + d^-(v)$. As each arc of a digraph contributes 1 to the sum of the outdegrees and 1 to the sum of indegrees, we have

$$\sum_{v \in V(D)} d^+(v) = \sum_{v \in V(D)} d^-(v) = m(D),$$

where $m(D)$ is the number of arcs of D.

A vertex of D is *isolated* if its degree is 0; it is *pendant* if its degree is 1. Thus, for a pendant vertex v, either $d^+(v) = 1$ and $d^-(v) = 0$, or $d^+(v) = 0$ and $d^-(v) = 1$.

Definitions 2.2.3. 1. A digraph D' is a *subdigraph* of a digraph D if $V(D') \subseteq V(D)$, $A(D') \subseteq A(D)$, and $I_{D'}$ is the restriction of I_D to $A(D')$.

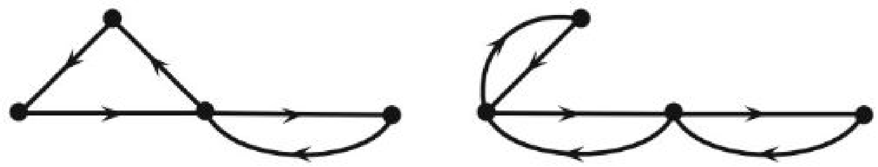


Fig. 2.2 A strong digraph (*left*) and a symmetric digraph (*right*)

2. A *directed walk* joining the vertex v_0 to the vertex v_k in D is an alternating sequence $W = v_0 a_1 v_1 a_2 v_2 \dots a_k v_k$, $1 \le i \le k$, with a_i incident out of v_{i-1} and incident into v_i. *Directed trails, directed paths, directed cycles*, and *induced subdigraphs* are defined analogously as for graphs.
3. A vertex v is *reachable* from a vertex u of D if there is a directed path in D from u to v.
4. Two vertices of D are *diconnected* if each is reachable from the other in D. Clearly, diconnection is an equivalence relation on the vertex set of D, and if the equivalence classes are $V_1, V_2, \dots, V_\omega$, the subdigraphs of D induced by $V_1, V_2, \dots, V_\omega$ are called the *dicomponents* of D.
5. A digraph is *diconnected* (also called *strongly-connected*) if it has exactly one dicomponent. A diconnected digraph is also called a *strong digraph.*
6. A digraph is *strict* if its underlying graph is simple. A digraph D is *symmetric* if, whenever (u, v) is an arc of D, then (v, u) is also an arc of D (see Fig. 2.2).

Exercise 2.1. How many orientations does a simple graph of m edges have?

Exercise 2.2. Let D be a digraph with no directed cycle. Prove that there exists a vertex whose indegree is 0. Deduce that there is an ordering $v_1, v_2, \dots, v_n$ of V such that, for $2 \le i \le n$, every arc of D with terminal vertex v_i has its initial vertex in $\{v_1, v_2, \dots, v_{i-1}\}$.

2.3 Tournaments

A digraph D is a *tournament* if its underlying graph is a complete graph. Thus, in a tournament, for every pair of distinct vertices u and v, either (u, v) or (v, u), but not both, is an arc of D. Figures 2.3a, b display all tournaments on three and four vertices, respectively.

The word "tournament" derives its name from the usual round-robin tournament. Suppose there are n players in a tournament and that every player is to play against every other player. The results of such a tournament can be represented by a tournament on n vertices, where the vertices represent the n players and an arc (u, v) represents the victory of player u over player v.

Suppose the players of a tournament have to be ranked. The corresponding digraph T, a tournament, could be used for such a ranking. The ranking of the vertices of T is as follows: One way of doing it is by looking at the sequence of outdegrees of T. This is because $d_T^+(v)$ stands for the number of players defeated by

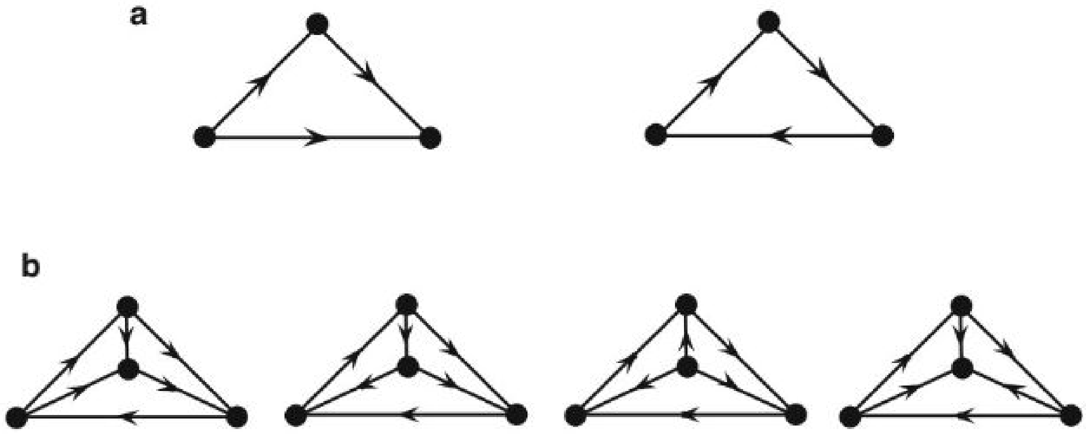


Fig. 2.3 Tournaments on (**a**) three and (**b**) four vertices

the player v. Another way of doing it is by finding a *directed Hamilton path,* that is, a spanning directed path in T. One could rank the players as per the sequence of this path so that each player defeats his or her successor. We now prove the existence of a directed Hamilton path in any tournament.

Theorem 2.3.1 (Rédei [165]). *Every tournament contains a directed Hamilton path.*

Proof. (By induction on the number of vertices n of the tournament.) The result can be directly verified for all tournaments having two or three vertices. Hence, suppose that the result is true for all tournaments on $n \geq 3$ vertices. Let T be a tournament on $n+1$ vertices $v_1, v_2, \ldots, v_{n+1}$. Now, delete v_{n+1} from T. The resulting subdigraph T' of T is a tournament on n vertices and hence by the induction hypothesis contains a directed Hamilton path. Assume that the Hamilton path is $v_1 v_2 \ldots v_n$, relabeling the vertices, if necessary.

If the arc joining v_1 and v_{n+1} has v_{n+1} as its tail, then $v_{n+1} v_1 v_2 \ldots v_n$ is a directed Hamilton path in T and the result stands proved (see Fig. 2.4a).

If the arc joining v_n and v_{n+1} is directed from v_n to v_{n+1}, then $v_1 v_2 \ldots v_n v_{n+1}$ is a directed Hamilton path in T (see Fig. 2.4b).

Now suppose that none of (v_{n+1}, v_1) and (v_n, v_{n+1}) is an arc of T. Hence, (v_1, v_{n+1}) and (v_{n+1}, v_n) are arcs of T—the first arc incident into v_{n+1} and the second arc incident out of v_{n+1}. Thus, as we pass on from v_1 to v_n, we encounter a reversal of the orientation of edges incident with v_{n+1}. Let v_i, $2 \leq i \leq n$, be the first vertex where this reversal takes place, so that (v_{i-1}, v_{n+1}) and (v_{n+1}, v_i) are arcs of T. Then $v_1 v_2 \ldots v_{i-1} v_{n+1} v_i v_{i+1} \ldots v_n$ is a directed Hamilton path of T (see Fig. 2.4c). □

Theorem 2.3.2 (Moon [141, 143]). *Every vertex of a diconnected tournament T on n vertices with $n \geq 3$ is contained in a directed k-cycle, $3 \leq k \leq n$. (T is then said to be vertex-pancyclic.)*

Proof. Let T be a diconnected tournament with $n \geq 3$ and u, a vertex of T. Let $S = N^+(u)$, the set of all outneighbors of u in T, and $S' = N^-(u)$, the set of all

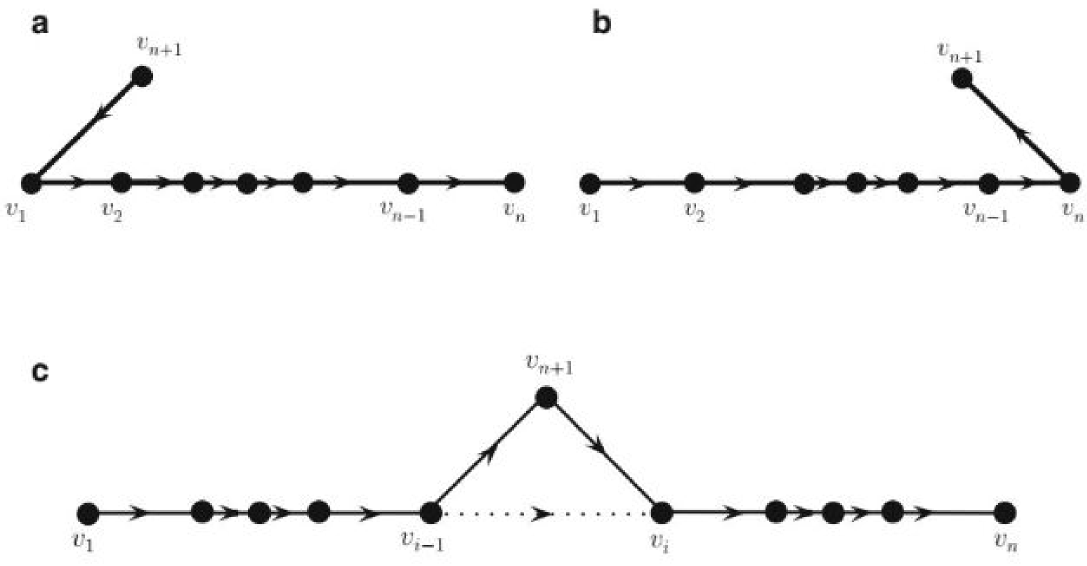


Fig. 2.4 Digraphs for proof of Theorem 2.3.1

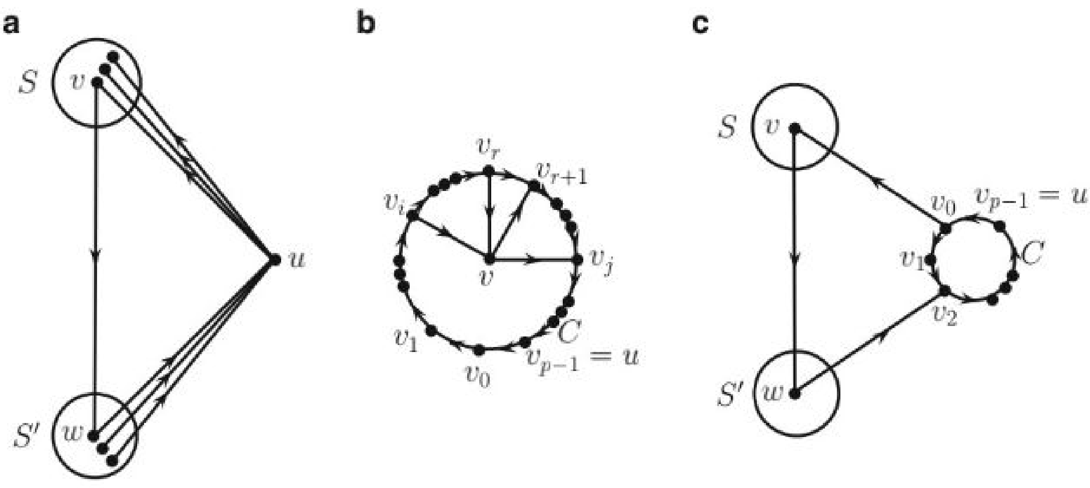


Fig. 2.5 Digraphs for proof of Theorem 2.3.2

inneighbors of u in T. As T is diconnected, none of S and S' is empty. If $[S, S']$ denotes the set of all arcs of T having their tails in S and heads in S', then $[S, S']$ is also nonempty for the same reason. If (v, w) is an arc of $[S, S']$, then (u, v, w, u) is a directed 3-cycle in T containing u. (see Fig. 2.5a.)

Suppose that u belongs to directed cycles of T of all lengths k, $3 \leq k \leq p$, where $p < n$. We shall prove that there is a directed $(p+1)$-cycle of T containing u.

Let $C : (v_0, v_1, \ldots, v_{p-1}, v_0)$ be a directed p-cycle containing u, where $v_{p-1} = u$. Suppose that v is a vertex of T not belonging to C such that for some i and j, $0 \leq i,\ j \leq p-1, i \neq j$, there exist arcs (v_i, v) and (v, v_j) of T (see Fig. 2.5b). Then there must exist arcs (v_r, v) and (v, v_{r+1}) of $A(T)$, $i \leq r \leq j-1$ (suffixes taken modulo p), and hence $(v_0, v_1, \ldots, v_r, v, v_{r+1}, \ldots, v_{p-1}, v_0)$ is a directed $(p+1)$-cycle containing u (see Fig. 2.5b).

If no such v exists, then for every vertex v of T not belonging to $V(C)$, either $(v_i, v) \in A(T)$ for every i, $0 \le i \le p-1$, or $(v, v_i) \in A(T)$ for every i, $0 \le i \le p-1$. Let $S = \{v \in V(T) \setminus V(C) : (v_i, v) \in A(T) \text{ for each } i, 0 \le i \le p-1\}$ and $S' = \{w \in V(T) \setminus V(C) : (w, v_i) \in A(T) \text{ for each } i, 0 \le i \le p-1\}$. The diconnectedness of T implies that none of S, S', and $[S, S']$ is empty. Let (v, w) be an arc of $[S, S']$. Then $(v_0, v, w, v_2, \ldots, v_{p-1}, v_0)$ is a directed $(p+1)$-cycle of T containing $v_{p-1} = u$ (see Fig. 2.5c). □

Remark 2.3.3. Theorem 2.3.2 shows, in particular, that every diconnected tournament is Hamiltonian; that is, it contains a directed spanning cycle.

Exercise 3.1. Show that every tournament T is diconnected or can be made into one by the reorientation of just one arc of T.

Exercise 3.2. Show that a tournament is diconnected if and only if it has a spanning directed cycle.

Exercise 3.3. Show that every tournament of order n has at most one vertex v with $d^+(v) = n-1$.

Exercise 3.4. Show that for each positive integer $n \ge 3$, there exists a non-Hamiltonian tournament of order n (that is, a tournament not containing a spanning directed cycle).

Exercise 3.5. Show that if a tournament contains a directed cycle, then it contains a directed cycle of length 3.

Exercise 3.6. Show that every tournament T contains a vertex v such that every other vertex of T is reachable from v by a directed path of length at most 2.

UNIT -2 CONNECTIVITY

3.1 Introduction

The connectivity of a graph is a "measure" of its connectedness. Some connected graphs are connected rather "loosely" in the sense that the deletion of a vertex or an edge from the graph destroys the connectedness of the graph. There are graphs at the other extreme as well, such as the complete graphs K_n. $n > 2$, which remain connected after the removal of any k vertices, \ <$k < n$ —

Consider a communication network. Any such network can be represented by a graph in which the vertices correspond to communication centers and the edges represent communication channels. In the communication network of Fig. 3.1a, any disruption in the communication center v will result in a communication breakdown, whereas in the network of Fig. 3.1b, at least two communication centers have to be disrupted to cause a breakdown. It is needless to stress the importance of maintaining reliable communication networks at all limes, especially during times of war, and the reliability of a communication network has a direct bearing on its connectivity.

In this chapter, we study the two graph parameters, namely, vertex connectivity and edge connectivity. We also introduce the parameter cyclical edge connectivity. We prove Mengefs theorem and several of its variations. In addition, the theorem of Ford and Fulkerson on flows in networks is established.

3.2 Vertex Cuts and Edges Cuts

We now introduce the notions of vertex cuts, edge cuts, vertex connectivity, and edge connectivity.

R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory,* Universitext, DOI 10.1007/978-1 -4614-4529-6_3, © Springer Science+Business Media New York 2012

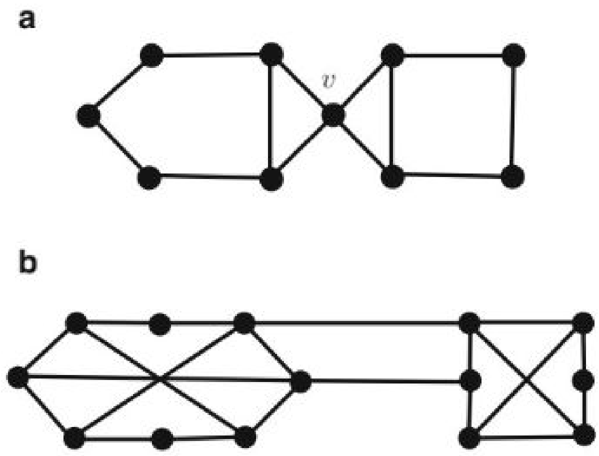


Fig. 3.1 Two types of communication networks

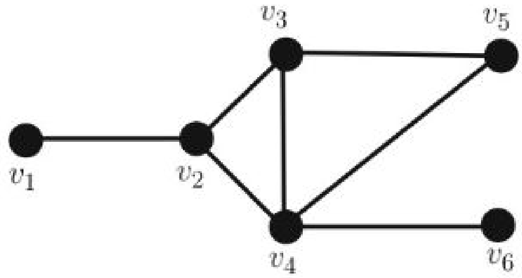


Fig. 3.2 Graph illustrating vertex cuts and edge cuts

Definitions 3.2.1. 1. A subset V' of the vertex set $V(G)$ of a connected graph G is a *vertex cut* of G if $G - V'$ is disconnected; it is a *k-vertex cut* if $|V'| = k$. V' is then called a *separating set of vertices* of G. A vertex v of G is a *cut vertex* of G if $\{v\}$ is a vertex cut of G.

2. Let G be a nontrivial connected graph with vertex set $V(G)$ and let S be a nonempty subset of $V(G)$. For $\bar{S} = V \backslash S \neq \emptyset$, let $[S, \bar{S}]$ denote the set of all edges of G that have one end vertex in S and the other in $\bar{S}$. A set of edges of G of the form $[S, \bar{S}]$ is called an *edge cut* of G. An edge e is a *cut edge* of G if $\{e\}$ is an edge cut of G. An edge cut of cardinality k is called a *k-edge cut* of G.

Example 3.2.2. For the graph of Fig. 3.2, $\{v_2\}$, and $\{v_3, v_4\}$ are vertex cuts. The edge subsets $\{v_3v_5, v_4v_5\}$, $\{v_1v_2\}$, and $\{v_4v_6\}$ are all edge cuts. Of these, v_2 is a cut vertex, and v_1v_2 and v_4v_6 are both cut edges. For the edge cut $\{v_3v_5, v_4v_5\}$, we may take $S = \{v_5\}$ so that $\bar{S} = \{v_1, v_2, v_3, v_4, v_6\}$.

Remarks 3.2.3. 1. If uv is an edge of an edge cut E', then all the edges having u and v as their ends also belong to E'.

2. No loop can belong to an edge cut.

Exercise 2.1. If $\{x, y\}$ is a 2-edge cut of a graph G, show that every cycle of G that contains x must also contain y.

Remarks 3.2.4. If G is connected and E' is a set of edges whose deletion results in a disconnected graph, then E' contains an edge cut of G. It is clear that if e is a cut edge of a connected graph G, then $G - e$ has exactly two components.

Remarks 3.2.5. Since the removal of a parallel edge of a connected graph does not result in a disconnected graph, such an edge cannot be a cut edge of the graph. A set of edges of a connected graph G whose deletion results in a disconnected graph is called a *separating set of edges.* In particular, any edge cut of a connected graph G is a separating set of edges of G.

We now characterize a cut vertex of G.

Theorem 3.2.6. *A vertex v of a connected graph G with at least three vertices is a cut vertex of G if and only if there exist vertices u and w of G distinct from v such that v is in every u-w path in G.*

Proof. If v is a cut vertex of G, then $G - v$ is disconnected and has at least two components, G_1 and G_2. Take $u \in V(G_1)$ and $w \in V(G_2)$. Then every u-w path in G must contain v, as otherwise u and w would belong to the same component of $G - v$.

Conversely, suppose that the condition of the theorem holds. Then the deletion of v destroys every u-w path in G, and hence u and w lie in distinct components of $G - v$. Therefore, $G - v$ is disconnected and v is a cut vertex of G. □

Theorems 3.2.7 and 3.2.8 characterize a cut edge of a graph.

Theorem 3.2.7. *An edge $e = xy$ of a connected graph G is a cut edge of G if and only if e belongs to no cycle of G.*

Proof. Let e be a cut edge of G and let $[S, \bar{S}] = \{e\}$ be the partition of V defined by $G - e$ so that one of x and y belongs to S, and the other to $\bar{S}$, say, $x \in S$ and $y \in \bar{S}$. If e belongs to a cycle of G, then $[S, \bar{S}]$ must contain at least one more edge, contradicting that $\{e\} = [S, \bar{S}]$. Hence, e cannot belong to a cycle.

Conversely, assume that e is not a cut edge of G. Then $G - e$ is connected, and hence there exists an x-y path P in $G - e$. Then $P \cup \{e\}$ is a cycle in G containing e. □

Theorem 3.2.8. *An edge $e = xy$ is a cut edge of a connected graph G if and only if there exist vertices u and v such that e belongs to every u-v path in G.*

Proof. Let $e = xy$ be a cut edge of G. Then $G - e$ has two components, say, G_1 and G_2. Let $u \in V(G_1)$ and $v \in V(G_2)$. Then, clearly, every u-v path in G contains e.

Conversely, suppose that there exist vertices u and v satisfying the condition of the theorem. Then there exists no u-v path in $G - e$ so that $G - e$ is disconnected. Hence, e is a cut edge of G. □

Remark 3.2.9. There exist graphs in which every edge is a cut edge. It follows from Theorem 3.2.7 that if G is a simple connected graph with at least one edge and without cycles, then every edge of G is a cut edge of G. A similar result is not true for cut vertices. Our next result shows that not every vertex of a connected graph (with at least two vertices) can be a cut vertex of G.

Theorem 3.2.10. *A connected graph G with at least two vertices contains at least two vertices that are not cut vertices.*

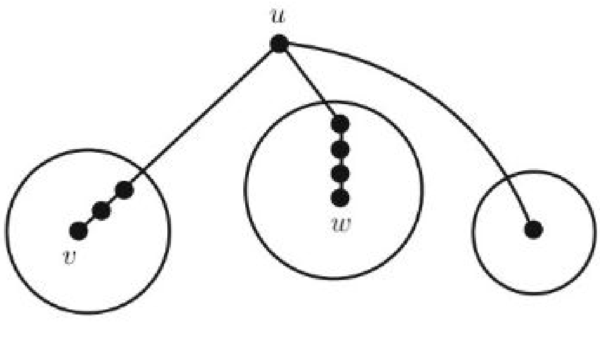


Fig. 3.3 Graph for proof of Theorem 3.2.10

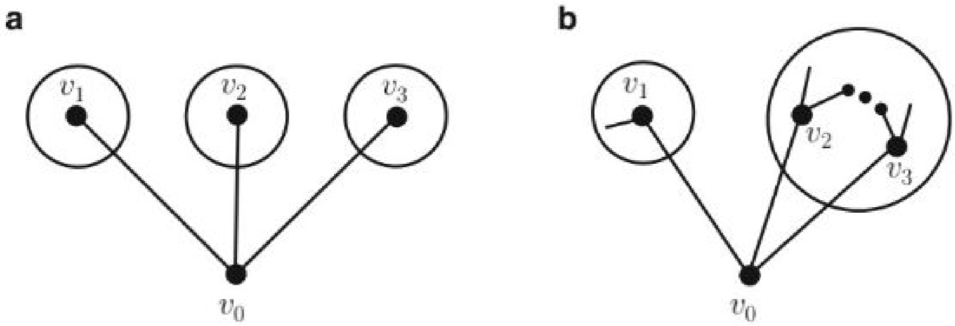


Fig. 3.4 Graph for proof of Proposition 3.2.11

Proof. First, suppose that $n(G) \geq 3$. Let u and v be vertices of G such that $d(u, v)$ is maximum. Then neither u nor v is a cut vertex of G. For if u were a cut vertex of G, $G - u$ would be disconnected, having at least two components. The vertex v belongs to one of these components. Let w be any vertex belonging to a component of $G - u$ not containing v. Then every v-w path in G must contain u (see Fig. 3.3). Consequently, $d(v, w) > d(v, u)$, contradicting the choice of u and v. Hence, u is not a cut vertex of G. Similarly, v is not a cut vertex of G.

If $n(G) = 2$, then K_2 is a spanning subgraph of G, and so no vertex of G is a cut vertex of G. This completes the proof of the theorem. □

Proposition 3.2.11. *A simple cubic (i.e., 3-regular) connected graph G has a cut vertex if and only if it has a cut edge.*

Proof. Let G have a cut vertex v_0. Let v_1, v_2, v_3 be the vertices of G that are adjacent to v_0 in G. Consider $G - v_0$, which has either two or three components. If $G - v_0$ has three components, no two of v_1, v_2, and v_3 can belong to the same component of $G - v_0$. In this case, each of v_0v_1, v_0v_2, and v_0v_3 is a cut edge of G. (See Fig. 3.4a.) In the case when $G - v_0$ has only two components, one of the vertices, say v_1, belongs to one component of $G - v_0$, and v_2 and v_3 belong to the other component. In this case, v_0v_1 is a cut edge. (See Fig. 3.4b.)

Conversely, suppose that $e = uv$ is a cut edge of G. Then the deletion of u results in the deletion of the edge uv. Since G is cubic, $G - u$ is disconnected. Accordingly, u is a cut vertex of G. □

Exercise 2.2. Find the vertex cuts and edge cuts of the graph of Fig. 3.2.

Exercise 2.3. Prove or disprove: Let G be a simple connected graph with $n(G) \geq 3$. Then G has a cut edge if and only if G has a cut vertex.

Exercise 2.4. Show that in a graph, the number of edges common to a cycle and an edge cut is even.

3.3 Connectivity and Edge Connectivity

We now introduce two parameters of a graph that in a way measure the connectedness of the graph.

Definition 3.3.1. For a nontrivial connected graph G having a pair of nonadjacent vertices, the minimum k for which there exists a k-vertex cut is called the *vertex connectivity* or simply the *connectivity* of G; it is denoted by $\kappa(G)$ or simply κ (kappa) when G is understood. If G is trivial or disconnected, $\kappa(G)$ is taken to be zero, whereas if G contains K_n as a spanning subgraph, $\kappa(G)$ is taken to be $n-1$.

A set of vertices and/or edges of a connected graph G is said to *disconnect* G if its deletion results in a disconnected graph.

When a connected graph G (on $n \geq 3$ vertices) does not contain K_n as a spanning subgraph, κ is the connectivity of G if there exists a set of κ vertices of G whose deletion results in a disconnected subgraph of G while no set of $\kappa - 1$ (or fewer) vertices has this property.

Exercise 3.1. Prove that a simple graph G with n vertices, $n \geq 2$, is complete if and only if $\kappa(G) = n - 1$.

Definition 3.3.2. The *edge connectivity* of a connected graph G is the smallest k for which there exists a k-edge cut (i.e., an edge cut having k edges). The edge connectivity of a trivial or disconnected graph is taken to be 0. The edge connectivity of G is denoted by $\lambda(G)$. If λ is the edge connectivity of a connected graph G, there exists a set of λ edges whose deletion results in a disconnected graph, and no subset of edges of G of size less than λ has this property.

Exercise 3.2. Prove that the deletion of edges of a minimum-edge cut of a connected graph G results in a disconnected graph with exactly two components. (Note that a similar result is not true for a minimum vertex cut.)

Definition 3.3.3. A graph G is r*-connected* if $\kappa(G) \geq r$. Also, G is r*-edge connected* if $\lambda(G) \geq r$.

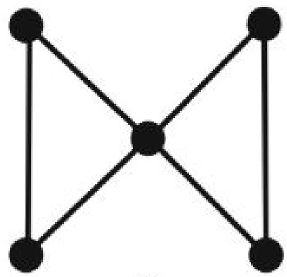
An r-connected (respectively, r-edge-connected) graph is also ℓ-connected (respectively, ℓ-edge connected) for each ℓ, $0 \leq \ell \leq r - 1$.

For the graph G of Fig. 3.5, $\kappa(G) = 1$ and $\lambda(G) = 2$.

We now derive inequalities connecting $\kappa(G)$, $\lambda(G)$, and $\delta(G)$.

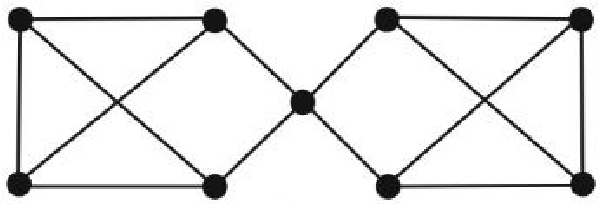
Theorem 3.3.4. *For any loopless connected graph* G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Fig. 3.5 A 1-connected graph



G

Fig. 3.6 Graph G with $\kappa = 1$, $\lambda = 2$ and $\delta = 3$



G

Proof. We observe that $\kappa = 0$ if and only if $\lambda = 0$. Also, $\delta = 0$ implies that $\kappa = 0$ and $\lambda = 0$. Hence we may assume that κ, λ, and δ are all at least 1. Let $\mathscr{E}$ be an edge cut of G with λ edges. Let u and v be the end vertices of an edge of $\mathscr{E}$. For each edge of $\mathscr{E}$ that does not have both u and v as end vertices, remove an end vertex that is different from u and v. If there are t such edges, at most t vertices have been removed. If the resulting graph, say H, is disconnected, then $\kappa \leq t < \lambda$. Otherwise, there will remain a subset of edges of E having u and v as end vertices, the removal of which from H would disconnect G. Hence, in addition to the already removed vertices, the removal of one of u and v will result in either a disconnected graph or a trivial graph. In the process, a set of at most $t + 1$ vertices has been removed and $\kappa \leq t + 1 \leq \lambda$.

Finally, it is clear that $\lambda \leq \delta$. In fact, if v is a vertex of G with $d_G(v) = \delta$, then the set $[\{v\}, V \setminus \{v\}]$ of δ edges of G incident at v forms an edge cut of G. Thus, $\lambda \leq \delta$. □

It is possible that the inequalities in Theorem 3.3.4 can be strict. See the graph G of Fig. 3.6, for which $\kappa = 1$, $\lambda = 2$, and $\delta = 3$.

Exercise 3.3. Prove or disprove: If H is a subgraph of G, then

(a) $\kappa(H) \leq \kappa(G)$ and
(b) $\lambda(H) \leq \lambda(G)$.

Exercise 3.4. Determine $\lambda(K_n)$.

Exercise 3.5. Determine the connectivity and edge connectivity of the Petersen graph P. (See graph P of Fig. 1.7. Note that P is a cubic graph.)

Theorem 3.3.5 gives a class of graphs for which $\kappa = \lambda$.

Theorem 3.3.5. *The connectivity and edge connectivity of a simple cubic graph G are equal.*

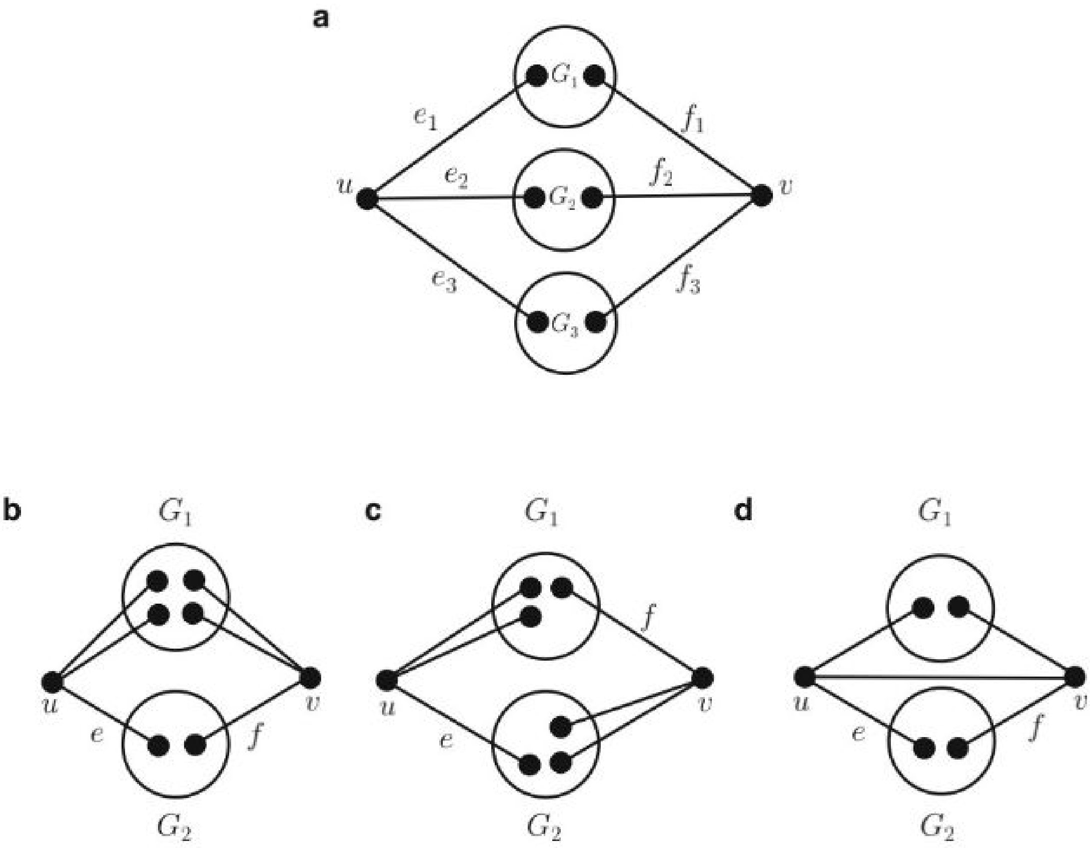


Fig. 3.7 Connected cubic graphs for proof of Theorem 3.3.5

Proof. We need only consider the case of a connected cubic graph. Again, since $\kappa \leq \lambda \leq \delta = 3$, we have only to consider the cases when $\kappa = 1, 2$, or 3. Now, Proposition 3.2.11 implies that for a simple cubic graph G, $\kappa = 1$ if and only if $\lambda = 1$.

If $\kappa = 3$, then by Theorem 3.3.4, $3 = \kappa \leq \lambda \leq \delta = 3$, and hence $\lambda = 3$.

We shall now prove that $\kappa = 2$ implies that $\lambda = 2$.

Suppose $\kappa = 2$ and $\{u, v\}$ is a 2-vertex cut of G. The deletion of $\{u, v\}$ results in a disconnected subgraph G' of G. Since each of u and v must be joined to each component of G', and since G is cubic, G' can have at most three components. If G' has three components, G_1, G_2, and G_3, and if e_i and f_i, $i = 1, 2, 3$, join, respectively, u and v with G_i, then each pair $\{e_i, f_i\}$ is an edge cut of G (see Fig. 3.7a).

If G' has only two components, G_1 and G_2, then each of u and v is joined to one of G_1 and G_2 by a single edge, say, e and f, respectively, so that $\{e, f\}$ is an edge cut of G (see Fig. 3.7b–d).

Hence, in either case there exists an edge cut consisting of two edges. As such, $\lambda \leq 2$. But by Theorem 3.3.4, $\lambda \geq \kappa = 2$. Hence $\lambda = 2$. Finally, the above arguments show that if $\lambda = 3$, then $\kappa = 3$, and if $\lambda = 2$, then $\kappa = 2$. □

Exercise 3.6. Give examples of cubic graphs G_1, G_2, and G_3 with $\kappa(G_1) = 1$, $\kappa(G_2) = 2$, and $\kappa(G_3) = 3$.

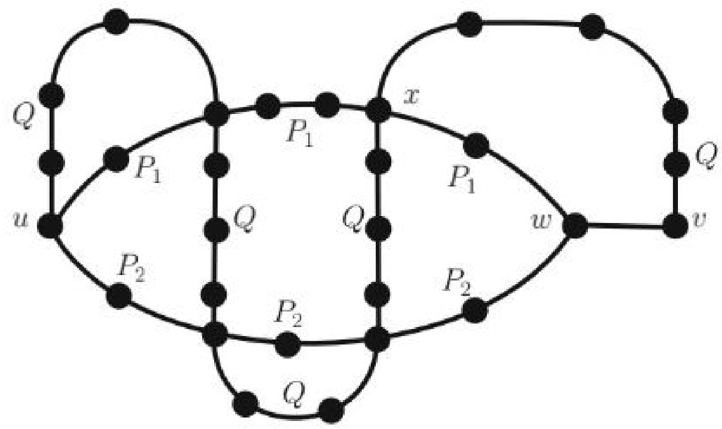


Fig. 3.8 Graph for proof of Theorem 3.3.7

Definition 3.3.6. A family of two or more paths in a graph G is said to be *internally disjoint* if no vertex of G is an internal vertex of more than one path in the family.

We now state and prove *Whitney's characterization theorem* of 2-connected graphs.

Theorem 3.3.7 (Whitney [193]). *A graph G with at least three vertices is 2-connected if and only if any two vertices of G are connected by at least two internally disjoint paths.*

Proof. Let G be 2-connected. Then G contains no cut vertex. Let u and v be two distinct vertices of G. We now use induction on $d(u, v)$ to prove that u and v are joined by two internally disjoint paths.

If $d(u, v) = 1$, let $e = uv$. As G is 2-connected and $n(G) \geq 3$, e cannot be a cut edge of G, since if e were a cut edge, at least one of u and v must be a cut vertex. By Theorem 3.2.7, e belongs to a cycle C in G. Then $C - e$ is a u-v path in G, internally disjoint from the path uv.

Now assume that any two vertices x and y of G with $d(x, y) = k-1, k \geq 2$, are joined by two internally disjoint x-y paths in G. Let $d(u, v) = k$. Let P be a u-v path of length k and w be the vertex of G just preceding v on P. Then $d(u, w) = k - 1$. By an induction hypothesis, there are two internally disjoint u-w paths, say P_1 and P_2, in G. As G has no cut vertex, $G - w$ is connected and hence there exists a u-v path Q in $G - w$. Q is clearly a u-v path in G not containing w. Let x be the vertex of Q such that the x-v section of Q contains only the vertex x in common with $P_1 \cup P_2$ (see Fig. 3.8).

We may suppose, without loss of generality, that x belongs to P_1. Then the union of the u-x section of P_1 and x-v section of Q and $P_2 \cup (wv)$ are two internally disjoint u-v paths in G. This gives the proof in one direction.

In the other direction, assume that any two distinct vertices of G are connected by at least two internally disjoint paths. Then G is connected. Further, G cannot contain a cut vertex, since if v were a cut vertex of G, there must exist vertices u and w such that every u-w path contains v (compare with Theorem 3.2.6), contradicting the hypothesis. Hence, G is 2-connected. □

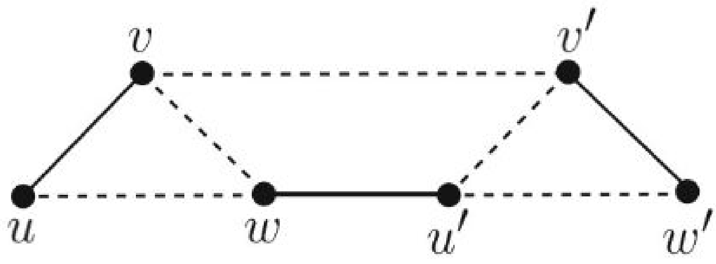


Fig. 3.9 Graph for Remark 3.3.9

Theorem 3.3.8. *A graph G with at least three vertices is 2-connected if and only if any two vertices of G lie on a common cycle.*

Proof. Let u and v be any two vertices of a 2-connected graph G. By Theorem 3.3.7, there exist two internally disjoint paths in G joining u and v. The union of these two paths is a cycle containing u and v.

Conversely, if any two vertices u and v lie on a cycle C, then C is the union of two internally disjoint u-v paths. Again, by Theorem 3.3.7, G is 2-connected. □

Remark 3.3.9. If G is 2-connected, if u and v are distinct vertices of G, and if P is a u-v path in G, it is not in general true that there exists another u-v path Q in G such that P and Q are internally disjoint. For example, in the 2-connected graph of Fig. 3.9, if P is the u-w' path $uwvv'u'w'$, there exists no u-w' path Q in G that is internally disjoint from P. However, there do exist two internally disjoint u-w' paths in G.

Exercise 3.7. (a) Show that a graph G with at least three vertices is 2-connected if and only if any vertex and any edge of G lie on a common cycle of G.
(b) Show that a graph G with at least three vertices is 2-connected if and only if any two edges of G lie on a common cycle.

Exercise 3.8. Prove that a graph is 2-connected if and only if for every pair of disjoint connected subgraphs G_1 and G_2, there exist two internally disjoint paths P_1 and P_2 of G between G_1 and G_2.

Exercise 3.9. *Edge form of Whitney's theorem:* Prove that a graph G with $n \geq 3$ is 2-edge connected if and only if any two distinct vertices of G are connected by at least two edge-disjoint paths in G. [Hint: Imitate the proof of Theorem 3.3.7, or pass on to $L(G)$.]

Exercise 3.10. (a) Disprove by a counterexample: If $\kappa(G) = k$, then $\kappa(L(G))=k$.
(b) Prove: $\lambda(G) \leq \kappa(L(G))$. Give an example of a graph G for which $\lambda(G) < \kappa(L(G))$.

Theorem 3.3.10. *In a 2-connected graph G, any two longest cycles have at least two vertices in common.*

Proof. Let $C_1 = u_1u_2 \dots u_ku_1$ and $C_2 = v_1v_2 \dots v_kv_1$ be two longest cycles in G. If C_1 and C_2 are disjoint, there exist (since G is 2-connected) two disjoint paths,

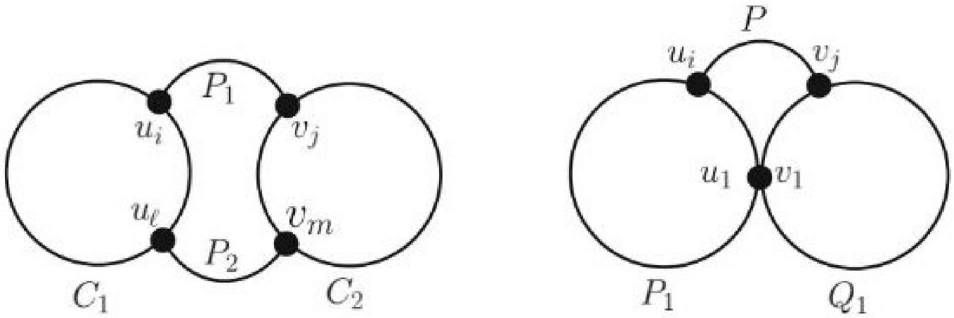
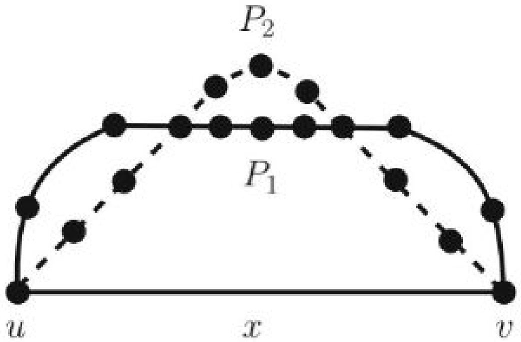


Fig. 3.10 Graphs for proof of Theorem 3.3.10

Fig. 3.11 Graph for proof of Theorem 3.3.11



say, P_1 joining u_i and v_j and P_2 joining u_ℓ and v_p, connecting C_1 and C_2 such that $u_i \neq u_\ell$ and $v_j \neq v_p$ (see Exercise 3.8). u_i and u_ℓ divide C_1 into two subpaths. Let L_1 be the longer of these subpaths. (If both subpaths are of equal length, we take either one of them to be L_1.) Let L_2 be defined in a similar manner in C_2. Then $L_1 \cup P_1 \cup L_2 \cup P_2$ is a cycle of length greater than that of C_1 (or C_2). Hence, C_1 and C_2 cannot be disjoint. (See Fig. 3.10.)

Suppose that C_1 and C_2 have exactly one vertex, say $u_1 = v_1$, in common. Since G is 2-connected, u_1 is not a cut vertex of G, and so there exists a path P with one end vertex u_i in $C_1 - u_1$ and the other end vertex v_j in $C_2 - v_1$, which is internally disjoint from $C_1 \cup C_2$. Let P_1 denote the longer of the two u_1-u_i sections of C_1, and Q_1 denote the longer of the two v_1-v_j sections of C_2. If the two sections of C_1 or of C_2 are of equal length, take any one of them. Then $P_1 \cup P \cup Q_1$ is a cycle longer than C_1 (or C_2). But this is impossible. Thus, C_1 and C_2 must have at least two vertices in common. □

Theorem 3.3.11 gives a simple characterization of 3-edge-connected graphs.

Theorem 3.3.11. *A connected simple graph G is 3-edge connected if and only if every edge of G is the (exact) intersection of the edge sets of two cycles of G.*

Proof. Let G be 3-edge connected and let $x = uv$ be an edge of G. Since $G - x$ is 2-edge connected, there exist two edge-disjoint u-v paths P_1 and P_2 in $G - x$ (see Exercise 3.9). Now, $P_1 \cup \{x\}$ and $P_2 \cup \{x\}$ are two cycles of G, the intersection of whose edge sets is precisely $\{x\}$ (see Fig. 3.11).

Conversely, suppose that for each edge $x = uv$ there exist two cycles C and C' such that $\{x\} = E(C) \cap E(C')$. G cannot have a cut edge since, by hypothesis, each edge belongs to two cycles and no cut edge can belong to a cycle; nor can G contain an edge cut consisting of two edges x and y, by Exercise 2.1. (Since any cycle that contains x also contains y, the intersection of any two such cycles must contain both x and y, a contradiction.) Hence, $\lambda(G) \geq 3$, and G is 3-edge connected. □

Chapter 4
Trees

4.1 Introduction

"Trees" form an important class of graphs. Of late, their importance has grown considerably in view of their wide applicability in theoretical computer science.

In this chapter, we present the basic structural properties of trees, their centers and centroids. In addition, we present two interesting consequences of the Tutte–Nash–Williams theorem on the existence of k pairwise edge-disjoint spanning trees in a simple connected graph. We also present Cayley's formula for the number of spanning trees in the labeled complete graph K_n. As applications, we present Kruskal's algorithm and Prim's algorithm, which determine a minimum-weight spanning tree in a connected weighted graph and discuss Dijkstra's algorithm, which determines a minimum-weight shortest path between two specified vertices of a connected weighted graph.

4.2 Definition, Characterization, and Simple Properties

Certain graphs derive their names from their diagrams. A "tree" is one such graph. Formally, a connected graph without cycles is defined as a *tree*. A graph without cycles is called an *acyclic graph* or a *forest*. So each component of a forest is a tree. A forest may consist of just a single tree! Figure 4.1 displays two pairs of isomorphic trees.

Remarks 4.2.1. 1. It follows from the definition that a forest (and hence a tree) is a simple graph.
2. A subgraph of a tree is a forest and a connected subgraph of a tree T is a *subtree* of T.

R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*,
Universitext, DOI 10.1007/978-1-4614-4529-6_4,

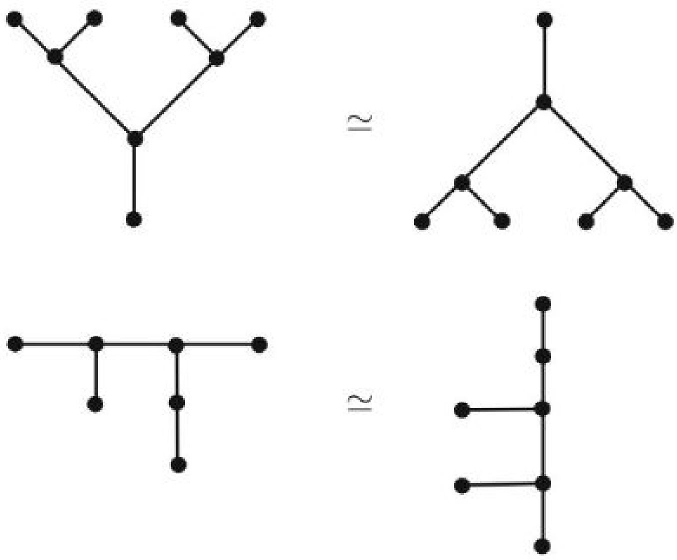


Fig. 4.1 Examples of isomorphic trees

In a connected graph, any two distinct vertices are connected by at least one path. Trees are precisely those simple connected graphs in which every pair of distinct vertices is joined by a unique path.

Theorem 4.2.2. *A simple graph is a tree if and only if any two distinct vertices are connected by a unique path.*

Proof. Let T be a tree. Suppose that two distinct vertices u and v are connected by two distinct u-v paths. Then their union contains a cycle (cf. Exercise 5.9, Chap. 1) in T, contradicting that T is a tree.

Conversely, suppose that any two vertices of a graph G are connected by a unique path. Then G is obviously connected. Also, G cannot contain a cycle, since any two distinct vertices of a cycle are connected by two distinct paths. Hence G is a tree.□

A spanning subgraph of a graph G, which is also a tree, is called a *spanning tree* of G. A connected graph G and two of its spanning trees T_1 and T_2 are shown in Fig. 4.2.

The graph G of Fig. 4.2 shows that a graph may contain more than one spanning tree; each of the trees T_1 and T_2 is a spanning tree of G.

A loop cannot be an edge of any spanning tree, since such a loop constitutes a cycle (of length 1). On the other hand, a cut edge of G must be an edge of every spanning tree of G. Theorem 4.2.3 shows that every connected graph contains a spanning tree.

Theorem 4.2.3. *Every connected graph contains a spanning tree.*

Proof. Let G be a connected graph. Let $\mathscr{C}$ be the collection of all connected spanning subgraphs of G. $\mathscr{C}$ is nonempty as $G \in \mathscr{C}$. Let $T \in \mathscr{C}$ have the fewest number of edges. Then T must be a spanning tree of G. If not, T would contain a cycle of G, and the deletion of any edge of this cycle would give a (spanning)

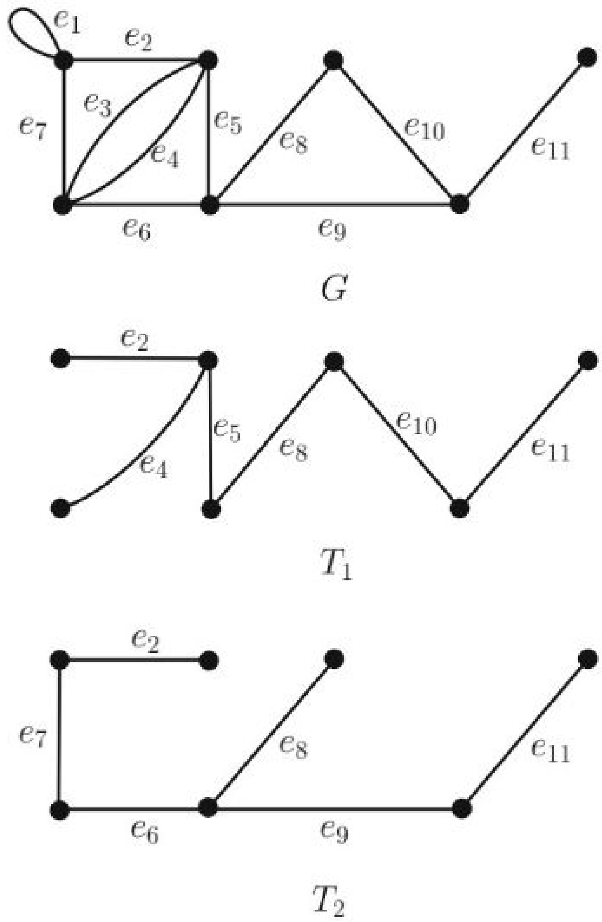


Fig. 4.2 Graph G and two of its spanning trees T_1 and T_2

subgraph in $\mathscr{C}$ having one edge less than that of T. This contradicts the choice of T. Hence, T has no cycles and is therefore a spanning tree of G. □

There is a nice relation between the number of vertices and the number of edges of any tree.

Theorem 4.2.4. *The number of edges in a tree on n vertices is* $n - 1$. *Conversely, a connected graph on n vertices and* $n - 1$ *edges is a tree.*

Proof. Let T be a tree. We use induction on n to prove that $m = n - 1$. When $n = 1$ or $n = 2$, the result is straightforward.

Now assume that the result is true for all trees on $(n - 1)$ or fewer vertices, $n \geq 3$. Let T be a tree with n vertices. Let $e = uv$ be an edge of T. Then uv is the unique path in T joining u and v. Hence the deletion of e from T results in a disconnected graph having two components T_1 and T_2. Being connected subgraphs of a tree, T_1 and T_2 are themselves trees. As $n(T_1)$ and $n(T_2)$ are less than $n(T)$, by an induction hypothesis, $m(T_1) = n(T_1) - 1$ and $m(T_2) = n(T_2) - 1$. Therefore, $m(T) = m(T_1) + m(T_2) + 1 = n(T_1) - 1 + n(T_2) - 1 + 1 = n(T_1) + n(T_2) - 1 = n(T) - 1$. Hence, the result is true for T. By induction, the result follows in one direction.

Conversely, let G be a connected graph with n vertices and $n-1$ edges. By Theorem 4.2.3, there exists a spanning tree T of G. T has n vertices and being a tree has $(n-1)$ edges. Hence $G = T$, and G is a tree. □

Exercise 2.1. Give an example of a graph with n vertices and $n-1$ edges that is not a tree.

Theorem 4.2.5. *A tree with at least two vertices contains at least two pendant vertices (i.e., end vertices or vertices of degree* 1*).*

Proof. Consider a longest path P of a tree T. The end vertices of P must be pendant vertices of T; otherwise, at least one of the end vertices of P has a second neighbor in P, and this yields a cycle, a contradiction. □

Corollary 4.2.6. *If* $\delta(G) \geq 2$, G *contains a cycle.*

Proof. If G has no cycles, G is a forest and hence $\delta(G) \leq 1$ by Theorem 4.2.5. □

Exercise 2.2. Show that a simple graph with ω components is a forest if and only if $m = n - \omega$.

Exercise 2.3. A vertex v of a tree T with at least three vertices is a cut vertex of T if and only if v is not a pendant vertex.

Exercise 2.4. Prove that every tree is a bipartite graph.

Our next result is a characterization of trees.

Theorem 4.2.7. *A connected graph G is a tree if and only if every edge of G is a cut edge of G.*

Proof. If G is a tree, there are no cycles in G. Hence, no edge of G can belong to a cycle. By Theorem 3.2.7, each edge of G is a cut edge of G. Conversely, if every edge of a connected graph G is a cut edge of G, then G cannot contain a cycle, since no edge of a cycle is a cut edge of G. Hence, G is a tree. □

Theorem 4.2.8. *A connected graph G with at least two vertices is a tree if and only if its degree sequence* $(d_1, d_2, \ldots, d_n)$ *satisfies the condition:* $\sum_{i=1}^{n} d_i = 2(n-1)$ *with* $d_i > 0$ *for each* i.

Proof. Let G be a tree. As G is connected and nontrivial, it can have no isolated vertex. Hence every term of the degree sequence of G is positive. Further, by Theorem 1.4.4, $\sum\limits_{i=1}^{n} d_i = 2m = 2(n-1)$.

Conversely, assume that the condition $\sum_{i=1}^{n} d_i = 2(n-1)$ holds. This implies that $m = n-1$ as $\sum_{i=1}^{n} d_i = 2m$. Now apply Theorem 4.2.4. □

Lemma 4.2.9. *If u and v are nonadjacent vertices of a tree T, then T + uv contains a unique cycle.*

Proof. If P is the unique u-v path in T, then $P + uv$ is a cycle in $T + uv$. It is unique, as the path P is unique in T. □

Example 4.2.10. Prove that if $m(G) = n(G)$ for a simple connected graph G, then G is unicyclic, that is, a graph containing exactly one cycle.

Proof. By Theorem 4.2.3, G contains a spanning tree T. As T has $n(G) - 1$ edges, $E(G) \backslash E(T)$ consists of a single edge e. Then $G = T \cup e$ is unicyclic. □

Exercise 2.5. If for a simple graph G, $m(G) \geq n(G)$, prove that G contains a cycle.

Exercise 2.6. Prove that every edge of a connected graph G that is not a loop is in some spanning tree of G.

Exercise 2.7. Prove that the following statements are equivalent:

(i) G is connected and unicyclic (i.e., G has exactly one cycle).
(ii) G is connected and $n = m$.
(iii) For some edge e of G, $G - e$ is a tree.
(iv) G is connected and the set of edges of G that are not cut edges forms a cycle.

Example 4.2.11. Prove that for a simple connected graph G, $L(G)$ is isomorphic to G if and only if G is a cycle.

Proof. If G is a cycle, then clearly $L(G)$ is isomorphic to G. Conversely, let $G \simeq L(G)$. Then $n(G) = n(L(G))$, and $m(G) = m(L(G))$. But since $n(L(G)) = m(G)$, we have $m(G) = n(G)$. By Example 4.2.10, G is unicyclic. Let $C = v_1 v_2 \ldots v_k v_1$ be the unique cycle in G. If $G \neq C$, there must be an edge $e \notin E(C)$ incident with some vertex v_i of C (as G is connected). Thus, there is a star with at least three edges at v_i. This star induces a clique of size at least 3 in $L(G)$ ($\simeq G$). This shows that there exists at least one more cycle in $L(G)$ distinct from the cycle corresponding to C in G. This contradicts the fact that $L(G) \simeq G$ (as G is unicyclic). □

4.4 Counting the Number of Spanning Trees

Counting the number of spanning trees in a graph occurs as a natural problem in many branches of science. Spanning trees were used by Kirchoff to generate a "cycle basis" for the cycles in the graphs of electrical networks. In this section, we consider the enumeration of spanning trees in graphs.

The number of spanning trees of a connected labeled graph G will be denoted by $\tau(G)$. If G is disconnected, we take $\tau(G) = 0$. There is a recursive formula for $\tau(G)$. Before we establish this formula, we shall define the concept of *edge contraction* in graphs.

Definition 4.4.1. An edge e of a graph G is said to be *contracted* if it is deleted from G and its ends are identified. The resulting graph is denoted by $G \circ e$.

Edge contraction is illustrated in Fig. 4.7.

If e is not a loop of G, then $n(G \circ e) = n(G) - 1$, $m(G \circ e) = m(G) - 1$, and $\omega(G \circ e) = \omega(G)$. For a loop e, $n(G \circ e) = n(G)$, $m(G \circ e) = m(G) - 1$, and $\omega(G \circ e) = \omega(G)$. Theorem 4.4.2 gives a recursive formula for $\tau(G)$.

Theorem 4.4.2. *If e is not a loop of a connected graph G, $\tau(G) = \tau(G - e) + \tau(G \circ e)$.*

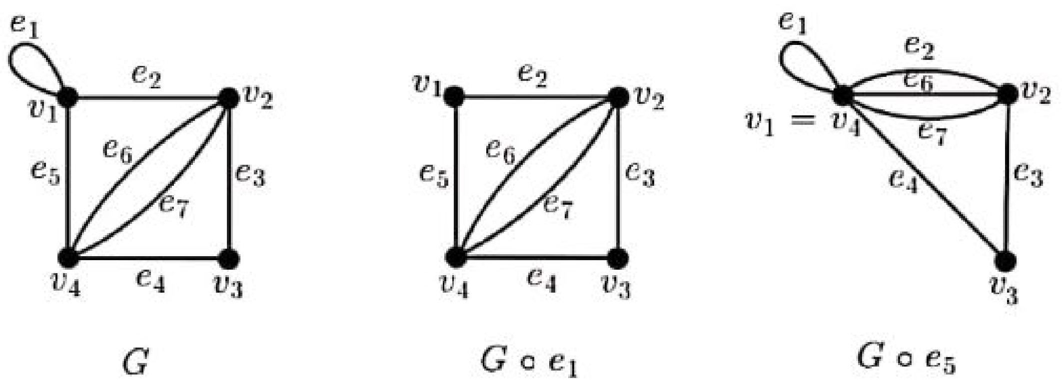
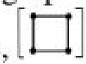


Fig. 4.7 Edge contraction

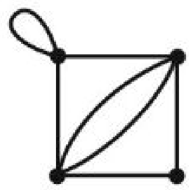
Proof. $\tau(G)$ is the sum of the number of spanning trees of G containing e and the number of spanning trees of G not containing e.

Since $V(G-e) = V(G)$, every spanning tree of $G-e$ is a spanning tree of G not containing e, and conversely, any spanning tree of G for which e is not an edge is also a spanning tree of $G-e$. Hence the number of spanning trees of G not containing e is precisely the number of spanning trees of $G-e$, that is, $\tau(G-e)$. If T is a spanning tree of G containing e, the contraction of e in both T and G results in a spanning tree $T \circ e$ of $G \circ e$.

Conversely, if T_0 is a spanning tree of $G \circ e$, there exists a unique spanning tree T of G containing e such that $T \circ e = T_0$. Thus, the number of spanning trees of G containing e is $\tau(G \circ e)$. Hence $\tau(G) = \tau(G-e) + \tau(G \circ e)$. □

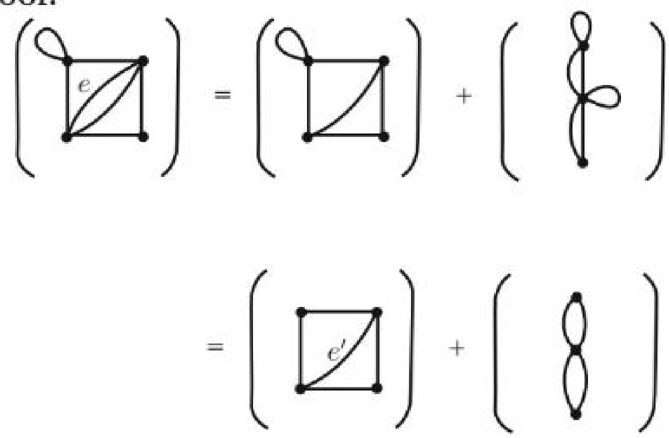
We illustrate below the use of Theorem 4.4.2 in calculating the number of spanning trees. In this illustration, each graph within parentheses stands for the number of its spanning trees. For example,  stands for the number of spanning trees of C_4.

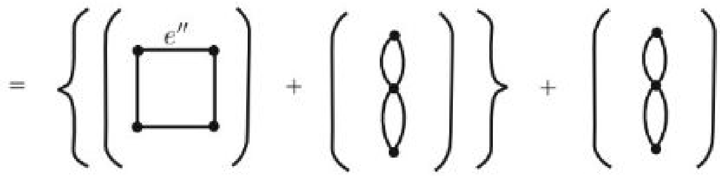
Example 4.4.3. Find $\tau(G)$ for the following graph G:

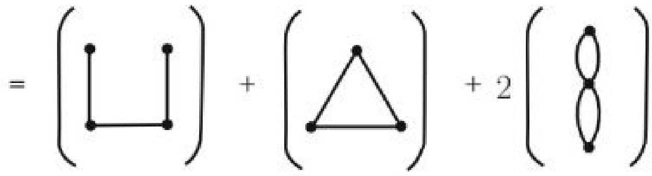


G

Proof.



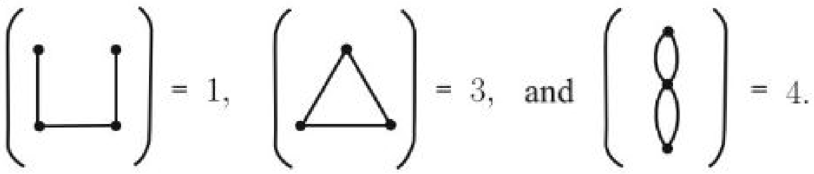




$$= 1 + 3 + 2(4)$$

$$= 12.$$

[By enumeration,



]

Hence $\tau(G) = 12$. □

We have seen in Sect. 3.2 that every connected graph has a spanning tree. When will it have k edge-disjoint spanning trees? An answer to this interesting question was given by both Tutte [181] and Nash-Williams [145] at just about the same time.

Theorem 4.4.4 (Tutte [181]; Nash-Williams [145]). *A simple connected graph G contains k pairwise edge-disjoint spanning trees if and only if for each partition $\mathscr{P}$ of $V(G)$ into p parts, the number $m(\mathscr{P})$ of edges of G joining distinct parts is at least $k(p-1)$, $2 \le p \le |V(G)|$.*

Proof. We prove only the easier part of the theorem (necessity of the condition). Suppose G has k pairwise edge-disjoint spanning trees. If T is one of them and if $\mathscr{P} = \{V_1, \ldots, V_p\}$ is a partition of $V(G)$ into p parts, then G must have at least $|\mathscr{P}| - 1$ edges of T. As this is true for each of the k pairwise edge-disjoint trees of G, the number of edges joining distinct parts of $\mathscr{P}$ is at least $k(p-1)$. □

For the proof of the converse part of the theorem, we refer the reader to the references cited.

As a consequence of Theorem 4.4.4, we obtain immediately at least one family of graphs that possesses the property stated in the theorem.

Corollary 4.4.5. *Every* $2k$*-edge-connected* ($k \geq 1$) *graph contains* k *pairwise edge-disjoint spanning trees.*

Proof. Let G be $2k$-edge connected, and let $\mathscr{P} = \{V_1, \ldots, V_p\}$ be a partition of V into p subsets. By hypothesis on G, there are at least $2k$ edges from each part V_i to $V \setminus V_i = \bigcup_{\substack{j=1 \\ j \neq i}}^{p} V_j$. The total number of such edges is at least kp (as each such edge is counted twice). Hence, $m(\mathscr{P}) \geq kp > k(p-1)$. Theorem 4.4.4 now ensures that there are at least k pairwise edge-disjoint spanning trees in G. □

Setting $k = 2$ in the above corollary, we get the result of Kundu.

Corollary 4.4.6 (Kundu [128]). *Every* 4*-edge-connected graph contains two edge-disjoint spanning trees.*

Corollary 4.4.7. *Every* 3*-edge-connected graph* G *has three spanning trees whose intersection is a spanning totally disconnected subgraph of* G.

Proof. Let G be a 3-edge-connected graph. Duplicate each edge of G by a parallel edge. The resulting graph, say, G', is 6-edge connected, and hence by Corollary 4.4.5, G' has three pairwise edge-disjoint spanning trees, say, T_1', T_2', and T_3'. Hence $E(T_1' \cap T_2' \cap T_3') = \phi$. Let T_i, $1 \leq i \leq 3$, be the tree obtained from T_i' by replacing any parallel edge of G' by its original edge in G. Then, clearly, T_1, T_2, and T_3 are three spanning trees of G with $E(T_1 \cap T_2 \cap T_3) = \phi$ because neither an edge of G nor its parallel edge can belong to all of T_1', T_2', and T_3'. □

4.5 Cayley's Formula

Cayley was the first mathematician to obtain a formula for the number of spanning trees of a labeled complete graph.

Theorem 4.5.1 (Cayley [33]). $\tau(K_n) = n^{n-2}$, *where* K_n *is a labeled complete graph on* n *vertices,* $n \geq 2$.

Before we prove Theorem 4.5.1, we establish two lemmas.

Lemma 4.5.2. *Let* $(d_1, \ldots, d_n)$ *be a sequence of positive integers with* $\sum_{i=1}^{n} d_i = 2(n-1)$. *Then there exists a tree* T *with vertex set* $\{v_1, \ldots, v_n\}$ *and* $d(v_i) = d_i$, $1 \leq i \leq n$.

Proof. It is easy to prove the result by induction on n. □

Lemma 4.5.3. *Let* $\{v_1, \ldots, v_n\}$, $n \geq 2$ *be given and let* $\{d_1, \ldots, d_n\}$ *be a sequence of positive integers such that* $\sum_{i=1}^{n} d_i = 2(n-1)$. *Then the number of trees with* $\{v_1, \ldots, v_n\}$ *as the vertex set in which* v_i *has degree* d_i, $1 \leq i \leq n$, *is* $\frac{(n-2)!}{(d_1-1)! \ldots (d_n-1)!}$.

Proof. We prove the result by induction on n. For $n = 2$, $2(n-1) = 2$, so that $d_1 + d_2 = 2$. Since $d_1 \geq 1$ and $d_2 \geq 1$, $d_1 = d_2 = 1$. Hence K_2 is the only tree in which v_i has degree d_i, $i = 1, 2$. So the result is true for $n = 2$. Now assume that the result is true for all positive integers up to $n-1$, $n \geq 3$. Let $\{d_1, \ldots, d_n\}$ be a sequence of positive integers such that $\sum_{i=1}^{n} d_i = 2(n-1)$, and let $\{v_1, \ldots, v_n\}$ be any set. If $d_i \geq 2$ for every i, $1 \leq i \leq n$, then $\sum_{i=1}^{n} d_i \geq 2n$. Hence, there exists an i, $1 \leq i \leq n$, for which $d_i = 1$. For the sake of definiteness, assume that $d_n = 1$. By Lemma 4.5.2, there exists a tree T with $V(T) = \{v_1, \ldots, v_n\}$ and degree of $v_i = d_i$. Let v_j be the unique vertex of T adjacent to v_n. Delete v_n from T. The resulting graph is a tree T' with $\{v_1, \ldots, v_{n-1}\}$ as its vertex set and $(d_1, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_{n-1})$ as its degree sequence.

In the opposite direction, given a tree T' with $\{v_1, \ldots, v_{n-1}\}$ as its vertex set and $(d_1, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_{n-1})$ as its degree sequence, a tree T with vertex set $\{v_1, \ldots, v_n\}$ and degree sequence $(d_1, \ldots, d_n)$, $d_n = 1$, can be obtained by introducing a new vertex v_n and taking $T = T' + v_j v_n$. Hence the number of trees with vertex set $\{v_1, \ldots, v_n\}$ and degree sequence $(d_1, \ldots, d_n)$ with $d_n =$ degree of $v_n = 1$ and v_n adjacent to v_j is the same as the number of trees with vertex set $\{v_1, \ldots, v_{n-1}\}$ and degree sequence $(d_1, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_{n-1})$. By the induction hypothesis, the latter number is equal to

$$\frac{(n-3)!}{(d_1-1)! \ldots (d_{j-1}-1)!\,(d_j-2)!\,(d_{j+1}-1)! \ldots (d_{n-1}-1)!}$$
$$= \frac{(n-3)!(d_j-1)}{(d_1-1)! \ldots (d_{j-1}-1)!\,(d_j-1)!\,(d_{j+1}-1)! \ldots (d_{n-1}-1)!}.$$

Summing over j, the number of trees with $\{v_1, \ldots, v_n\}$ as its vertex set and $(d_1, \ldots, d_n)$ as its degree sequence is

$$\sum_{j=1}^{n-1} \frac{(n-3)!\,(d_j-1)}{(d_1-1)! \ldots (d_{n-1}-1)!}$$
$$= \frac{(n-3)!}{(d_1-1)! \ldots (d_{n-1}-1)!} \sum_{j=1}^{n-1} (d_j - 1)$$
$$= \frac{(n-3)!}{(d_1-1)! \ldots (d_{n-1}-1)!} \left[\left(\sum_{j=1}^{n-1} d_j \right) - (n-1) \right]$$
$$= \frac{(n-3)!}{(d_1-1)! \ldots (d_{n-1}-1)!} [(2n-3) - (n-1)]$$
$$= \frac{(n-3)!}{(d_1-1)! \ldots (d_{n-1}-1)!} (n-2)$$

$$= \frac{(n-2)!}{(d_1-1)! \ \ldots \ (d_{n-1}-1)!}$$
$$= \frac{(n-2)!}{(d_1-1)! \ \ldots \ (d_n-1)!} \text{ (recall that } d_n = 1).$$

This completes the proof of Lemma 4.5.2. □

Proof of theorem 4.5.1. The total number of trees T_n with vertex set $\{v_1, \ldots, v_n\}$ is obtained by summing over all possible sequences $(d_1, \ldots, d_n)$ with $\sum_{i=1}^{n} d_i = 2n-2$. Hence,

$$\tau(K_n) = \sum_{d_i \geq 1} \frac{(n-2)!}{(d_1-1)! \ \ldots \ (d_n-1)!} \text{ with } \sum_{i=1}^{n} d_i = 2n-2$$
$$= \sum_{k_i \geq 0} \frac{(n-2)!}{k_1! \ \ldots \ k_n!} \text{ with } \sum_{i=1}^{n} k_i = n-2, \text{ where } k_i = d_i - 1, 1 \leq i \leq n.$$

Putting $x_1 = x_2 = \cdots = x_n = 1$ and $m = n-2$ in the multinomial expansion $(x_1 + x_2 + \cdots + x_n)^m = \sum_{k_i \geq 0} \frac{x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}}{k_1! k_2! \ldots k_n!} m!$ with $(k_1 + k_2 + \cdots + k_n) = m$, we get $n^{n-2} = \sum_{k_i \geq 0} \frac{(n-2)!}{k_1! k_2! \ldots k_n!}$ with $(k_1 + k_2 + \cdots + k_n) = n-2$. Thus, $\tau(K_n) = n^{n-2}$. □

UNIT -3

Independent Sets and Matchings

5.1 Introduction

Vertex-independent sets and vertex coverings as also edge-independent sets and edge coverings of graphs occur very naturally in many practical situations and hence have several potential applications. In this chapter, we study the properties of these sets. In addition, we discuss matchings in graphs and, in particular, in bipartite graphs. Matchings in bipartite graphs have varied applications in operations research. We also present two celebrated theorems of graph theory, namely, Tulte's 1-factor theorem and Hall's matching theorem. All graphs considered in this chapter are loopless.

5.2 Vertex-Independent Sets and Vertex Coverings

Definition 5.2.1. A subset *S* of the vertex set K of a graph *G* is called *independent* if no two vertices of *S* are adjacent in *G. S* C V is a *maximum independent set* of G if G has no independent set *S'* with IS^7! > |S|. A *maximal independent set* of *G* is an independent set that is not a proper subset of another independent set of *G.*

For example, in the graph of Fig. 5.1, {w. v, *w}* is a maximum independent set and (x, y} is a maximal independent set that is not maximum.

Definition 5.2.2. A subset AT of V is called a *covering* of *G* if every edge of *G* is incident with at least one vertex of *K.* A covering *K* is *minimum* if there is no covering *K'* of *G* such that \K^f\ < |; it is *minimal* if there is no covering *K*\ of *G* such that *K*\ is a proper subset of *K.*

In the graph *W5* of Fig. 5.2, {vi. V2, V3, V4, V5} is a covering of *W5* and (V|. V3. V4. is a minimal covering. Also, the set {x, *y]* is a minimum covering of the graph of Fig. 5.1.

R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory,* Universitext, DOI 10.1007/978-1 -4614-4529-6_3, © Springer Science+Business Media New York 2012

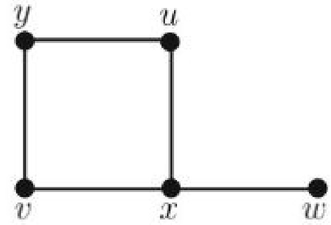


Fig. 5.1 Graph with maximum independent set $\{u, v, w\}$ and maximal independent set $\{x, y\}$

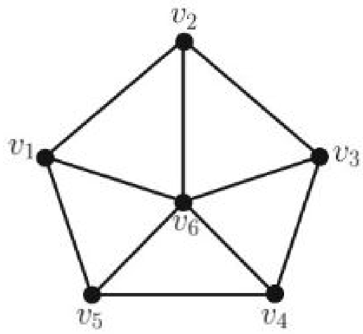


Fig. 5.2 Wheel W_5

The concepts of covering and independent sets of a graph arise very naturally in practical problems. Suppose we want to store a set of chemicals in different rooms. Naturally, we would like to store incompatible chemicals, that is, chemicals that are likely to react violently when brought together, in distinct rooms. Let G be a graph whose vertex set represents the set of chemicals and let two vertices be made adjacent in G if and only if the corresponding chemicals are incompatible. Then any set of vertices representing compatible chemicals forms an independent set of G.

Now consider the graph G whose vertices represent the various locations in a factory and whose edges represent the pathways between pairs of such locations. A light source placed at a location supplies light to all the pathways incident to that location. A set of light sources that supplies light to all the pathways in the factory forms a covering of G.

Theorem 5.2.3. *A subset S of V is independent if and only if $V \setminus S$ is a covering of G.*

Proof. S is independent if and only if no two vertices in S are adjacent in G. Hence, every edge of G must be incident to a vertex of $V \setminus S$. This is the case if and only if $V \setminus S$ is a covering of G. □

Definition 5.2.4. The number of vertices in a maximum independent set of G is called the *independence number* (or the *stability number*) of G and is denoted by $\alpha(G)$. The number of vertices in a minimum covering of G is the *covering number* of G and is denoted by $\beta(G)$. We denote these numbers simply by α and β when there is no confusion.

Corollary 5.2.5. *For any graph G, $\alpha + \beta = n$.*

Proof. Let S be a maximum independent set of G. By Theorem 5.2.3, $V \backslash S$ is a covering of G and therefore $|V \backslash S| = n - \alpha \geq \beta$. Similarly, let K be a minimum covering of G. Then $V \backslash K$ is independent and so $|V \backslash K| = n - \beta \leq \alpha$. These two inequalities together imply that $n = \alpha + \beta$. □

5.3 Edge-Independent Sets

Definitions 5.3.1. 1. A subset M of the edge set E of a loopless graph G is called *independent* if no two edges of M are adjacent in G.
2. A *matching* in G is a set of independent edges.
3. An *edge covering* of G is a subset L of E such that every vertex of G is incident to some edge of L. Hence, an edge covering of G exists if and only if $\delta > 0$.
4. A matching M of G is *maximum* if G has no matching M' with $|M'| > |M|$. M is *maximal* if G has no matching M' strictly containing M. $\alpha'(G)$ is the cardinality of a maximum matching and $\beta'(G)$ is the size of a minimum edge covering of G.
5. A set S of vertices of G is said to be *saturated* by a matching M of G or *M-saturated* if every vertex of S is incident to some edge of M. A vertex v of G is *M-saturated* if $\{v\}$ is M-saturated. v is *M-unsaturated* if it is not M-saturated.

For example, in the wheel W_5 (Fig. 5.2), $M = \{v_1v_2, v_4v_6\}$ is a maximal matching; $\{v_1v_5, v_2v_3, v_4v_6\}$ is a maximum matching and a minimum edge covering; the vertices v_1, v_2, v_4, and v_6 are M-saturated, whereas v_3 and v_5 are M-unsaturated.

Remark 5.3.2. The edge analog of Theorem 5.2.3 is not true, however. For instance, in the graph G of Fig. 5.3, the set $E' = \{e_3, e_4\}$ is independent, but $E \backslash E' = \{e_1, e_2, e_5\}$ is not an edge covering of G. Also, $E'' = \{e_1, e_3, e_4\}$ is an edge covering of G, but $E \backslash E''$ is not independent in G. Again, E' is a matching in G that saturates v_2, v_3, v_4 and v_5 but does not saturate v_1.

Theorem 5.3.3. *For any graph G for which $\delta > 0$, $\alpha' + \beta' = n$.*

Proof. Let M be a maximum matching in G so that $|M| = \alpha'$. Let U be the set of M-unsaturated vertices in G. Since M is maximum, U is an independent set of vertices with $|U| = n - 2\alpha'$. Since $\delta > 0$, we can pick one edge for each vertex in

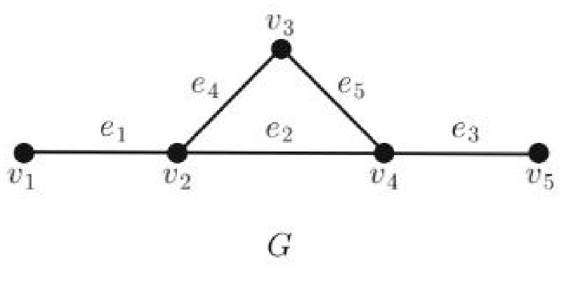


Fig. 5.3 Graph illustrating edge relationships

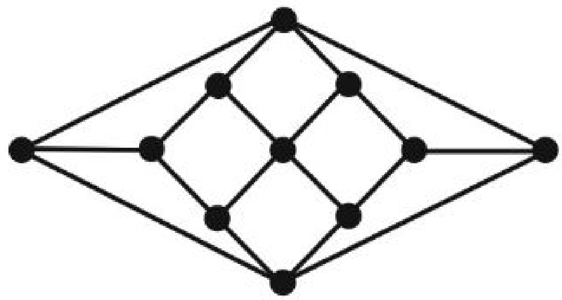


Fig. 5.4 Herschel graph

U incident with it. Let F be the set of edges thus chosen. Then $M \cup F$ is an edge covering of G. Hence, $|M \cup F| = |M| + |F| = \alpha' + n - 2\alpha' \geq \beta'$, and therefore

$$n \geq \alpha' + \beta'. \tag{5.1}$$

Now let L be a minimum edge covering of G so that $|L| = \beta'$. Let $H = G[L]$ be the edge subgraph of G defined by L, and let M_H be a maximum matching in H. Denote the set of M_H-unsaturated vertices in H by U. As L is an edge covering of G, H is a spanning subgraph of G. Consequently, $|L| - |M_H| = |L \backslash M_H| \geq |U| = n - 2|M_H|$ and so $|L| + |M_H| \geq n$. But since M_H is a matching in G, $|M_H| \leq \alpha'$. Thus,

$$n \leq |L| + |M_H| \leq \beta' + \alpha'. \tag{5.2}$$

Inequalities (5.1) and (5.2) imply that $\alpha' + \beta' = n$. □

Exercise 3.1. Determine the values of the parameters α, α', β, and β' for

1. K_n,
2. The Petersen graph P,
3. The Herschel graph (see Fig. 5.4).

Exercise 3.2. For any graph G with $\delta > 0$, prove that $\alpha \leq \beta'$ and $\alpha' \geq \beta$.

Exercise 3.3. Show that for a bipartite graph G, $\alpha\,\beta \geq m$ and that equality holds if and only if G is complete.

5.4 Matchings and Factors

Definition 5.4.1. A *matching* of a graph G is (as given in Definition 5.3.1) a set of independent edges of G. If $e = uv$ is an edge of a matching M of G, the end vertices u and v of e are said to be *matched* by M.

If M_1 and M_2 are matchings of G, the edge subgraph defined by $M_1 \Delta M_2$, the symmetric difference of M_1 and M_2, is a subgraph H of G whose components are paths or even cycles of G in which the edges alternate between M_1 and M_2.

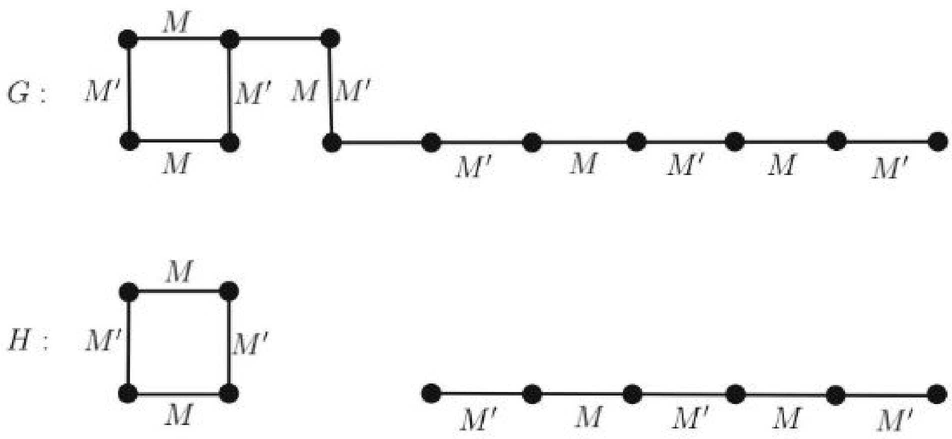


Fig. 5.5 Graphs for proof of Theorem 5.4.4

Definition 5.4.2. An *M-augmenting path* in G is a path in which the edges alternate between $E \backslash M$ and M and its end vertices are M-unsaturated. An *M-alternating path* in G is a path whose edges alternate between $E \backslash M$ and M.

Example 5.4.3. In the graph G of Fig. 5.2, $M_1 = \{v_1v_2, v_3v_4, v_5v_6\}$, $M_2 = \{v_1v_2, v_3v_6, v_4v_5\}$, and $M_3 = \{v_3v_4, v_5v_6\}$ are matchings of G. Moreover, $G[M_1 \Delta M_2]$ is the even cycle $(v_3v_4v_5v_6v_3)$. The path $v_2v_3v_4v_6v_5v_1$ is an M_3-augmenting path in G.

Maximum matchings have been characterized by Berge [19].

Theorem 5.4.4. *A matching M of a graph G is maximum if and only if G has no M-augmenting path.*

Proof. Assume first that M is maximum. If G has an M-augmenting path P : $v_0v_1v_2 \ldots v_{2t+1}$ in which the edges alternate between $E \backslash M$ and M, then P has one edge of $E \backslash M$ more than that of M. Define

$$M' = (M \cup \{v_0v_1, v_2v_3, \ldots, v_{2t}v_{2t+1}\}) \backslash \{v_1v_2, v_3v_4, \ldots, v_{2t-1}v_{2t}\}.$$

Clearly, M' is a matching of G with $|M'| = |M| + 1$, which is a contradiction since M is a maximum matching of G.

Conversely, assume that G has no M-augmenting path. Then M must be maximum. If not, there exists a matching M' of G with $|M'| > |M|$. Let H be the edge subgraph $G[M \Delta M']$ defined by the symmetric difference of M and M'. Then the components of H are paths or even cycles in which the edges alternate between M and M'. Since $|M'| > |M|$, at least one of the components of H must be a path starting and ending with edges of M'. But then such a path is an M-augmenting path of G, contradicting the assumption (see Fig. 5.5). □

Definition 5.4.5. A *factor* of a graph G is a spanning subgraph of G. A *k-factor* of G is a factor of G that is k-regular. Thus, a *1-factor* of G is a matching that saturates

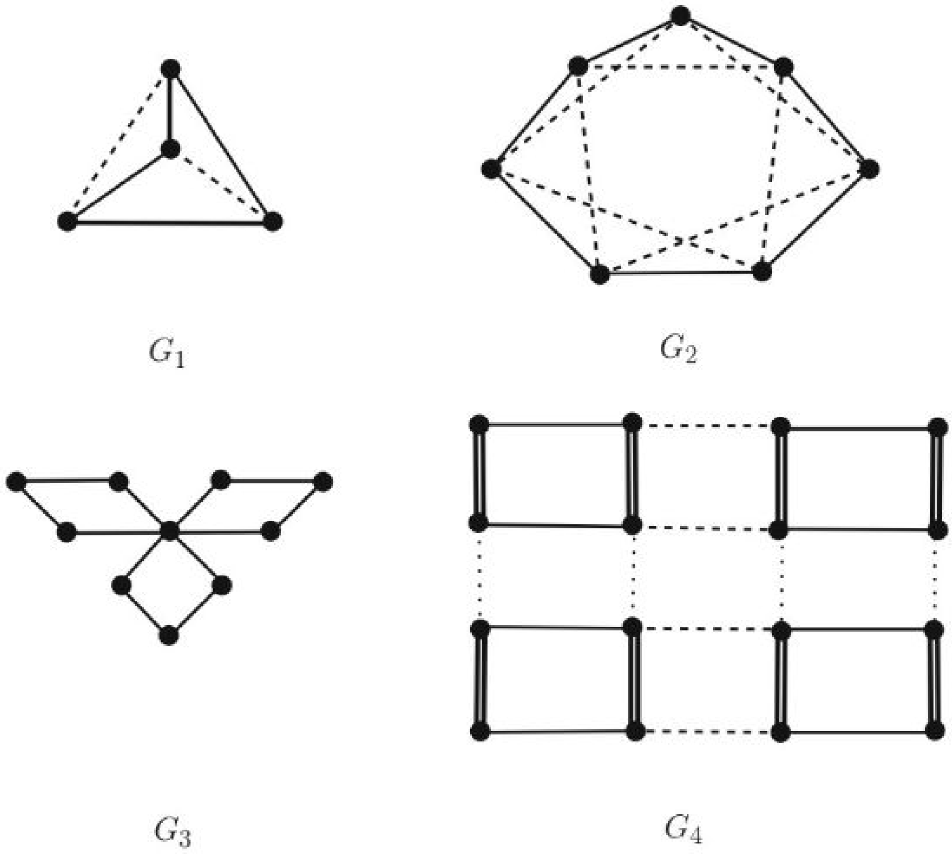


Fig. 5.6 Graphs illustrating factorability

all the vertices of G. For this reason, a 1-factor of G is called a *perfect matching* of G. A 2-factor of G is a factor of G that is a disjoint union of cycles of G. A graph G is *k-factorable* if G is an edge-disjoint union of k-factors of G.

Example 5.4.6. In Fig. 5.6, G_1 is 1-factorable and G_2 is 2-factorable, whereas G_3 has neither a 1-factor nor a 2-factor. The dotted, solid, and ordinary lines of G_1 give the three distinct 1-factors, and the dotted and ordinary lines of G_2 give its two distinct 2-factors.

Exercise 4.1. Give an example of a cubic graph having no 1-factor.

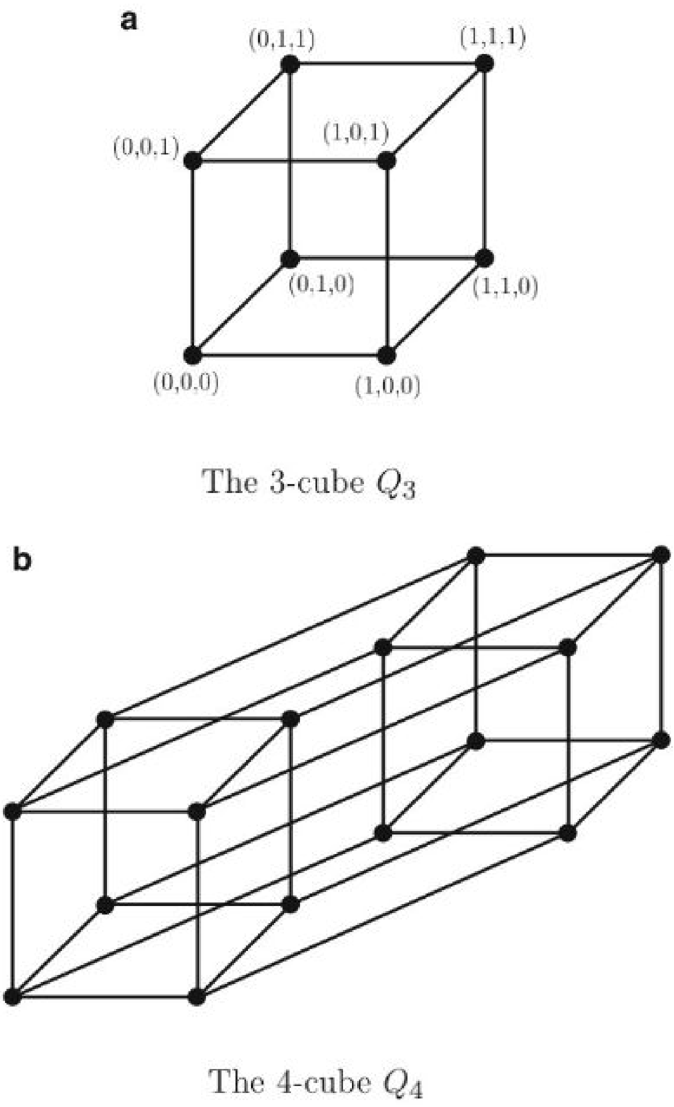
Exercise 4.2. Show that $K_{n,n}$ and K_{2n} are 1-factorable.

Exercise 4.3. Show that the number of 1-factors of

(i) $K_{n,n}$ is $n!$,
(ii) K_{2n} is $\frac{(2n)!}{2^n\, n!}$.

Exercise 4.4. The *n-cube* Q_n is the graph whose vertices are binary n-tuples. Two vertices of Q_n are adjacent if and only if they differ in exactly one place. Show that Q_n ($n \geq 2$) has a perfect matching. (The 3-cube Q_3 and the 4-cube Q_4 are displayed in Fig. 5.7.) It is easy to see that $Q_n \simeq K_2 \square K_2 \square \ldots \square K_2$ (n times).

Fig. 5.7 (**a**) 3-cube Q_3 and (**b**) 4-cube Q_4



Exercise 4.5. Show that the Petersen graph P is not 1-factorable. (Hint: Look at the possible types of 1-factors of P.)

Exercise 4.6. Show that every tree has at most one perfect matching.

Exercise 4.7*. Show that if a 2-edge-connected graph has a 1-factor, then it has at least two distinct 1-factors.

Exercise 4.8. Show that the graph G_4 of Fig. 5.6 is not 1-factorable.

An Application to Physics 5.4.7. In crystal physics, a crystal is represented by a three-dimensional lattice in which each face corresponds to a two-dimensional lattice. Each vertex of the lattice represents an atom of the crystal, and an edge between two vertices represents the bond between the two corresponding atoms.

In crystallography, one is interested in obtaining an analytical expression for certain surface properties of crystals consisting of diatomic molecules (also called dimers). For this, one must find the number of ways in which all the atoms of the crystal can be paired off as molecules consisting of two atoms each. The problem is clearly equivalent to that of finding the number of perfect matchings of the corresponding two-dimensional lattice.

Two different dimer coverings (perfect matchings) of the lattice defined by the graph G_4 are exhibited in Fig. 5.6—one in solid lines and the other in parallel lines.

Chapter 6
Eulerian and Hamiltonian Graphs

6.1 Introduction

The study of Eulerian graphs was initiated in the 18th century and that of Hamiltonian graphs in the 19th century. These graphs possess rich structures; hence, their study is a very fertile field of research for graph theorists. In this chapter, we present several structure theorems for these graphs.

6.2 Eulerian Graphs

Definition 6.2.1. An *Euler trail* in a graph G is a spanning trail in G that contains all the edges of G. An *Euler tour* of G is a closed Euler trail of G. G is called *Eulerian* (Fig. 6.1a) if G has an *Euler tour*. It was Euler who first considered these graphs, and hence their name.

It is clear that an Euler tour of G, if it exists, can be described from any vertex of G. Clearly, every Eulerian graph is connected.

Euler showed in 1736 that the celebrated *Königsberg bridge problem* has no solution. The city of Königsberg (now called Kaliningrad) has seven bridges linking two islands A and B and the banks C and D of the Pregel (now called Pregalya) River, as shown in Fig. 6.2.

The problem was to start from any one of the four land areas, take a stroll across the seven bridges, and get back to the starting point without crossing any bridge a second time. This problem can be converted into one concerning the graph obtained by representing each land area by a vertex and each bridge by an edge. The resulting graph H is the graph of Fig. 6.1b. The Königsberg bridge problem will have a solution provided that this graph H is Eulerian. But this is not the case since it has vertices of odd degrees (see Theorem 6.2.2).

Eulerian graphs admit, among others, the following two elegant characterizations, Theorems 6.2.2 and 6.2.3*.

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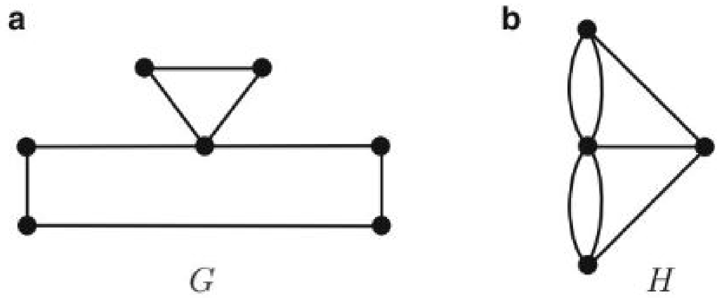


Fig. 6.1 (**a**) Eulerian graph G; (**b**) non-Eulerian graph H

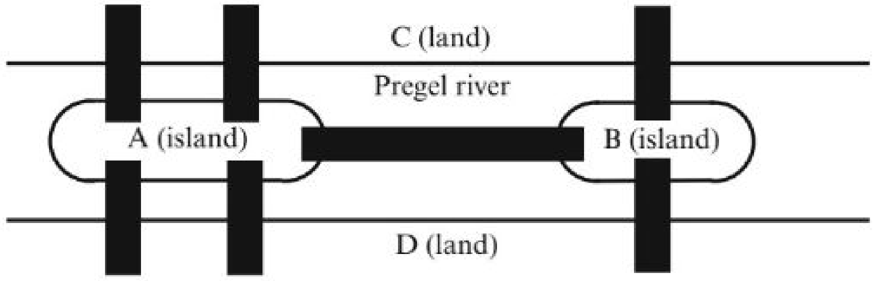


Fig. 6.2 Königsberg bridge problem

Theorem 6.2.2. *For a nontrivial connected graph G, the following statements are equivalent:*

(i) G is Eulerian.
(ii) The degree of each vertex of G is an even positive integer.
(iii) G is an edge-disjoint union of cycles.

Proof. (i) $\Rightarrow$ (ii): Let T be an Euler tour of G described from some vertex $v_0 \in V(G)$. If $v \in V(G)$, and $v \neq v_0$, then every time T enters v, it must move out of v to get back to v_0. Hence two edges incident with v are used during a visit to v, and therefore, $d(v)$ is even. At v_0, every time T moves out of v_0, it must get back to v_0. Consequently, $d(v_0)$ is also even. Thus, the degree of each vertex of G is even.

(ii) $\Rightarrow$ (iii): As $\delta(G) \geq 2$, G contains a cycle C_1 (Exercise 11.11 of Chap. 1). In $G \backslash E(C_1)$, remove the isolated vertices if there are any. Let the resulting subgraph of G be G_1. If G_1 is nonempty, each vertex of G_1 is again of even positive degree. Hence $\delta(G_1) \geq 2$, and so G_1 contains a cycle C_2. It follows that after a finite number, say r, of steps, $G \backslash E(C_1 \cup \ldots \cup C_r)$ is totally disconnected. Then G is the edge-disjoint union of the cycles $C_1, C_2, \ldots, C_r$.

(iii) $\Rightarrow$ (i): Assume that G is an edge-disjoint union of cycles. Since any cycle is Eulerian, G certainly contains an Eulerian subgraph. Let G_1 be a longest closed trail in G. Then G_1 must be G. If not, let $G_2 = G \backslash E(G_1)$. Since G is an edge-disjoint union of cycles, every vertex of G is of even degree ≥ 2. Further, since G_1 is Eulerian, each vertex of G_1 is of even degree ≥ 2. Hence each vertex of G_2 is of even degree. Since G_2 is not totally disconnected and G is connected, G_2 contains a cycle C having a vertex v in common with G_1. Describe the Euler tour of G_1

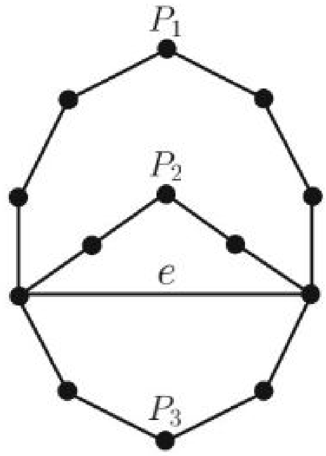


Fig. 6.3 Eulerian graph with edge e belonging to three cycles

starting and ending at v and follow it by C. Then $G_1 \cup C$ is a closed trail in G longer than G_1. This contradicts the choice of G_1, and so G_1 must be G. Hence G is Eulerian. □

If $G_1, \ldots, G_r$ are subgraphs of a graph G that are pairwise edge-disjoint and their union is G, then this fact is denoted by writing $G = G_1 \oplus \ldots \oplus G_r$. In the above equation, if $G_i = C_i$, a cycle of G for each i, then $G = C_1 \oplus \ldots C_r$. The set of cycles $S = \{C_1, \ldots, C_r\}$ is then called a *cycle decomposition* of G. Thus, Theorem 6.2.2 implies that *a connected graph is Eulerian if and only if it admits a cycle decomposition.*

There is yet another characterization of Eulerian graphs due to McKee [138] and Toida [175]. Our proof is based on Fleischner [63, 64].

Theorem 6.2.3*. *A graph G is Eulerian if and only if each edge e of G belongs to an odd number of cycles of G.*

For instance, in Fig. 6.3, e belongs to the three cycles $P_1 \cup e$, $P_2 \cup e$, and $P_3 \cup e$.

Proof. Denote by γ_e the number of cycles of G containing e. Assume that γ_e is odd for each edge e of G. Since a loop at any vertex v of G is in exactly one cycle of G and contributes 2 to the degree of v in G, we may suppose that G is loopless.

Let $S = \{C_1, \ldots, C_p\}$ be the set of cycles of G. Replace each edge e of G by γ_e parallel edges and replace e in each of the γ_e cycles containing e by one of these parallel edges, making sure that none of the parallel edges is repeated. Let the resulting graph be G_0 and let the new set of cycles be $S_0 = \{C_1^0, \ldots, C_p^0\}$. Clearly, S_0 is a cycle decomposition of G_0. Hence, by Theorem 6.2.2, G_0 is Eulerian. But then $d_{G_0}(v) \equiv 0 \pmod 2$ for each $v \in V(G_0) = V(G)$. Moreover, $d_G(v) = d_{G_0}(v) - \sum_e (\gamma_e - 1)$, where e is incident at v in G and hence $d_G(v) \equiv 0 \pmod 2$, γ_e being odd for each $e \in E(G)$. Thus, G is Eulerian.

Conversely, assume that G is Eulerian. We proceed by induction on $n = |V(G)|$. If $n = 1$, each edge is a loop and hence belongs to exactly one cycle of G.

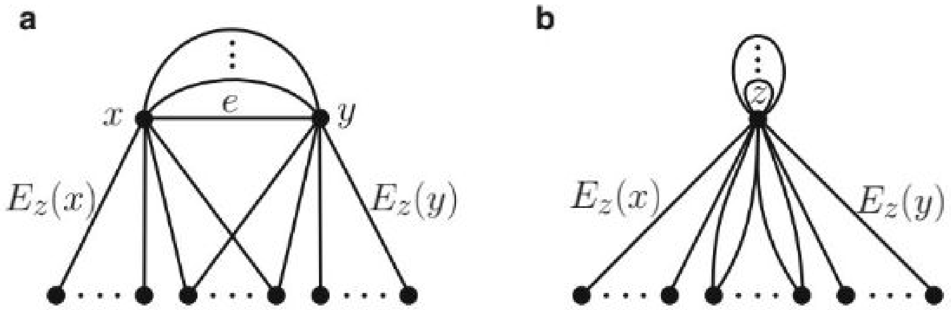


Fig. 6.4 Graph for proof of Theorem 6.2.3

Assume the result for graphs with fewer than n ($\geq$ 2) vertices. Let G be a graph with n vertices. Let $e = xy$ be an edge of G and let $\lambda(e)$ be the multiplicity of e in G.

The graph $G \circ e$ obtained from G by contracting the edge e (cf. Sect. 4.4 of Chap. 4) is also Eulerian. Denote by z the new vertex of $G \circ e$ obtained by identifying the vertices x and y of G. The set of edges incident with z in $G \circ e$ is partitioned into three subsets (see Fig. 6.4):

1. $E_z(x)$ = set of edges arising out of edges of G incident with x but not with y
2. $E_z(y)$ = set of edges arising out of edges of G incident with y but not with x
3. $E_z(xy)$ = set of $\lambda(e) - 1$ loops of $G \circ e$ corresponding to the edges parallel to e in G

Let $k = |E_z(x)|$. Since G is Eulerian,

$$k + \lambda(e) = d_G(x) \equiv 0 (\text{mod}\, 2). \tag{6.1}$$

Let Γ_f and $\Gamma(e_i, e_j)$ denote, respectively, the number of cycles in $G \circ e$ containing the edge f and the pair (e_i, e_j) of edges. Since $|V(G \circ e)| = n - 1$, and since $G \circ e$ is Eulerian by the induction assumption, Γ_f is odd for each edge f of $G \circ e$. Now, any cycle of G containing e either consists of e and an edge parallel to e in G (and there are $\lambda(e) - 1$ of them) or contains e, an edge e_i of $E_z(x)$, and an edge e'_j of $E_z(y)$. These correspond in $G \circ e$, respectively, to a loop at z and to a cycle containing the edges of $G \circ e$ that correspond to the edges e_i and e'_j of G. By abuse of notation, we denote these corresponding edges of $G \circ e$ also by e_i and e'_j, respectively. Moreover, any cycle of $G \circ e$ containing an edge e_i of $E_z(x)$ will also contain either an edge e_j of $E_z(x)$ or an edge e'_j of $E_z(y)$, but not both. A cycle of the former type is counted once in Γ_{e_i} and once in Γ_{e_j}, and these will not give rise to cycles in G containing e. Thus,

$$\gamma_e = (\lambda(e) - 1) + \sum_{e_i \in E_z(x)} \Gamma_{e_i} - \sum_{\substack{\{i,j\} \\ i \neq j \\ e_i,\, e_j \in E_z(x)}} \Gamma(e_i, e_j).$$

Now, by the induction hypothesis, $\Gamma_{e_i} \equiv 1 \pmod 2$ for each e_i, and $\Gamma(e_i, e_j) = \Gamma(e_j, e_i)$ in the last sum on the right, and hence this latter sum is even. Thus, $\gamma_e \equiv (\lambda(e) - 1) + k \pmod 2 \equiv 1 \pmod 2$ by relation (6.1). □

A consequence of Theorem 6.2.3 is a result of Bondy and Halberstam [26], which gives yet another characterization of Eulerian graphs.

Corollary 6.2.4*. *A graph is Eulerian if and only if it has an odd number of cycle decompositions.*

Proof. In one direction, the proof is trivial. If G has an odd number of cycle decompositions, then it has at least one, and hence G is Eulerian.

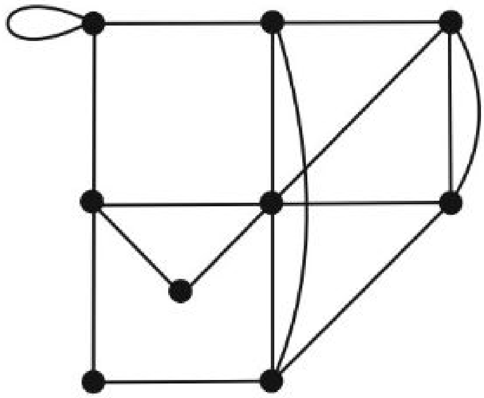
Conversely, assume that G is Eulerian. Let $e \in E(G)$ and let $C_1, \ldots, C_r$ be the cycles containing e. By Theorem 6.2.3, r is odd. We proceed by induction on $m = |E(G)|$ with G Eulerian.

If G is just a cycle, then the result is true. Assume then that G is not a cycle. This means that for each i, $1 \leq i \leq r$, by the induction assumption, $G_i = G - E(C_i)$ has an odd number, say s_i, of cycle decompositions. (If G_i is disconnected, apply the induction assumption to each of the nontrivial components of G_i.) The union of each of these cycle decompositions of G_i and C_i yields a cycle decomposition of G. Hence the number of cycle decompositions of G containing C_i is s_i, $1 \leq i \leq r$. Let $s(G)$ denote the number of cycle decompositions of G. Then

$$\begin{aligned} s(G) &= \sum_{i=1}^{r} s_i \equiv r \pmod 2 \text{ (since } s_i \equiv 1 \pmod 2\text{)} \\ &\equiv 1 \pmod 2. \end{aligned}$$

□

Exercise 2.1. Find an Euler tour in the graph G below.



G

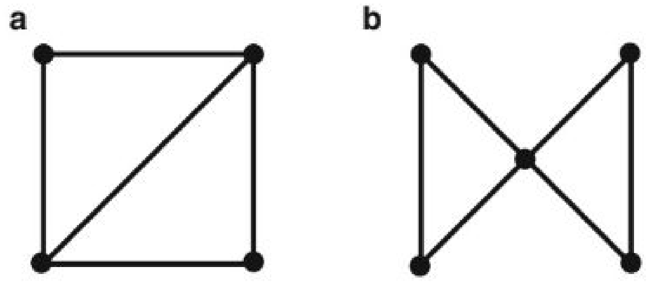


Fig. 6.5 (**a**) Hamiltonian graph; (**b**) non-Hamiltonian but traceable graph

Exercise 2.2. Does there exist an Eulerian graph with

(i) An even number of vertices and an odd number of edges?
(ii) An odd number of vertices and an even number of edges? Draw such a graph if it exists.

Exercise 2.3. Prove that a connected graph is Eulerian if and only if each of its blocks is Eulerian.

Exercise 2.4. If G is a connected graph with $2k(k > 0)$ vertices of odd degree, show that $E(G)$ can be partitioned into k open (i.e., not closed) trails.

Exercise 2.5. Prove that a connected graph is Eulerian if and only if each of its edge cuts has an even number of edges.

6.3 Hamiltonian Graphs

Definition 6.3.1. A graph is called *Hamiltonian* if it has a spanning cycle (see Fig. 6.5a). These graphs were first studied by Sir William Hamilton, a mathematician. A spanning cycle of a graph G, when it exists, is often called a *Hamilton cycle* (or *Hamiltonian cycle*) of G.

Definition 6.3.2. A graph G is called *traceable* if it has a spanning path of G (see Fig. 6.5b). A spanning path of G is also called a *Hamilton path* (or *Hamiltonian path*) of G.

6.3.1 Hamilton's "Around the World" Game

Hamilton introduced these graphs in 1859 through a game that used a solid dodecahedron (Fig. 6.6). A dodecahedron has 20 vertices and 12 pentagonal faces. At each vertex of the solid, a peg was attached. The vertices were marked Amsterdam, Ann Arbor, Berlin, Budapest, Dublin, Edinburgh, Jerusalem, London, Melbourne, Moscow, Novosibirsk, New York, Paris, Peking, Prague, Rio di Janeiro,

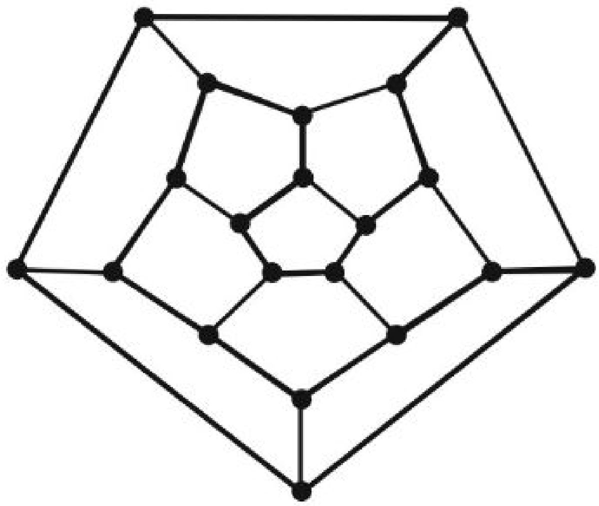


Fig. 6.6 Solid dodecahedron for Hamilton's "Around the World" problem

Rome, San Francisco, Tokyo, and Warsaw. Further, a string was also provided. The object of the game was to start from any one of the vertices and keep on attaching the string to the pegs as we move from one vertex to another along a particular edge with the condition that we have to get back to the starting city without visiting any intermediate city more than once. In other words, the problem asks one to find a Hamilton cycle in the graph of the dodecahedron (see Fig. 6.6). Hamilton solved this problem as follows: When a traveler arrives at a city, he has the choice of taking the edge to his right or left. Denote the choice of taking the edge to the right by R and that of taking the edge to the left by L. Let 1 denote the operation of staying where he is.

Define the product O_1O_2 of two operations O_1 and O_2 as O_1 followed by O_2. For example, LR denotes going left first and then going right. Two sequences of operations are *equal* if, after starting at a vertex, the two sequences lead to the same vertex. The product defined above is associative but not commutative. Further, it is clear (see Fig. 6.6) that

$$\begin{aligned} R^5 &= L^5 = 1 \\ RL^2R &= LRL, \\ LR^2L &= RLR, \\ RL^3R &= L^2, \text{ and} \\ LR^3L &= R^2. \end{aligned}$$

These relations give

$$\begin{aligned} 1 &= R^5 = R^2R^3 = (LR^3L)R^3 - (LR^3)(LR^3) = (LR^3)^2 = (LR^2R)^2 \\ &= (L(LR^3L)R)^2 = (L^2R^3LR)^2 = (L^2((LR^3L)R)LR)^2 = (L^3R^3LRLR)^2 \\ &= LLLRRRLRLRLLLRRRLRLR. \end{aligned} \tag{6.2}$$

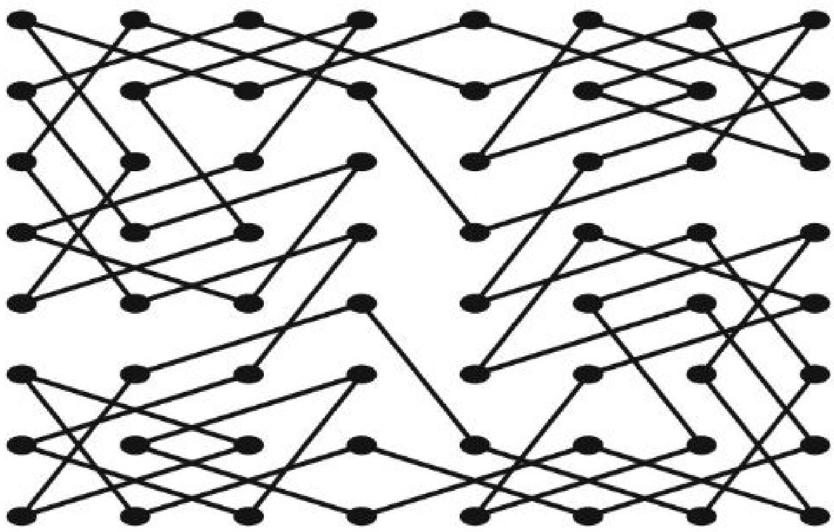


Fig. 6.7 A knight's tour in a chessboard

The last sequence of operations contains 20 operations and contains no partial sequence equal to 1. Hence, this sequence must represent a Hamilton cycle. Thus, starting from any vertex and following the sequence of operations (6.2), we do indeed get a Hamilton cycle of the graph of Fig. 6.6.

Knight's Tour in a Chessboard 6.3.3. The knight's tour problem is the problem of determining a closed tour through all 64 squares of an 8×8 chessboard by a knight with the condition that the knight does not visit any intermediate square more than once. This is equivalent to finding a Hamilton cycle in the corresponding graph of 64 $(= 8 \times 8)$ vertices in which two vertices are adjacent if and only if the knight can move from one vertex to the other following the rules of the chess game. Figure 6.7 displays a knight's tour.

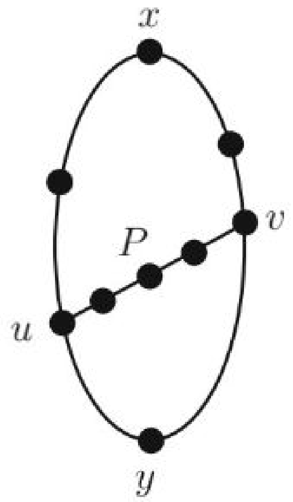
Even though Eulerian graphs admit an elegant characterization, no decent characterization of Hamiltonian graphs is known as yet. In fact, it is one of the most difficult unsolved problems in graph theory. (Actually, it is an NP-complete problem; see reference [71].) Many sufficient conditions for a graph to be Hamiltonian are known; however, none of them happens to be an elegant necessary condition.

We begin with a necessary condition. Recall that $\omega(H)$ stands for the number of components of the graph H.

Theorem 6.3.4. *If G is Hamiltonian, then for every nonempty proper subset S of V, $\omega(G - S) \leq |S|$.*

Proof. Let C be a Hamilton cycle in G. Then, since C is a spanning subgraph of G, $\omega(G - S) \leq \omega(C - S)$. If $|S| = 1$, $C - S$ is a path, and therefore $\omega(C - S) = 1 = |S|$. The removal of a vertex from a path P results in one or two components, according to whether the removed vertex is an end vertex or an internal vertex of P.

Fig. 6.8 Theta graph



Hence, by induction, the number of components in $C - S$ cannot exceed $|S|$. This proves that $\omega(G - S) \leq \omega(C - S) \leq |S|$. □

It follows directly from the definition of a Hamiltonian graph or from Theorem 6.3.4 that any Hamiltonian graph must be 2-connected. [If G has a cut vertex v, then taking $S = \{v\}$, we see that $\omega(G - S) > |S|$.] The converse, however, is not true. For example, the theta graph of Fig. 6.8 is 2-connected but not Hamiltonian. Here, P stands for a u-v path of any length ≥ 2 containing neither x nor y.

Exercise 3.1. Show by means of an example that the condition in Theorem 6.3.4 is not sufficient for G to be Hamiltonian.

Exercise 3.2. Use Theorem 6.3.4 to show that the Herschel graph (shown in Fig. 5.4) is non-Hamiltonian.

Exercise 3.3. Do Exercise 3.2 by using Theorem 1.5.10 (characterization theorem for bipartite graphs).

If a cubic graph G has a Hamilton cycle C, then $G \backslash E(C)$ is a 1-factor of G. Hence, for a cubic graph G to be Hamiltonian, G must have a 1-factor F such that $G \backslash E(F)$ is a Hamilton cycle of G. Now, the Petersen graph P (shown in Fig. 1.7) has two different types of 1-factors (see Fig. 6.9), and for any such 1-factor F of P, $P \backslash E(F)$ consists of two disjoint 5-cycles. Hence P is non-Hamiltonian.

Theorem 6.3.5 is a basic result due to Ore [150] which gives a sufficient condition for a graph to be Hamiltonian.

Theorem 6.3.5 (Ore [150]). *Let G be a simple graph with $n \geq 3$ vertices. If, for every pair of nonadjacent vertices u, v of G, $d(u) + d(v) \geq n$, then G is Hamiltonian.*

Proof. Suppose that G satisfies the condition of the theorem, but G is not Hamiltonian. Add edges to G (without adding vertices) and get a supergraph G^* of G such that G^* is a maximal simple graph that satisfies the condition of the

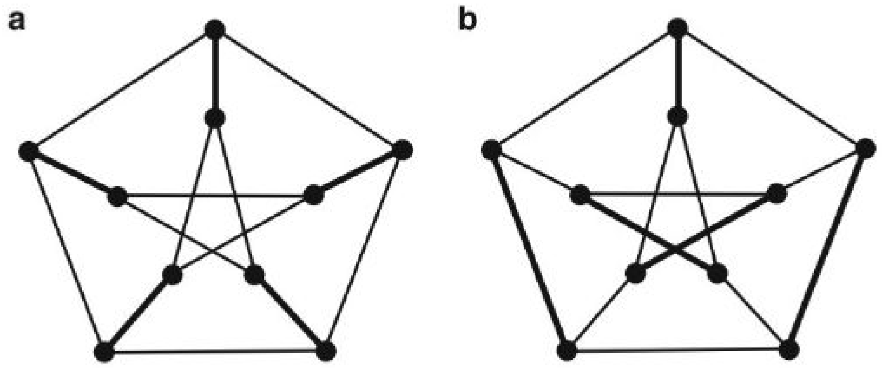


Fig. 6.9 Petersen graph. The solid edges form a 1-factor of P

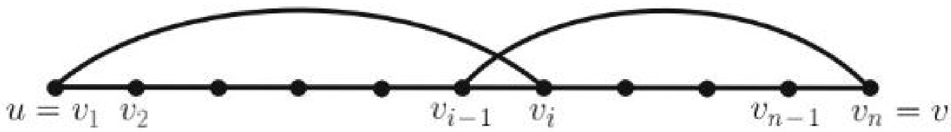


Fig. 6.10 Hamilton path for proof of Theorem 6.3.5

theorem, but G^* is non-Hamiltonian. Such a graph G^* must exist since G is non-Hamiltonian while the complete graph on $V(G)$ is Hamiltonian. Hence, for any pair u and v of nonadjacent vertices of G^*, $G^* + uv$ must contain a Hamilton cycle C. This cycle C would certainly contain the edge $e = uv$. Then $C - e$ is a Hamilton path $u = v_1 v_2 v_3 \ldots v_n = v$ of G^* (see Fig. 6.10).

Now, if $v_i \in N(u)$, $v_{i-1} \notin N(v)$; otherwise, $v_1\, v_2 \ldots v_{i-1}\, v_n\, v_{n-1}\, v_{n-2} \ldots v_{i+1}$ $v_i\, v_1$ would be a Hamilton cycle in G^*. Hence, for each vertex v_i adjacent to u, the vertex v_{i-1} of $V - \{v\}$ is nonadjacent to v. But then

$$d_{G^*}(v) \leq (n-1) - d_{G^*}(u).$$

This gives that $d_{G^*}(u) + d_{G^*}(v) \leq n - 1$, and therefore $d_G(u) + d_G(v) \leq n - 1$, a contradiction. □

Corollary 6.3.6 (Dirac [54]). *If G is a simple graph with $n \geq 3$ and $\delta \geq \frac{n}{2}$, then G is Hamiltonian.* □

Corollary 6.3.7. *Let G be a simple graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n - 1$ for every pair of nonadjacent vertices u and v of G, then G is traceable.*

Proof. Choose a new vertex w and let G' be the graph $G \vee \{w\}$. Then each vertex of G has its degree increased by one, and therefore in G', $d(u) + d(v) \geq n + 1$ for every pair of nonadjacent vertices. Since $|V(G')| = n + 1$, by Theorem 6.3.5, G' is Hamiltonian. If C' is a Hamilton cycle of G', then $C' - w$ is a Hamilton path of G. Thus, G is traceable. □

Exercise 3.4. Show by means of an example that the conditions of Theorem 6.3.5 and its Corollary 6.3.6 are not necessary for a simple connected graph to be Hamiltonian.

Exercise 3.5. Show that if a cubic graph G has a spanning closed trail, then G is Hamiltonian.

Exercise 3.6. Prove that the n-cube Q_n is Hamiltonian for every $n \geq 2$.

Exercise 3.7. Prove that the wheel W_n is Hamiltonian for every $n \geq 4$.

Exercise 3.8. Prove that a simple k-regular graph on $2k-1$ vertices is Hamiltonian.

Exercise 3.9. For any vertex v of the Petersen graph P, show that $P - v$ is Hamiltonian. (A non-Hamiltonian graph G with this property, namely, for any vertex v of G the subgraph $G - v$ of G is Hamiltonian, is called a hypo-Hamiltonian graph. In fact, P is the lowest-order graph with this property.)

Exercise 3.10. For any vertex v of the Petersen graph P, show that a Hamilton path exists starting at v.

Exercise 3.11. If $G = G(X, Y)$ is a bipartite Hamiltonian graph, show that $|X| = |Y|$.

Exercise 3.12. Let G be a simple graph on $2k$ vertices with $\delta(G) \geq k$. Show that G has a perfect matching.

Exercise 3.13. Prove that a simple graph of order n with n even and $\delta \geq \frac{(n+2)}{2}$ has a 3-factor.

Bondy and Chvátal [25] observed that the proof of Theorem 6.3.5 is essentially based on the following result.

Theorem 6.3.8. *Let G be a simple graph of order $n \geq 3$ vertices. Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian for every pair of nonadjacent vertices u and v with $d(u) + d(v) \geq n$.*

The last result has been instrumental for Bondy and Chvátal to define the closure of a graph G.

Definition 6.3.9. The *closure* of a graph G, denoted $\text{cl}(G)$, is defined to be that supergraph of G obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair exists.

This recursive definition does not stipulate the order in which the new edges are added. Hence, we must first show that the definition does not depend upon the order of the newly added edges. Figure 6.11 explains the construction of $\text{cl}(G)$.

Theorem 6.3.10. *The closure $cl(G)$ of a graph G is well defined.*

Proof. Let G_1 and G_2 be two graphs obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least n until no such pair exists. We have to prove that $G_1 = G_2$.

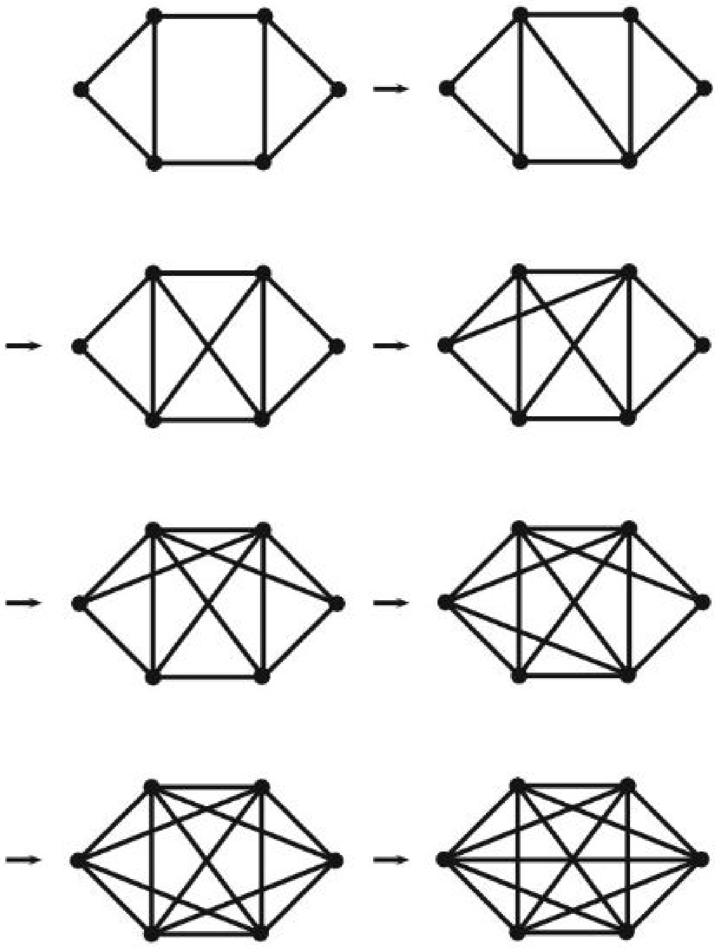


Fig. 6.11 Closure of a graph

Let $\{e_1, \ldots, e_p\}$ and $\{f_1, \ldots, f_q\}$ be the sets of new edges added to G in these sequential orderings to get G_1 and G_2, respectively. We want to show that each e_i is some f_j (and therefore belongs to G_2) and that each f_k is some e_l (and therefore belongs to G_1). Let e_i be the first edge in $\{e_1, \ldots, e_p\}$ not belonging to G_2. Then $\{e_1, \ldots, e_{i-1}\}$ are all in both G_1 and G_2, and $uv = e_i \notin E(G_2)$. Let $H = G + \{e_1, \ldots, e_{i-1}\}$. Then H is a subgraph of both G_1 and G_2. By the way cl(G) is defined,

$$d_H(u) + d_H(v) \geq n,$$

and hence,

$$d_{G_2}(u) + d_{G_2}(v) \geq n.$$

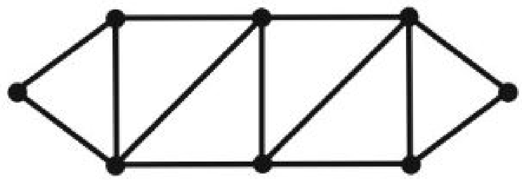
But this is a contradiction since u and v are nonadjacent vertices of G_2, and G_2 is a closure of G. Thus $e_i \in E(G_2)$ for each i and similarly, $f_k \in E(G_1)$ for each k. □

An immediate consequence of Theorem 6.3.8 is the following.

Theorem 6.3.11. *If cl(G) is Hamiltonian, then G is Hamiltonian.*

Corollary 6.3.12. *If cl(G) is complete, then G is Hamiltonian.*

Exercise 3.14. Determine the closure of the following graph.



We conclude this section with a result of Chvátal and Erdös [39].

Theorem 6.3.13 (Chvátal and Erdös). *If, for a simple 2-connected graph G, $\alpha \leq \kappa$, then G is Hamiltonian. (α is the independence number of G and κ is the connectivity of G.)*

Proof. Suppose $\alpha \leq \kappa$ but G is not Hamiltonian. Let $C : v_0 v_1 \ldots v_{p-1}$ be a longest cycle of G. We fix this orientation on C. By Dirac's theorem (Exercise 6.4 of Chap. 3), $p \geq \kappa$. Let $v \in V(G) \setminus V(C)$. Then by Menger's theorem (see also Exercise 6.3 of Chap. 3), there exist κ internally disjoint paths $P_1, \ldots, P_\kappa$ from v to C. Let $v_{i_1}, v_{i_2}, \ldots, v_{i_\kappa}$ be the end vertices (with suffixes in the increasing order) of these paths on C. No two of the consecutive vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_\kappa}, v_{i_1}$ can be adjacent vertices of C, since otherwise we get a cycle of G longer than C. Hence, between any two consecutive vertices of $\{v_{i_1}, v_{i_2}, \ldots, v_{i_\kappa}, v_{i_1}\}$, there exists at least one vertex of G. Let u_{i_j} be the vertex next to v_{i_j} in the v_{i_j}-$v_{i_{j+1}}$ path along C (see Fig. 6.12a).

We claim that $\{u_{i_1}, \ldots, u_{i_\kappa}\}$ is an independent set of G. Suppose u_{i_j} is adjacent to u_{i_m}, $m > j$ (suffixes taken modulo κ); then

$$u_{i_j} \ldots v_{i_{j+1}} \ldots v_{i_m} P_m^{-1} v P_j v_{i_j} \ldots v_{i_{j-1}} \ldots u_{i_m} u_{i_j}$$

is a cycle of G longer than C, a contradiction.

Further, $\{v, u_{i_1} \ldots, u_{i_\kappa}\}$ is also an independent set of G. [Otherwise, $v u_{i_m} \in E(G)$ for some m. See Fig. 6.12b. Then

$$v u_{i_m} \ldots v_{i_{m+1}} \ldots v_{i_\kappa} \ldots v_{i_1} \ldots v_{i_m} P_m^{-1} v$$

is a cycle longer than C, a contradiction.] But this implies that $\alpha > \kappa$, a contradiction to our hypothesis. Thus G is Hamiltonian. □

This theorem, although interesting, is not powerful in that for the cycle C_n, $\kappa = 2$ while $\alpha = \left\lfloor \frac{n}{2} \right\rfloor$ and hence increases with n.

A graph G with at least three vertices is *Hamiltonian-connected* if any two vertices of G are connected by a Hamilton path in G. For example, for $n \geq 3$, K_n is Hamiltonian-connected, whereas for $n \geq 4$, C_n is not Hamiltonian-connected.

Theorem 6.3.14. *If G is a simple graph with $n \geq 3$ vertices such that $d(u)+d(v) \geq n+1$ for every pair of nonadjacent vertices of G, then G is Hamiltonian-connected.*

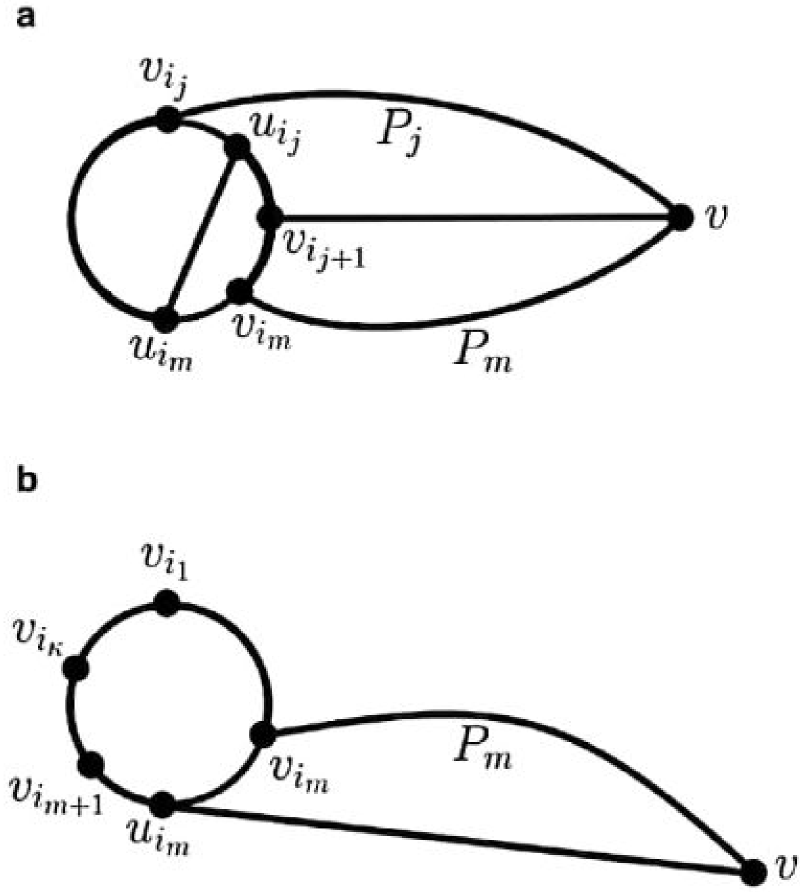


Fig. 6.12 Graphs for proof of Theorem 6.3.13

Proof. Let u and v be any two vertices of G. Our aim is to show that a Hamilton path exists from u to v in G.

Choose a new vertex w, and let $G^* = G \cup \{wu, wv\}$. We claim that $\text{cl}(G^*) = K_{n+1}$. First, the recursive addition of the pairs of nonadjacent vertices u and v of G with $d(u) + d(v) \geq n + 1$ gives K_n. Further, each vertex of K_n is of degree $n - 1$ in K_n and $d_{G^*}(w) = 2$. Hence, $\text{cl}(G^*) = K_{n+1}$. So by Corollary 6.3.12, G^* is Hamiltonian. Let C be a Hamilton cycle in G^*. Then $C - w$ is a Hamilton path in G from u to v. □

UNIT - 4 Graph Colorings

7.1 Introduction

Graph theory would not be what it is today if there had been no coloring problems. In fact, a major portion of the 20th-century research in graph theory has its origin in the four-color problem. (See Chap. 8 for details.)

In this chapter, we present the basic results concerning vertex colorings and edge colorings of graphs. We present two important theorems on graph colorings, namely. Brooks' theorem and Vizing's theorem. We also present a brief discussion on "snarks" and Kirkman's schoolgirl problem. In addition, a detailed description of the Mycielskian of a graph is also presented.

7.2 Vertex Colorings

7.2.1 Applications of Graph Coloring

We begin with a practical application of graph coloring known as the *storage problem*. Suppose a university's Department of Chemistry wants to store its chemicals. It is quite probable that some chemicals cause violent reactions when brought together. Such chemicals are *incompatible chemicals*. For safe storage, incompatible chemicals should be kept in distinct rooms. The easiest way to accomplish this is, of course, to store one chemical in each room. But this is certainly not the best way of doing it since we will be using more rooms than are really needed (unless, of course, all the chemicals are mutually incompatible!). So we ask: What is the minimum number of rooms required to store all the chemicals so that in each room only compatible chemicals are stored?

We convert the above storage problem into a problem in graphs. Form a graph $G = (V, E)$ by making V correspond bijectively to the set of available chemicals and making u adjacent to v if and only if the chemicals corresponding to u and R. Balakrishnan and K. Ranganathan. *A Textbook of Graph Theory,* 143

Universitext, DOI 10.1007/978-1 -4614-4529-6_7,

v are incompatible. Then, any set of compatible chemicals correspond to a set of independent vertices of G. Thus, a safe storing of chemicals corresponds to a partition of V into independent subsets of G. The cardinality of such a minimum partition of V is then the required number of rooms. The minimum cardinality is called the *chromatic number* of the graph G.

Definition 7.2.1. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of independent subsets that partition the vertex set of G. Any such minimum partition is called a *chromatic partition* of $V(G)$.

The storage problem just described is actually a vertex coloring problem of G. A *vertex coloring* of G is a map $f : V \to S$, where S is a set of distinct colors; it is *proper* if adjacent vertices of G receive distinct colors of S. This means that if $uv \in E(G)$, then $f(u) \neq f(v)$. Thus, $\chi(G)$ is the minimum cardinality of S for which there exists a proper vertex coloring of G by colors of S. Clearly, in any proper vertex coloring of G, the vertices that receive the same color are independent. The vertices that receive a particular color make up a *color class*. This allows an equivalent way of defining the chromatic number.

Definition 7.2.2. The *chromatic number* of a graph G is the minimum number of colors needed for a proper vertex coloring of G. G is k*-chromatic* if $\chi(G) = k$.

Definition 7.2.3. A k*-coloring* of a graph G is a vertex coloring of G that uses at most k colors.

Definition 7.2.4. A graph G is said to be k*-colorable* if G admits a *proper* vertex coloring using at most k colors.

In considering the chromatic number of a graph, only the adjacency of vertices is taken into account. Hence, multiple edges and loops may be discarded while considering chromatic numbers, unless needed otherwise. As a consequence, we may restrict ourselves to simple graphs when dealing with (vertex) chromatic numbers.

It is clear that $\chi(K_n) = n$. Further, $\chi(G) = 2$ if and only if G is bipartite having at least one edge. In particular, $\chi(T) = 2$ for any tree T with at least one edge (since any tree is bipartite). Further (see Fig. 7.1),

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases} \tag{7.1}$$

Exercise 2.1. Prove $\chi(G) = 2$ if and only if G is a bipartite graph with at least one edge.

Exercise 2.2. Determine the chromatic number of

(i) The Petersen graph
(ii) Wheel W_n (see Sect. 1.7, Chap. 1)
(iii) The Herschel graph (see Fig. 5.4)
(iv) The Grötzsch graph (see Fig. 7.6)

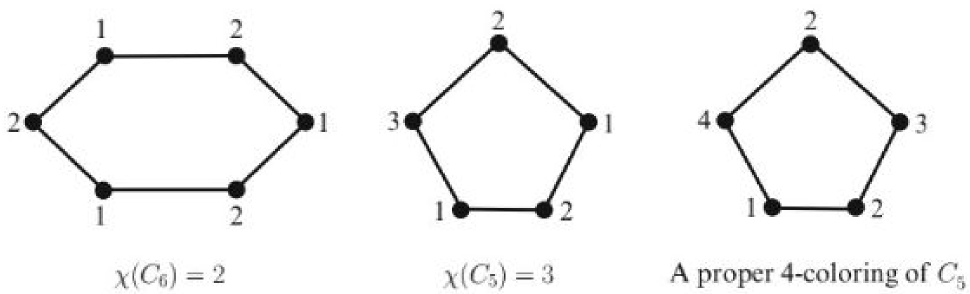


Fig. 7.1 Illustration of proper vertex coloring

We next consider another application of graph coloring. Let S be a set of students. Each student of S is to take a certain number of examinations for which he or she has registered. Undoubtedly, the examination schedule must be such that all students who have registered for a particular examination will take it at the same time.

Let $\mathbb{P}$ be *the set of examinations* and for $p \in \mathbb{P}$, let $S(p)$ be *the set of students* who have to take the examination p. Our aim is to draw up an examination schedule involving only the minimum number of days on the supposition that papers a and b can be given on the same day provided they have no common candidate and that no candidate shall have more than one examination on any day.

Form a graph $G = G(\mathbb{P}, E)$, where $a, b \in \mathbb{P}$ are adjacent if and only if $S(a) \cap S(b) \neq \emptyset$. Then each proper vertex coloring of G yields an examination schedule with the vertices in any color class representing the schedule on a particular day. Thus, $\chi(G)$ gives the minimum number of days required for the examination schedule.

Exercise 2.3. Draw up an examination schedule involving the minimum number of days for the following problem:

Set of students	Examination subjects
S_1	Algebra, real analysis, and topology
S_2	Algebra, operations research, and complex analysis
S_3	Real analysis, functional analysis, and complex analysis
S_4	Algebra, graph theory, and combinatorics
S_5	Combinatorics, topology, and functional analysis
S_6	Operations research, graph theory, and coding theory
S_7	Operations research, graph theory, and number theory
S_8	Algebra, number theory, and coding theory
S_9	Algebra, operations research, and real analysis

Exercise 2.4. If G is k-regular, prove that $\chi(G) \geq \frac{n}{n-k}$.

Theorem 7.2.5 gives upper and lower bounds for the chromatic number of a graph G in terms of its independence number and order.

Theorem 7.2.5. *For any graph G with n vertices and independence number* α,

$$\frac{n}{\alpha} \leq \chi \leq n - \alpha + 1.$$

Proof. There exists a chromatic partition $\{V_1, V_2, \ldots, V_\chi\}$ of V. Since each V_i is independent, $|V_i| \leq \alpha$, $1 \leq i \leq \chi$. Hence, $n = \sum_{i=1}^{\chi} |V_i| \leq \alpha\, \chi$, and this gives the inequality on the left.

To prove the inequality on the right, consider a maximum independent set S of α vertices. Then the subsets of $V \backslash S$ of cardinality 1 together with S yield a partition of V into $(n - \alpha) + 1$ independent subsets. □

Remark 7.2.6. Unfortunately, none of the above bounds is a good one. For example, if G is the graph obtained by connecting C_{2r} with a disjoint K_{2r} ($r \geq 2$), by an edge, we have $n = 4r$, $\alpha = r + 1$, and $\chi = 2r$, and the above inequalities become $\frac{4r}{r+1} \leq 2r \leq 3r$. For a simple graph G, the number $\chi^c = \chi^c(G) = \chi(G^c)$, the chromatic number of G^c is the minimum number of subsets in a partition of $V(G)$ into subsets each inducing a complete subgraph of G. Bounds on the sum and product of $\chi(G)$ and $\chi^c(G)$ were obtained by Nordhaus and Gaddum [148] (see also reference [93]), as given in Theorem 7.2.7.

Theorem 7.2.7 (Nordhaus and Gaddum [148]). *For any simple graph G*,

$$2\sqrt{n} \leq \chi + \chi^c \leq n + 1, \text{and} n \leq \chi\, \chi^c \leq \left(\frac{n+1}{2}\right)^2.$$

Proof. Let $\chi(G) = k$ and let $V_1, V_2, \ldots, V_k$ be the k color classes in a chromatic partition of G. Then $\sum_{i=1}^{k} |V_i| = n$, and so $max_{1 \leq i \leq k} |V_i| \geq \frac{n}{k}$. Since each V_i is an independent set of G, it induces a complete subgraph in G^c. Hence, $\chi^c \geq \max_{1 \leq i \leq k} |V_i|$, and so $\chi\, \chi^c = k\, \chi^c \geq k \circ \max_{1 \leq i \leq k} |V_i| \geq k \circ \frac{n}{k} = n$. Further, since the arithmetic mean of χ and χ^c is greater than or equal to their geometric mean, $\frac{\chi + \chi^c}{2} \geq \sqrt{\chi\, \chi^c} \geq \sqrt{n}$. Hence, $\chi + \chi^c \geq 2\sqrt{n}$. This establishes both the lower bounds.

To show that $\chi + \chi^c \leq n + 1$, we use induction on n. When $n = 1$, $\chi = \chi^c = 1$, and so we have equality in this case. So assume that $\chi + \chi^c \leq (n - 1) + 1 = n$ for all graphs G having $n - 1$ vertices, $n \geq 2$. Let H be any graph with n vertices, and let v be any vertex of H. Then $G = H - v$ is a graph with $n - 1$ vertices and $G^c = (H - v)^c = H^c - v$. By the induction assumption, $\chi(G) + \chi(G^c) \leq n$.

Now $\chi(H) \leq \chi(G) + 1$ and $\chi(H^c) \leq \chi(G^c) + 1$. If either $\chi(H) \leq \chi(G)$ or $\chi(H^c) \leq \chi(G^c)$, then $\chi(H) + \chi(H^c) \leq \chi(G) + \chi(G^c) + 1 \leq n + 1$. Suppose then $\chi(H) = \chi(G) + 1$ and $\chi(H^c) = \chi(G^c) + 1$. $\chi(H) = \chi(G) + 1$ implies that removal of v from H decreases the chromatic number, and hence $d_H(v) \geq \chi(G)$. [If $d_H(v) < \chi(G)$, then in any proper coloring of G with $\chi(G)$ colors at most $\chi(G) - 1$ colors would have been used to color the neighbors of v in G, and hence v can be given one of the left-out colors, and therefore we have a coloring of H with $\chi(G)$ colors. Hence, $\chi(H) = \chi(G)$, a contradiction.] For a similar reason, $\chi(H^c) = \chi(G^c) + 1$ implies that $n - 1 - d_H(v) = d_{H^c}(v) \geq \chi(G^c)$; thus, $\chi(G) + \chi(G^c) \leq d_H(v) + n - 1 - d_H(v) = n - 1$. This implies, however, that $\chi(H) + \chi(H^c) = \chi(G) + \chi(G^c) + 2 \leq n + 1$.

Finally, applying the inequality $\sqrt{\chi\,\chi^c} \le \frac{\chi+\chi^c}{2}$, we get $\chi\,\chi^c \le (\frac{\chi+\chi^c}{2})^2 \le (\frac{n+1}{2})^2$. □

Note 7.2.8. Since the publication of Theorem 7.2.7, there had been similar results for other graph parameters (see, for instance, [115] for the domination number γ). All these results have now come to be known as Nordhaus–Gaddum inequalities, with reference to the parameters in question.

Exercise 2.5. For a simple graph G, prove that $\chi(G^c) \ge \alpha(G)$.

Exercise 2.6. Prove $\chi(G) \le \ell + 1$, where ℓ is the length of a longest path in G. For each positive integer ℓ, give a graph G with chromatic number $\ell + 1$ and in which any longest path has length ℓ.

Exercise 2.7. Which numbers can be chromatic numbers of unicyclic graphs? Draw a unicyclic graph on 15 vertices with $\Delta = 3$ and having each of these numbers as its chromatic number.

Exercise 2.8. If G is connected and $m \le n$, show that $\chi(G) \le 3$.

Exercise 2.9. Let G_n be the graph defined by $V(G_n) = \{(i,j) : 1 \le i < j \le n\}$, and $E(G_n) = \{\big((i,j)(k,l)\big) : i < j = k < l\}$. Prove

(i) $\omega(G_n) = 2$.
(ii) $\chi(G_n) = \lceil \log_2 n \rceil$. [Note that $\chi(G_n) \to \infty$ as $n \to \infty$.]

Exercise 2.10. Prove that $\chi(G \Box H) = \max(\chi(G), \chi(H))$.

Exercise 2.11. Prove $\chi(G \times H) \le \min(\chi(G),\ \chi(H)))$ (A celebrated conjecture of Hedetniemi [104] states that $\chi(G \times H) = \min(\chi(G),\ \chi(H)))$.

7.3 Critical Graphs

Definition 7.3.1. A graph G is called *critical* if for every proper subgraph H of G, $\chi(H) < \chi(G)$. Equivalently, $\chi(G - e) < \chi(G)$ for each edge e of G. Also, G is k-*critical* if it is k-chromatic and critical.

Remarks 7.3.2. If $\chi(G) = 1$, then G is either trivial or totally disconnected. Hence, G is 1-critical if and only if G is K_1. Again, $\chi(G) = 2$ implies that G is bipartite and has at least one edge. Hence, G is 2-critical if and only if G is K_2. For an odd cycle C, $\chi(C) = 3$, and if G contains an odd cycle C properly, G cannot be 3-critical.

Exercise 3.1. Prove that any critical graph is connected.

Exercise 3.2. Prove that for any graph G, $\chi(G - v) = \chi(G)$ or $\chi(G) - 1$ for any $v \in V$, and $\chi(G - e) = \chi(G)$ or $\chi(G) - 1$ for any $e \in E$.

Exercise 3.3. Show that if G is k-critical, $\chi(G - v) = \chi(G - e) = k - 1$ for any $v \in V$ and $e \in E$.

Exercise 3.4. [If $\chi(G-e) < \chi(G)$ for any e of G, G is sometimes called *edge-critical*, and if $\chi(G-v) < \chi(G)$ for any vertex v of G, G is called *vertex-critical*.] Show that a nontrivial connected graph is vertex-critical if it is edge-critical. Disprove the converse by a counterexample.

Exercise 3.5. Show that a graph is 3-critical if and only if it is an odd cycle. It is clear that any k-chromatic graph contains a k-critical subgraph. (This is seen by removing vertices and edges in succession, whenever possible, without diminishing the chromatic number.)

Theorem 7.3.3. *If G is k-critical, then $\delta(G) \geq k-1$.*

Proof. Suppose $\delta(G) \leq k-2$. Let v be a vertex of minimum degree in G. Since G is k-critical, $\chi(G-v) = \chi(G)-1 = k-1$ (see Exercise 3.3). Hence, in any proper $(k-1)$-coloring of $G-v$, at most $(k-2)$ colors would have been used to color the neighbors of v in G. Thus, there is at least one color, say c, that is left out of these $k-1$ colors. If v is given the color c, a proper $(k-1)$-coloring of G is obtained. This is impossible since G is k-chromatic. Hence, $\delta(G) \geq (k-1)$. □

Corollary 7.3.4. *For any graph G, $\chi(G) \leq 1 + \Delta(G)$.*

Proof. Let G be a k-chromatic graph, and let H be a k-critical subgraph of G. Then $\chi(H) = \chi(G) = k$. By Theorem 7.3.3, $\delta(H) \geq k-1$, and hence $k \leq 1 + \delta(H) \leq 1 + \Delta(H) \leq 1 + \Delta(G)$. □

Exercise 3.6. Give another proof of Corollary 7.3.4 by using induction on $n = |V(G)|$.

Exercise 3.7. If $\chi(G) = k$, show that G contains at least k vertices each of degree at least $k-1$.

Exercise 3.8. Prove or disprove: If G is k-chromatic, then G contains a K_k.

Exercise 3.9. Prove: Any k ($\geq$ 2)-critical graph contains a $(k-1)$-critical subgraph.

Exercise 3.10. For each of the graphs G of Exercise 2.2, find a critical subgraph H of G with $\chi(H) = \chi(G)$.

Exercise 3.11. Prove that the wheel $W_{2n-1} = C_{2n-1} \vee K_1$ is a 4-critical graph for each $n \geq 2$. Does a similar statement apply to W_{2n}?

Theorem 7.3.5. *In a critical graph G, no vertex cut is a clique.*

Proof. Suppose G is a k-critical graph and S is a vertex cut of G that is a clique of G (i.e., a complete subgraph of G). Let H_i, $1 \leq i \leq r$, be the components of $G \backslash S$, and let $G_i = G[V(H_i) \cup S]$. Then each G_i is a proper subgraph of G and hence admits a proper $(k-1)$-coloring. Since S is a clique, its vertices must receive distinct colors in any proper $(k-1)$-coloring of G_i. Hence, by fixing the colors for the vertices of S, and coloring for each i the remaining vertices of G_i so as to give a proper $(k-1)$-coloring of G_i, we obtain a proper $(k-1)$-coloring of G. This contradicts the fact that G is k-chromatic (see Fig. 7.2). □

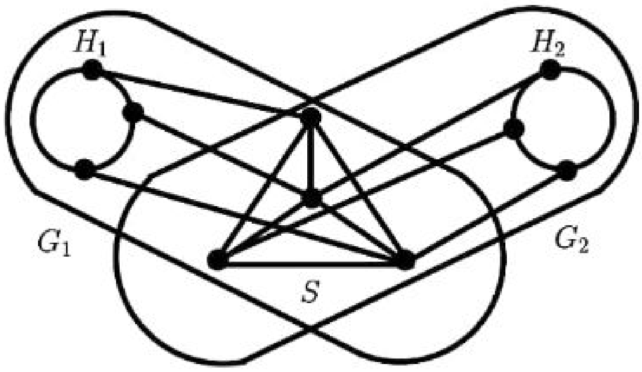


Fig. 7.2 $G[S] \simeq K_4$ $(r = 2)$

Corollary 7.3.6. *Every critical graph is a block.*

Exercise 3.12.* Prove that every k-critical graph is $(k-1)$-edge connected (Dirac [53]).

Exercise 3.13. Show by means of an example that criticality is essential in Exercise 3.12; that is, a k-chromatic graph need not be $(k-1)$-edge connected.

7.3.1 Brooks' Theorem

We next consider *Brooks'* [31] *theorem.* Recall Corollary 7.3.4, which states that $\chi(G) \leq 1 + \Delta(G)$. If G is an odd cycle, $\chi(G) = 3 = 1 + 2 = 1 + \Delta(G)$, and if G is a complete graph, say K_k, $\chi(G) = k = 1 + (k-1) = 1 + \Delta(G)$. That these are the only extremal families of graphs for which $\chi(G) = 1 + \Delta(G)$ is the assertion of Brooks' theorem.

Theorem 7.3.7 (Brooks' theorem). *If a connected graph G is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.*

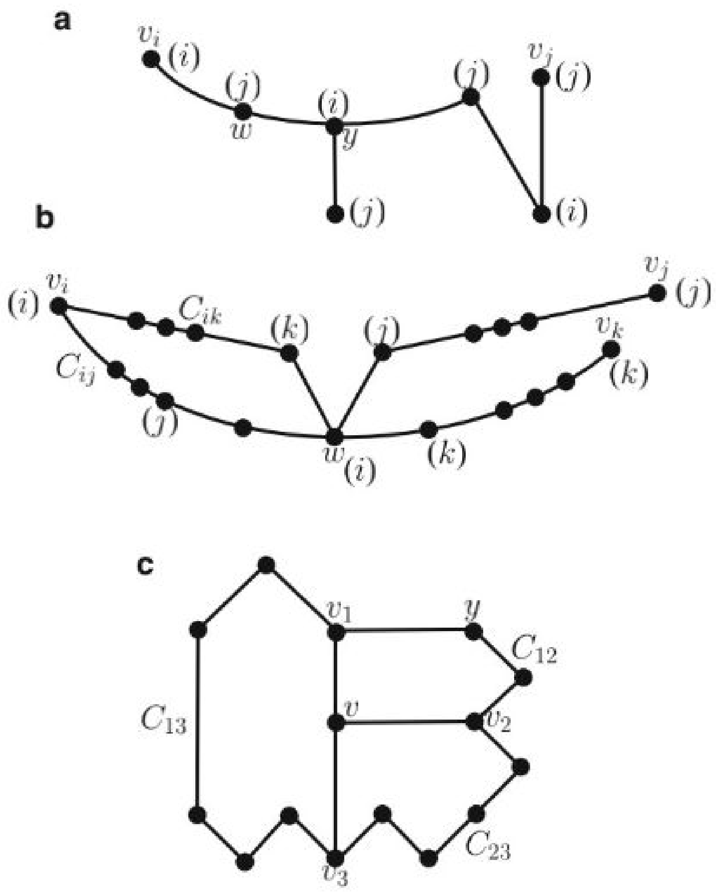
Proof. If $\Delta(G) \leq 2$, then G is either a path or a cycle. For a path G (other than K_1 and K_2), and for an even cycle G, $\chi(G) = 2 = \Delta(G)$. According to our assumption, G is not an odd cycle. So let $\Delta(G) \geq 3$.

The proof is by contradiction. Suppose the result is not true. Then there exists a minimal graph G of maximum degree $\Delta(G) = \Delta \geq 3$ such that G is not Δ-colorable, but for any vertex v of G, $G - v$ is Δ-colorable.

Claim 1. Let v be any vertex of G. Then in any proper Δ-coloring of $G - v$, all the Δ colors must be used for coloring the neighbors v in G. Otherwise, if some color i is not represented in $N_G(v)$, then v could be colored using i, and this would give a Δ-coloring of G, a contradiction to the choice of G. Thus, G is a Δ-regular graph satisfying Claim 1.

For $v \in V(G)$, let $N(v) = \{v_1, v_2, \ldots, v_\Delta\}$. In a proper Δ-coloring of $G - v = H$, let v_i receive color i, $1 \leq i \leq \Delta$. For $i \neq j$, let H_{ij} be the subgraph of H induced by the vertices receiving the ith and jth colors.

Fig. 7.3 Graphs for proof of Theorem 7.3.7 (The numbers inside the parentheses denote the vertex colors)



Claim 2. v_i and v_j belong to the same component of H_{ij}. Otherwise, the colors i and j can be interchanged in the component of H_{ij} that contains the vertex v_j. Such an interchange of colors once again yields a proper Δ-coloring of H. In this new coloring, both v_i and v_j receive the same color, namely, i, a contradiction to Claim 1. This proves Claim 2.

Claim 3. If C_{ij} is the component of H_{ij} containing v_i and v_j, then C_{ij} is a path in H_{ij}. As before, $N_H(v_i)$ contains exactly one vertex of color j. Further, C_{ij} cannot contain a vertex, say y, of degree at least 3; for, if y is the first such vertex on a $v_i - v_j$ path in C_{ij} that has been colored, say, with i, then at least three neighbors of y in C_{ij} have the color j. Hence, we can recolor y in H with a color different from both i and j, and in this new coloring of H, v_i and v_j would belong to distinct components of H_{ij} (see Fig. 7.3a). (Note that by our choice of y, any $v_i - v_j$ path in H_{ij} must contain y.) But this contradicts Claim 3.

Claim 4. $C_{ij} \cap C_{ik} = \{v_i\}$ for $j \neq k$. Indeed, if $w \in C_{ij} \cap C_{ik}$, $w \neq v_i$, then w is adjacent to two vertices of color j on C_{ij} and two vertices of color k on C_{ik} (see Fig. 7.3b). Again, we can recolor w in H by giving a color different from the colors of the neighbors of w in H. In this new coloring of H, v_i and v_j belong to distinct components of H_{ij}, a contradiction to Claim 2. This completes the proof of Claim 4.

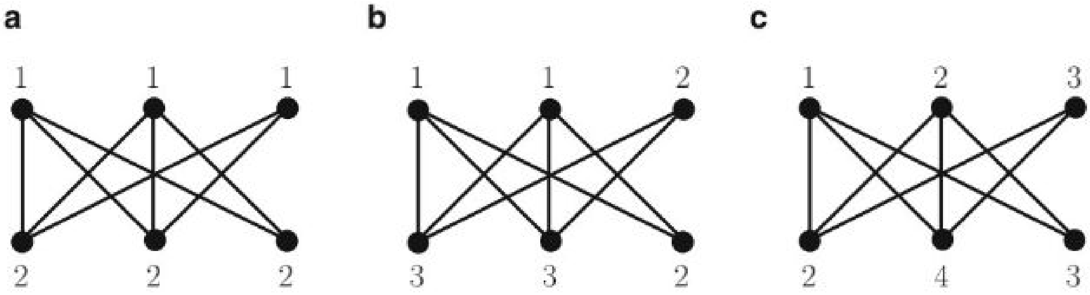


Fig. 7.4 Different colorings of $K_{3,3} - e$

We are now in a position to complete the proof of the theorem. By hypothesis, G is not complete. Hence, G has a vertex v, and a pair of nonadjacent vertices v_1 and v_2 in $N_G(v)$ (see Exercise 5.11, Chap. 1). Then the $v_1 - v_2$ path C_{12} in H_{12} of $H = G - v$ contains a vertex y ($\neq v_2$) adjacent to v_1. Naturally, y would receive color 2. Since $\Delta \geq 3$, by Claim 1, there exists a vertex $v_3 \in N_G(v)$. Now interchange colors 1 and 3 in the path C_{13} of H_{13}. This would result in a new coloring of $H = G - v$. Denote the v_i-v_j path in H under this new coloring by C'_{ij} (see Fig. 7.3c). Then $y \in C'_{23}$ since v_1 receives color 3 in the new coloring (whereas y retains color 2). Also, $y \in C_{12} - v_1 \subset C'_{12}$. Thus, $y \in C'_{23} \cap C'_{12}$. This contradicts Claim 4 (since $y \neq v_2$), and the proof is complete. □

7.3.2 *Other Coloring Parameters*

There are several other vertex coloring parameters of a graph G. We now mention three of them. Let f be a k-coloring (not necessarily proper) of G, and let $(V_1, V_2, \ldots, V_k)$ be the color classes of G induced by f. Coloring f is *pseudocomplete* if between any two distinct color classes, there is at least one edge of G. f is *complete* if it is pseudocomplete and each V_i, $1 \leq i \leq k$, is an independent set of G. Thus, $\chi(G)$ is the minimum k for which G has a complete k-coloring f.

Definition 7.3.8. The *achromatic number* $\mathrm{a}(G)$ of a graph G is the maximum k for which G has a complete k-coloring.

Definition 7.3.9. The *pseudoachromatic number* $\psi(G)$ of G is the maximum k for which G has a pseudocomplete k-coloring.

Example 7.3.10. Figure 7.4 gives (a) a chromatic, (b) an achromatic, and (c) a pseudoachromatic coloring of $K_{3,3} - e$.

It is clear that for any graph G, $\chi(G) \leq \mathrm{a}(G) \leq \psi(G)$.

Exercise 3.14. Let G be a graph and H a subgraph of G. Prove that $\chi(H) \leq \chi(G)$ and $\psi(H) \leq \psi(G)$. Show by means of an example that $\mathrm{a}(H) \leq \mathrm{a}(G)$ need not always be true.

Exercise 3.15. Prove

(i) $\psi(\psi - 1) \leq 2m$.
(ii) $\psi(K_a \vee K_b^c) = a + 1$.

From (ii) deduce that for any graph, $\psi \leq n - \alpha + 1$.

Exercise 3.16. If G has a complete coloring using k colors, prove that $k \leq \frac{1+\sqrt{1+8m}}{2}$. ($m$ = size of G).

Exercise 3.17. Prove that for a complete bipartite graph G, $\mathrm{a}(G) = 2$.

Exercise 3.18. What is the minimum number of edges that a connected graph with pseudoachromatic number ψ can have? Construct one such tree.

Exercise 3.19. If G is a subgraph of H, prove that $\psi(G) \leq \psi(H)$.

Exercise 3.20. Prove: $\psi(K_{n,n}) = n + 1$.

7.3.3 b-Colorings

Definition 7.3.11. A b-*coloring* of a graph G is a proper coloring with the additional property that each color class contains a color-dominating vertex (c.d.v.), that is, a vertex that has a neighbor in all the other color classes. The b-*chromatic number* of G is the largest k such that G has a b-coloring using k colors; it is denoted by $\mathrm{b}(G)$.

The concept of b-coloring was introduced by Irving and Manlove [111].

Exercise 3.21 guarantees the existence of the b-chromatic number for any graph G and shows that $\chi(G) \leq \mathrm{b}(G)$. Note that $\mathrm{b}(K_n) = n$ while $\mathrm{b}(K_{m,n}) = 2$.

Exercise 3.21. Show that the chromatic coloring of a graph G is a b-coloring of G.

Exercise 3.22. Prove that $K_{n,n} - F$, $n \geq 2$, where F is a 1-factor of $K_{n,n}$, has a b-coloring using 2 colors and n colors but none with k colors for any k in $2 < k < n$.

Exercise 3.23. Prove $\mathrm{b}(G) \leq 1 + \Delta(G)$. A better upper bound for $\mathrm{b}(G)$ is given in the next exercise.

Exercise 3.24. Let $d_1 \leq d_2 \leq \ldots \leq d_n$ be the degree sequence of the graph G with vertex set $V = \{v_1, \ldots, v_n\}$, and $d_i = d(v_i)$, $1 \leq i \leq n$. Let $M(G) = \max\{i : d_i \geq i - 1, 1 \leq i \leq n\}$. Prove that $\mathrm{b}(G) \leq M(G)$. Show further that the number of vertices of degree at least $M(G)$ in G is at most $M(G)$.

Exercise 3.25. Let Q_p be the hypercube of dimension p. Prove $\mathrm{b}(Q_1) = \mathrm{b}(Q_2)=2$, and $\mathrm{b}(Q_3) = 4$. [A result of Kouider and Mahéo [125] states that for $p \geq 3$, $\mathrm{b}(Q_p) = p + 1$.]

We complete this section by presenting a result of Kratochvíl, Tuza, and Voigt [126] that characterizes graphs with b-chromatic number 2. Let G be a bipartite graph with bipartition (X, Y). A vertex $x \in X$ (respectively, $y \in Y$) is called a *full vertex* (or a *charismatic vertex*) of X (respectively, Y) if it is adjacent to all the vertices of Y (respectively, X).

Theorem 7.3.12 ([126]). *Let G be a nontrivial connected graph. Then* $\mathrm{b}(G) = 2$ *if and only if G is bipartite and has a full vertex in each part of the bipartition.*

Proof. Suppose G is bipartite and has a full vertex in each part, say $x \in X$ and $y \in Y$. Naturally, in any b-coloring, the color class containing x, say W_1, is a subset of X and that containing y, say W_2, is a subset of Y. If G has a third color class W_3 disjoint from W_1 and W_2, then W_3 must have a c.d.v. adjacent to a vertex of W_1 and a vertex of W_2. This is impossible, as G is bipartite. Therefore, $\mathrm{b}(G) = 2$.

Conversely, let $\mathrm{b}(G) = 2$. Then $\chi(G) = 2$ and therefore G is bipartite. Let (X, Y) be the bipartition of G. Assume that G does not have a full vertex in at least one part, say, X. Let $x_1 \in X$. As x_1 is not a full vertex, there exists a vertex $y_1 \in Y$ to which it is not adjacent. Let X_1 be the maximal subset of X such that $V_1 = X_1 \cup \{y_1\}$ is independent in G. Now choose a new vertex $x_2 \in X \backslash X_1$. Again, as X has no full vertex, we can find a $y_2 \in Y \backslash \{y_1\}$ to which x_2 is not adjacent. Let X_2 be the maximal subset of $X \backslash X_1$ such that $V_2 = X_2 \cup \{y_2\}$ is independent in G. In this way, all the vertices of X would be exhausted and let $V_1, V_2, \ldots, V_k$ be the independent sets thus formed. Also, let Y_0 denote the set of uncovered vertices of Y, if any. Since G is connected, $G \neq \langle V_i \cup V_j \rangle$, and $G \neq \langle V_l \cup Y_0 \rangle$, $i, j, l \in \{1, 2, \ldots, k\}$. Hence, $k \geq 2$ when $Y_0 \neq \emptyset$ and $k \geq 3$ when $Y_0 = \emptyset$. Thus, the partition $V = V_1 \cup V_2 \cup \ldots \cup V_k \cup \{V_{k+1} = Y_0\}$ has at least 3 parts. If each of these parts has a c.d.v., we get a contradiction to the fact that $\mathrm{b}(G) = 2$. If not, assume that the class V_l has no c.d.v. Then for each vertex x of V_l, there exists a color class V_j, $j \neq l$, having no neighbor of x. Then x could be moved to the class V_j. In this way, the vertices in V_l can be moved to the other V_i's without disturbing independence. Let us call the new classes $V_1', V_2', \ldots, V_{l-1}', V_{l+1}', \ldots, V_{k+1}'$. If each of these color classes contains a c.d.v., we get a contradiction as $k \geq 3$. Otherwise, argue as before and reduce the number of color classes. As G is connected, successive reductions should end up in at least three classes, contradicting the hypothesis that $\mathrm{b}(G) = 2$. □

A description of several other coloring parameters can be found in Jensen and Toft [116].

7.4 Homomorphisms and Colorings

Homomorphisms of graphs generalize the concept of graph colorings.

Definition 7.4.1. Let G and H be simple graphs. A *homomorphism* from G to H is a map $f : V(G) \to V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. The map f is an *isomorphism* if f is bijective and $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$.

vertices S_i and S_j are adjacent if $v_i v_j \in E(H)$. This defines a natural isomorphism $\tilde{f} : G/f \simeq H$.

A consequence of the above remarks is the fact that a complete k-coloring of G is just a homomorphism of G onto K_k. Recall that both the chromatic and achromatic colorings are complete colorings. We now establish the coloring interpolation theorem for the complete coloring.

Theorem 7.4.7 (Interpolation theorem for complete coloring). *If a graph G admits a complete k-coloring and a complete l-coloring, then it admits a complete i-coloring for every i between k and l.*

Proof. Let $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_l$ be the color partitions in the two complete colorings. We assume without loss of generality that $k < l$. Clearly, it suffices to construct a complete $(k+1)$-coloring of G. For each $i = 0, 1, 2, \ldots, l$, let $C_i = \bigcup_{1 \le j \le i} B_j$. Let Θ_i denote the partition of $V(G)$ by the nonempty sets of the sequence $B_1, B_2, \ldots, B_i$; $A_1 - C_i, A_2 - C_i, \ldots, A_k - C_i$. The partition Θ_0 has parts $A_1, A_2, \ldots, A_k$; the partition Θ_l has parts $B_1, B_2, \ldots, B_l$ (since $C_l = V(G)$, $A_i - C_l = \emptyset$ for each j). Hence, $G/\Theta_0 \simeq K_k$ and $G/\Theta_l \simeq K_l$. Hence, there must exist a first suffix j, $0 < j \le l$, such that G/Θ_j is not k-colorable. By the choice of j, this implies that G/Θ_j is $(k+1)$-colorable since we can simply color B_j by the $(k+1)$-st color, and hence by Lemma 7.4.2, G is $(k+1)$-colorable. (Just compose the two onto homomorphisms $G \to G/\Theta_j \to K_{k+1}$.) □

Exercise 3.22 shows that an interpolation theorem similar to that of complete coloring does not hold good for the b-coloring.

Exercise 4.1. Let $f : G \to H$ be a graph homomorphism and let $x, y \in V(G)$. Prove $d_H(x, y) \le d_G(x, y)$.

Exercise 4.2. Assume that there exists a homomorphism from G onto C_k, where k is odd. Show that G must contain an odd cycle. Show by means of an example that a similar statement need not hold good if k is even.

Exercise 4.3. Prove that there exists a homomorphism from C_{2l+1} to C_{2k+1} if and only if $l \le k$.

7.5 Triangle-Free Graphs

Definition 7.5.1. A graph G is *triangle-free* if G contains no K_3.

Remark 7.5.2. Triangle-free graphs cannot contain a K_k, $k \ge 3$, either. It is obvious that if a graph G contains a clique of size k, then $\chi(G) \ge k$. However, the converse is not true. That is, if the chromatic number of G is large, then G need not contain a clique of large size. The construction of triangle-free k-chromatic graphs, for $k \ge 3$, was raised in the middle of the 20th century. In answer to this question, Mycielski [144] developed an interesting graph transformation known as the *Mycielskian* of a graph.

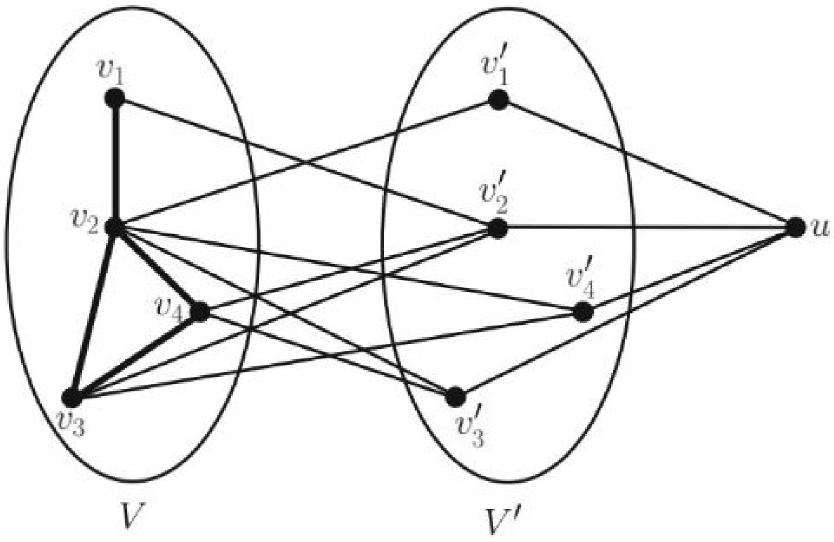


Fig. 7.5 $\mu(K_{1,3} + e)$

Definition 7.5.3. Let G be a finite simple connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *Mycielskian* $\mu(G)$ of G is defined as follows: The vertex set $V(\mu(G))$ of $\mu(G)$ is the disjoint union $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and the edge set of $\mu(G)$ is $E(\mu(G)) = E \cup \{x'y : xy \in E\} \cup \{x'u : x' \in V'\}$.

We denote $V(\mu(G))$ by the triad $\{V, V', u\}$. For $x \in V$, we call $x' \in V'$, the twin of x in $\mu(G)$, and vice versa, and u, the root of $\mu(G)$. Figure 7.5 displays the Mycielskian $\mu(K_{1,3} + e)$.

Remark 7.5.4. The following facts about $\mu(G)$, where G is of order n and size m, are obvious:

(i) $|V(\mu(G))| = 2n + 1$.
(ii) For each $v \in V$, $d_{\mu(G)}(v) = 2d_G(v)$.
(iii) For each $v' \in V'$, $d_{\mu(G)}(v') = d_G(v) + 1$.
(iv) $d_{\mu(G)}(u) = n$.

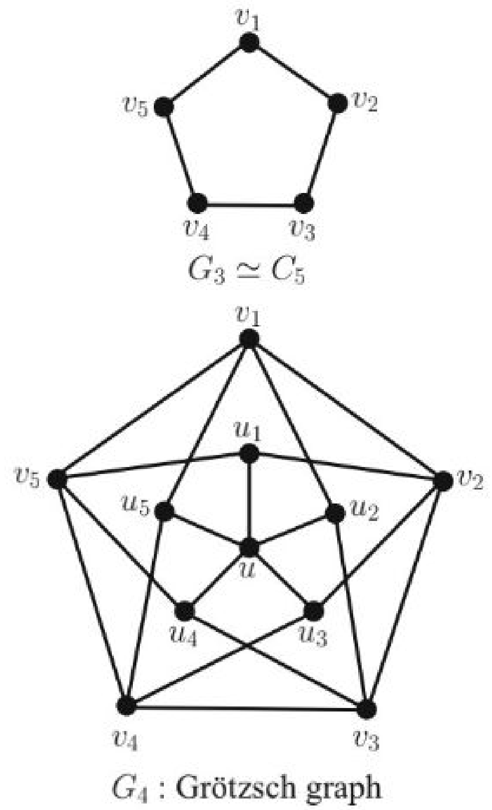
We now establish some basic results concerning the Mycielskian.

Theorem 7.5.5. $\chi(\mu(G)) = \chi(G) + 1$.

Proof. Assume that $\chi(G) = k$. Consider a proper (vertex) k-coloring c of G using the colors, say, $1, 2, \dots, k$. We now give a proper $(k + 1)$-coloring c' for $\mu(G)$. For $v \in V$, set $c'(v) = c(v)$. For the twin $v' \in V'$, set $c'(v') = c(v)$. For the root u of $\mu(G)$, set $c'(u) = k + 1$. Then c' is a proper coloring for $\mu(G)$ using $k + 1$ colors and therefore $\chi(\mu(G)) \leq k + 1$. [c' is proper because for any edge xy', $c'(x) = c(x) \neq c(y) = c'(y')$.] We now show that it is actually $k + 1$.

Suppose $\mu(G)$ has a proper k-coloring c'' using the colors $1, 2, \dots, k$. Assume, without loss of generality, that $c''(u) = 1$. Then for any $v' \in V'$, $c''(v') \neq 1$. Recolor each vertex of V that has been colored by 1 in c'' by the color of its twin under c''.

Fig. 7.6 The Grötzsch graph, $\mu(C_5)$



Then this gives a proper coloring of V using the $k-1$ colors $2, 3, \ldots, k$. This is impossible as $\chi(G) = k$. This proves that $\chi(\mu(G)) = k + 1 = \chi(G) + 1$. □

Theorem 7.5.6. *If G triangle-free, then $\mu(G)$ is also triangle-free.*

Proof. Assume that G is triangle-free. If $\mu(G)$ contains a triangle, it can only be of the form vwz', where $v \in V$, $w \in V$, and $z' \in V'$, so that vz' and wz' are edges of $\mu(G)$. This means, by the definition of $\mu(G)$, that vz and wz are edges of G and hence vwz is a triangle in G, a contradiction. □

Theorem 7.5.7 (Mycielski [144]). *For any positive integer p, there exists a triangle-free graph with chromatic number p.*

Proof. For $p = 1, 2$, the result is trivial. [For $p = 1$, take $G = K_1$, and for $p = 2$, take $G = K_2$. For $p = 3$, take $G = \mu(K_2)$. $\mu(K_2) = C_5$ is triangle-free and $\chi(C_5) = 3$.] For $p \geq 3$, by Theorems 7.5.5 and 7.5.6, the iterated Mycielskian $\mu^{p-2}(K_2) = \mu(\mu^{p-3}(K_2))$ is triangle-free and has chromatic number p. □

Remark 7.5.8. The graph $\mu^2(K_2) = \mu(C_5)$ is the Grötzsch graph of Fig. 7.6.

Theorem 7.5.9. *If G is critical, then so is $\mu(G)$.*

Proof. Assume that G is k-critical. Since by Theorem 7.4.5, $\chi(\mu(G)) = k + 1$, we have to show that $\mu(G)$ is $(k + 1)$-critical.

Start with a $(k+1)$-coloring c with colors $1, 2, \ldots, k+1$ of $\mu(G)$ with vertex set $\{V, V', u\}$.

We first show that $\chi(\mu(G) - u) = k$. Without loss of generality, assume that $c(u) = 1$. Then 1 is not represented in V'. Let S be the set of vertices receiving the color 1 in V under c. Recolor each vertex v of S by the color of its twin $v' \in V'$. This gives a proper coloring of $\mu(G)-u$ using k colors and hence $\chi(\mu(G)-u) = k$. [Recall that adjacency of v and w in G implies adjacency of vw' and $v'w$ in $\mu(G)$.]

Next remove a vertex v' of V' from $\mu(G)$. Without loss of generality, assume that $c(u) = 1$ and $c(v') = 2$. Now recolor the vertices of $G - v$ by the $k - 1$ colors $3, \ldots, k, k+1$ (this is possible as G is k-critical) and recolor the vertices of $V' - v'$, if necessary, by the colors of their twins in $V - v$. Also, give color 1 to v. This coloring of $\mu(G) - v'$ misses the color 2 and gives a proper k-coloring to $\mu(G) - v'$.

Lastly, we give a k-coloring to $\mu(G) - v$, $v \in V$. Color the vertices of $G - v$ by $1, 2, \ldots, k-1$ so that the resulting coloring of $G - v$ is proper. Let A be the subset of $G - v$ whose vertices have received color 1 in this new coloring and $A' \subset V'$ denote the set of twins of the vertices in A. Now color the vertices of $(V' \backslash A') - v'$ by the colors of their twins in G, the vertices of $A' \cup \{v'\}$ by color k, and u by color 1. This coloring is a proper coloring of $\mu(G) - v$, which misses the color $k+1$ in the list $\{1, 2, \ldots, k+1\}$. Thus, $\mu(G)$ is $(k+1)$-critical. □

Remark 7.5.10. Apply Theorem 7.5.12 to observe that for each $k \geq 1$, there exists a k-critical triangle-free graph. Not every k-critical graph is triangle-free; for example, the complete graph K_k $(k \geq 3)$ is $k-$critical but is not triangle-free.

Lemma 7.5.11. *Let $f : G \to H$ be a graph isomorphism of G onto H. Then $f(N_G(x)) = N_H(f(x))$. Further, $G - x \simeq H - f(x)$, and $G - N_G[x] \simeq H - N_H[f(x)]$ under the restriction maps of f to the respective domains.*

Proof. The proof follows from the definition of graph isomorphism. □

Theorem 7.5.12 ([13]). *For connected graphs G and H, $\mu(G) \simeq \mu(H)$ if and only if $G \simeq H$.*

Proof. If $G \simeq H$, then trivially $\mu(G) \simeq \mu(H)$. So assume that G and H are connected and that $\mu(G) \simeq \mu(H)$. When $n = 2$ or 3, the result is trivial. So assume that $n \geq 4$. If G is of order n, then $\mu(G)$ and $\mu(H)$ are both of order $2n+1$, and so H is also of order n. Let $f : \mu(G) \to \mu(H)$ be the given isomorphism, where $V(\mu(G))$ and $V(\mu(H))$ are given by the triads (V_1, V_1', u_1) and (V_2, V_2', u_2), respectively.

We look at the possible images of the root u_1 of $\mu(G)$ under f. Both u_1 and u_2 are vertices of degree n. If $f(u_1) = u_2$, then by Lemma 7.5.11, $G = \mu(G) - N[u_1] \simeq \mu(H) - N[u_2] = H$.

Next we claim that $f(u_1) \notin V_2$. Suppose $f(u_1) \in V_2$. Since $d_{\mu(H)}(f(u_1)) = d_{\mu(G)}(u_1) = n$, it follows from the definition of the Mycielskian that in $\mu(H)$, $\frac{n}{2}$ neighbors of $f(u_1)$ belong to V_2 while another $\frac{n}{2}$ neighbors (the twins) belong to V_2'. (This forces n to be even.) These n neighbors of $f(u_1)$ form an independent subset of $\mu(H)$. Then $H' = \mu(H) - N_{\mu(H)}[f(u_1)] \simeq \mu(G) - N_{\mu(G)}[u_1] = G$.

Now if $x \in V_2$ is adjacent to $f(u_1)$ in $\mu(H)$, then x is adjacent to $f(u_1)'$, the twin of $f(u_1)$ belonging to V_2' in $\mu(H)$. Further, $d_{H'}(f(u_1)') = 1 = d_G(v)$, where $v \in V_1$ (the vertex set of G) corresponds to $f(u_1)'$ in $\mu(H)$. But then $d_{\mu(G)}(v) = 2$, while $d_{\mu(H)}(f(u)') = \frac{n}{2} + 1 > 2$, as $n \geq 4$. Hence, this case cannot arise.

Finally, suppose that $f(u_1) \in V_2'$. Set $f(u_1) = y'$. Then y, the twin of y' in $\mu(H)$, belongs to V_2. As $d_{\mu(G)}(u_1) = n$, $d_{\mu(H)}(y') = n$. The vertex y' has $n-1$ neighbors in V_2, say, $x_1, x_2, \ldots, x_{n-1}$. Then $N_H(y) = \{x_1, x_2, \ldots, x_{n-1}\}$, and hence y is also adjacent to $x_1', x_2', \ldots, x_{n-1}'$ in V_2'. Further, as $N_{\mu(G)}(u_1)$ is independent, $N_{\mu(H)}(y')$ is also independent. Therefore, $H =$ star $K_{1,n-1}$ consisting of the edges $yx_1, yx_2, \ldots, yx_{n-1}$. Moreover, $G = \mu(G) - N[u_1] \simeq \mu(H) - N[y'] =$ star $K_{1,n-1}$ consisting of the edges $yx_1', yx_2', \ldots, yx_{n-1}'$. Thus, $G \simeq K_{1,n-1} \simeq H$. □

7.6 Edge Colorings of Graphs

7.6.1 *The Timetable Problem*

Suppose in a school there are r teachers, $T_1, T_2, \ldots, T_r$, and s classes, $C_1, C_2, \ldots, C_s$. Each teacher T_i is expected to teach the class C_j for p_{ij} periods. It is clear that during any particular period, no more than one teacher can handle a particular class and no more than one class can be engaged by any teacher. Our aim is to draw up a timetable for the day that requires only the minimum number of periods. This problem is known as the "timetable problem."

To convert this problem into a graph-theoretic one, we form the bipartite graph $G = G(T, C)$ with bipartition (T, C), where T represents the set of teachers T_i and C represents the set of classes C_j. Further, T_i is made adjacent to C_j in G with p_{ij} parallel edges if and only if teacher T_i is to handle class C_j for p_{ij} periods. Now color the edges of G so that no two adjacent edges receive the same color. Then the edges in a particular color class, that is, the edges in that color, form a matching in G and correspond to a schedule of work for a particular period. Hence, the minimum number of periods required is the minimum number of colors in an edge coloring of G in which adjacent edges receive distinct colors; in other words, it is the edge-chromatic number of G. We now present these notions as formal definitions.

Definition 7.6.1. An *edge coloring* of a loopless graph G is a function $\pi : E(G) \to S$, where S is a set of distinct colors; it is *proper* if no two adjacent edges receive the same color. Thus, a proper edge coloring π of G is a function $\pi : E(G) \to S$ such that $\pi(e) \neq \pi(e')$ whenever edges e and e' are adjacent in G, and it is a proper k-edge coloring of G if $|S| = k$.

Definition 7.6.2. The minimum k for which a loopless graph G has a proper k-edge coloring is called the *edge-chromatic number* or *chromatic index* of G. It is denoted by $\chi'(G)$. G is *k-edge-chromatic* if $\chi'(G) = k$.

Further, if an edge uv is colored by color c, we say that c is represented at both u and v. If G has a proper k-edge coloring, $E(G)$ is partitioned into k edge-disjoint matchings.

It is clear that for any (loopless) graph G, $\chi'(G) \geq \Delta(G)$ since the $\Delta(G)$ edges incident at a vertex v of maximum degree $\Delta(G)$ must all receive distinct colors. For bipartite graphs, however, equality holds.

Theorem 7.6.3 (König). *If G is a bipartite graph, $\chi'(G) = \Delta(G)$.*

Proof. The proof is by induction on the size (i.e., number of edges) m of G. The result is true for $m = 1$. Assume the result for bipartite graphs of size at most $m-1$. Let G have m edges. Let $e = uv \in E(G)$. Then $G-e$ has [since $\Delta(G-e) \leq \Delta(G)$] a proper Δ-edge coloring, say c. Out of these Δ colors, suppose that one particular color is not represented at both u and v. Then in this coloring the edge uv can be colored with this color, and a proper Δ-edge coloring of G is obtained.

In the other case (that is, in the case in which each of the Δ colors is represented either at u or at v in $G-e$), since the degrees of u and v in $G-e$ are at most $\Delta - 1$, there exists a color out of the Δ colors that is not represented in $G-e$ at u, and similarly there exists a color not represented at v. Thus, if color j is not represented at u in c, then j is represented at v in c, and if color i is not represented at v in c, then i is represented at u in c. Since G is bipartite and u and v are not in the same parts of the bipartition, there can exist no u-v path in G in which the colors alternate between i and j.

Let P be a maximal path in $G-e$ starting from u in which the colors of the edges alternate between i and j. Interchange the colors i and j in P. This would still yield a proper edge coloring of $G-e$ using the Δ colors in which color i is not represented at both u and v. Now color the edge uv by the color i. This results in a proper Δ-edge coloring of G. □

Exercise 6.1. Disprove the converse of Theorem 7.6.3 by a counterexample.

Next, we determine the chromatic index of the complete graphs.

Theorem 7.6.4. $\chi'(K_n) = \begin{cases} n-1 & \textit{if } n \textit{ is even,} \\ n & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. (Berge) Since K_n is regular of degree $n-1$, $\chi'(K_n) \geq n-1$.

Case 1. n is even. We show that $\chi'(K_n) \leq n-1$ by exhibiting a proper $(n-1)$-edge coloring of K_n. Label the n vertices of K_n as $0, 1, \ldots, n-1$. Draw a circle with center at 0 and place the remaining $n-1$ numbers on the circumference of the circle so that they form a regular $(n-1)$-gon (Fig. 7.7). Then the $\frac{n}{2}$ edges $(0,1)$, $(2, n-1)$, $(3, n-2)$, $\ldots$, $(\frac{n}{2}, \frac{n}{2}+1)$ form a 1-factor of K_n. These $\frac{n}{2}$ edges are the thick edges of Fig. 7.7. Rotation of these edges through the angle $\frac{2\pi}{n-1}$ in succession gives $(n-1)$ edge-disjoint 1-factors of K_n. This would account for $\frac{n}{2}(n-1)$ edges and hence all the edges of K_n. (Actually, the above construction displays a 1-factorization of K_n when n is even.) Each 1-factor can be assigned a distinct color. Thus, $\chi'(K_n) \leq n-1$. This proves the result in Case 1.

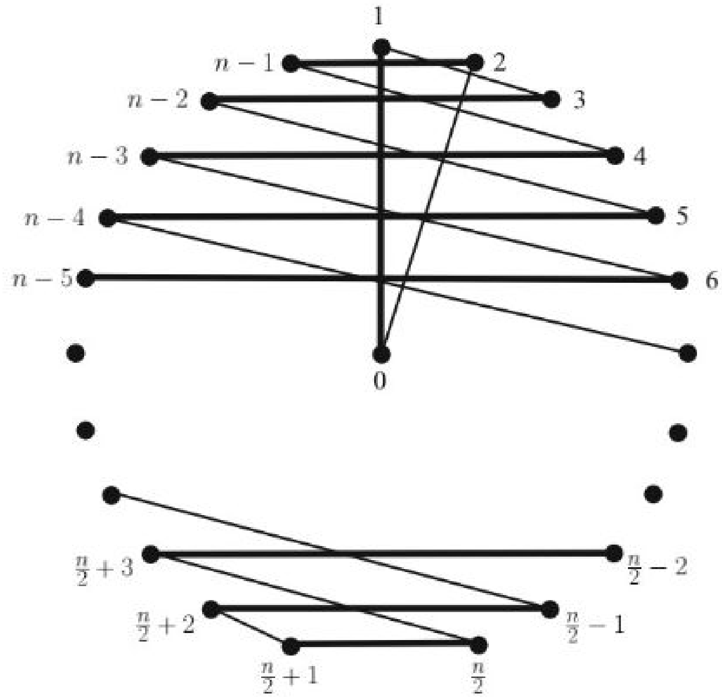


Fig. 7.7 Graph for proof of Theorem 7.6.4

Case 2. n is odd. Take a new vertex and make it adjacent to all the n vertices of K_n. This gives K_{n+1}. By Case 1, $\chi'(K_{n+1}) = n$. The restriction of this edge coloring to K_n yields a proper n-edge coloring of K_n. Hence, $\chi'(K_n) \leq n$. However, K_n cannot be edge colored properly with $n-1$ colors. This is because the size of any matching of K_n can contain no more than $\frac{n-1}{2}$ edges, and hence $n-1$ matchings of K_n can contain no more than $\frac{(n-1)^2}{2}$ edges. But K_n has $\frac{n(n-1)}{2}$ edges. Thus, $\chi'(K_n) \geq n$, and hence $\chi'(K_n) = n$. □

Exercise 6.2. Show that a Hamiltonian cubic graph is 3-edge-chromatic.

Exercise 6.3. Show that the Petersen graph is 4-edge-chromatic.

Exercise 6.4. Show that the Herschel graph (see Fig. 5.4) is 4-edge-chromatic.

Exercise 6.5. Determine the edge-chromatic number of the Grötzsch graph (Fig. 7.6).

Exercise 6.6. Show that a simple cubic graph with a cut edge is 4-edge-chromatic.

Exercise 6.7. Describe a proper k-edge coloring of a k-regular bipartite graph.

Exercise 6.8. Show that any bipartite graph G of maximum degree Δ is a subgraph of a Δ-regular bipartite graph. Hence, furnish an alternative proof of Theorem 7.6.3, using Exercise 6.7.

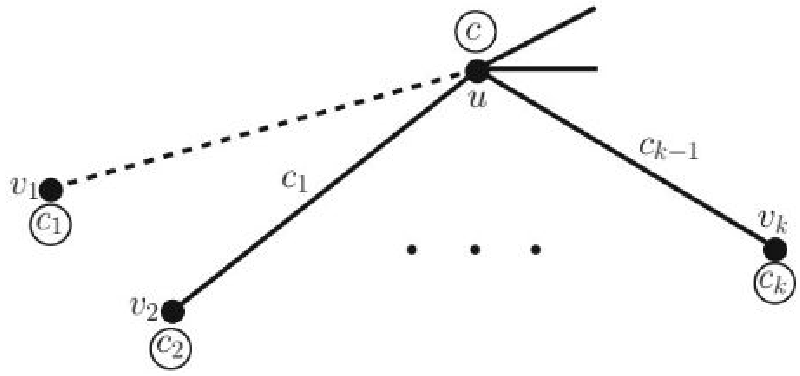


Fig. 7.8 Graph for proof of Theorem 7.6.5

7.6.2 Vizing's Theorem

Although it is true that for any loopless graph G, $\chi'(G) \geq \Delta(G)$, it turns out that for any simple graph G, $\chi'(G) \leq 1 + \Delta(G)$. This major result in edge coloring of graphs was established by Vizing [183] and independently by Gupta [81].

Theorem 7.6.5 (Vizing-Gupta). *For any simple graph* G, $\Delta(G) \leq \chi'(G) \leq 1 + \Delta(G)$.

Proof. In a proper edge coloring of G, $\Delta(G)$, colors are to be used for the edges incident at a vertex of maximum degree in G. Hence, $\chi'(G) \geq \Delta(G)$.

We now prove that $\chi'(G) \leq 1 + \Delta$, where $\Delta = \Delta(G)$.

If G is not $(1 + \Delta)$-edge-colorable, choose a subgraph H of G with a maximum possible number of edges such that H is $(1 + \Delta)$-edge-colorable. We derive a contradiction by showing that there exists a subgraph H_0 of G that is $(1 + \Delta)$-edge-colorable and has one edge more than H.

By our assumption, G has an edge $uv_1 \notin E(H)$. Since $d(u) \leq \Delta$, and $1 + \Delta$ colors are being used in H, there is a color c that is not represented at u (i.e., not used for any edge of H incident at u). For the same reason, there is a color c_1 not represented at v_1. (See Fig. 7.8, where the color not represented at a particular vertex is enclosed in a circle and marked near the vertex.)

There must be an edge, say uv_2 of H, colored c_1; otherwise, uv_1 can be assigned the color c_1, and $H \cup (uv_1)$, which has one edge more than H, would have a proper $(1 + \Delta)$-edge coloring. Again, there is a color, say c_2, not represented at v_2. Then as above, there is an edge uv_3 colored c_2 and there is a color, say c_3, not represented at v_3.

In this way, we construct a sequence of edges $\{uv_1, uv_2, \ldots, uv_k\}$ such that color c_i is not represented at vertex v_i, $1 \leq i \leq k$, and the edge uv_{j+1} receives the color c_j, $1 \leq j \leq k - 1$ (see Fig. 7.8).

Suppose at some stage, say the rth stage, where $1 \leq r \leq k$, c (the missing color at u) is not represented at v_r. We then "cascade" (i.e., shift in order) the colors $c_1, \ldots, c_{r-1}$ from $uv_2, uv_3, \ldots, uv_r$ to $uv_1, uv_2, \ldots, uv_{r-1}$. Under this new coloring,

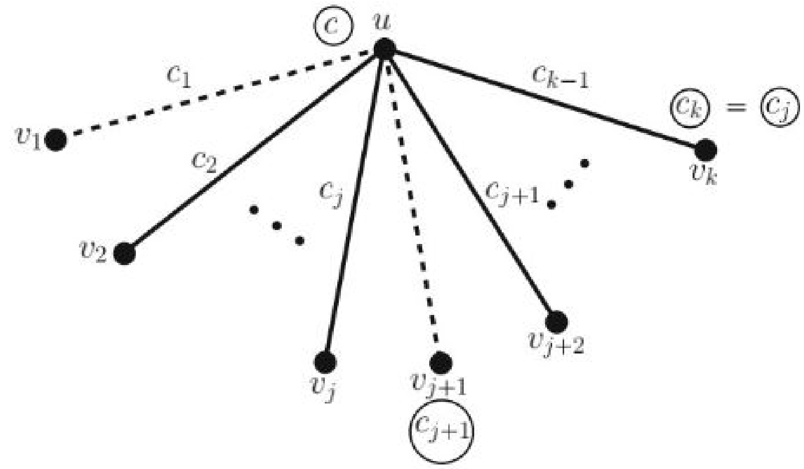


Fig. 7.9 Another graph for proof of Theorem 7.6.5

c is not represented both at u and at v_r, and therefore we can color uv_r with c. This yields a proper $(1 + \Delta)$-edge coloring to $H \cup (uv_1)$, contradicting the choice of H. Hence, we may assume that c is represented at each of the vertices $v_1, v_2, \ldots, v_k$.

Now we need to know why the sequence of edges uv_i, $1 \le i \le k$, had stopped. There are two possible reasons. Either there is no edge incident to u that is colored c_k, or the color $c_k = c_j$ for some $j < k-1$ and so has already been represented at u. Note that the sequence must stop at some finite stage since $d(u)$ is finite; however, it may as well stop before all the edges incident to u are exhausted.

If c_k is not represented at u in H, then we can cascade as before so that uv_i gets color c_i, $1 \le i \le k-1$, and then color uv_k with color c_k. Once again, we have a contradiction to our assumption on H.

Thus, we must have $c_k = c_j$ for come $j < k - 1$. In this case, cascade the colors $c_1, c_2, \ldots, c_j$ so that uv_i has color c_i, $1 \le i \le j$, and leave uv_{j+1} uncolored (Fig. 7.9). Let $S = (H \cup (uv_1)) - uv_{j+1}$. Then S and H have the same number of edges.

Now consider S_{cc_j}, the subgraph of S defined by the edges of S with colors c and c_j. Clearly, each component of S_{cc_j} is either an even cycle or a path in which the adjacent edges alternate with colors c and c_j.

Now, c is represented at each of the vertices $v_1, v_2, \ldots, v_k$, and in particular at v_{j+1} and v_k. But c_j is not represented at v_{j+1} and v_k, since we have just moved c_j to uv_j, and $c_j = c_k$ is not represented at v_k. Hence in S_{cc_j}, the degrees of v_{j+1} and v_k are both equal to 1. Moreover, c_j is represented at u, but c is not. Therefore, u also has degree 1 in S_{cc_j}. As each component of S_{cc_j} is either a path or an even cycle, not all of u, v_{j+1}, and v_k can be in the same component of S_{cc_j} (since a nontrivial path has only two vertices of degree 1).

If u and v_{j+1} are in different components of S_{cc_j}, interchange the colors c and c_{j+1} in the component containing v_{j+1}. Then c is not represented at both u and v_{j+1}, and so we can color the edge uv_{j+1} with c. This gives a $(1+\Delta)$-edge coloring to the graph $S \cup (uv_{j+1})$.

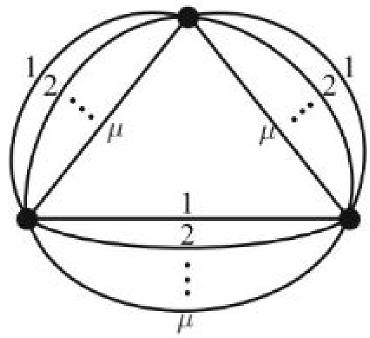


Fig. 7.10 Graph illustrating the generalized Vizing's theorem

Suppose then u and v_{j+1} are in the same components of S_{cc_j}. Then, necessarily, v_k is not in this component. Interchange c and c_j in the component containing v_k. In this case, *further* cascade the colors so that uv_i has color c_i, $1 \leq i \leq k-1$. Now color uv_k with color c.

Thus, we have extended our edge coloring of S with $1 + \Delta$ colors to one more edge of G. This contradiction proves that $H = G$, and thus $\chi'(G) \leq 1 + \Delta$. □

Actually, Vizing proved a more general result than the one given above. Let G be any loopless graph and let μ denote the maximum number of edges joining two vertices in G. Then the generalized Vizing's theorem states that $\Delta \leq \chi' \leq \Delta + \mu$. This theorem is the best possible in that there are graphs with $\chi' = \Delta + \mu$. For example, let G be the graph of Fig. 7.10. Since any two edges of G are adjacent, $\chi' = m(G) = 3\mu = \Delta + \mu$. For a proof of the generalized Vizing's theorem, see Yap [194].

Definition 7.6.6. Graphs for which $\chi' = \Delta$ are called *Class* 1 graphs and those for which $\chi' = 1 + \Delta$ are called *Class* 2 graphs.

Example 7.6.7. Bipartite graphs are of class 1 (see Theorem 7.6.3), whereas the Petersen graph (see Exercise 6.3) and any simple cubic graph with a cut edge (see Exercise 6.6) are of class 2.

For details relating to graphs of class 1 and class 2, see [62, 194].

Exercise 6.9. Let G be a simple Δ-edge-chromatic critical graph [i.e., G is of class 1 and for every edge e of G, $\chi'(G - e) < \chi'(G)$]. Prove that if $uv \in E(G)$, then $d(u) + d(v) \geq \Delta + 2$.

We now return to the timetable problem. Following are some examples of such a problem.

Problem 1. In a social health checkup scheme, specialist physicians are to visit various health centers. Given the places each physician has to visit and also the time interval of his or her visit, how can we fit in an itinerary? The assumption is that each health center can accommodate only one doctor at a time.

Problem 2. Mobile laboratories are to visit various schools in a city. Given the places each lab has to visit and also the time interval (period) of visits in a day, how can we fit in a timetable for the laboratories?

Problem 3. In an educational institution, as is well known, teachers have to instruct various classes. Given the various classes each teacher has to instruct in a day, how can we fit in a timetable? It is presumed that a teacher can teach only one class at a time and that each class could be taught by only one teacher at a time!

We shall now discuss Problem 3. Let $x_1, x_2, \ldots, x_n$ denote the teachers and $y_1, y_2, \ldots, y_m$ the classes. Let t_{ij} denote the number of periods for which teacher x_i has to meet class y_j. How can we draw up a timetable? If there are constraints on the availability of classrooms, what is the minimum number of periods required to implement a timetable? If the number of periods in a day is specified, what is the minimum number of rooms required to implement the timetable? All these problems could be analyzed by using a suitable graph.

Let $G(T, C)$ be a bipartite graph formed with $T = \{x_1, x_2, \ldots, x_p\}$ and $C = \{y_1, y_2, \ldots, y_q\}$ as the bipartition and in which there are t_{ij} parallel edges with x_i and y_j as their common ends. If T denotes the set of teachers and C the set of classrooms, a teaching assignment for a period determines a matching in the bipartite graph G. Conversely, any matching in G corresponds to a teaching assignment for one period. The edges of G could be partitioned into Δ edge-disjoint matchings (see Theorem 7.6.3). Corresponding to the Δ matchings, a Δ-period timetable can be drawn up.

Let N be the total number of periods to be taught by all teachers put together. Then, on average, N/Δ classes are to be taught per period. Hence, at least $\lceil N/\Delta \rceil$ rooms are necessary to implement a Δ-period timetable. We present below a method for drawing up such a timetable. For this, we need Lemma 7.6.8.

Lemma 7.6.8. *Let M and N be disjoint matchings of a graph G with $|M| > |N|$. Then there are disjoint matchings M' and N' of G with $|M'| = |M| - 1$ and $|N'| = |N| + 1$ and with $M' \cup N' = M \cup N$.*

Proof. Consider the subgraph $H = G[M \cup N]$. Each component of H is either an even cycle or a path with edges alternating between M and N. Since $|M| > |N|$, some path component P of H must have its initial and terminal edges in M. Let $P = v_0 e_1 v_1 e_2 v_2 \ldots e_{2r+1} v_{2r+1}$.

Now set

$$M' = (M \setminus \{e_1, e_3, \ldots, e_{2r+1}\}) \cup \{e_2, e_4, \ldots, e_{2r}\}$$

and

$$N' = (N \setminus \{e_2, e_4, \ldots, e_{2r}\}) \cup \{e_1, e_3, \ldots, e_{2r+1}\}.$$

Then M' and N' are disjoint matchings of G satisfying the conditions of the lemma. □

Theorem 7.6.9. *If G is a bipartite graph (with m edges), and if $m \geq t \geq \Delta$, then there exist t disjoint matchings $M_1, M_2, \ldots, M_t$ of G such that*

$$E = M_1 \cup M_2 \cup \ldots \cup M_t$$

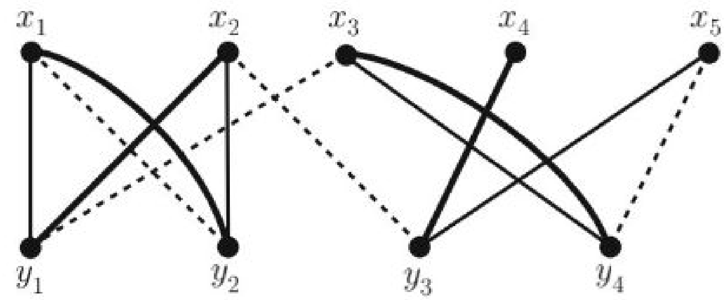


Fig. 7.11 Bipartite graph corresponding to Problem 1

and, for $1 \leq i \leq t$,

$$\lfloor m/t \rfloor \leq |M_i| \leq \lceil m/t \rceil .$$

(In other words, any connected bipartite graph G is equitably t-edge-colorable, where $m \geq t \geq \Delta$.)

Proof. By Theorem 7.6.3, $\chi' = \Delta$. Hence, $E(G)$ can be partitioned into Δ matchings $M'_1, M'_2, \ldots, M'_\Delta$. So for $t \geq \Delta$, there exist disjoint matchings $M'_1, M'_2, \ldots, M'_t$, where $M'_i = \emptyset$ for $\Delta + 1 \leq i \leq t$, and

$$E = M'_1 \cup M'_2 \cup \ldots M'_t.$$

Now repeatedly apply Lemma 7.6.8 to pairs of matchings that differ by more than one in size. This would eventually result in matchings $M_1, M_2, \ldots, M_t$ of G satisfying the condition stated in the theorem. □

Coming back to our timetable problem, if the number of rooms available, say r, is less than N/Δ (so that $N/r > \Delta$), then the number of periods is to be correspondingly increased. Hence, starting with an edge partition of $E(G)$ into matchings $M'_1, M'_2, \ldots, M'_\Delta$, we apply Lemma 7.6.8 repeatedly to get an edge partition of $E(G)$ into disjoint matchings $M_1, M_2, \ldots, M_{\lceil N/r \rceil}$. This partition gives a $\lceil N/r \rceil$-period timetable that uses r rooms.

Illustration. The teaching assignments of five professors, x_1, x_2, x_3, x_4, x_5, in the mathematics department of a particular university are given by the following array:

	I Year	II Year	III Year	IV Year
	y_1	y_2	y_3	y_4
x_1	1	2	–	–
x_2	1	1	1	–
x_3	1	–	–	2
x_4	–	–	1	–
x_5	–	–	1	1

The bipartite graph G corresponding to the above problem is shown in Fig. 7.11. Each of the sets of edges drawn by the ordinary lines, dashed lines, and thick lines

Table 7.1 Timetable

		Period		
		I	II	III
Professor:	x_1	y_1	y_2	y_2
	x_2	y_2	y_3	y_1
	x_3	y_4	y_1	y_4
	x_4	–	–	y_3
	x_5	y_3	y_4	–

gives a matching in G. The three matchings cover the edges of G. Hence, they can be the basis of a three-period timetable. The corresponding timetable is given in Table 7.1.

In each period, four classes are to be met. Hence, at least four rooms are needed to implement this timetable. Here $\Delta = 3$ and $N = 12$. Consequently, G could be covered by three matchings each containing $\lfloor 12/3 \rceil$ or $\lceil 12/3 \rceil$ edges, that is, exactly four edges. This gives the edge partition

$$M' = \{M_1', M_2', M_3'\},$$

where

$$M_1' = \{x_1y_1, x_2y_2, x_3y_4, x_5y_3\},$$
$$M_2' = \{x_1y_2, x_2y_3, x_3y_1, x_5y_4\},$$

and

$$M_3' = \{x_1y_2, x_2y_1, x_3y_4.x_4y_3\}.$$

Now, take $M'' = \{M_1', M_2', M_3', M_4' = \emptyset\}$, and apply Lemma 7.6.8. This gives an edge partition $M = \{M_1, M_2, M_3, M_4\}$, where $M_1 = \{x_1y_1, x_2y_2, x_3y_4\}$, $M_2 = \{x_1y_2, x_2y_3, x_5y_4\}$, $M_3 = \{x_2y_1, x_3y_4, x_4y_3\}$, and $M_4 = \{x_5y_3, x_3y_1, x_1y_2\}$. The above partition yields a four-period timetable using three rooms.

7.9 Chromatic Polynomials

In 1946, Birkhoff and Lewis [23] introduced the chromatic polynomial of a graph in their attempt to tackle the four-color problem (see Chap. 8) through algebraic techniques.

For a graph G and a given set of λ colors, the function $f(G;\lambda)$ is defined to be the number of ways of (vertex) coloring G properly using the λ colors. Hence, $f(G;\lambda) = 0$ when G has no proper λ-coloring. Clearly, the minimum λ for which $f(G;\lambda) > 0$ is the chromatic number $\chi(G)$ of G.

It is easy to see that $f(K_n;\lambda) = \lambda(\lambda-1)\ldots(\lambda-n+1)$ for $\lambda \geq n$. This is because any vertex of K_n can be colored by any one of the given λ colors. After coloring a vertex of K_n, a second vertex of K_n can be colored by any one of the remaining $(\lambda - 1)$ colors, and so on. In particular, $f(K_3; \lambda) = \lambda(\lambda - 1)(\lambda - 2)$. Also, $f(K_n^c; \lambda) = \lambda^n$.

Let $e = uv$ be any edge of G. Recall (see Sect. 4.3, Chap. 4) that the graph $G \circ e$ is obtained from G by contracting the edge e. Theorem 7.9.1 presents a simple reduction formula to compute $f(G; \lambda)$.

Theorem 7.9.1. *Let G be any graph. Then* $f(G; \lambda) = f(G - e; \lambda) - f(G \circ e; \lambda)$ *for any edge e of G.*

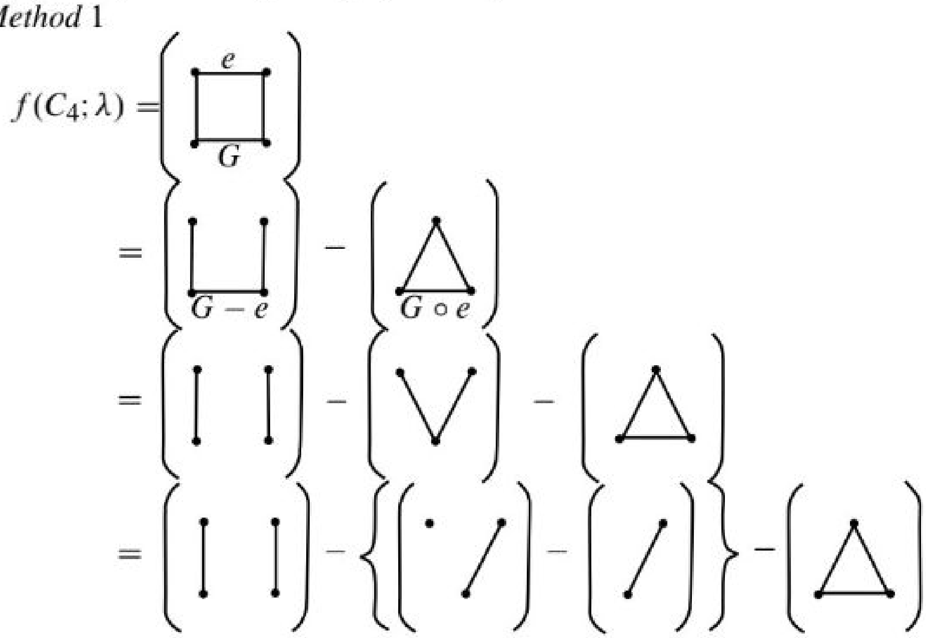
Proof. $f(G - e; \lambda)$ denotes the number of proper colorings of $G - e$ using λ colors. Hence, it is the sum of the number of proper colorings of $G - e$ in which u and v receive the same color and the number of proper colorings of $G - e$ in which u and v receive distinct colors. The former number is $f(G \circ e; \lambda)$, and the latter number is $f(G; \lambda)$. □

Exercise 9.1. If G and H are disjoint graphs, show that

$$f(G \cup H; \lambda) = f(G; \lambda) f(H; \lambda).$$

Theorem 7.9.1 could be used recursively to determine $f(G; \lambda)$ for graphs of small size by taking the given graph on n vertices as G and successively deleting edges until we end up with the totally disconnected graph K_n^c. It can also be determined by taking the given graph as $G - e$ and recursively adding a new edge e until we end up with the complete graph K_n. For a fixed n, when $m(G)$, the number of edges of G is small, the first method is preferable, and when it is large, the second method is preferable. These two methods are illustrated for the graph C_4. [Here the function $f(G; \lambda)$ is represented by the graph itself.]

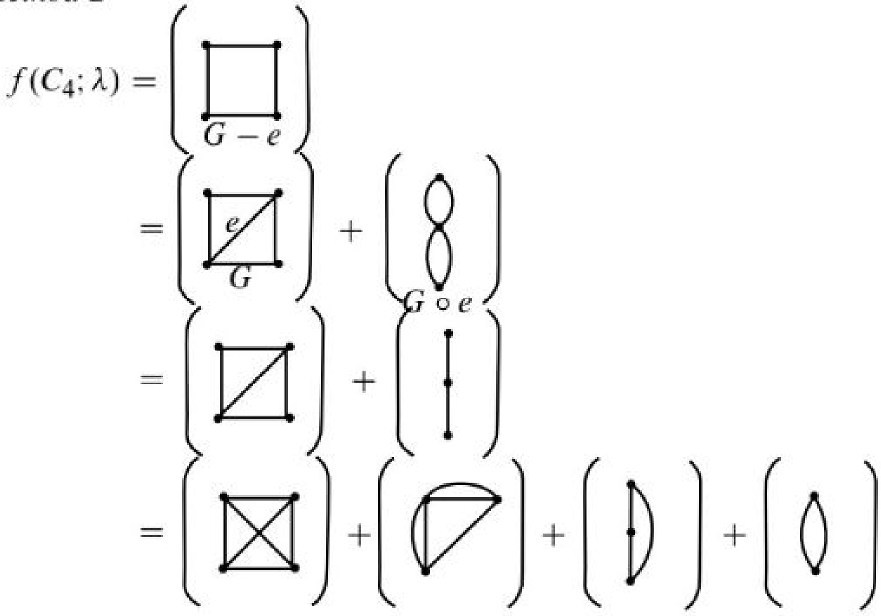
Method 1



$$= (\lambda(\lambda - 1)^2 - \{\lambda^2(\lambda - 1) - \lambda(\lambda - 1)\} - \lambda(\lambda - 1)(\lambda - 2)$$
$$= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda.$$

Method 2

4



$$
\begin{aligned}
&= f(K_4;\lambda) + f(K_3;\lambda) + f(K_3;\lambda) + f(K_2;\lambda)\\
&= f(K_4;\lambda) + 2f(K_3;\lambda) + f(K_2;\lambda)\\
&= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + 2\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1)\\
&= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda.
\end{aligned}
$$

The function $f(C_4;\lambda)$ computed above is a monic polynomial with integer coefficients of degree $n = 4$ in which the coefficient of $\lambda^3 = -4 = -m$, the constant term is zero, and the coefficients alternate in sign. That this is the case with all such functions $f(G;\lambda)$ is the content of Theorem 7.9.2. For this reason, the function $f(G;\lambda)$ is called the *chromatic polynomial* of the graph G.

Theorem 7.9.2. *For a simple graph G of order n and size m, $f(G;\lambda)$ is a monic polynomial of degree n in λ with integer coefficients and constant term zero. In addition, its coefficients alternate in sign and the coefficient of λ^{n-1} is $-m$.*

Proof. The proof is by induction on m. If $m = 0$, G is K_n^c and $f(K_n^c;\lambda) = \lambda^n$, and if $m = 1$, G is K_2 and $f(K_2;\lambda) = \lambda^2 - \lambda$, and the statement of the theorem is trivially true in these cases. Suppose now that the theorem holds for all graphs with fewer than m edges, where $m \geq 2$. Let G be any simple graph of order n and size m, and let e be any edge of G. Both $G - e$ and $G \circ e$ (after removal of multiple edges, if necessary) are simple graphs with at most $m - 1$ edges, and hence, by the induction hypothesis,

$$f(G-e;\lambda) = \lambda^n - a_0\lambda^{n-1} + a_1\lambda^{n-2} - \ldots + (-1)^{n-1}a_{n-2}\lambda,$$

and

$$f(G\circ e;\lambda) = \lambda^{n-1} - b_1\lambda^{n-2} + \ldots + (-1)^{n-2}b_{n-2}\lambda,$$

where $a_0, \ldots, a_{n-2}; b_1, \ldots, b_{n-2}$ are nonnegative integers (so that the coefficients alternate in sign), and a_0 is the number of edges in $G - e$, which is $m - 1$. By Theorem 7.9.1, $f(G;\lambda) = f(G-e;\lambda) - f(G\circ e;\lambda)$, and hence

$$f(G;\lambda) = \lambda^n - (a_0+1)\lambda^{n-1} + (a_1+b_1)\lambda^{n-2} - \ldots + (-1)^{n-1}(a_{n-2}+b_{n-2})\lambda.$$

Since $a_0 + 1 = m$, $f(G;\lambda)$ has all the stated properties. □

Theorem 7.9.3. *A simple graph G on n vertices is a tree if and only if* $f(G;\lambda) = \lambda(\lambda-1)^{n-1}$.

Proof. Let G be a tree. We prove that $f(G;\lambda) = \lambda(\lambda-1)^{n-1}$ by induction on n. If $n = 1$, the result is trivial. So assume the result for trees with at most $n-1$ vertices, $n \geq 2$. Let G be a tree with n vertices, and e be a pendent edge of G. By Theorem 7.9.1, $f(G;\lambda) = f(G-e;\lambda) - f(G\circ e;\lambda)$. Now, $G-e$ is a forest with two component trees of orders $n-1$ and 1, and hence $f(G-e;\lambda) = (\lambda(\lambda-1)^{n-2})\lambda$ (see Exercise 9.1). Since $G \circ e$ is a tree with $n-1$ vertices, $f(G\circ e;\lambda) = \lambda(\lambda-1)^{n-2}$. Thus, $f(G;\lambda) = (\lambda(\lambda-1)^{n-2})\lambda - \lambda(\lambda-1)^{n-2} = \lambda(\lambda-1)^{n-1}$.

Conversely, assume that G is a simple graph with $f(G;\lambda) = \lambda(\lambda-1)^{n-1} = \lambda^n - (n-1)\lambda^{n-1} + \ldots + (-1)^{n-1}\lambda$. Hence, by Theorem 7.9.2, G has n vertices and $n-1$ edges. Further, the last term, $(-1)^{n-1}\lambda$, ensures that G is connected (see Exercise 9.2). Hence, G is a tree (see Theorem 4.2.4). □

Remark 7.9.4. Theorem 7.9.3 shows that the chromatic polynomial of a graph G does not fix the graph uniquely up to isomorphism. For example, even though the graphs $K_{1,3}$ and P_4 are not isomorphic, they have the same chromatic polynomial, namely, $\lambda(\lambda-1)^3$.

Exercise 9.2. If G has ω components, show that λ^{ω} is a factor of $f(G;\lambda)$.

Exercise 9.3. Show that there exists no graph with the following polynomials as chromatic polynomial (i) $\lambda^5 - 4\lambda^4 + 8\lambda^3 - 4\lambda^2 + \lambda$; (ii) $\lambda^4 - 3\lambda^3 + \lambda^2$; (iii) $\lambda^7 - \lambda^6 + 1$.

Exercise 9.4. Find a graph G whose chromatic polynomial is $\lambda^5 - 6\lambda^4 + 11\lambda^3 - 6\lambda^2$.

Exercise 9.5. Show that for the cycle C_n of length n, $f(C_n;\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)$, $n \geq 3$.

Exercise 9.6. Show that for any graph G, $f(G \vee K_1;\lambda) = \lambda f(G;\lambda-1)$, and hence prove that $f(W_n;\lambda) = \lambda(\lambda-2)^n + (-1)^n\lambda(\lambda-2)$.

Notes

A good reference for graph colorings is the book by Jensen and Toft [116]. The book by Fiorini and Wilson [62] concentrates on edge colorings. Theorem 7.5.7 (Mycielski's theorem) has also been proved independently by Blanche Descartes [50] as well as by Zykov [195]. For a complete description of graph homomorphisms, see [105].

The proof of Brooks' theorem given in this chapter is based on the proof given by Fournier [67] (see also references [27] and [106]).

The term "snark" was given to the snark graph by Martin Gardner after the unusual creature that is described in Lewis Carroll's poem, T*he Hunting of the Snark*. A detailed account of the snarks, including their constructions, can be found in the interesting book by Holton and Sheehan [106].

UNIT -5

Planarity

8.1 Introduction

The study of planar and nonplanar graphs and, in particular, the several attempts to solve the *four-color conjecture* have contributed a great deal to the growth of graph theory. Actually, these efforts have been instrumental lo the development of algebraic, topological, and computational techniques in graph theory.

In this chapter, we present some of the basic results on planar graphs. In particular, the two important characterization theorems for planar graphs, namely, Wagner's theorem (same as the Harary-Tutte theorem) and Kuratowski's theorem, are presented. Moreover, the nonhamiltonicity of the Tutte graph on 46 vertices (see Fig. 8.28 and also the front wrapper) is explained in detail.

8.2 Planar and Nonplanar Graphs

Definition 8.2.1. A graph G is *planar* if there exists a drawing of G in the plane in which no two edges intersect in a point other than a vertex of G. where each edge is a Jordan arc (that is, a simple arc). Such a drawing of a planar graph G is called a *plane representation* of G. In this case, we also say that G has been embedded in the plane. A *plane graph* is a planar graph that has already been embedded in the plane.

Example 8.2.2. There exist planar as well as nonplanar graphs. In Fig. 8.1, a planar graph and two of its plane representations are shown. Note that all trees are planar as also are cycles and wheels. The Petersen graph is nonplanar (a proof of this result is given later in this chapter.).

Before proceeding further, let us recall here the celebrated Jordan curve theorem. If J is any closed Jordan curve in the plane, the complement of J (with respect

R. Balakrishnan and K. Ranganathan. *A Textbook of Graph Theory,*
Universitext, DOI 10.1007/978-1 -4614-4529-6_8,

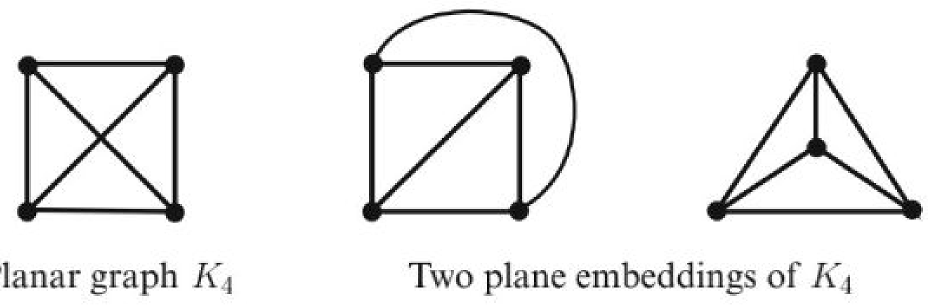
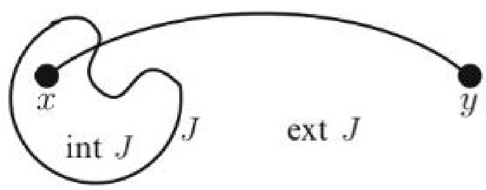


Fig. 8.1 A planar graph with two plane embeddings

Fig. 8.2 Arc connecting point x in int J with point y in ext J



to the plane) is partitioned into two disjoint open connected subsets of the plane, one of which is bounded and the other unbounded. The bounded subset is called the *interior* of J and is denoted by int J. The unbounded subset is called the *exterior* of J and is denoted by ext J. The *Jordan curve theorem* (of topology) states that if J is any closed Jordan curve in the plane, any arc joining a point of int J and a point of ext J must intersect J at some point (see Fig. 8.2) (the proof of this result, although intuitively obvious, is tedious).

Let G be a plane graph. Then the union of the edges (as Jordan arcs) of a cycle C of G form a closed Jordan curve, which we also denote by C. A plane graph G divides the rest of the plane (i.e., plane minus the edges and vertices of G), say π, into one or more faces, which we define below. We define an equivalence relation $\sim$ on π.

Definition 8.2.3. We say that for points A and B of π, $A \sim B$ if and only if there exists a Jordan arc from A to B in π. Clearly, $\sim$ is an equivalence relation on π. The equivalence classes of the above equivalence relation are called the *faces* of G.

Remark 8.2.4. 1. We claim that a connected graph is a tree if and only if it has only one face. Indeed, since there are no cycles in a tree T, the complement of a plane embedding of T in the plane is connected (in the above sense), and hence a tree has only one face. Conversely, it is clear that if a connected plane graph has only one face, then it must be a tree.

2. Any plane graph has exactly one unbounded face. The unbounded face is also referred to as the exterior face of the plane graph. All other faces, if any, are bounded. Figure 8.3 represents a plane graph with seven faces.

The distinction between bounded and unbounded faces of a plane graph is only superfluous, as there exists a plane representation G_1 of a plane graph G in which any specified face of G_1 becomes the unbounded face, as is shown below. (This of

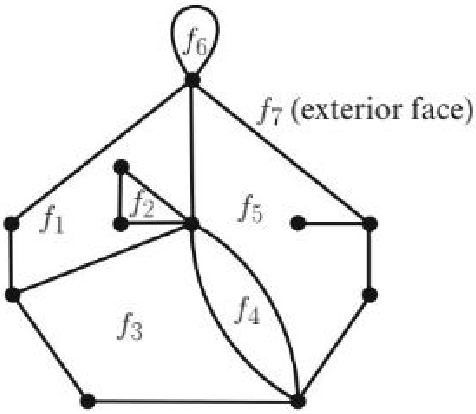


Fig. 8.3 A plane graph with seven faces

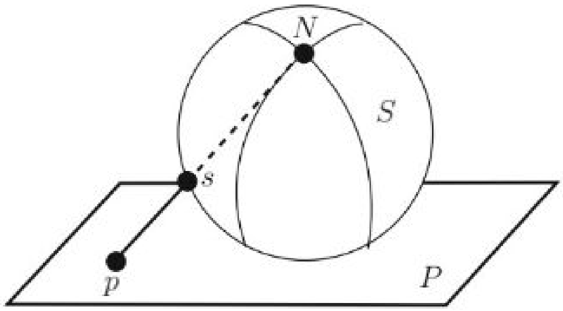


Fig. 8.4 Stereographic projection of the sphere S from N

course means that there exists a plane representation of G such that any specified vertex or edge belongs to the unbounded face.) We consider embeddings of a graph on a sphere. A graph is *embeddable on a sphere* S if it can be drawn on the surface of S so that its edges intersect only at its vertices. Such a drawing, if it exists, is called an embedding of G on S. Embeddings on a sphere are called *spherical embeddings*. What we have given here is only a naive definition. For a more rigorous description of spherical embeddings, see [79].

To prove the next theorem, we need to recall the notion of stereographic projection. Let S be a sphere resting on a plane P so that P is a tangent plane to S. Let N be the "north pole," the point on the sphere diametrically opposite the point of contact of S and P. Let the straight line joining N and a point s of $S \setminus \{N\}$ meet P at p. Then the mapping $\eta : S\setminus\{N\} \to P$ defined by $\eta(s) = p$ is called the *stereographic projection* of S from N (see Fig. 8.4).

Theorem 8.2.5. *A graph is planar if and only if it is embeddable on a sphere.*

Proof. Let a graph G be embeddable on a sphere and let G' be a spherical embedding of G. The image of G' under the stereographic projection η of the sphere from a point N of the sphere not on G' is a plane representation of G on P. Conversely, if G'' is a plane embedding of G on a plane P, then the inverse of the stereographic projection of G'' on a sphere touching the plane P gives a spherical embedding of G. □

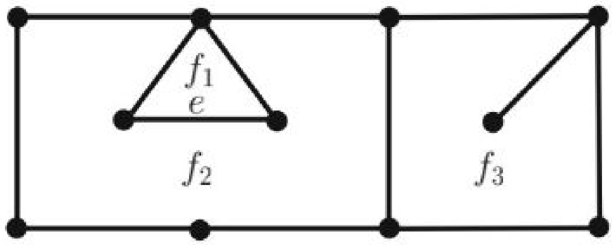


Fig. 8.5 Plane graph with four faces

Theorem 8.2.6. *(a) Let G be a plane graph and f be a face of G. Then there exists a plane embedding of G in which f is the exterior face.*

(b) Let G be a planar graph. Then G can be embedded in the plane in such a way that any specified vertex (or edge) belongs to the unbounded face of the resulting plane graph.

Proof. (a) Let n be a point of int f. Let $G' = \sigma(G)$ be a spherical embedding of G and let $N = \sigma(n)$. Let η be the stereographic projection of the sphere with N as the north pole. Then the map $\eta\sigma$ (σ followed by η) gives a plane embedding of G that maps f onto the exterior face of the plane representation $(\eta\sigma)(G)$ of G.

(b) Let f be a face containing the specified vertex (respectively, edge) in a plane representation of G. Now, by part (a) of the theorem, there exists a plane embedding of G in which f becomes the exterior face. The specified vertex (respectively, edge) then becomes a vertex (respectively, edge) of the new unbounded face. □

Remark 8.2.7. 1. Let G be a connected plane graph. Each edge of G belongs to one or two faces of G. A cut edge of G belongs to exactly one face, and conversely, if an edge belongs to exactly one face of G, it must be a cut edge of G. An edge of G that is not a cut edge belongs to exactly two faces and conversely.

2. The union of the vertices and edges of G incident with a face f of G is called the *boundary* of f and is denoted by $b(f)$. The vertices and edges of a plane graph G belonging to the boundary of a face of G are said to be *incident* with that face. If G is connected, the boundary of each face is a closed walk in which each cut edge of G is traversed twice. When there are no cut edges, the boundary of each face of G is a closed trail in G. (See, for instance, face f_1 of Fig. 8.3.) However, if G is a disconnected plane graph, then the edges and the vertices incident with the exterior face will not define a trail.

3. The number of edges incident with a face f is defined as the *degree* of f. In counting the degree of a face, a cut edge is counted twice. Thus, each edge of a plane graph G contributes two to the sum of the degrees of the faces. It follows that if $\mathcal{F}$ denotes the set of faces of a plane graph G, then $\sum_{f \in \mathcal{F}} d(f) = 2m(G)$, where $d(f)$ denotes the degree of the face f.

In Fig. 8.5, $d(f_1) = 3$, $d(f_2) = 9$, $d(f_3) = 6$, and $d(f_4) = 8$.

Theorem 8.2.8 connects the planarity of G with the planarity of its blocks.

Theorem 8.2.8. *A graph G is planar if and only if each of its blocks is planar.*

Proof. If G is planar, then each of its blocks is planar, since a subgraph of a planar graph is planar. Conversely, suppose that each block of G is planar. We now use induction on the number of blocks of G to prove the result. Without loss of generality, we assume that G is connected. If G has only one block, then G is planar.

Now suppose that G has k planar blocks and that the result is true for all connected graphs having $(k-1)$ planar blocks. Choose any end block B_0 of G and delete from G all the vertices of B_0 except the unique cut vertex, say v_0, of G in B_0. The resulting connected subgraph G' of G contains $(k-1)$ planar blocks. Hence, by the induction hypothesis, G' is planar. Let $\tilde{G}'$ be a plane embedding of G' such that v_0 belongs to the boundary of the unbounded face, say f' (refer to Theorem 8.2.6). Let $\tilde{B}_0$ be a plane embedding of B_0 in f' so that v_0 is in the boundary of the exterior face of $\tilde{B}_0$. Then (by the identification of v_0 in the two embeddings), $\tilde{G}' \cup \tilde{B}_0$ is a plane embedding of G. □

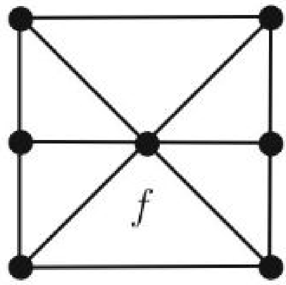
Remark 8.2.9. In testing for the planarity of a graph G, one may delete multiple edges and loops of G, if any. This is so because if a graph H is nonplanar, the removal of loops and parallel edges of H results in a subgraph of H, which is also nonplanar. Also, by Theorem 8.2.8, G can be assumed to be a block and hence 2-connected. If G has a vertex of degree 2, say v_0, and vv_0v' is the path formed by the two edges incident with v_0, contraction of vv_0 and deletion of the multiple edges (if any) thus formed again result in a planar graph. Let G' be the graph obtained from G by performing such contractions successively at vertices of degree 2 and deleting the resulting multiple edges. Then G is planar if and only if G' is planar. From these observations, it is clear that in designing a planarity algorithm (i.e., an algorithm to test planarity), it suffices to consider only 2-connected simple graphs with minimum degree at least 3. (For a planarity algorithm, see [49].)

Exercise 2.1. Show that every graph with at most three cycles is planar.

Exercise 2.2. Find a simple graph G with degree sequence (4, 4, 3, 3, 3, 3) such that

(a) G is planar.
(b) G is nonplanar.

Exercise 2.3. Redraw the following planar graph so that the face f becomes the exterior face.



8.3 Euler Formula and Its Consequences

We have noted that a planar graph may have more than one plane representation (see Fig. 8.1). A natural question that would arise is whether the number of faces is the same in each such representation. The answer to this question is provided by the Euler formula.

Theorem 8.3.1 (Euler formula). *For a connected plane graph G, $n - m + \mathfrak{f} = 2$, where n, m, and $\mathfrak{f}$ denote the number of vertices, edges, and faces of G, respectively.*

Proof. We apply induction on $\mathfrak{f}$.

If $\mathfrak{f} = 1$, then G is a tree and $m = n - 1$. Hence, $n - m + \mathfrak{f} = 2$.

Now assume that the result is true for all plane graphs with $\mathfrak{f} - 1$ faces, $\mathfrak{f} \geq 2$, and suppose that G has $\mathfrak{f}$ faces. Since $\mathfrak{f} \geq 2$, G is not a tree, and hence contains a cycle C. Let e be an edge of C. Then e belongs to exactly two faces, say f_1 and f_2, of G and the deletion of e from G results in the formation of a single face from f_1 and f_2 (see Fig. 8.5). Also, since e is not a cut edge of G, $G - e$ is connected. Further, the number of faces of $G - e$ is $\mathfrak{f} - 1$. So applying induction to $G - e$, we get $n - (m - 1) + (\mathfrak{f} - 1) = 2$, and this implies that $n - m + \mathfrak{f} = 2$. This completes the proof of the theorem. □

Below are some of the consequences of the Euler formula.

Corollary 8.3.2. *All plane embeddings of a given planar graph have the same number of faces.*

Proof. Since $\mathfrak{f} = m - n + 2$, the number of faces depends only on n and m, and not on the particular embedding. □

Corollary 8.3.3. *If G is a simple planar graph with at least three vertices, then $m \leq 3n - 6$.*

Proof. Without loss of generality, we can assume that G is a simple connected plane graph. Since G is simple and $n \geq 3$, each face of G has degree at least 3. Hence, if $\mathcal{F}$ denotes the set of faces of G, $\sum_{f \in \mathcal{F}} d(f) \geq 3\mathfrak{f}$. But $\sum_{f \in \mathcal{F}} d(f) = 2m$. Consequently, $2m \geq 3\mathfrak{f}$, so that $\mathfrak{f} \leq \frac{2m}{3}$.

By the Euler formula, $m = n + \mathfrak{f} - 2$. Now $\mathfrak{f} \leq \frac{2m}{3}$ implies that $m \leq n + \left(\frac{2m}{3}\right) - 2$. This gives $m \leq 3n - 6$. □

The above result is not valid if $n = 1$ or 2. Also, the condition of Corollary 8.3.3 is not sufficient for the planarity of a simple connected graph as the Petersen graph shows. For the Petersen graph, $m = 15$, $n = 10$, and hence $m \leq 3n - 6$, but the graph is not planar (see Corollary 8.3.7 below).

Example 8.3.4. Show that the complement of a simple planar graph with 11 vertices is nonplanar.

Solution. Let G be a simple planar graph with $n(G) = 11$. Since G is planar, $m(G) \leq 3n - 6 = 27$. If G^c were also planar, then $m(G^c) \leq 3n - 6 = 27$. On the one hand, $m(G) + m(G^c) \leq 27 + 27 = 54$, whereas, on the other hand, $m(G) + m(G^c) = m(K_{11}) = \binom{11}{2} = 55$. Hence, we arrive at a contradiction. This contradiction proves that G^c is nonplanar. □

Corollary 8.3.5. *For any simple planar graph G, $\delta(G) \leq 5$.*

Proof. If $n \leq 6$, then $\Delta(G) \leq 5$. Hence $\delta(G) \leq \Delta(G) \leq 5$, proving the result for such graphs. So assume that $n \geq 7$. By Corollary 8.3.3, $m \leq 3n - 6$. Now, $\delta n \leq \sum_{v \in V(G)} d_G(v) = 2m \leq 2(3n - 6) = 6n - 12$. Hence $n(\delta - 6) \leq -12$. Consequently, $\delta - 6$ is negative, implying that $\delta \leq 5$. □

Recall that the *girth* of a graph G is the length of a shortest cycle in G.

Theorem 8.3.6. *If the girth k of a connected plane graph G is at least 3, then $m \leq \frac{k(n-2)}{(k-2)}$.*

Proof. Let $\mathcal{F}$ denote the set of faces and $\mathfrak{f}$, as before, denote the number of faces of G. If $f \in \mathcal{F}$, then $d(f) \geq k$. Since $2m = \sum_{f \in \mathcal{F}} d(f)$, we get $2m \geq k\mathfrak{f}$. By Theorem 8.3.1, $\mathfrak{f} = 2 - n + m$. Hence, $2m \geq k(2 - n + m)$, implying that $m(k-2) \leq k(n-2)$. Thus, $m \leq \frac{k(n-2)}{(k-2)}$. □

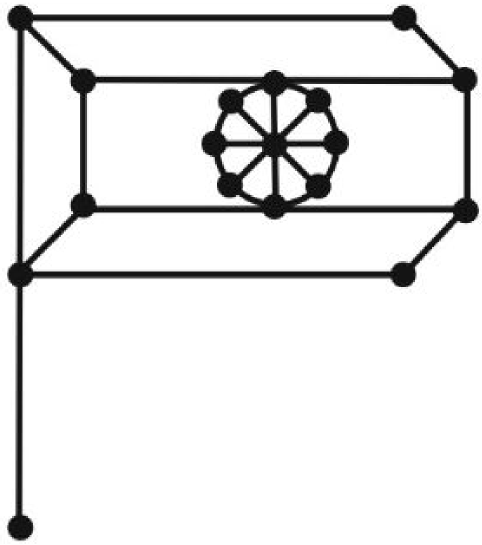
Corollary 8.3.7. *The Petersen graph P is nonplanar.*

Proof. The girth of the Petersen graph P is 5, $n(P) = 10$, and $m(P) = 15$. Hence, if P were planar, $15 \leq \frac{5(10-2)}{5-2}$, which is not true. Hence, P is nonplanar. □

Exercise 3.1. Show that every simple bipartite cubic planar graph contains a C_4.

Exercise 3.2. A nonplanar graph G is called *planar-vertex-critical* if $G - v$ is planar for every vertex v of G. Prove that a planar-vertex-critical graph must be 2-connected.

Exercise 3.3. Verify Euler's formula for the plane graph P.



Exercise 3.4. Let G be a simple plane cubic graph having eight faces. Determine $n(G)$. Draw two such graphs that are nonisomorphic.

Exercise 3.5. Prove that if G is a simple connected planar bipartite graph, then $m \leq 2n - 4$, where $n \geq 3$.

Exercise 3.6. Prove that a simple planar graph (with at least four vertices) has at least four vertices each of degree 5 at most.

Exercise 3.7. If G is a nonplanar graph, show that it has either five vertices of degree at least 4, or six vertices of degree at least 3.

Exercise 3.8. Prove that a simple planar graph with minimum degree at least five contains at least 12 vertices. Give an example of a simple planar graph on 12 vertices with minimum degree 5.

Exercise 3.9. Show that there is no 6-connected planar graph.

Exercise 3.10. Let G be a plane graph of order n and size m in which every face is bounded by a k-cycle. Show that $m = \frac{k(n-2)}{(k-2)}$.

Definition 8.3.8. A graph G is *maximal planar* if G is planar, but for any pair of nonadjacent vertices u and v of G, $G + uv$ is nonplanar.

Remark 8.3.9. Any planar graph is a spanning subgraph of a maximal planar graph. Indeed, if $\tilde{G}$ is a plane embedding of a planar graph G with at least three vertices, and if $e = uv$ is a cut edge of $\tilde{G}$ embedded in a face f of $\tilde{G}$, it is clear that there exists a vertex w on the boundary of f such that the edge uw or vw can be drawn in f so that either $\tilde{G} + (vw)$ or $\tilde{G} + (uw)$ is also a plane graph (see Fig. 8.6a). Further, if C_0 is any cycle bounding a face f_0 of a plane graph H, then edges can be drawn in int C_0 without crossing each other so that f_0 is divided into triangles (see Fig. 8.6b).

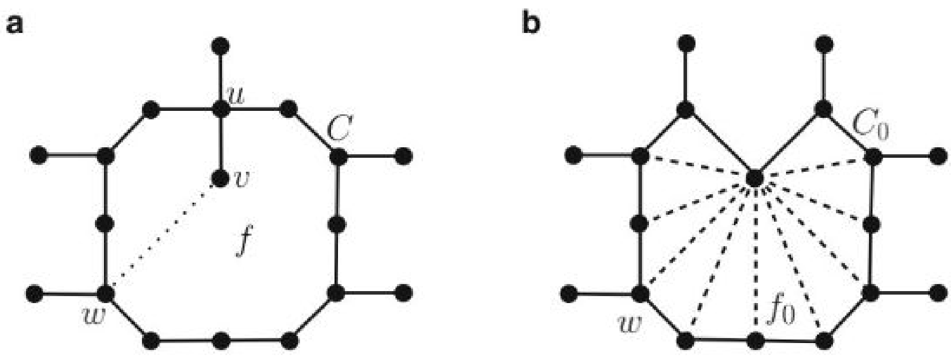


Fig. 8.6 Procedure to get maximal planar graphs

Definition 8.3.10. A *plane triangulation* is a plane graph in which each of its faces is bounded by a triangle. A plane triangulation of a plane graph G is a plane triangulation H such that G is a spanning subgraph of H.

Remark 8.3.11. Remark 8.3.9 shows that a plane embedding of a simple maximal planar graph is a plane triangulation.

Note that any simple plane graph is a subgraph of a simple maximal plane graph and hence is a spanning subgraph of some plane triangulation. Thus, to any simple plane graph G that is not already a plane triangulation, we can add a set of new edges to obtain a plane triangulation. The set of new edges thus added need not be unique.

Figure 8.7a is a simple plane graph G and Fig. 8.7b is a plane triangulation of G; Fig. 8.7c is a plane triangulation of G isomorphic to the graph of Fig. 8.7b having only straight-line edges. (A result of Fáry [60] states that every simple planar graph has a plane embedding in which each edge is a straight line.)

Exercise 3.11. Embed the 3-cube Q_3 (see Exercise 4.4 of Chap. 5) in a maximal planar graph having the same vertex set as Q_3. Count the number of new edges added.

Exercise 3.12. Prove that for a simple maximal planar graph on $n \geq 3$ vertices, $m = 3n - 6$.

Exercise 3.13. Use Exercise 3.12 to show that for any simple planar graph, $m \leq 3n - 6$.

Exercise 3.14. Show that every plane triangulation of order $n \geq 4$ is 3-connected.

Exercise 3.15. Let G be a maximal planar graph with $n \geq 4$. Let n_i denote the number of vertices of degree i in G. Then prove that $3n_3 + 2n_4 + n_5 = 12 + n_7 + 2n_8 + 3n_9 + 4n_{10} + \dots$. (Hint: Use the fact that $n = n_3 + n_4 + n_5 + n_6 + \dots$.)

Exercise 3.16. Generalize the Euler formula for disconnected plane graphs.

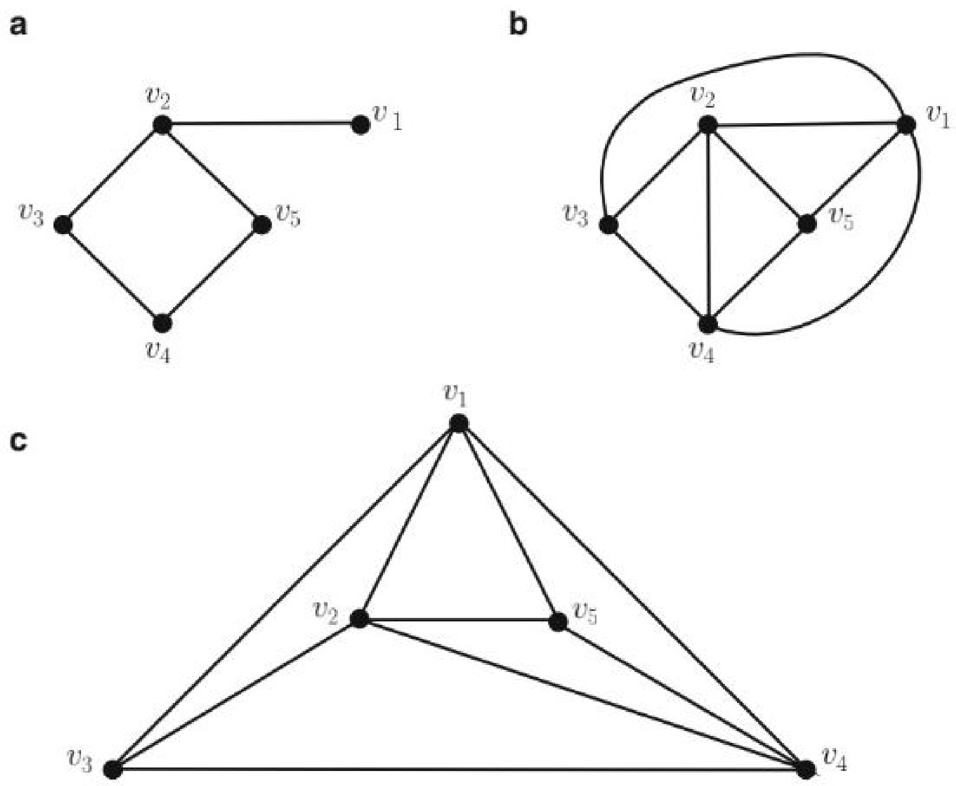


Fig. 8.7 (**a**) Graph G and (**b**), (**c**) are plane triangulations of G

8.4 K_5 and $K_{3,3}$ are Nonplanar Graphs

In this section we prove that K_5 and $K_{3,3}$ are nonplanar. These two graphs are basic in Kuratowski's characterization of planar graphs (see Theorem 8.7.5 given later in this chapter). For this reason, they are often referred to as the two *Kuratowski graphs*.

Theorem 8.4.1. *K_5 is nonplanar.*

First proof. This proof uses the Jordan curve theorem. Assume the contrary, namely, K_5 is planar. Let v_1, v_2, v_3, v_4, and v_5 be the vertices of K_5 in a plane representation of K_5. The cycle $C = v_1v_2v_3v_4v_1$ (as a closed Jordan curve) divides the plane into two faces, namely, the interior and the exterior of C. The vertex v_5 must belong either to int C or to ext C. Suppose that v_5 belongs to int C (a similar proof holds if v_5 belongs to ext C). Draw the edges v_5v_1, v_5v_2, v_5v_3, and v_5v_4 in int C. Now there remain two more edges v_1v_3 and v_2v_4 to be drawn. None of these can be drawn in int C, since it is assumed that K_5 is planar. Thus, v_1v_3 lies in ext C. Then one of v_2 and v_4 belongs to the interior of the closed Jordan curve $C_1 = v_1v_5v_3v_1$ and the other to its exterior (see Fig. 8.8). Hence, v_2v_4 cannot be drawn without violating planarity. □

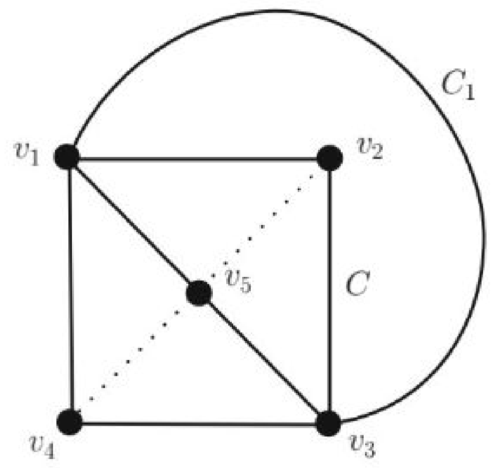


Fig. 8.8 Graph for first proof of Theorem 8.4.1

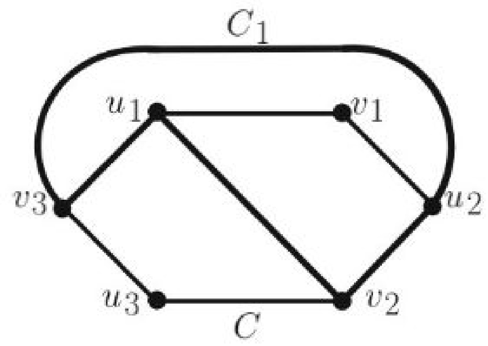


Fig. 8.9 Graph for first proof of Theorem 8.4.3

Remark 8.4.2. The first proof of Theorem 8.4.1 shows that all the edges of K_5 except one can be drawn in the plane without violating planarity. Hence for any edge e of K_5, $K_5 - e$ is planar.

Second proof. If K_5 were planar, it follows from Theorem 8.3.6 that $10 \leq \frac{3(5-2)}{(3-2)}$, which is not true. Hence K_5 is nonplanar. □

Theorem 8.4.3. *$K_{3,3}$ is nonplanar.*

First proof. The proof is by the use of the Jordan curve theorem. Suppose that $K_{3,3}$ is planar. Let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3\}$ be the bipartition of $K_{3,3}$ in a plane representation of the graph. Consider the cycle $C = u_1v_1u_2v_2u_3v_3u_1$. Since the graph is assumed to be planar, the edge u_1v_2 must lie either in the interior of C or in its exterior. For the sake of definiteness, assume that it lies in int C (a similar proof holds if one assumes that the edge u_1v_2 lies in ext C). Two more edges remain to be drawn, namely, u_2v_3 and u_3v_1. None of these can be drawn in int C without crossing the edge u_1v_2. Hence, both of them are to be drawn in ext C. Now draw u_2v_3 in ext C. Then one of v_1 and u_3 belongs to the interior of the closed Jordan curve $C_1 = u_1v_2u_2v_3u_1$ and the other to the exterior of C_1 (see Fig. 8.9). Hence, the edge v_1u_3 cannot be drawn without violating planarity. This shows that $K_{3,3}$ is nonplanar. □

Second proof. Suppose $K_{3,3}$ is planar. Let $\mathfrak{f}$ be the number of faces of $G = K_{3,3}$ in a plane embedding of G and $\mathcal{F}$, the set of faces of G. As the girth of $K_{3,3}$ is 4, we have $m = \frac{1}{2}\sum_{f \in \mathcal{F}} d(f) \geq \frac{4\mathfrak{f}}{2} = 2\mathfrak{f}$. By Theorem 8.3.1, $n - m + \mathfrak{f} = 2$. For $K_{3,3}$, $n = 6$, and $m = 9$. Hence, $\mathfrak{f} = 2 + m - n = 5$. Thus, $9 \geq 2.5 = 10$, a contradiction. □

Exercise 4.1. Give yet another proof of Theorem 8.4.3.

Exercise 4.2. Find the maximum number of edges in a planar complete tripartite graph with each part of size at least 2.

Remark 8.4.4. As in the case of K_5, for any edge e of $K_{3,3}$, $K_{3,3} - e$ is planar. Observe that the graphs K_5 and $K_{3,3}$ have some features in common.

1. Both are regular graphs.
2. The removal of a vertex or an edge from each graph results in a planar graph.
3. Contraction of an edge results in a planar graph.
4. K_5 is a nonplanar graph with the smallest number of vertices, whereas $K_{3,3}$ is a nonplanar graph with the smallest number of edges. (Hence, any nonplanar graph must have at least five vertices and nine edges.)

8.5 Dual of a Plane Graph

Let G be a plane graph. One can form out of G a new graph H in the following way. Corresponding to each face f of G, take a vertex f^* and corresponding to each edge e of G, take an edge e^*. Then edge e^* joins vertices f^* and g^* in H if and only if edge e is common to the boundaries of faces f and g in G. (It is possible that f may be the same as g.) The graph H is then called the *dual* (or more precisely, the geometric dual) of G (see Fig. 8.10). If e is a cut edge of G embedded in face f of G, then e^* is a loop at f^*. H is a planar graph and there exists a natural way of embedding H in the plane. Vertex f^*, corresponding to face f, is placed in face f of G. Edge e^*, joining f^* and g^*, is drawn so that e^* crosses e once and only once and crosses no other edge. This procedure is illustrated in Fig. 8.11. This embedding is the canonical embedding of H. H with this canonical embedding is denoted by G^*. Any two embeddings of H, as described above, are isomorphic.

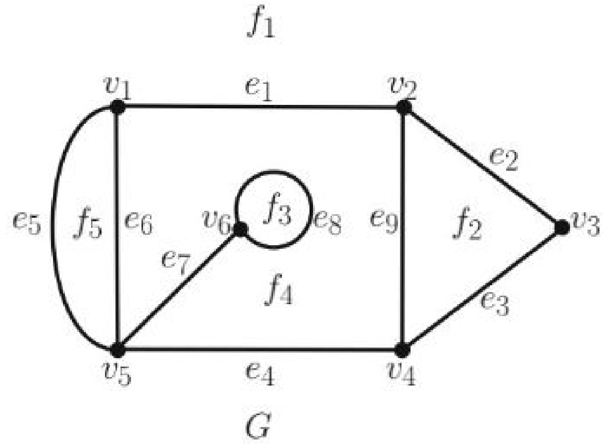
The definition of the dual implies that $m(G^*) = m(G)$, $n(G^*) = \mathfrak{f}(G)$, and $d_{G^*}(f^*) = d_G(f)$, where $d_G(f)$ denotes the degree of the face f of G.

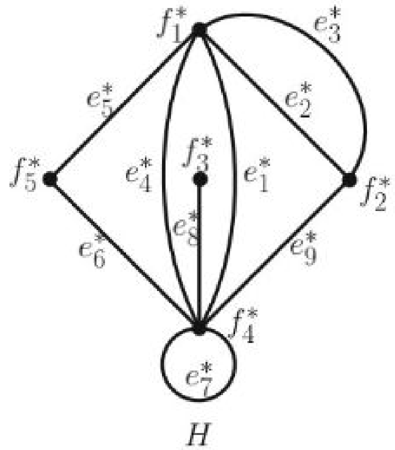
From the manner of construction of G^*, it follows that

(i) An edge e of a plane graph G is a cut edge of G if and only if e^* is a loop of G^*, and it is a loop of G if and only if e^* is a cut edge of G^*.
(ii) G^* is connected whether G is connected or not (see graphs G and G^* of Fig. 8.12).

The canonical embedding of the dual of G^* is denoted by G^{**}. It is easy to check that G^{**} is isomorphic to G if and only if G is connected. Graph isomorphism

Fig. 8.10 A plane graph G and its dual H





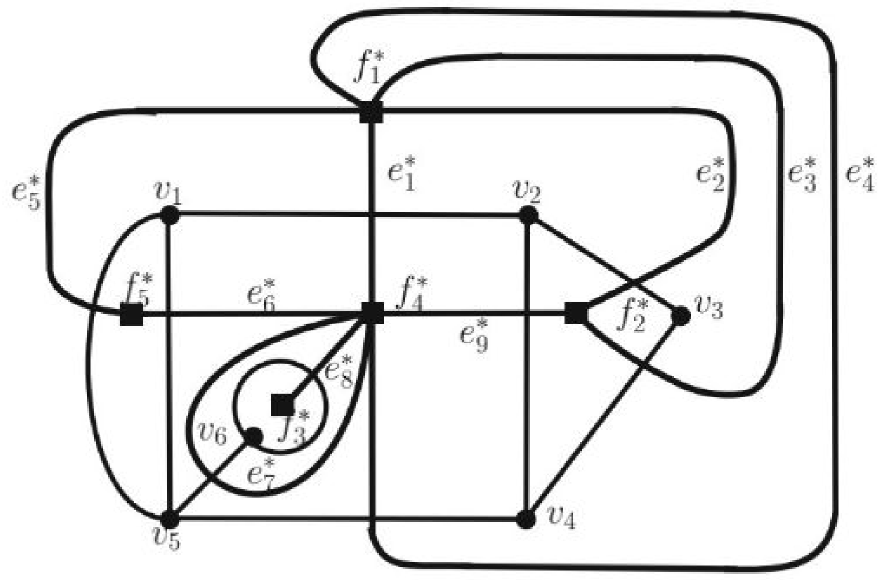
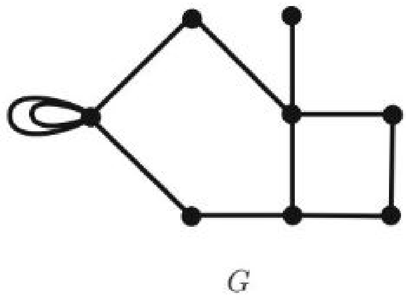


Fig. 8.11 Procedure for drawing the dual graph

does not preserve duality; that is, isomorphic plane graphs may have nonisomorphic duals. The graphs G and H of Fig. 8.13 are isomorphic plane graphs, but $G^* \not\simeq H^*$. G has a face of degree 5, whereas no face of H has degree 5. Hence, G^* has a vertex of degree 5, whereas H^* has no vertex of degree 5. Consequently, $G^* \not\simeq H^*$.

Exercise 5.1. Draw the dual of

(i) The Herschel graph (graph of Fig. 5.4).
(ii) The graph G given below:



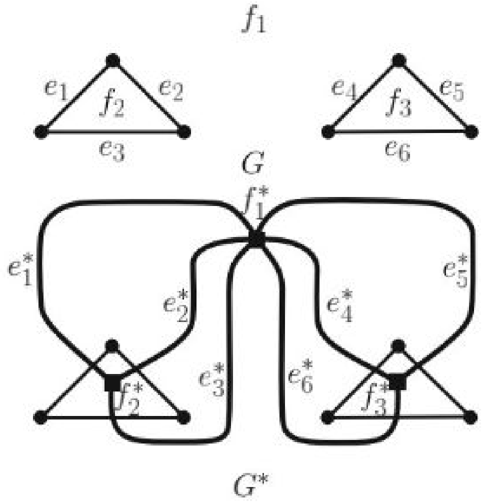


Fig. 8.12 A disconnected graph G and its (connected) dual G^*

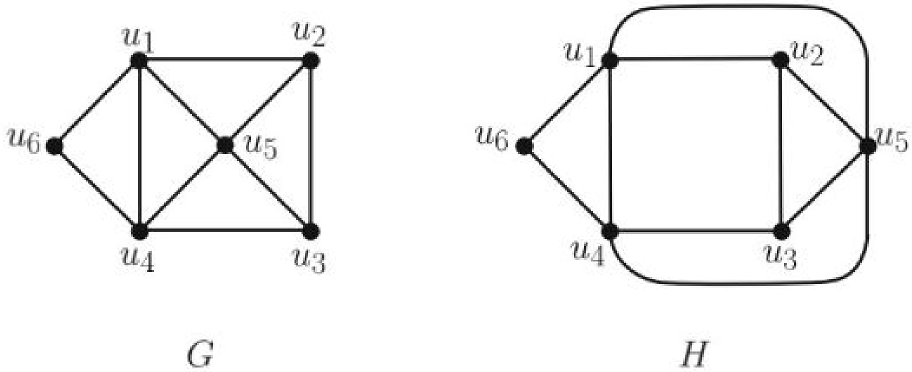


Fig. 8.13 Isomorphic graphs G and H for which $G^* \not\simeq H^*$

Exercise 5.2. A plane graph G is called *self-dual* if $G \simeq G^*$. Prove the following:

(i) All wheels W_n ($n \geq 3$) are self-dual.
(ii) For a self-dual graph, $2n = m + 2$.

Exercise 5.3. Construct two infinite families of self-dual graphs.

8.6 The Four-Color Theorem and the Heawood Five-Color Theorem

What is the minimum number of colors required to color the world map of countries so that no two countries having a common boundary receive the same color? This simple-looking problem manifested itself into one of the most challenging problems of graph theory, popularly known as the four-color conjecture (4CC).

The geographical map of the countries of the world is a typical example of a plane graph. An assignment of colors to the faces of a plane graph G so that no two faces having a common boundary containing at least one edge receive the same color is a *face coloring* of G. The face-chromatic number $\chi^*(G)$ of a plane graph G is the minimum k for which G has a face coloring using k colors. The problem of coloring a map so that no two adjacent countries receive the same color can thus be transformed into a problem of face coloring of a plane graph G. The face coloring of G is closely related to the vertex coloring of the dual G^* of G. The fact that two faces of G are adjacent in G if and only if the corresponding vertices of G^* are adjacent in G^* shows that G is k-face-colorable if and only if G^* is k-vertex-colorable.

It was young Francis Guthrie who conjectured, while coloring the district map of England, that four colors were sufficient to color the world map so that adjacent countries receive distinct colors. This conjecture was communicated by his brother to De Morgan in 1852. Guthrie's conjecture is equivalent to the statement that any plane graph is 4-face-colorable. The latter statement is equivalent to the conjecture: Every planar graph is 4-vertex-colorable.

After the conjecture was first published in 1852, many attempted to settle it. In the process of settling the conjecture, many equivalent formulations of this conjecture were found. Assaults on the conjecture were made using such varied branches of mathematics as algebra, number theory, and finite geometries. The solution found the light of the day when Appel, Haken, and Koch [8] of the University of Illinois established the validity of the conjecture in 1976 with the aid of computers (see also [6, 7]). The proof includes, among other things, 10^{10} units of operations, amounting to a staggering 1200 hours of computer time on a high-speed computer available at that time.

Although the computer-oriented proof of Appel, Haken, and Koch settled the conjecture in 1976 and has stood the test of time, a theoretical proof of the four-color problem is still to be found.

Even though the solution of the 4CC has been a formidable task, it is rather easy to establish that every planar graph is 6-vertex-colorable.

Theorem 8.6.1. *Every planar graph is 6-vertex-colorable.*

Proof. The proof is by induction on n, the number of vertices of the graph. The result is trivial for planar graphs with at most six vertices. Assume the result for planar graphs with $n-1$, $n \geq 7$, vertices. Let G be a planar graph with n vertices. By Corollary 8.3.5, $\delta(G) \leq 5$, and hence G has a vertex v of degree at most 5. By hypothesis, $G - v$ is 6-vertex-colorable. In any proper 6-vertex coloring of $G - v$, the neighbors of v in G would have used only at most five colors, and hence v can be colored by an unused color. In other words, G is 6-vertex-colorable. □

It involves some ingenious arguments to reduce the upper bound for the chromatic number of a planar graph from 6 to 5. The upper bound 5 was obtained by Heawood [103] as early as 1890.

Theorem 8.6.2 (Heawood's five-color theorem). *Every planar graph is 5-vertex-colorable.*

Proof. The proof is by induction on $n(G) = n$. Without loss of generality, we assume that G is a connected plane graph. If $n \leq 5$, the result is clearly true. Hence, assume that $n \geq 6$ and that any planar graph with fewer than n vertices is 5-vertex-colorable. G being planar, $\delta(G) \leq 5$ by Corollary 8.3.5, and so G contains a vertex v_0 of degree not exceeding 5. By the induction hypothesis, $G - v_0$ is 5-vertex-colorable.

If $d(v_0) \leq 4$, at most four colors would have been used in coloring the neighbors of v_0 in G in a 5-vertex coloring of $G - v_0$. Hence, an unused color can then be assigned to v_0 to yield a proper 5-vertex coloring of G.

If $d(v_0) = 5$, but only four or fewer colors are used to color the neighbors of v_0 in a proper 5-vertex coloring of $G - v_0$, then also an unused color can be assigned to v_0 to yield a proper 5-vertex coloring of G.

Hence assume that the degree of v_0 is 5 and that in every 5-coloring of $G - v_0$, the neighbors of v_0 in G receive five distinct colors. Let v_1, v_2, v_3, v_4, and v_5 be the neighbors of v_0 in a cyclic order in a plane embedding of G. Choose some proper 5-coloring of $G - v_0$ with colors, say, $c_1, c_2, \ldots, c_5$. Let $\{V_1, V_2, \ldots, V_5\}$ be the color partition of $G - v_0$, where the vertices in V_i are colored c_i, $1 \leq i \leq 5$. Assume further that $v_i \in V_i$, $1 \leq i \leq 5$.

Let G_{ij} be the subgraph of $G - v_0$ induced by $V_i \cup V_j$. Suppose v_i and v_j, $1 \leq i, j \leq 5$, belong to distinct components of G_{ij}. Then the interchange of the colors c_i and c_j in the component of G_{ij} containing v_i would give a recoloring of $G - v_0$ in which only four colors are assigned to the neighbors of v_0. But this is against our assumption. Hence, v_i and v_j must belong to the same component of G_{ij}. Let $P_{i,j}$

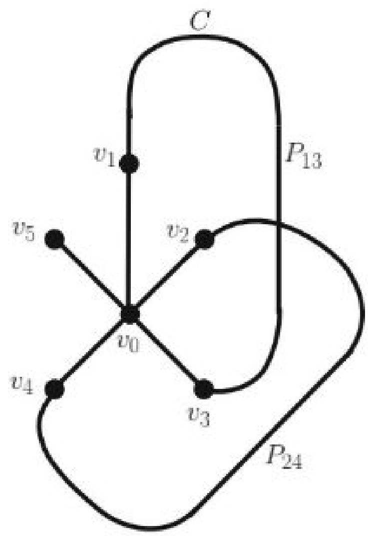


Fig. 8.14 Graph for proof of Theorem 8.6.2

be a v_i-v_j path in G_{ij}. Let C denote the cycle $v_0 v_1 P_{13} v_3 v_0$ in G (Fig. 8.14). Then C separates v_2 and v_4; that is, one of v_2 and v_4 must lie in int C and the other in ext C. In Fig. 8.14, $v_2 \in$ int C and $v_4 \in$ ext C. Then P_{24} must cross C at a vertex of C. But this is clearly impossible since no vertex of C receives either of the colors c_2 and c_4. Hence this possibility cannot arise, and G is 5-vertex-colorable. □

Note that the bound 4 in the inequality $\chi(G) \leq 4$ for planar graphs G is best possible since K_4 is planar and $\chi(K_4) = 4$.

Exercise 6.1. Show that a planar graph G is bipartite if and only if each of its faces is of even degree in any plane embedding of G.

Exercise 6.2. Show that a connected plane graph G is bipartite if and only if G^* is Eulerian. Hence, show that a connected plane graph is 2-face-colorable if and only if it is Eulerian.

Exercise 6.3. Prove that a Hamiltonian plane graph is 4-face-colorable and that its dual is 4-vertex-colorable.

Exercise 6.4. Show that a plane triangulation has a 3-face coloring if and only if it is not K_4. (Hint: Use Brooks' theorem.)

Remark 8.6.3. (Grötzsch): If G is a planar graph that contains no triangle, then G is 3-vertex-colorable.

8.7 Kuratowski's Theorem

Definition 8.7.1. 1. A *subdivision of an edge* $e = uv$ of a graph G is obtained by introducing a new vertex w in e, that is, by replacing the edge $e = uv$ of G by the path uwv of length 2 so that the new vertex w is of degree 2 in the resulting graph (see Fig. 8.15a).

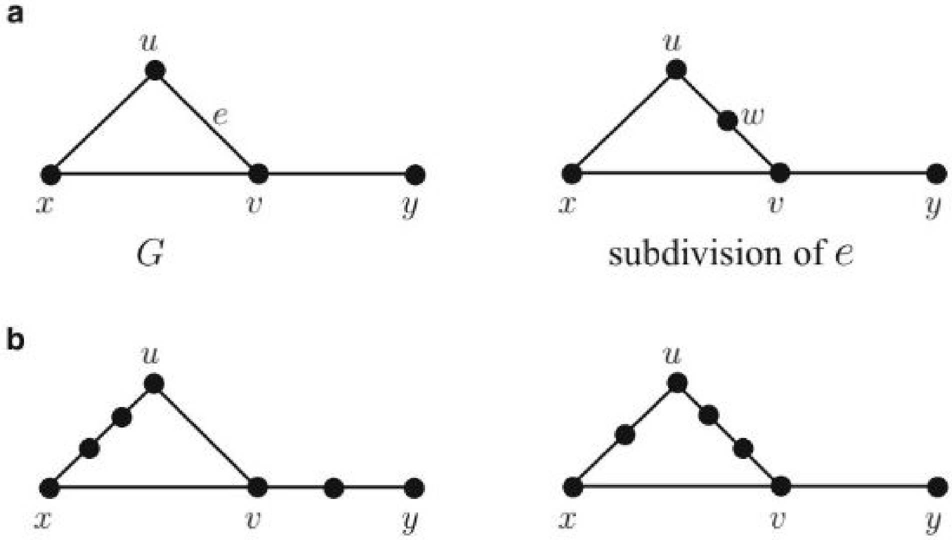


Fig. 8.15 (**a**) Subdivision of edge e of graph G, (**b**) two homeomorphs of graph G

2. A *homeomorph* or a *subdivision of a graph* G is a graph obtained from G by applying a finite number of subdivisions of edges in succession (see Fig. 8.15b). G itself is regarded as a subdivision of G.
3. Two graphs G_1 and G_2 are called *homeomorphic* if they are both homeomorphs of some graph G. Clearly, the graphs of Fig. 8.15b are homeomorphic, even though neither of the two graphs is a homeomorph of the other.

Kuratowski's theorem [129] characterizing planar graphs was one of the major breakthrough results in graph theory of the 20th century. As mentioned earlier, while examining planarity of graphs, we need only consider simple graphs since the presence of loops and multiple edges does not affect the planarity of graphs. Consequently, *a graph is planar if and only if its underlying simple graph is planar.* We therefore consider in this section only (finite) simple graphs. We recall that for any edge e of a graph G, $G - e$ is the subgraph of G obtained by deleting the edge e, whereas $G \circ e$ denotes the contraction of e. We always discard isolated vertices when edges get deleted and remove the new multiple edges when edges get contracted. More generally, for a subgraph H of G, $G \circ H$ denotes the graph obtained by the successive contractions of all the edges of H in G. The resulting graph is independent of the order of contraction. Moreover, if G is planar, then $G \circ e$ is planar; consequently, $G \circ H$ is planar. In other words, if $G \circ H$ is nonplanar for some subgraph H of G, then G is also nonplanar. Further, any two homeomorphic graphs are contractible to the same graph.

Definition 8.7.2. If $G \circ H = K$, we call K a *contraction* of G; we also say that G is *contractible* to K. G is said to be *subcontractible* to K if G has a subgraph H contractible to K. We also refer to this fact by saying that K is a *subcontraction* of G.

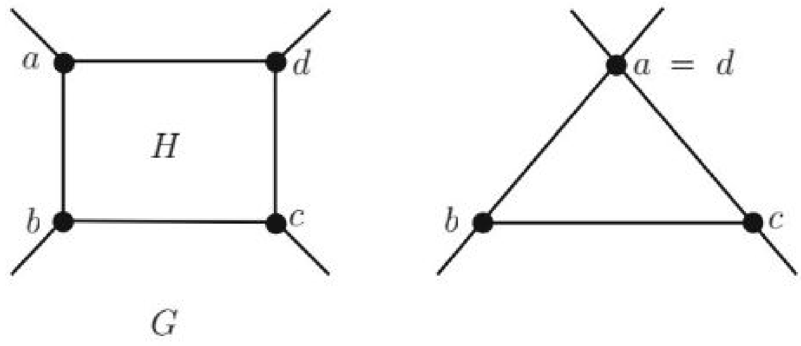


Fig. 8.16 Graph G subcontractible to triangle abc

Example 8.7.3. For instance, in Fig. 8.16, graph G is subcontractible to the triangle abc. (Take H to be the cycle $abcd$ and contract the edge ad in H. By abuse of notation, the new vertex is denoted by a or d.) We note further that if G' is a homeomorph of G, then contraction of one of the edges incident at each vertex of degree 2 in $V(G')\backslash V(G)$ results in a graph homeomorphic to G.

Our first aim is to prove the following result, which was established by Wagner [186] and, independently, by Harary and Tutte [96].

Theorem 8.7.4 ([96,186]). *A graph is planar if and only if it is not subcontractible to K_5 or $K_{3,3}$.*

As a consequence, we establish Kuratowski's characterization theorem for planar graphs.

Theorem 8.7.5 (Kuratowski [129]). *A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.*

The proofs of Theorems 8.7.4 and 8.7.5, as presented here, are due to Fournier [68]. Recall that any subgraph and any contraction of a planar graph are both planar.

Definition 8.7.6. A simple connected nonplanar graph G is *irreducible* if, for each edge e of G, $G \circ e$ is planar.

For instance, both K_5 and $K_{3,3}$ are irreducible.

Proof of theorem 8.7.4. If G has a subgraph G_0 contractible to K_5 or $K_{3,3}$, then since K_5 and $K_{3,3}$ are nonplanar, G_0 and therefore G are nonplanar.

We now prove the converse. Assume that G is a simple connected nonplanar graph. By Theorem 8.2.8, at least one block of G is nonplanar. Hence, assume that G is a simple 2-connected nonplanar graph. We now show that G has a subgraph contractible to K_5 or $K_{3,3}$.

Keep contracting edges of G (and delete the new multiple edges, if any, at each stage of the contraction) until a (2-connected) irreducible (nonplanar) graph H results. Clearly, $\delta(H) \geq 3$. Now, if e and f are any two distinct edges of G, then $(G \circ e) - f = (G - f) \circ e$. Hence, the graph H may as well be obtained by

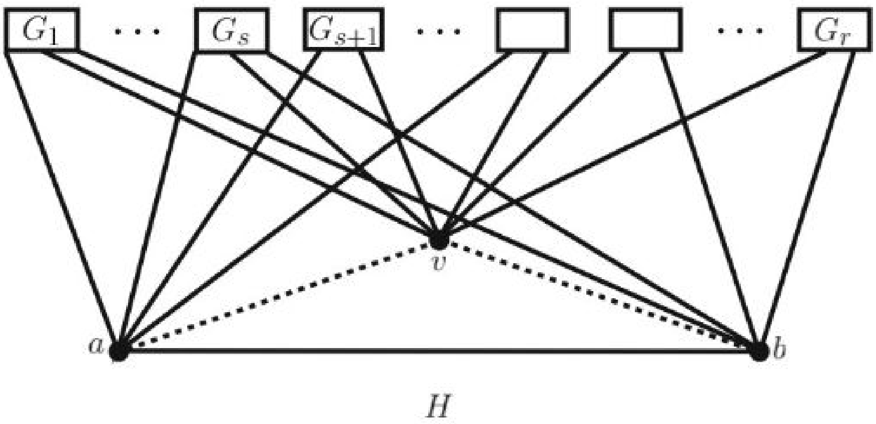


Fig. 8.17 Graph H for case 1 of proof of Theorem 8.7.4

deleting a set (which may be empty) of edges of G, resulting in a subgraph G_0 of G and then contracting a subgraph of G_0. We now complete the proof of the theorem by showing that H has a subgraph K homeomorphic (and hence contractible) to K_5 or $K_{3,3}$. In this case, G has the subgraph G_0, which is contractible to K_5 or $K_{3,3}$.

Let $e = ab \in E(H)$ and $H' = H - \{a, b\}$. Then H' is connected. If not, $\{a, b\}$ is a vertex cut of H. Let $G'_1, \ldots, G'_r$ be the components of H'. As H is irreducible, $H - V(G'_r)$ is planar, and there exists a plane embedding of H' in which the edge ab is in the exterior face. As G'_r is planar, G'_r can be embedded in this exterior face of H'. This would make H a planar graph, a contradiction. Thus, H' is connected.]

Case 1. H' has a cut vertex v. Let $G_1, G_2, \ldots, G_r$ ($r \geq 2$) be the components of $H' - \{v\}$, and let $G_1, G_2, \ldots, G_s$, $0 \leq s \leq r$, be those components that are connected to both a and b. (see Fig. 8.17). If $r > s$, then each of $G_{s+1}, \ldots, G_r$ is connected to only one of a or b. Assume that G_r is connected to b and not to a. From the plane representation of $G \circ (G_{s+1} \cup \ldots \cup G_r)$, the contraction of G obtained by contracting the edges of $G_{s+1}, \ldots, G_r$, we can obtain a plane representation of H' (see Fig. 8.17). [In fact, if G_r is contracted to the vertex w_r, then as the subgraph $A_r = \langle v, b, v(G_r) \rangle$ of H' is planar, the pair of edges $\{vw_r, w_r b\}$ can be replaced by the planar subgraph A_r and so on.] Hence this case cannot arise. Consequently, $r = s$. If $r = s = 2$, the plane embeddings of $H' \circ G_1$ and $H' \circ G_2$ yield a plane embedding of H', a contradiction (see Fig. 8.18). Consequently, $r = s \geq 3$. In this case, H' contains a homeomorph of $K_{3,3}$ (see Fig. 8.19), with $\{w_1, w_2, w_3 \,;\, a, b, v\}$ being the vertex set of $K_{3,3}$. (Other possibilities for w_1, w_2, w_3 will also yield a homeomorph of $K_{3,3}$.)

Case 2. H' is 2-connected. Then H' contains a cycle C of length at least 3. Consider a plane embedding of $H \circ e$ (where $e = ab$, as above). If c denotes the new vertex to which a and b get contracted, $(H \circ e) - c = H'$. We may therefore suppose without loss of generality that c is in the interior of the cycle C in the plane embedding of $H \circ e$.

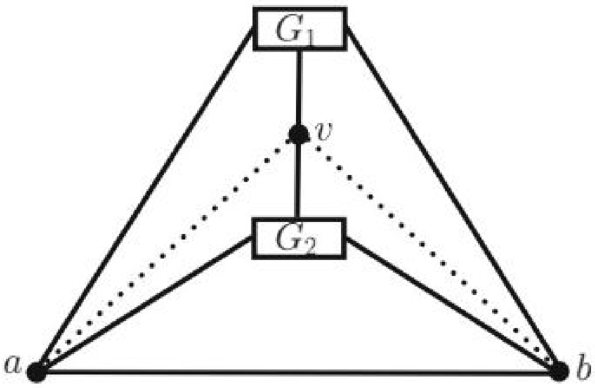


Fig. 8.18 Plane embedding for case 1 of proof of Theorem 8.7.4

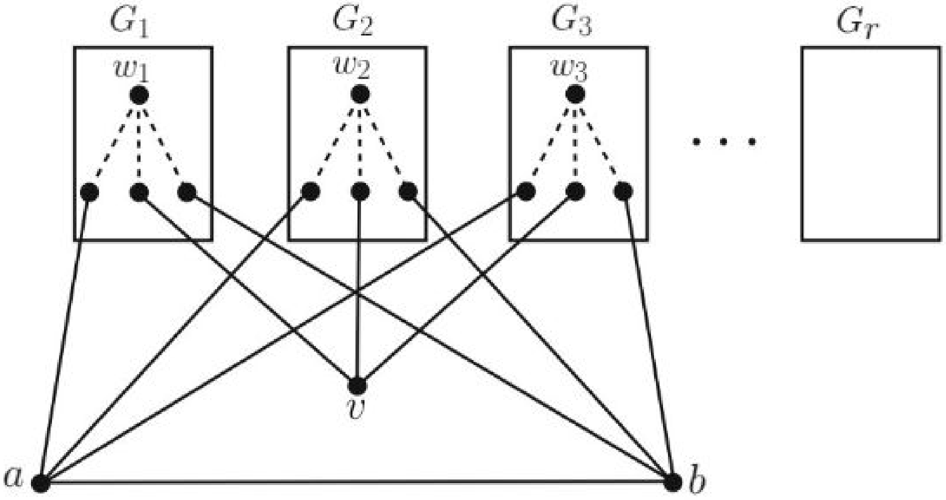


Fig. 8.19 Homeomorph for case 1 of proof of Theorem 8.7.4

Now, the edges of $H \circ e$ incident to c arise out of edges of H incident to a or b. There arise three possibilities with reference to the positions of the edges of $H \circ e$ incident to c relative to the cycle C.

(i) Suppose the edges incident to c occur so that the edges incident to a and the edges incident to b in H are consecutive around c in a plane embedding of $H \circ e$, as shown in Fig. 8.20a. Since H is a minimal nonplanar graph, the paths from c to C can only be single edges. Then the plane representation of $H \circ e$ gives a plane representation of H, as in Fig. 8.20b, a contradiction. So this possibility cannot arise.

(ii) Suppose there are three edges of $H \circ e$ incident with c, with each edge corresponding to a pair of edges of H, one incident to a and the other to b, as in Fig. 8.21a. Then H contains a subgraph contractible to K_5, as shown in Fig. 8.21b.

We are now left with only one more possibility.

(iii) There are four edges of $H \circ e$ incident to c, and they arise alternately out of edges incident to a and b in H, as in Fig. 8.22a. Then there arises in H

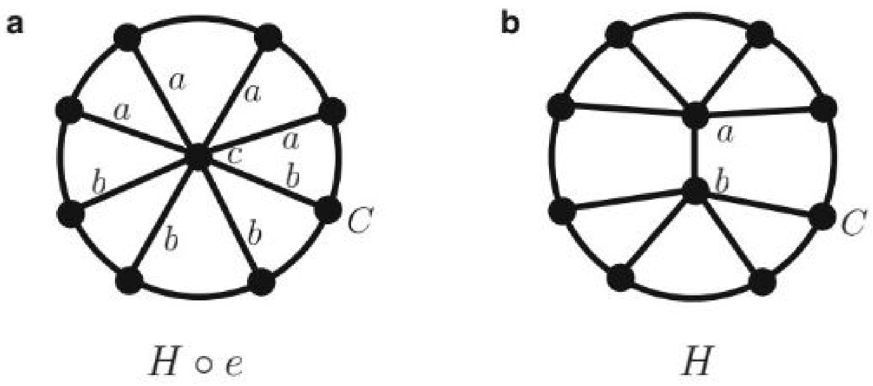


Fig. 8.20 First configuration for case 2 of proof of Theorem 8.7.4. Edges incident to a and b are marked a and b, respectively

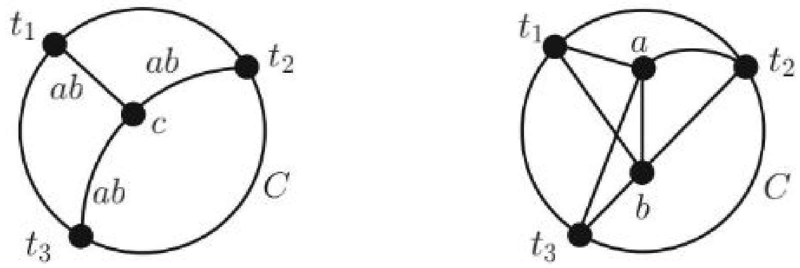


Fig. 8.21 Second configuration for case 2 of proof of Theorem 8.7.4. Edges incident to both a and b are marked ab

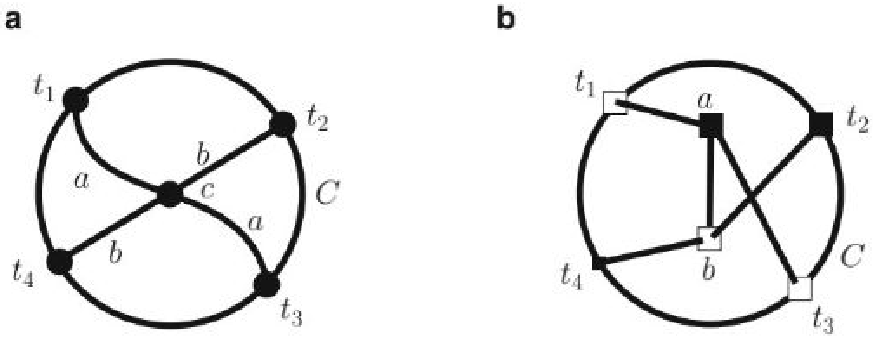


Fig. 8.22 Third configuration for case 2 of proof of Theorem 8.7.4

a homeomorph of $K_{3,3}$, as shown in Fig. 8.22b. The sets $X = \{a, t_2, t_4\}$ and $Y = \{b, t_1, t_3\}$ are the sets of the bipartition of this homeomorph of $K_{3,3}$. □

We now proceed to prove Theorem 8.7.5.

Proof of theorem 8.7.5. The "sufficiency" part of the proof is trivial. If G contains a homeomorph of either K_5 or $K_{3,3}$, G is certainly nonplanar, since a homeomorph of a planar graph is planar.

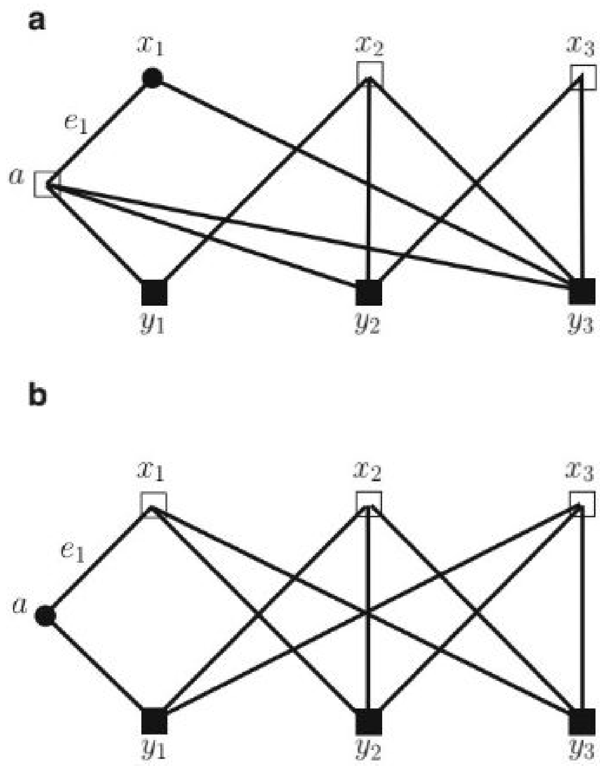


Fig. 8.23 Graphs for proof of Theorem 8.7.5

Assume that G is connected and nonplanar. Remove edges from G one after another until we get an edge-minimal connected nonplanar subgraph G_0 of G; that is, G_0 is nonplanar and for any edge e of G, $G_0 - e$ is planar. Now contract the edges in G_0 incident with vertices of degree at most 2 in some order. Let us denote the resulting graph by G_0'. Then G_0' is nonplanar, whereas $G_0' - e$ is planar for any edge e of G_0', and the minimum degree of G_0' is at least 3. We now have to show that G_0' contains a homeomorph of K_5 or $K_{3,3}$.

By Theorem 8.7.4, G_0' is subcontractible to K_5 or $K_{3,3}$. This means that G_0' contains a subgraph H that is contractible to K_5 or $K_{3,3}$. As $G_0' - e$ is planar for any edge e of G_0', $G_0' = H$. Thus, G_0' itself is contractible to K_5 or $K_{3,3}$. If G_0' is either K_5 or $K_{3,3}$, we are done. Assume now that G_0' is neither K_5 nor $K_{3,3}$. Let $e_1, e_2, \ldots, e_r$ be the edges of G_0', when contracted in order, that result in a K_5 or $K_{3,3}$.

First, let us assume that $r = 1$, so that $G_0' \circ e_1$ is either K_5 or $K_{3,3}$. Suppose that $G_0' \circ e_1 = K_{3,3}$ with $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ as the partite sets of vertices. Suppose that x_1 is the vertex obtained by identifying the ends of e_1. We may then take $e_1 = x_1 a$ (by abuse of notation), where a is a vertex distinct from the x_i's and y_j's (Fig. 8.23a). If a is adjacent to all of y_1, y_2 and y_3, then $\{a, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ form a bipartition of a $K_{3,3}$ in G_0'. If a is adjacent to only one or two of $\{y_1, y_2, y_3\}$ (Fig. 8.23b), then again G_0' contains a homeomorph of $K_{3,3}$.

Next, let us assume that $G_0' \circ e_1 = K_5$ with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$. Suppose that v_1 is the vertex obtained by identifying the ends of e_1. As before, we may take $e_1 = v_1 a$, where $a \notin \{v_1, v_2, v_3, v_4, v_5\}$. If a is adjacent to

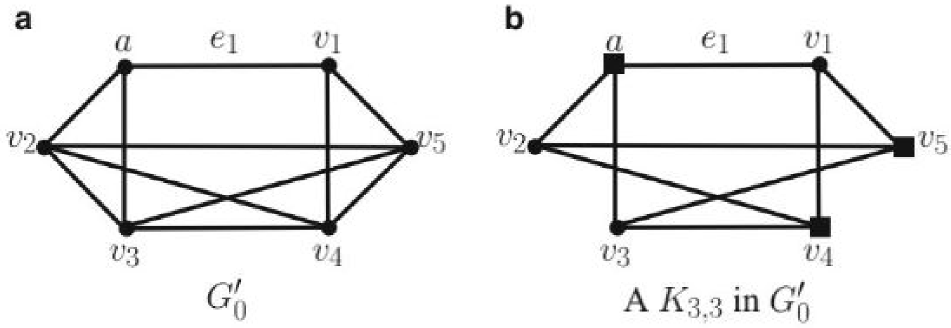
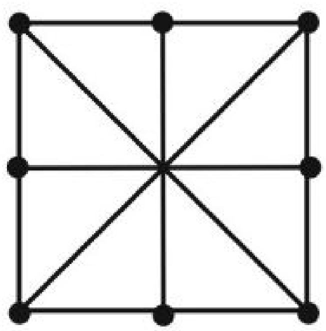


Fig. 8.24 Graphs for proof of Theorem 8.7.5

all of $\{v_2, v_3, v_4, v_5\}$, then $G_0' - v_1$ is a K_5, contradiction to the fact that any proper subgraph of G_0' is planar. If a is adjacent to only three of $\{v_2, v_3, v_4, v_5\}$, say v_2, v_3, and v_4, then the edge-induced subgraph of G_0' induced by the edges $av_1, av_2, av_3, av_4, v_1v_5, v_2v_3, v_2v_4, v_2v_5, v_3v_4, v_3v_5$, and v_4v_5 is a homeomorph of K_5. In this case, G_0' also contains a homeomorph of $K_{3,3}$. Since $d_{G_0'}(v_1) \geq 3$, v_1 is adjacent to at least one of v_2, v_3, and v_4, say v_2. Then the edge-induced subgraph of G_0' induced by the edges in $\{av_1, av_3, av_4, v_1v_2, v_2v_3, v_2v_4, v_1v_5, v_3v_5, v_4v_5\}$ is a $K_{3,3}$, with $\{a, v_4, v_5\}$ and $\{v_1, v_2, v_3\}$ forming the bipartition. We now consider the case when a is adjacent to only two of v_2, v_3, v_4 and v_5, say v_2 and v_3. Then, necessarily, v_1 is adjacent to v_4 and v_5 (since on contraction of the edge v_1a, v_1 is adjacent to v_2, v_3, v_4, and v_5). In this case G_0' also contains a $K_{3,3}$ (see Fig. 8.24b). Finally, the case when a is adjacent to at most one of v_2, v_3, v_4, and v_5 cannot arise since the degree of a is at least 3 in G_0'. Thus, in any case, we have proved that when $r = 1$, G_0' contains a homeomorph of $K_{3,3}$. The result can now easily be seen to be true by induction on r. Indeed, if $H_2 = H_1 \circ e$ and H_2 contains a homeomorph of $K_{3,3}$, then H_1 contains a homeomorph of $K_{3,3}$. □

The nonplanarity of the Petersen graph (Fig. 8.25a) can be established by showing that it is contractible to K_5 (see Fig. 8.25b) or by showing that it contains a homeomorph of $K_{3,3}$ (see Fig. 8.25c).

Exercise 7.1. Prove that the following graph is nonplanar.



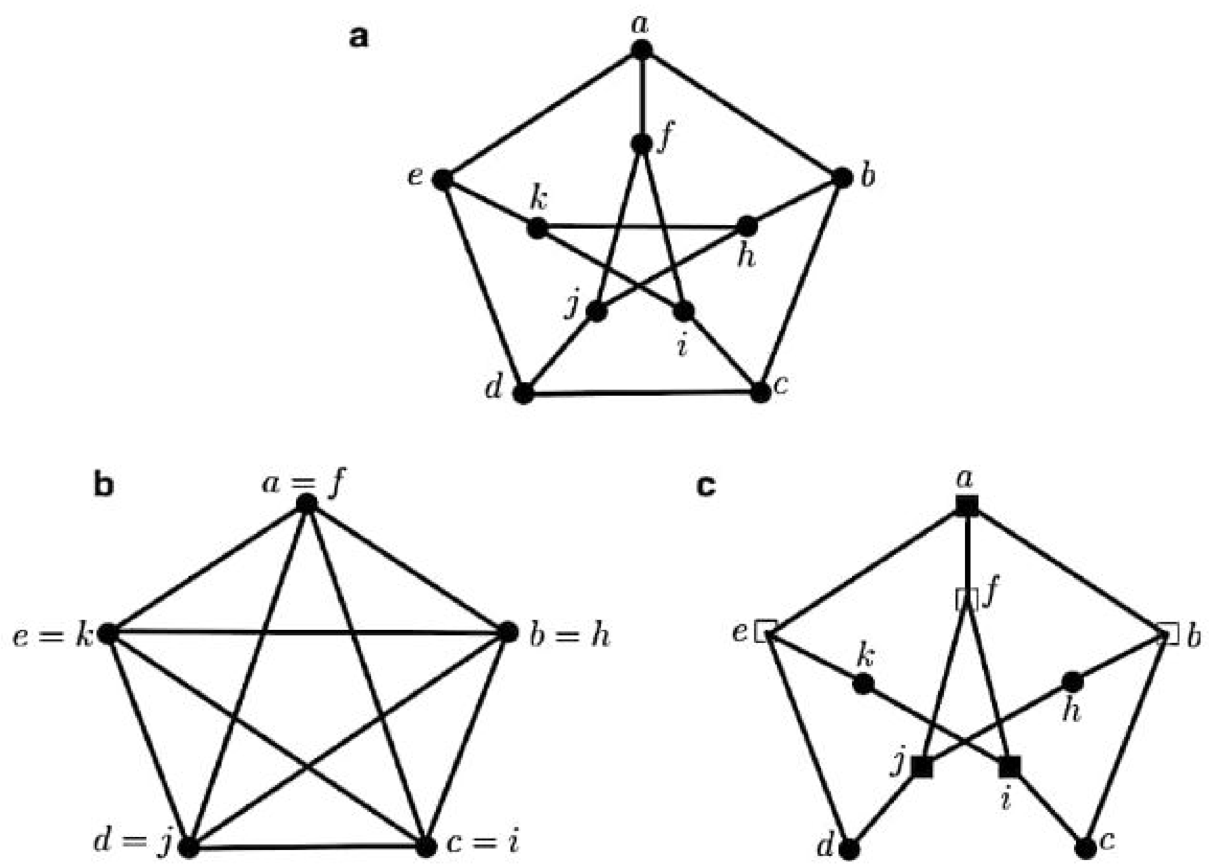


Fig. 8.25 Nonplanarity of the Petersen graph. **(a)** The Petersen graph P, **(b)** contraction of P to K_5, **(c)** A subdivision of $K_{3,3}$ in P